1.(a) PDF: $f(x) = x e^{-x^2/2}$

CDF: $F(x) = \int_0^x f(x)\,dx$

$= \int_0^x x e^{-x^2/2} = 1 - e^{-x^2/2}$

$P(1 < X < 3) = F(3) - F(1) = e^{-1/2} - e^{-9/2} \simeq 0.6$

(b) $P(X < q_j) = F(q_j)$

$\Rightarrow F(q_j) = j/4$

$\Rightarrow 1 - e^{-q_j^2/2} = j/4$

$\Rightarrow 1 - j/4 = e^{-q_j^2/2} \Rightarrow -\frac{q_j^2}{2} = \log_e(1 - j/4)$

$\Rightarrow q_j = \sqrt{2 \log_e\left(\frac{1}{1 - j/4}\right)}$

so, $q_1 \simeq 0.76 \quad (\sqrt{2 \log_e(4/3)})$

$q_2 \simeq 1.18 \quad (\sqrt{2 \log_e(2)})$

$q_3 \simeq 1.66 \quad (\sqrt{2 \log_e(4)})$

2. (a)

$$P(X \leq x \mid X > a) = \frac{P(X \leq x, X > a)}{P(X > a)}$$

$$= \frac{F(x) - F(a)}{1 - F(a)}$$

(b) Let $G(x)$ be conditional CDF and $g(x)$ be conditional PDF given $X > a$

$\Rightarrow$ $g(x) = G'(x)$

$$G(x) = \frac{F(x) - F(a)}{1 - F(a)}$$

$$\Rightarrow g(x) = G'(x) = \frac{F'(x)}{1 - F(a)} = \frac{f(x)}{1 - F(a)}$$

(c) $F(x)$ is a valid CDF $\Rightarrow$ $F(a) \leq 1$

$\Rightarrow$ $1 - F(a) \geq 0$

$f(x)$ is a valid PDF $\Rightarrow$ $f(x) \geq 0$

Hence, $\frac{f(x)}{1 - F(a)} \geq 0$

Now given $X > a$, we'll integrate conditional PDF from a to ∞

$$\Rightarrow \int_a^\infty \frac{f(x)}{1 - F(a)} dx = \frac{1}{1 - F(a)} \int_a^\infty f(x)\, dx = \frac{1}{1 - F(a)} [F(x)]_a^\infty$$

$$F(\infty) = 1 \Rightarrow \int_a^\infty \frac{f(x)}{1 - F(a)} = \frac{1 - F(a)}{1 - F(a)} = 1$$

3. (a) $E(A) = E(\pi R^2)$

$= \pi \, E(R^2)$

From LOTUS, $= \pi \int_0^1 R^2 \, dR = \pi \left[\frac{R^3}{3}\right]_0^1$

$= \frac{\pi}{3}$

$Var(A) = E(A^2) - (E(A))^2$

$E(A^2) = E(\pi^2 R^4) = \pi^2 E(R^4)$

From LOTUS, $\pi^2 \int_0^1 R^4 \, dR = \pi^2 \left[\frac{R^5}{5}\right]_0^1$

$= \frac{\pi^2}{5}$

$\Rightarrow Var(A) = \frac{\pi^2}{5} - \left(\frac{\pi}{3}\right)^2 = \frac{4\pi^2}{45}$

(b) CDF: $P(A < a) \Rightarrow P(\pi R^2 < a)$

$\Rightarrow P\left(R < \sqrt{\frac{a}{\pi}}\right)$

Since, $R \sim Unif(0,1)$

$\Rightarrow P(R < \sqrt{a/\pi}) = \sqrt{\frac{a}{\pi}}$

Hence CDF is $\sqrt{\frac{a}{\pi}}$

PDF: $f(a) = F'(a) = \frac{1}{\sqrt{\pi}} \times \frac{1}{2\sqrt{a}} = \frac{1}{2\sqrt{a\pi}}$

