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HW1 STAT-110

1\. (a) First digit can be filled by 8 different digits (2 to 9) and rest of the digits can be filled in 10 (0 to 9) ways.

Hence total number of ways = 8×10^6

(b) Total number possible phone numbers which start with 911 = 10^4 — (i)

Total number of possible phone numbers which do not start with 911 = total possible numbers − (i)

$$= 8 \times 10^6 - 10^4 = 799 \times 10^4$$

(2) There are 10 pairs.

□ □ □ .... □

Each block above represents a pair. For a given pair of players in each block, blocks can be arranged in 10! ways. And all of them are same set of two players, hence same matches. Hence for total of 20! ways of ordering players, each set of matches repeated 10! times, and since each pair can be ordered in 2 ways amongst themselves (because for both the orders each player in a pair gets white and black pieces once).

Hence total number of ways $= \frac{20!}{10!}$

3.(a)

W W W D D L L

Above configuration can be arranged in $\frac{7!}{3!2!2!} = 35$ ways.

(b) A - 4 points and B - 3 points can be achieved in following ways.

$4W\ 3L - \frac{7!}{4!3!} = 35$

$3W\ 2D\ 2L - \frac{7!}{3!2!2!} = 210$

$2W\ 4D\ 1L - \frac{7!}{2!4!1!} = 105$

$1W\ 6D - \frac{7!}{1!6!} = 7$

Hence total number of ways $= 35+210+105+7$
$= 357$

~~(c) Match will last 7 games if last game is a win or a draw. Compliment of this event is if last game is a loss (let's call it A^c)~~

(c) Match will last 7 games if Last game is a win or a draw.

This can happen in following ways.

$3W\ 3L\ \underline{W} = \frac{6!}{3!\,3!} = 20$

$2W\ 2D\ 2L\ \underline{W} = \frac{6!}{2!\,2!\,2!} = 90$

$3W\ 1D\ 2L\ \underline{D} = \frac{6!}{3!\,1!\,2!} = 60$

$1W\ 4D\ 1L\ \underline{W} = \frac{6!}{4!\,1!\,1!} = 30$

$2W\ 3D\ 1L\ \underline{D} = \frac{6!}{2!\,3!\,1!} = 60$

$1W\ 6D = \frac{7!}{6!\,1!} = 7$

Hence total number of ways $= 20+90+60+30+60+7$

$= 267$

(4) Let's say we have two groups A and B. Each group A and B has n members and only members of A are eligible to to become president. We want to choose n people from A and B and one of them is chosen as president.

Left hand side is that there are n ways to select a president from A and then we select $(n-1)$ people from remaining $(2n-1)$ from A and B.

Right hand side is we choose k from group A and $(n-k)$ from B and hence there are k ways to select president, $\forall\ 1 \le k \le n$.

Sum over $1 \le k \le n$ will give us the answer.

$$= \sum_{k=1}^{n} k\binom{n}{k}\binom{n}{n-k} = \sum_{k=}^{n} k\binom{n}{k}^2 \quad \left(\binom{n}{n-k} = \binom{n}{k}\right)$$

(5) Number of ways in which keys of elevator can be pressed = Number of ways keys are pressed if all three are going to different floors + two of them going to same floor and one to a different floor + all three going to same floor

$$= \binom{9}{3} + \binom{9}{2} + \binom{9}{1} = 84 + 36 + 9 = 129$$

~~These are~~

Following are the 7 possibilities of three people going to consecutive floors

$\{(2,3,4), (3,4,5), (4,5,6), (5,6,7), (6,7,8),$

$(7,8,9), (8,9,10)\}$

Hence probability of 3 consecutive floors

$$= \frac{7}{129}$$

(6) Let X_j is probability that none was born in season j $\forall\ 1 \le j \le 4$

Hence $P(X_1 \cup X_2 \cup X_3 \cup X_4)$ = Probababibility that there is at least one season in which there is no birth day.

From symmetry,

$$P(X_j) = \left(\frac{3}{4}\right)^7 \quad \forall\ 1 \le j \le 4$$

$$P(X_i \cap X_j) = \left(\frac{2}{4}\right)^7 \quad \forall\ 1 \le i, j \le 4 \text{ and } j \ne i$$

$$P(X_i \cap X_j \cap X_k) = \left(\frac{1}{4}\right)^7 \quad \forall\ 1 \le i, j, k \le 4 \text{ and } i \ne j \ne k$$

$$P(X_1 \cap X_2 \cap X_3 \cap X_4) = 0 \quad (\text{not possible})$$

From inclusion-exclusion,

$$P(X_1 \cup X_2 \cup X_3 \cup X_4) = 4 \times \left(\frac{3}{4}\right)^7 - \binom{4}{2}\left(\frac{2}{4}\right)^7 + \binom{4}{1}\left(\frac{1}{4}\right)^7$$

$$= \frac{4 \times 3^7 - 6 \times 2^7 + 4}{4^7} = \frac{499}{1024}$$

Hence Probability that all 4 seasons has at least one birthday $= 1 - P(X_1 \cup X_2 \cup X_3 \cup X_4)$

$$= 1 - \frac{499}{1024} = \frac{525}{1024} \simeq 0.51$$

7. (a) $$P(K|R) = \frac{P(K \cap R)}{P(R)}$$

Now, $P(K \cap R) = P(K)$, because if Fred knows the answer then he will get it right

$$P(R) = P(R|K)\,P(K) + P(R/K^c)\,P(K^c)$$

(From LOTP)

$P(R|K) = 1$

$P(K^c) = 1-P$

$P(R/K^c) = \frac{1}{n}$ ($\frac{1}{n}$ is the probability that his guess is correct given Fred does not answer).

