Stat S-110 Homework 1 Solutions, Summer 2015

The following problems are from Chapters 1-2 of *Introduction to Probability* by Joe Blitzstein and Jessica Hwang. We write BH 2.7, for example, to denote Exercise 7 of Chapter 2 of the book.

1. (BH 1.2) (a) How many 7-digit phone numbers are possible, assuming that the first digit can't be a 0 or a 1?

(b) Re-solve (a), except now assume also that the phone number is not allowed to start with 911 (since this is reserved for emergency use, and it would not be desirable for the system to wait to see whether more digits were going to be dialed after someone has dialed 911).

Solution:

(a) By the multiplication rule, there are $8 \cdot 10^6$ possibilities.

(b) There are 10^4 phone numbers in (a) that start with 911 (again by the multiplication rule, since the first 3 digits are 911 and the remaining 4 digits are unconstrained). Excluding these and using the result of (a), the number of possibilities is

$$8 \cdot 10^6 - 10^4 = 7990000.$$

2. (BH 1.6) There are 20 people at a chess club on a certain day. They each find opponents and start playing. How many possibilities are there for how they are matched up, assuming that in each game it *does* matter who has the white pieces (in a chess game, one player has the white pieces and the other player has the black pieces)?

Solution: There are $\frac{20!}{2^{10} \cdot 10!}$ ways to determine who plays whom without considering color, by the multiplication rule or the result of Example 1.5.4 (Partnerships). For each game, there are 2 choices for who has the white pieces, so overall the number of possibilities is

$$\frac{2^{10} \cdot 20!}{2^{10} \cdot 10!} = \frac{20!}{10!} = 670442572800.$$

Alternatively, imagine a long table with 10 chessboards in a row, with 10 chairs on each side of the table (for the chess players to sit in). Chess pieces are set up at each board, oriented so that the white pieces are all on one side of the table and the black pieces are all on the other side of the table. There are 20! ways for the chess players to be assigned to chairs, and once the players are seated in their chairs, a configuration for the chess games has been determined. This procedure overcounts by a factor of 10! since, after matching up the players, we can permute the players on one side of the table in any way, as long as we also permute the players on the other side of the table in the same way. So again the number of possibilities is $\frac{20!}{10!}$.

3. (BH 1.7) Two chess players, A and B, are going to play 7 games. Each game has three possible outcomes: a win for A (which is a loss for B), a draw (tie), and a loss for A (which is a win for B). A win is worth 1 point, a draw is worth 0.5 points, and a loss is worth 0 points.

(a) How many possible outcomes for the individual games are there, such that overall player A ends up with 3 wins, 2 draws, and 2 losses?

(b) How many possible outcomes for the individual games are there, such that A ends up with 4 points and B ends up with 3 points?

(c) Now assume that they are playing a best-of-7 match, where the match will end when either player has 4 points or when 7 games have been played, whichever is first. For example, if after 6 games the score is 4 to 2 in favor of A, then A wins the match and they don't play a 7th game. How many possible outcomes for the individual games are there, such that the match lasts for 7 games and A wins by a score of 4 to 3?

Solution:

(a) Writing W for win, D for draw, and L for loss (for player A), an outcome of the desired form is any permutation of WWWDDLL. So there are

$$\frac{7!}{3!2!2!} = 210$$

possible outcomes of the desired form.

(b) To end up with 4 points, A needs to have one of the following results: (i) 4 wins and 3 losses; (ii) 3 wins, 2 draws, and 2 losses; (iii) 2 wins, 4 draws, and 1 loss; or (iv) 1 win and 6 draws. Reasoning as in (a) and adding up these possibilities, there are

$$\frac{7!}{4!3!} + \frac{7!}{3!2!2!} + \frac{7!}{2!4!1!} + \frac{7!}{1!6!} = 357$$

possible outcomes of the desired form.

(c) For the desired outcomes, either (i) player A is ahead 3.5 to 2.5 after 6 games and then draws game 7, or (ii) the match is tied (3 to 3) after 6 games and then player A wins game 7. Reasoning as in (b), there are

$$\frac{6!}{3!1!2!} + \frac{6!}{2!3!1!} + \frac{6!}{1!5!} = 126$$

possibilities of type (i) and

$$\frac{6!}{3!3!} + \frac{6!}{2!2!2!} + \frac{6!}{1!4!1!} + 1 = 141$$

possibilities of type (ii), so overall there are

$$126 + 141 = 267$$

possible outcomes of the desired form.