4. (a) for $U \sim Unif(0,1)$, $\mu = \frac{1}{2}$ and $\sigma = \frac{1}{\sqrt{12}}$

$$P(|X-\mu| < \sigma) \Rightarrow P(\mu-\sigma < X < \mu+\sigma)$$

$$\Rightarrow P\left(\left(\frac{1}{2} - \frac{1}{\sqrt{12}}\right) < X < \left(\frac{1}{2} + \frac{1}{\sqrt{12}}\right)\right)$$

$$\Rightarrow P(0.211 < X < 0.788) \simeq 57.6\%$$

$$P(|X-\mu| < 2\sigma) \Rightarrow P(-0.07 < X < 1.076) = 100\%$$

Simillarly, $P(|X-\mu| < 3\sigma) = 100\%$

So rule is 57.6 - 100 - 100 %

(b) For $X \sim Expo(1)$, $\mu = 1$ and $\sigma^2 = 1 \Rightarrow \sigma = 1$

$$\Rightarrow P(|X-\mu| < \sigma) = P(|X-1| < 1) = P(0 < X < 2)$$

$$= F(2) = 1 - e^{-2} \simeq 86.46\%$$

$$P(|X-\mu| < 2\sigma) = P(|X-1| < 2) = P(-1 < X < 3) = P(0 < X < 3)$$

$$= P(X < 3) = F(3) = 1 - e^{-3} \simeq 95\%$$

Simillarly, $P(|X-\mu| < 3\sigma) = P(X < 4) = 1 - e^{-4} \simeq 98.16\%$

~~So rule is 86.46 - 95 - 98~~

So rule is 86.46 - 95 - 98.16 %

(c) Let $Y \sim \text{Expo}(\lambda)$ and r.v. $X = Y\lambda$ $\left(Y = \frac{X}{\lambda} \right)$

so, $X \sim \text{Expo}(1)$

For $Y \sim \text{Expo}(\lambda)$, $\mu = \frac{1}{\lambda}$ and $\sigma = \frac{1}{\lambda}$

Now, $P(|Y-\mu| < \sigma) = P\left(\left|Y - \frac{1}{\lambda}\right| < \frac{1}{\lambda}\right)$

Since $\lambda > 0$, $P(|\lambda Y - 1| < 1) = P(|X-1| < 1)$

And same is true for 2σ and 3σ

i.e. $P\left(\left|Y - \frac{1}{\lambda}\right| < \frac{2}{\lambda}\right) = P(|X-1| < 2)$

and, $P\left(\left|Y - \frac{1}{\lambda}\right| < \frac{3}{\lambda}\right) = P(|X-1| < 3)$

Hence same rule applies for $Y \sim \text{Expo}(\lambda)$ for all λ

so, $\text{Expo}(1/2)$ and same rule as $\text{Expo}(1)$

(5) CDF : $F(x) = P(X \leq x)$

$= P(\max(U_1, U_2 \dots U_n) \leq x)$

$= P(U_1 \leq x, U_2 \leq x, \dots U_n \leq x)$

Since $U_1, U_2 \dots U_n$ are i.i.d Unif(0,1)

$P(U_1 \leq x, U_2 \leq x \dots U_n \leq x)$

$= P(U_1 \leq x) P(U_2 \leq x) \dots P(U_n \leq x)$

$= x^n$

PDF: $f(x) = F'(x) = n x^{n-1}$

$EX = \int_0^1 x n x^{n-1} dx = n \int_0^1 x^n dx$

$= \frac{n}{n+1} \left[x^{n+1} \right]_0^1$

$= \frac{n}{n+1}$

6. (a) PDF: $f(x) = \frac{a}{x^{(a+1)}} = a x^{(-a-1)}$

CDF: $F(x) = \int_1^x f(x)\,dx = \int_1^x a x^{-(a+1)}\,dx$

$= a \int_1^x a x^{-(a+1)}\,dx = -\frac{a}{a}\left[x^{-a}\right]_1^x$

$= 1 - x^{-a}$

$F(x) = (1 - x^{-a})$

$F(x)$ is a valid CDF because it is strictly increasing function ($F'(x) = a x^{-(a+1)} > 0$) $\forall\ x \geq 1$

and at lower bound (i.e. $x = 1$)

