1. (a) Let T be time interval between 1 and 1:30

Let r.v. A is time taken by Alice, B is time taken by Bob and C be time taken by Carl to arrive.

$A \sim (0, T)$ $\quad B \sim (0, T)$ $\quad C \sim (0, T)$

We want to find

$P(C < A < T, C < B < T)$

so integrating joint PDF over given region $= \frac{1}{T^3}\int_0^T \int_c^T \int_c^T da\, db\, dc$

$= \frac{1}{T^3}\int_0^T \int_c^T (T-c)\, db\, dc = \frac{1}{T^3}\int_0^T (T-c)^2\, dc$

$= -\frac{1}{T^3}\left[\frac{(T-c)^3}{3}\right]_0^T = \frac{1}{T^3} \times \frac{T^3}{3} = \frac{1}{3}$

(b) $C = T/3$ and $(C+10) < B < T$

and $(C+10) < A < T$

$10 = T/3 \Rightarrow$ $\quad 2T/3 < B < T$ and $2T/3 < A < T$

We want to find $P\left(\frac{2T}{3} < B < T, \frac{2T}{3} < A < T \mid C = T/3\right)$

$= P\left(\frac{2T}{3} < B < T, \frac{2T}{3} < A < T\right)$ (as they are indedendant)

Integrating over given region

$\frac{1}{T^2}\int_{2T/3}^T \int_{2T/3}^T da\, db = \frac{1}{T^2} \times \frac{T}{3} \times \frac{T}{3} = \frac{1}{9}$

(C) Atleast A or B arrives in the interval $[2T/3, T]$ and other one arrives in $[T/3, T]$

From LOTP and symmetry we want to find

$$2P(2T/3 < A < T, T/3 < B < T \mid C = T/3)$$

$$= 2P(2T/3 < A < T, T/3 < B < T)$$

$$= \frac{2}{T^2}\int_{T/3}^{T}\int_{2T/3}^{T} da\, db = \frac{2}{T^2} \times \frac{2T}{3} \times \frac{T}{3} = \frac{4}{9}$$

2.(a) Support of X is $[0, L]$

Given $X = x$, support for Y is $[0, x]$

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)\, f_x(x)$$

$$= \frac{1}{x} \times \frac{1}{L} = \frac{1}{Lx}$$

(b) To get marginal PDF of X we integrate joint PDF over support of Y

$$= \int_0^x \frac{1}{Lx} = \frac{1}{L} \Rightarrow X \sim \text{Unif}(0, L)$$

(c)
$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \frac{\frac{1}{Lx}}{\frac{1}{L}} = \frac{1}{x}$$

$\Rightarrow$ $(Y|X) \sim \text{Unif}(0, x)$

(d) For $Y = y$, X will be in $[y, L]$

$$\Rightarrow f_Y(y) = \int_y^L f(x,y)\, dx$$

$$= \frac{1}{L}\int_y^L \frac{1}{x}\, dx$$

$$= \frac{1}{L} \log_e\left(\frac{L}{y}\right)$$

(e)
$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{\frac{1}{Lx}}{\frac{1}{L}\log_e\left(\frac{L}{y}\right)}$$

$$= \frac{1}{x \log_e\left(\frac{L}{y}\right)}$$

3.(a) X, Y, Z have Multinomial distribution.

$$P(X=i, Y=j, Z=k) = \frac{5!}{i!\,j!\,k!}\left(\frac{1}{13}\right)^i\left(\frac{1}{13}\right)^j\left(\frac{11}{13}\right)^k$$

$$= \frac{5!}{i!\,j!\,k!}\frac{11^k}{(13)^{i+j+k}} = \frac{5!\;11^k}{i!\,j!\,k!\;13^5}$$

(b) To get $P(X=i, Y=j)$ we marginalize over all possible $Z=k$. k varies from 0 to 5.