So, $$P(K|R) = \frac{P}{P + \frac{(1-P)}{n}}$$

(b) $$\frac{P}{P + \frac{1-P}{n}} \geq P \Rightarrow n \geq nP + (1-P) \Rightarrow n \geq 1$$ (which is true)

Hence, $P(K|R) \geq P$

Probability that Fred gets a question right and he knows the answer is equal to probability that he knows the answer because if Fred knows the answer implies that he will get it right. A P is denormalized by probability that Fred gets it right.
Hence intuitively, $P(K|R) \geq P$

$P(K|R) = P$ when $n = 1$

(8) Let, D = ~~A~~ coin is double headed.

H = Heads all 7 times.

From Baye's theorem,

$$P(D|H) = \frac{P(H|D)\,P(D)}{P(H)}$$

$P(H|D) = 1$ (implied)

$P(D) = 0.01$

$P(H) = P(H|D)\,P(D) + P(H/D^c)\,P(D^c)$ (LOTP)

$= 0.01 + \left(\frac{1}{2}\right)^7 \times 0.99$ $\left(\begin{array}{l} P(H/D^c) = \frac{1}{2^7} \\ P(D^c) = 0.99 \end{array} \right)$

$$\Rightarrow P(D|H) = \frac{0.01}{0.01 + \frac{0.99}{2^7}} \approx 0.56$$

9. (a) ~~Probabi~~ Let, A = A coin is double headed

P(A) = Probability that given coin is the coin suspected to be double headed × Probability that the suspected coin is double headed

$$= \frac{1}{100} \times \frac{1}{2} = 0.005$$

Similar to ~~8(i)~~ problem 8,
Let, B = Heads all 7 times)

$$P(A|B) = \frac{0.005}{0.005 + \frac{0.995}{2^7}}$$

$$\approx 0.39$$

10. (a) $P(A_2 / A_1) = 0.8$

$P(A_2^c / A_1) = 0.2$

$P(A_3 | A_1) = P(A_3 | A_2, A_1) P(A_2/A_1) + P(A_3 | A_2^c, A_1) P(A_2^c/A_1)$

(From LOTP)

Now, $P(A_3 | A_2, A_1) = P(A_3 | A_2)$ and $P(A_3 | A_2^c, A_1) = P(A_3 | A_2^c)$

(because A_3 and A_1 are conditionally indedependant)

$\Rightarrow \quad P(A_3 | A_1) = 0.8 \times 0.8 + 0.2 \times 0.3$

$= 0.64 + 0.06$

$= 0.7$

(b) From LOTP,

$P(A_2) = P(A_2 | A_1) P(A_1) + P(A_2 | A_1^c) P(A_1^c)$

$= 0.8 \times 0.75 + 0.3 \times 0.25$

$= 0.675$

Again from LOTP,

$P(A_3) = P(A_3 | A_2) P(A_2) + P(A_3 | A_2^c) P(A_2^c)$

$= 0.8 \times 0.675 + 0.3 \times 0.325$

$= 0.6375$

(11) $P(A|W) = 0.6$

From Baye's theorem,

$$P(W/A) = \frac{P(A|W)\,P(W)}{P(A)}$$

$$0.7 = \frac{0.6 \times \left(P(W|A)P(A) + P(W|A^c)\,P(A^c) \right)}{P(A)}$$

let, $P(A) = x$

$$\Rightarrow \quad 0.7x = \frac{0.6 \times \left(0.7x + 0.3(1-x)\right)}{x}$$

$$\Rightarrow \quad 0.7x = 0.42x + 0.18 - 0.18x$$

$$\Rightarrow \quad 0.46x = 0.18$$

$$\Rightarrow \quad x = \frac{18}{46} = \frac{9}{23}$$

$$P(A) = \frac{9}{23}$$

12. (a) For Company B,

Overall success rate = $P(T|D)P(D) + P(T^c/D^c)P(D^c)$

where, T = test is positive

D = person has disease

$P(T|D) = 0$ because it always says negative.

$\Rightarrow$ Overall success rate = $1 \times 0.99 = 0.99 > 0.95$

Hence claim is true.

(b) Company's B results are not reliable when person has disease, which also means that overall success rate is not the only index of reliability. Test should give reliable results in all cases, i.e. whether a person has disease or not.

(c) From the question,

$$0.99 = P(T|D)P(D) + P(T^c/D^c)P(D^c)$$

Let, $P(T/D) = P(T^c|D^c) = x$

$$\Rightarrow \quad 0.99x + 0.01x = 0.99$$

$$\Rightarrow \quad x = 0.99$$

Now, $P(T^c/D^c) = 1 \Rightarrow P(T|D) \times 0.01 + 0.99 = 0.99$
$\Rightarrow P(T|D) = 0$

In this case $P(T|D)$ is irrelevant and their goal is always achieved.

Now, $P(T|D) = 1$

$$\Rightarrow \quad 0.01 + 0.99\, P(T^c|D^c) = 0.99$$

$$\Rightarrow \quad P(T^c|D^c) = \frac{0.98}{0.99} \simeq 0.98989$$