$F(1) = 0$

and $\lim_{x \to \infty} F(x) = 1$ (at upper bound $F(x) = 1$)

(b) From univerasility of the uniform we can generate r.v.s U from $U \sim Unif(0,1)$ and plug U into $F^{-1}(U)$ (inverse of pareto CDF)

$U = 1 - x^{-a} \Rightarrow x^a = \frac{1}{1-U} \Rightarrow x = \left(\frac{1}{1-U}\right)^{1/a}$

So, $F^{-1}(U) = \left(\frac{1}{1-U}\right)^{1/a}$ $F^{-1}(0) = 1$ and $F^{-1}(1) = \infty$ Hence support is also good.

And Hence we can generate $X \sim Pareto(a)$ from U by $X = F^{-1}(U)$.

(7) $\epsilon \sim N(0, 0.04)$

$\Rightarrow \mu = 0, \quad \sigma^2 = 0.04, \quad \sigma = 0.2$

we want to find $P(|\epsilon| \leq 0.4)$

$= P(-0.4 \leq \epsilon \leq 0.4)$

$= \Phi(z_1) - \Phi(z_2)$

where, $z \sim N(0,1)$

and $z_1 = \dfrac{0.4 - \cancel{\mu}}{\sigma} = \dfrac{0.4}{0.2} = 2$

$z_2 = \dfrac{-0.4 - \mu}{\sigma} = \dfrac{-0.4}{0.2} = -2$

$\Rightarrow \Phi(z_1) - \Phi(z_2) = \Phi(2) - \Phi(-2)$

and $\Phi(-2) = 1 - \Phi(2)$

$= 2\Phi(2) - 1$

As we see we want probability of an r.v. within 2 times standard deviation from mean and its a normal distribution.

Hence from 68-95-99.7% rule

required probability = 95% = 0.95

8.(a) Let X_1 be r.v. for time taken by Walter and X_2 for time taken by Carl.

We want to find $P(X_2 < X_1)$

$$= P((X_2 - X_1) < 0)$$

Let $X = (X_2 - X_1)$

So, $X \sim N(\mu', \sigma')$

where, $\mu' = c - w \quad (\mu_{Carl} - \mu_{Walter})$

and $\sigma'^2 = \sigma^2 + (2\sigma)^2 \quad (\sigma_{Carl}^2 + \sigma_{Walter}^2)$

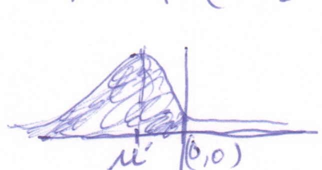
$= 5\sigma^2$

$\Rightarrow \sigma' = \sigma\sqrt{5}$

And $P(X_2 - X_1 < 0) = \Phi\left(\frac{0-\mu'}{\sigma\sqrt{5}}\right)$

$$= \Phi\left(\frac{w-c}{\sigma\sqrt{5}}\right)$$

(b) We know that in normal distribution probability of r.v. less than mean is 1/2. So if mean lies on negative side of x-axis then $P(X_2 - X_1 < 0)$ is more than half.



$P(X_2 - X_1 < 0)$ is the area of shaded region which is more than 1/2 because area till μ' itself is 1/2.