$$P(X=i, Y=j) = \sum_{k=0}^{5} \frac{5!\;11^k}{i!\,j!\,k!\;13^5}$$

$$= \frac{5!}{i!\,j!\,13^5}\sum_{k=0}^{5}\frac{11^k}{k!} \approx \frac{5!}{i!\,j!\,13^5}e^{11}$$

(c) i kings can be chosen in $\binom{4}{i}$ ways

j queen can be chosen in $\binom{4}{j}$ ways

k other cards can be chosen in $\binom{44}{k}$ ways

5 cards can be chosen in $\binom{52}{5}$ ways

$$\Rightarrow P(X=i, Y=j, Z=k) = \frac{\binom{4}{i}\binom{4}{j}\binom{44}{k}}{\binom{52}{5}}$$

(4) Let x be r.v. for total number of matches and I_j be indicator r.v. for jth person match.

$\Rightarrow$ $X = I_1 + I_2 \cdots I_n = \sum_{j=1}^{n} I_j$

$\Rightarrow$ $Var(X) = Var\left(\sum_{j=1}^{n} I_j\right)$

$= Var(I_1) + Var(I_2) \cdots Var(I_n) + 2 \sum_{i<j} Cov(I_i, I_j)$

from symmetry,

$Var(I_1) = Var(I_2) \cdots = Var(I_n)$

and, $Cov(I_1, I_2) = Cov(I_i, I_j)$ for all i, j and $i \neq j$

$\Rightarrow$ $Var(X) = n\, Var(I_1) + 2 \times \binom{n}{2} Cov(I_1, I_2)$

$Var(I_1) = E(I_1^2) - (EI_1)^2$

$= E(I_1) - (EI_1)^2 = \frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2}$

$Cov(I_1, I_2) = E(I_1 I_2) - E(I_1)E(I_2)$

$= \frac{1}{n} \times \frac{1}{n-1} - \frac{1}{n^2} = \frac{n-n+1}{n^2(n-1)} = \frac{1}{n^2(n-1)}$

$\Rightarrow$ $Var(X) = \frac{n \times (n-1)}{n^2} + 2 \times \frac{n(n-1)}{2} \times \frac{1}{n^2(n-1)}$

$= \frac{n-1}{n} + \frac{1}{n} = 1$

$\Rightarrow$ Standard deviation of $X = \sqrt{Var(X)} = 1$

(5) Let I_1 be indicato r.v. that ship of length 2 is missed, I_2 for ship of length 3 missed and I_3 for ship of length 4 missed.

X is r.v. for # ships missed.

$\Rightarrow \quad X = I_1 + I_2 + I_3$

$\Rightarrow \quad EX = E(I_1 + I_2 + I_3) = EI_1 + EI_2 + EI_3$

EI_1 = Probability that ship of length 2 is missed.

We can chose 5 grids from 98 remaining grids where ship of length 2 does not reside in order to miss the ship.

$$\Rightarrow EI_1 = \frac{\binom{98}{5}}{\binom{100}{5}} = \frac{98!}{93!\cancel{5!}} \times \frac{95!\cancel{5!}}{100!}$$

$$= \frac{94 \times 95}{99 \times 100} \simeq 0.902$$

Simillarly, $$EI_2 = \frac{\binom{97}{5}}{\binom{100}{5}} = \frac{97!}{92!} \times \frac{95!}{100!} \simeq 0.856$$

$$EI_3 = \frac{\binom{96}{5}}{\binom{100}{5}} = \frac{96!}{91!} \times \frac{95!}{100!} \simeq 0.812$$

$$\Rightarrow EX = EI_1 + EI_2 + EI_3 = 0.902 + 0.856 + 0.812 \simeq 2.57$$

Let Y is r.v. for # ships hit

$\Rightarrow \quad X + Y = 3 \quad \Rightarrow \quad EY = E(3) - EX = 3 - 2.57 \simeq 0.43$