4. (BH 1.17) Give a story proof that

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1},$$

for all positive integers n.

Hint: Consider choosing a committee of size n from two groups of size n each, where only one of the two groups has people eligible to become president.

Solution:

Imagine that there are n juniors and n seniors in a certain club. A committee of size n is chosen, and one of these people becomes president. Suppose though that the president must be a senior. Letting k be the number of seniors on the committee, there are $\binom{n}{k}$ ways to choose the seniors, $\binom{n}{n-k} = \binom{n}{k}$ ways to choose the juniors, and after these choices are made there are k choices of president. So the overall number of possibilities is the left-hand side of the identity.

Alternatively, we can choose the president *first* (as any of the n seniors), and then choose any $n-1$ of the remaining $2n-1$ people to form the rest of the committee. This gives the right-hand side of the identity.

5. (BH 1.21) Three people get into an empty elevator at the first floor of a building that has 10 floors. Each presses the button for their desired floor (unless one of the others has already pressed that button). Assume that they are equally likely to want to go to floors 2 through 10 (independently of each other). What is the probability that the buttons for 3 consecutive floors are pressed?

Solution: The number of possible outcomes for who is going to which floor is 9^3. There are 7 possibilities for which buttons are pressed such that there are 3 consecutive floors: $(2, 3, 4), (3, 4, 5), \ldots, (8, 9, 10)$. For each of these 7 possibilities, there are 3! ways to choose who is going to which floor. So by the naive definition, the probability is

$$\frac{3! \cdot 7}{9^3} = \frac{42}{729} \approx 0.0576.$$

6. (BH 1.49) For a group of 7 people, find the probability that all 4 seasons (winter, spring, summer, fall) occur at least once each among their birthdays, assuming that all seasons are equally likely.

Solution: Let A_i be the event that there are no birthdays in the ith season (with respect to some ordering of the seasons). The probability that all seasons occur at least once is $1 - P(A_1 \cup A_2 \cup A_3 \cup A_4)$. Note that $A_1 \cap A_2 \cap A_3 \cap A_4 = \emptyset$ (the most extreme case is when everyone is born in the same season). By inclusion-exclusion and symmetry,

$$P(A_1 \cup A_2 \cup A_3 \cup A_4) \quad = \quad 4P(A_1) - \binom{4}{2} P(A_1 \cap A_2) + \binom{4}{3} P(A_1 \cap A_2 \cap A_3).$$

We have $P(A_1) = (3/4)^7, P(A_1 \cap A_2) = (2/4)^7, P(A_1 \cap A_2 \cap A_3) = (1/4)^7$, so

$$1 - P(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - \left(4\left(\frac{3}{4}\right)^7 - \frac{6}{2^7} + \frac{4}{4^7} \right) = \frac{525}{1024} \approx 0.513.$$

7. (BH 2.4) Fred is answering a multiple-choice problem on an exam, and has to choose one of n options (exactly one of which is correct). Let K be the event that he knows the answer, and R be the event that he gets the problem right (either through knowledge or through luck). Suppose that if he knows the right answer he will definitely get the problem right, but if he does not know then he will guess completely randomly. Let $P(K) = p$.

(a) Find $P(K|R)$ (in terms of p and n).

(b) Show that $P(K|R) \geq p$, and explain why this makes sense intuitively. When (if ever) does $P(K|R)$ equal p?

Solution:

(a) By Bayes' rule and the law of total probability,

$$P(K|R) = \frac{P(R|K)P(K)}{P(R|K)P(K) + P(R|K^c)P(K^c)} = \frac{p}{p + (1-p)/n}.$$

(b) By the above, $P(K|R) \geq p$ is equivalent to $p + (1-p)/n \leq 1$, which is a true statement since $p + (1-p)/n \leq p + 1 - p = 1$. This makes sense intuitively since getting the question right should increase our confidence that Fred knows the answer. Equality holds if and only if one of the extreme cases $n = 1$ or $p = 0$ or $p = 1$ holds. If $n = 1$, it's not really a multiple-choice problem, and Fred getting the problem right is completely uninformative; if $p = 0$ or $p = 1$, then whether Fred knows the answer is a foregone conclusion, and no evidence will make us more (or less) sure that Fred knows the answer.

8. (BH 2.6) A hat contains 100 coins, where 99 are fair but one is double-headed (always landing Heads). A coin is chosen uniformly at random. The chosen coin is flipped 7 times, and it lands Heads all 7 times. Given this information, what is the probability that the chosen coin is double-headed? (Of course, another approach here would be to *look at both sides of the coin*—but this is a metaphorical coin.)