$\Rightarrow \mu' < 0 \Rightarrow c - w < 0 \Rightarrow c < w$

(c) For Walter,

$$P(X_1 < w+10) = \Phi\left(\frac{w+10-w}{\sigma}\right)$$
$$= \Phi\left(\frac{10}{\sigma}\right)$$

For Carl,

$$P(X_2 < w+10) = \Phi\left(\frac{w+10-c}{2\sigma}\right)$$

From the question,

$$\Phi\left(\frac{w+10-c}{2\sigma}\right) > \Phi\left(\frac{10}{\sigma}\right)$$

Since Φ is an increasing function

$$\Rightarrow \quad \frac{w+10-c}{2\sigma} > \frac{10}{\sigma}$$

$$\Rightarrow \quad w-c > 10$$

(9) Let T_j be time taken by student j for $j = 1, 2, 3$ and T be earliest time all 3 have completed the homework.

Hence, $T = \max(T_1, T_2, T_3)$

CDF: $F(t) = P(T < t)$

$= P(\max(T_1, T_2, T_3) < t)$

Since T_1, T_2, T_3 are independant

$P(\max(T_1, T_2 T_3) < t) = P(T_1 < t, T_2 < t, T_3 < t)$

$= P(T_1 < t)\, P(T_2 < t)\, P(T_3 < t)$

$= (1 - e^{-\lambda t})^3$ (where $\lambda = \frac{1}{6}$)

PDF: $f(t) = F'(t)$

$= 3\lambda e^{-\lambda t}(1 - e^{-\lambda t})^2$

$E(T) = \int_0^\infty t f(t)\, dt$

$= \int_0^\infty 3\lambda t \,(e^{-\lambda t} + e^{-3\lambda t} - 2e^{-2\lambda t})\, dt$

$= 3\int_0^\infty t\lambda e^{-\lambda t} dt + 3\int_0^\infty t\lambda e^{-3\lambda t} dt - 6\int_0^\infty t\lambda e^{-2\lambda t} dt$

$= \frac{3}{\lambda} + 3\int_0 ^\infty \underset{\substack{\downarrow \\ (3t = x)}}{\frac{x}{3}} \lambda e^{-x\lambda} \frac{dx}{3} - 6\int_0^\infty \underset{\substack{\downarrow \\ (2t = y)}}{\frac{y}{2}} \lambda e^{-y\lambda} \frac{dy}{2}$

$= \frac{3}{\lambda} + \frac{3}{9}\int_0^\infty x\lambda e^{-\lambda x} dx - \frac{6}{4}\int_0^y y\lambda e^{-\lambda y} dy$

$= \frac{3}{\lambda} + \frac{3}{9}\frac{1}{\lambda} - \frac{6}{4}\frac{1}{\lambda} = \frac{1}{\lambda}\left(3 + \frac{1}{3} - \frac{3}{2}\right) = \frac{1}{\lambda} \times \frac{18 + 2 - 9}{6}$

$= \frac{1}{\lambda} \times \frac{11}{6} = 11$ (Replacing $\frac{1}{\lambda}$ by 6)

Hence on an average all three are expected to finish homework after 11 hours from starting.

10.(a) CDF of Y: $P(Y<t) = P(e^X < t) = P(X < \log(t))$

$$= \Phi\left(\frac{\log(t) - \mu}{\sigma}\right)$$

CDF of X: $\Phi\left(\frac{x-\mu}{\sigma}\right)$

For these to have same values for a t and x following criteria should meet.

$$\frac{\log(t) - \mu}{\sigma} = \frac{x - \mu}{\sigma}$$

$$\Rightarrow t = e^x$$

Hence, to get 1/2 as probability for median t should be e^μ, because μ is median for X. So student A is right.