Now, $\text{Var}(X) = \text{Var}(I_1 + I_2 + I_3)$

$$= \text{Var}(I_1) + \text{Var}(I_2) + \text{Var}(I_3) + 2\big(\text{Cov}(I_1, I_2) + \text{Cov}(I_2, I_3) + \text{Cov}(I_1, I_3)\big)$$

$$\text{Var}(I_1) = EI_1^2 - (EI_1)^2 = EI_1 - (EI_1)^2 = (0.902) - (0.902)^2 \approx 0.089$$

Simillarly, $\text{Var}(I_2) = 0.856 - (0.856)^2 \approx 0.123$

$$\text{Var}(I_3) = 0.812 - (0.812)^2 \approx 0.152$$

$$\text{Cov}(I_1, I_2) = E(I_1 I_2) - EI_1 EI_2$$

$E(I_1 I_2) =$ Probability that both ships of length 2 and 3 are missed $= \dfrac{\binom{95}{5}}{\binom{100}{5}}$ (chose 5 grids from remaining 95 grids)

$$= \frac{95!}{90!} \times \frac{95!}{100!} \approx 0.77$$

Simillarly, $E(I_1 I_3) = \dfrac{\binom{94}{5}}{\binom{100}{5}} \approx 0.73$

$$E(I_2 I_3) = \frac{\binom{93}{5}}{\binom{100}{5}} \approx 0.69$$

$\Rightarrow$ $\text{Cov}(I_1, I_2) = 0.77 - 0.902 \times 0.856 = -0.0021$

$\text{Cov}(I_1, I_3) = 0.73 - 0.902 \times 0.812 = -0.0024$

$\text{Cov}(I_2, I_3) = 0.69 - 0.856 \times 0.812 = -0.005$

$\Rightarrow$ $\text{Var}(X) = 0.089 + 0.123 + 0.152 - 2(0.0021 + 0.0024 + 0.005)$

$$\approx 0.345$$

$\Rightarrow$ $\text{Var}(Y) = \text{Var}(3 - X) = \text{Var}(X) = 0.345$

(6) Probability that a person makes purchase $= P = \frac{1}{2} \times \frac{1}{10} + \frac{1}{3} \times \frac{9}{10} = 0.35$

$\Rightarrow$ Probability that a person does not make purchase $= q = 1 - 0.35 = 0.65$

X = # People make purchase

Y = # People who does not make purchase

$\Rightarrow (X+Y) \sim Pois(100)$

As X and Y are independant and from chicken-egg problem,

$$P(X=x \mid Y=42) = P(X=x)$$

and $X \sim Pois(100 \times 0.35) \sim Pois(35)$

Hence, $P(X=x \mid Y=42) = P(X=x) = \dfrac{35^{x} e^{-35x}}{x!}$

7.(a) Let X be r.v. for # choices of 4 courses

From LOTUS, $EX = \sum_{k=4}^{\infty} \binom{k}{4} P(X=k) = \sum_{k=4}^{\infty} \frac{\cancel{k!}}{(k-4)!\,4!} \times \frac{e^{-\lambda}\lambda^k}{\cancel{k!}}$

$$= \frac{e^{-\lambda}}{4!} \sum_{k=4}^{\infty} \frac{\lambda^k}{k-4!} = \frac{e^{-\lambda}\lambda^4}{4!} \sum_{k=4}^{\infty} \frac{\lambda^{(k-4)}}{(k-4)!}$$

$$= \frac{e^{-\lambda}\lambda^4}{4!} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$= \frac{e^{-\lambda}\lambda^4 e^{\lambda}}{4!} = \frac{\lambda^4}{4!} = \frac{\lambda^4}{24}$$