Solution: Let A be the event that the chosen coin lands Heads all 7 times, and B be the event that the chosen coin is double-headed. Then

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)} = \frac{0.01}{0.01 + (1/2)^7 \cdot 0.99} = \frac{128}{227} \approx 0.564.$$

9. (BH 2.7) A hat contains 100 coins, where *at least* 99 are fair, but there may be one that is double-headed (always landing Heads); if there is no such coin, then all 100 are fair. Let D be the event that there is such a coin, and suppose that $P(D) = 1/2$. A coin is chosen uniformly at random. The chosen coin is flipped 7 times, and it lands Heads all 7 times.

(a) Given this information, what is the probability that one of the coins is double-headed?

(b) Given this information, what is the probability that the chosen coin is double-headed?

Solution:

(a) Let A be the event that the chosen coin lands Heads all 7 times, and C be the event that the chosen coin is double-headed. By Bayes' rule and LOTP,

$$P(D|A) = \frac{P(A|D)P(D)}{P(A|D)P(D) + P(A|D^c)P(D^c)}.$$

We have $P(D) = P(D^c) = 1/2$ and $P(A|D^c) = 1/2^7$, so the only remaining ingredient that we need to find is $P(A|D)$. We can do this using LOTP with extra conditioning (it would be useful to know whether the *chosen* coin is double-headed, not just whether *somewhere* there is a double-headed coin, so we condition on whether or not C occurs):

$$P(A|D) = P(A|D,C)P(C|D) + P(A|D,C^c)P(C^c|D) = \frac{1}{100} + \frac{1}{2^7} \cdot \frac{99}{100}.$$

Plugging in these results, we have

$$P(D|A) = \frac{227}{327} = 0.694.$$

(b) By LOTP with extra conditioning (it would be useful to know whether there *is* a double-headed coin),

$$P(C|A) = P(C|A,D)P(D|A) + P(C|A,D^c)P(D^c|A),$$

with notation as in (a). But $P(C|A,D^c) = 0$, and we already found $P(D|A)$ in (a). Also,

$$P(C|A,D) = \frac{128}{227},$$

as shown in the previous exercise (conditioning on D and A puts us exactly in the setup of that exercise). Thus,

$$P(C|A) = \frac{128}{227} \cdot \frac{227}{327} = \frac{128}{327} \approx 0.391.$$

10. (BH 2.10) Fred is working on a major project. In planning the project, two milestones are set up, with dates by which they should be accomplished. This serves as a way to track Fred's progress. Let A_1 be the event that Fred completes the first milestone on time, A_2 be the event that he completes the second milestone on time, and A_3 be the event that he completes the project on time.

Suppose that $P(A_{j+1}|A_j) = 0.8$ but $P(A_{j+1}|A_j^c) = 0.3$ for $j = 1, 2$, since if Fred falls behind on his schedule it will be hard for him to get caught up. Also, assume that the second milestone supersedes the first, in the sense that once we know whether he is on time in completing the second milestone, it no longer matters what happened with the first milestone. We can express this by saying that A_1 and A_3 are conditionally independent given A_2 and they're also conditionally independent given A_2^c.

(a) Find the probability that Fred will finish the project on time, given that he completes the first milestone on time. Also find the probability that Fred will finish the project on time, given that he is late for the first milestone.

(b) Suppose that $P(A_1) = 0.75$. Find the probability that Fred will finish the project on time.

Solution:

(a) We need to find $P(A_3|A_1)$ and $P(A_3|A_1^c)$. To do so, let's use LOTP to condition on whether or not A_2 occurs:

$$P(A_3|A_1) = P(A_3|A_1, A_2)P(A_2|A_1) + P(A_3|A_1, A_2^c)P(A_2^c|A_1).$$

Using the conditional independence assumptions, this becomes

$$P(A_3|A_2)P(A_2|A_1) + P(A_3|A_2^c)P(A_2^c|A_1) = (0.8)(0.8) + (0.3)(0.2) = 0.7.$$

Similarly,

$$P(A_3|A_1^c) = P(A_3|A_2)P(A_2|A_1^c)+P(A_3|A_2^c)P(A_2^c|A_1^c) = (0.8)(0.3)+(0.3)(0.7) = 0.45.$$

(b) By LOTP and Part (a),

$$P(A_3) = P(A_3|A_1)P(A_1) + P(A_3|A_1^c)P(A_1^c) = (0.7)(0.75) + (0.45)(0.25) = 0.6375.$$

11. (BH 2.11) An *exit poll* in an election is a survey taken of voters just after they have voted. One major use of exit polls has been so that news organizations can try to figure out as soon as possible who won the election, before the votes are officially counted. This has been notoriously inaccurate in various elections, sometimes because of *selection bias*: the sample of people who are invited to and agree to participate in the survey may not be similar enough to the overall population of voters.