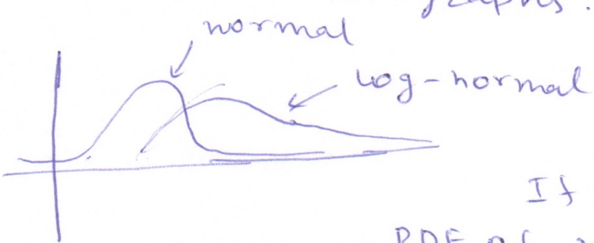
(b) As we saw in part (a)

CDF of Y: $\Phi\left(\frac{\log(t) - \mu}{\sigma}\right)$

$\Rightarrow$ PDF of Y: $\frac{1}{t}\Phi'\left(\frac{\log(t) - \mu}{\sigma}\right) = \frac{1}{t\sqrt{2\pi}} e^{-\frac{(\log(t)-\mu)^2}{2\sigma^2}}$

PDF of X: $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}$

If we plot two graphs.



~~We see that maxima of two graphs are different~~

If we take derivative of PDF of Y we find that maxima (mode) is not e^μ. Hence Student B is wrong.

(c) As we saw in part (b) nature of two graphs (PDF of X and Y) does not have any relation with how X and Y are related (i.e $Y = e^x$). And also mode of Y does not occur at $Y = \mu$.

Hence student C is wrong.

11. (a) $M_W(t) = M_{X^2+Y^2}(t)$

$= E\left(e^{t(X^2+Y^2)}\right)$

$= E\left(e^{tX^2} e^{tY^2}\right) = E\left(e^{tX^2}\right) E\left(e^{tY^2}\right)$

(because X and Y are i.i.ds)

$= M_{X^2}(t) M_{Y^2}(t)$

$= \dfrac{1}{1-2t}$

(b) if $Y \sim \text{Expo}(\lambda)$

then $M_Y(t) = \dfrac{\lambda}{\lambda - t}$

And MGF uniquely identifies distribution

so $\dfrac{1}{1-2t}$ is exponential distribution.

$\dfrac{\lambda}{\lambda - t} = \dfrac{1}{1-2t}$

$\Rightarrow \dfrac{1}{1-\frac{t}{\lambda}} = \dfrac{1}{1-2t} \Rightarrow \dfrac{1}{\lambda} = 2 \Rightarrow \lambda = \dfrac{1}{2}$

Hence distribution is $\text{Expo}(1/2)$

12. (a) $P(Y > s+t \mid Y > s) = \dfrac{P(Y > s+t,\, Y > s)}{P(Y > s)}$

$$= \frac{P(Y > s+t)}{P(Y > s)}$$

$$= \frac{P(X^{\beta} > s+t)}{P(X^{\beta} > s)} = \frac{P\left(X > (s+t)^{1/\beta}\right)}{P(X > s^{1/\beta})}$$

$$= \frac{e^{-(s+t)^{1/\beta}}}{e^{-s^{1/\beta}}} = e^{s^{1/\beta} - (s+t)^{1/\beta}}$$

$$= e^{(s^{1/3} - (s+t)^{1/3})}$$

$$\neq P(Y > t)$$

Hence Weibull does not have memoryless property.

(b)

$E(Y) = E(X^3)$

$X \sim N(0,1)$

$M_X(t) = \dfrac{1}{1-t} = \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} n! \dfrac{t^n}{n!}$

From taylor series, $M_X(t) = \sum_{n=0}^{\infty} M^{(n)}(0) \dfrac{t^n}{n!}$

Hence comparing coefficients

$M^{(n)}(0) = E(X^n) = n!$

$\Rightarrow E(X^3) = 3! = 6$

$\Rightarrow E(Y) = 6$

$E(Y^2) = E(X^6) = 6!$

$Var(Y) = E(Y^2) - (EY)^2 = 6! - 6^2 = 684$

$E(Y^n) = E(X^{3n}) = 3n!$

(c) $M_Y(t) = E(e^{tY})$

$$= E\left(e^{tX^3}\right)$$

$$= \int_0^\infty e^{tx^3} e^{-x}\, dx$$

$$= \int_0^\infty e^{tx^3 - x}\, dx$$

As we see for $t > 0$ the above integral diverges and we cannot compute a finite value of the integral.
Hence MGF of Y does not exist.