~~(b) $P(X_{am}=a, X_{pm}=b) = \sum_n P(X_{am}=a, X_{pm}=b)\, P(N=$~~

(b) $P(X_{am}=a, X_{pm}=b) = \sum_n P(X_{am}=a, X_{pm}=b \mid N=n)\, P(N=n)$

(From LOTP)

where N = # of courses

$\Rightarrow X_{am} + X_{pm} = N$

Hence, $P(X_{am}=a, P_{pm}=b \mid N=n) = 0 \quad \forall n \neq a+b$

So we are left with

$$P(X_{am}=a, X_{pm}=b \mid N=a+b)\, P(N=a+b)$$

$= P(X_{am}=a \mid N=a+b)\, P(N=a+b)$ (If $X_{am}=a$ and given $N=a+b$, $X_{pm}=b$ is implied)

Now for fixed N, $X_{am} \sim \text{Bin}(N=n, 1/2)$

i.e. $(X_{am} \mid N=n) \sim \text{Bin}(n, 1/2)$

$$P(N=a+b) = \frac{\lambda^{a+b} e^{-\lambda}}{(a+b)!}$$

$$\Rightarrow P(X_{am}=a, X_{pm}=b) = \binom{a+b}{a}\left(\frac{1}{2}\right)^{a+b} \lambda^{a+b} \frac{e^{-\lambda}}{(a+b)!}$$

$$= \frac{1}{a!\,b!}\left(\frac{1}{2}\right)^a\left(\frac{1}{2}\right)^b \lambda^a \lambda^b e^{-\frac{\lambda}{2}} e^{-\frac{\lambda}{2}}$$

$$= \frac{\left(\frac{\lambda}{2}\right)^a e^{-\frac{\lambda}{2}}}{a!} \times \frac{\left(\frac{\lambda}{2}\right)^b e^{-\frac{\lambda}{2}}}{b!}$$

$$= \frac{\left(\frac{\lambda}{2}\right)^{a+b} e^{-\lambda}}{a!\,b!}$$

(c) In part (b) we observe that

$X_{am} \sim \text{Pois}\left(\frac{\lambda}{2}\right)$ (Also analogous to
$X_{pm} \sim \text{Pois}\left(\frac{\lambda}{2}\right)$ Chicken-egg problem)

And, $X_1 + X_2 = X_{am}$

$X_3 + X_4 = X_{pm}$

since X_1, X_2, X_3, X_4 are independant r.v.s

$\Rightarrow X_1 \sim \text{Pois}\left(\frac{\lambda}{4}\right)$, $X_2 \sim \text{Pois}\left(\frac{\lambda}{4}\right)$, $X_3 \sim \text{Pois}\left(\frac{\lambda}{4}\right)$

$X_4 \sim \text{Pois}\left(\frac{\lambda}{4}\right)$

$$P(X_1 = a, X_2 = b, X_3 = c, X_4 = d) = \frac{\left(\frac{\lambda}{4}\right)^a \times e^{-\frac{\lambda}{4}}}{a!} \times \frac{\left(\frac{\lambda}{4}\right)^b e^{-\frac{\lambda}{4}}}{b!} \times \frac{\left(\frac{\lambda}{4}\right)^c e^{-\frac{\lambda}{4}}}{c!} \times \frac{\left(\frac{\lambda}{4}\right)^d e^{-\frac{\lambda}{4}}}{d!}$$

$$= \frac{\left(\frac{\lambda}{4}\right)^{a+b+c+d} e^{-\lambda}}{a!\,b!\,c!\,d!}$$

Expected # of non-conflicting courses

$= E(X_1 + X_2 + X_3 + X_4)$

$= EX_1 + EX_2 + EX_3 + EX_4 = 4 \times \frac{\lambda}{4} = \lambda$

$$\frac{EX(a)}{EX(c)} = \frac{\lambda^4}{24 \times \lambda} = \frac{\lambda^3}{24}$$