Consider an election with two candidates, Candidate A and Candidate B. Every voter is invited to participate in an exit poll, where they are asked whom they voted for; some accept and some refuse. For a randomly selected voter, let A be the event that they voted for A, and W be the event that they are willing to participate in the exit poll. Suppose that $P(W|A) = 0.7$ but $P(W|A^c) = 0.3$. In the exit poll, 60% of the respondents say they voted for A (assume that they are all honest), suggesting a comfortable victory for A. Find $P(A)$, the true proportion of people who voted for A.

Solution: We have $P(A|W) = 0.6$ since 60% of the respondents voted for A. Let $p = P(A)$. Then

$$0.6 = P(A|W) = \frac{P(W|A)P(A)}{P(W|A)P(A) + P(W|A^c)P(A^c)} = \frac{0.7p}{0.7p + 0.3(1-p)}.$$

Solving for p, we obtain

$$P(A) = \frac{9}{23} \approx 0.391.$$

So actually A received fewer than half of the votes!

12. (BH 2.13) Company A has just developed a diagnostic test for a certain disease. The disease afflicts 1% of the population. As defined in Example 2.3.9, the *sensitivity* of the test is the probability of someone testing positive, given that they have the disease, and the *specificity* of the test is the probability that of someone testing negative, given that they don't have the disease. Assume that, as in Example 2.3.9, the sensitivity and specificity are both 0.95.

Company B, which is a rival of Company A, offers a competing test for the disease. Company B claims that their test is faster and less expensive to perform than Company A's test, is less painful (Company A's test requires an incision), and yet has a higher overall success rate, where overall success rate is defined as the probability that a random person gets diagnosed correctly.

(a) It turns out that Company B's test can be described and performed very simply: no matter who the patient is, diagnose that they do not have the disease. Check whether Company B's claim about overall success rates is true.

(b) Explain why Company A's test may still be useful.

(c) Company A wants to develop a new test such that the overall success rate is higher than that of Company B's test. If the sensitivity and specificity are equal, how high does the sensitivity have to be to achieve their goal? If (amazingly) they can get the sensitivity equal to 1, how high does the specificity have to be to achieve their goal? If (amazingly) they can get the specificity equal to 1, how high does the sensitivity have to be to achieve their goal?

Solution:

(a) For Company B's test, the probability that a random person in the population is diagnosed correctly is 0.99, since 99% of the people do not have the disease. For a random member of the population, let C be the event that Company A's test yields the correct result, T be the event of testing positive in Company A's test, and D be the event of having the disease. Then

$$\begin{aligned} P(C) &= P(C|D)P(D) + P(C|D^c)P(D^c) \\ &= P(T|D)P(D) + P(T^c|D^c)P(D^c) \\ &= (0.95)(0.01) + (0.95)(0.99) \\ &= 0.95, \end{aligned}$$

which makes sense intuitively since the sensitivity and specificity of Company A's test are both 0.95. So Company B is correct about having a higher overall success rate.

(b) Despite the result of (a), Company A's test may still provide very useful information, whereas Company B's test is uninformative. If Fred tests positive on Company A's test, Example 2.3.9 shows that his probability of having the disease increases from 0.01 to 0.16 (so it is still fairly unlikely that he has the disease, but it is much more likely than it was before the test result; further testing may well be advisable). In contrast, Fred's probability of having the disease does not change after undergoing Company's B test, since the test result is a foregone conclusion.

(c) Let s be the sensitivity and p be the specificity of A's new test. With notation as in the solution to (a), we have

$$P(C) = 0.01s + 0.99p.$$

If $s = p$, then $P(C) = s$, so Company A needs $s > 0.99$.

If $s = 1$, then $P(C) = 0.01 + 0.99p > 0.99$ if $p > 98/99 \approx 0.9899$.

If $p = 1$, then $P(C) = 0.01s + 0.99$ is automatically greater than 0.99 (unless $s = 0$, in which case both companies have tests with sensitivity 0 and specificity 1).