8. (a) From chicken-egg problem,

$X = Y + Z$

$Y \sim \text{Pois}(p\lambda)$

$Z \sim \text{Pois}((1-p)\lambda)$

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$$

$$= \frac{\text{Cov}(Y+Z, Y)}{\sqrt{\lambda \lambda p}}$$

$$= \frac{\text{Cov}(Y, Y) + \text{Cov}(Y, Z)}{\lambda\sqrt{p}}$$

$$= \frac{\lambda p + 0}{\lambda \sqrt{p}} = \sqrt{p}$$

($\text{Cov}(Y, Z) = 0$ and Y and Z are indepedant and $\text{Cov}(Y,Y) = \text{Var}(Y) = \lambda p$)

(b) Let S be r.v. for # non-statistics major

$S \sim \text{Bin}(n, 1-r)$

$$P(Y=y, Z=z, S=s) = P(Y=y, Z=z, S=s \mid S=s)\, P(S=s)$$

$$= P(Y=y \mid S=s)\, P(S=s)$$

(Because given $S=s$, $S=s$ and $Z = n-s-y$ is implied)

Now, $(Y=y \mid S=s) \sim \text{Bin}(n-s, p)$

$$\Rightarrow P(Y=y, Z=z, S=s) = \binom{n-s}{y} p^y (1-p)^{n-y-s} \binom{n}{s} (1-r)^s r^{n-s}$$

$$= \frac{n!}{s!\,y!\,(n-s-y)!} \cdot p^y (1-p)^{n-y-s} (1-r)^s r^{n-s}$$

(c)

$$= \frac{n!}{y!\,z!\,s!} p^y (1-p)^z (1-r)^s r^{y+z}$$

$Y \sim \text{Bin}(n, pr) \Rightarrow P(Y=y) = \binom{n}{y} (pr)^y (1-pr)^{n-y}$

$Z \sim \text{Bin}(n, (1-p)r) \Rightarrow P(Z=z) = \binom{n}{z} ((1-p)r)^z (1-r+pr)^{n-z}$

$S \sim \text{Bin}(n, 1-r) \Rightarrow P(S=s) = \binom{n}{s} (1-r)^s r^{n-s}$

(c) $X \sim Bin(n, r)$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

$$= \cancel{Cov(Z+X, X)}$$

$$= \frac{Cov(Z+Y, Y)}{\sqrt{nr(1-r) \times n\,pr(1-pr)}}$$

$$= \frac{Cov(Y,Y) + Cov(Z,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

($Cov(Z,Y) = 0$ because Z and Y are independent)

$$= \frac{Var\,Y}{\sqrt{Var(X)\,Var(Y)}}$$

$$= \sqrt{\frac{Var(Y)}{Var(X)}} = \sqrt{\frac{pr(1-pr)}{p(1-p)}} = \sqrt{\frac{r(1-pr)}{1-p}}$$

(9) $R^2 = X^2 + Y^2$, $X = \sqrt{R^2}\cos\theta$

$Y = \sqrt{R^2}\sin\theta$

$\cancel{\theta = \cos\left(\frac{X}{\sqrt{X^2+Y^2}}\right)}$

$$\theta = \cos^{-1}\left(\frac{X}{\sqrt{X^2+Y^2}}\right) = \sin^{-1}\left(\frac{Y}{\sqrt{X^2+Y^2}}\right)$$

$$f_{R^2,\theta}(r^2,\theta) = \frac{\partial(x,y)}{\partial(r^2,\theta)} f_{X,Y}(x,y)$$

$$\frac{\partial(x,y)}{\partial(r^2,\theta)} = \begin{pmatrix} \frac{\cos\theta}{2R} & -R\sin\theta \\ \frac{\sin\theta}{2R} & R\cos\theta \end{pmatrix}$$

$$\left|\frac{\partial(x,y)}{\partial(r^2,\theta)}\right| = \left|\frac{\cos^2\theta}{2} + \frac{\sin^2\theta}{2}\right| = 1/2$$

$$\Rightarrow f_{R^2,\theta}(r^2,\theta) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \times \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \times \frac{1}{2}$$

$$= \frac{1}{4\pi} e^{-\frac{(x^2+y^2)}{2}}$$

$$= \frac{1}{4\pi} e^{-r^2/2}$$

As θ does not appear in joint PDF, it must be uniform and $\theta \in [0, 2\pi] \Rightarrow$ $f_\theta(\theta) = \frac{1}{2\pi}$

$$\Rightarrow f_{R^2}(r^2) = \frac{1}{2} e^{-r^2/2}$$

~~As joint PDF is product of marginal PDF~~

$R^2 \sim \text{Expo}(1/2)$

$\theta \sim \text{Unif}(0, 2\pi)$

10.(a) Let $A = \frac{T_1}{T_1+T_2}$ and $B = \frac{T_2}{T_1+T_2}$

$A + B = 1 \Rightarrow B = 1 - A$

$\Rightarrow \frac{A}{B} = \frac{A}{1-A}$

From Post office - Bank Problem,

A and (T_1+T_2) are independant r.vs

$\Rightarrow \frac{A}{1-A}$ and (T_1+T_2) are also independant

(b) $P(T_1 < T_2) = P(T_1 - T_2 < 0) \Rightarrow T_1 \in [0, T_2],\ T_2 \in [0, \infty]$

$$= \int_0^\infty \int_0^{t_2} (\lambda_1 e^{-\lambda_1 t_1} dt_1)\, \lambda_2 e^{-\lambda_2 t_2} dt_2$$

$$= \lambda_1 \lambda_2 \int_0^\infty e^{-\lambda_2 t_2} dt_2 \int_0^{t_2} e^{-\lambda_1 t_1} dt_1$$

$$= \lambda_2 \int_0^\infty e^{-\lambda_2 t_2} (1 - e^{-\lambda_1 t_2})\, dt_2$$

$$= 1 - \lambda_2 \int_0^\infty e^{-(\lambda_1+\lambda_2)t_2} dt$$

$$= 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

if $(\lambda_1 = \lambda_2) \Rightarrow P(T_1 < T_2) = \frac{1}{2}$ which make sense from symmetry

(c) We want to find,

$EX = E(\min(T_1, T_2)) + E(T_1) P(T_1 < T_2) + E(T_2) P(T_2 < T_1)$

We know that $\min(T_1, T_2) \sim \mathrm{Expo}(\lambda_1 + \lambda_2)$

$$\Rightarrow EX = \frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1} \times \frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_2} \times \frac{\lambda_2}{\lambda_2+\lambda_1}$$

$$= \frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1+\lambda_2} = \frac{3}{\lambda_1+\lambda_2}$$

(ii) From the question,

$$P(p > r \mid X = n) = c$$

where X be r.v. for # successful trials

$$\Rightarrow P(p > r \mid X = n) = \frac{P(X = n \mid p > r)\, P(p > r)}{P(X = n)}$$

$P(X = k \mid p) \sim Bin(n, p)$

$\Rightarrow P(X = n \mid p) = p^n$

$P(p > r) = (1 - r)$

$$P(X = n) = \int_0^1 p^n \, dp = \left(\frac{p^{n+1}}{n+1}\right)_0^1 = \frac{1}{n+1}$$

$$\Rightarrow P(p > r \mid X = n) = \frac{p^n (1 - r)}{1/n+1}$$

$$= p^n (n+1)(1 - r)$$

$$\Rightarrow c = p^n (n+1)(1 - r)$$

12. (a) There are $(n+1)$ possible spots where Y_{new} can go, and to be in $[Y_{(j)}, Y_{(k)}]$ there are $(k-j)$ possibilities.

Hence, as all positions are equally likely

$$\Rightarrow P(Y_{new} \in [Y_{(j)}, Y_{(k)}]) = \frac{k-j}{n+1}$$

(b) We want a interval of $[Y_{(j)}, Y_{(j+95)}]$

And since there are $n = 99$ r.v.s

We want 3 r.v.s out of this interval.