KOLHAPUR INSTITUTE OF TECHNOLOGY'S,

COLLEGE OF ENGINEERING (AUTONOMOUS), KOLHAPUR

DEPARTMENT OF COMPUTER SCIENCE AND BUSINESS SYSYTEMS

Second Year B.Tech. (SEM - III)

STATISTICAL METHODS (UCBBS0303)

Unit 1 Multivariate Normal Distribution

Introduction:

The Multivariate Normal Distribution is a generalization of the univariate normal density to p ≥ 2 dimensions.

Suppose we have (p x 1) random vector $\underline{X}$ of the observations on several variables. Let (p x 1) vector $\underline{\mu}$ represents the expected value of random vector $\underline{X}$ also (p x p) matrix Σ is the variance covariance matrix of the random vector $\underline{X}$

Then the p-dimensional normal density for the random vector $\underline{X}' = [X_1, X_2, ...X_p]$ has the following form,

$$f(\underline{X}) = \frac{1}{\left(\sqrt{2\pi}\right)^p \sqrt{|\Sigma|}} \exp\left[-\frac{1}{2}(\underline{x}-\underline{\mu})' \Sigma^{-1} (\underline{x}-\underline{\mu})\right]$$

Where,

$\underline{X}$ is a (p x 1) dimension random vector. $-\infty < \underline{x_i} < \infty \quad , \quad i = 1, 2, ...p$

$|\Sigma|$ denotes determinant of variance covariance matrix Σ (Σ is called a dispersion matrix)

In notation $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$

Definition:

A random vector $\underline{X}$ of order (p x 1) with mean vector $\underline{\mu}$ and variance covariance matrix Σ is said to follows multivariate normal distribution if its probability density function is given by,

$$f(\underline{X}) = \frac{1}{\left(\sqrt{2\pi}\right)^p \sqrt{|\Sigma|}} \exp\left[-\frac{1}{2}(\underline{x}-\underline{\mu})' \Sigma^{-1} (\underline{x}-\underline{\mu})\right] \qquad -\infty < \underline{X} < \infty \;\&\; -\infty < \underline{\mu} < \infty$$

Where,

$|\Sigma|$ denotes determinant of variance covariance matrix Σ (Σ is called a dispersion matrix)

Σ^{-1} denotes inverse of variance covariance matrix Σ (Σ is called a dispersion matrix)

In notation $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$

i.e. random vector $\underline{X}$ distributed multivariate normal distribution with mean vector $\underline{\mu}$ and variance covariance matrix Σ.

Standard Normal Variate (S.N.V.):

In univariate normal distribution we know that, if $X \sim N(\mu, \sigma^2)$ then S.N.V. Z is defined as, $Z = \dfrac{x-\mu}{\sigma} \sim N(0,1)$ similarly in multivariate normal distribution, if $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$ then $\underline{Z} \sim N_p(\underline{0}, I_{p\times p})$ where $\underline{0} = [0, 0, ...0]_{1\times p}$ and I is (p x p) identity matrix.

$$I = \begin{bmatrix} 1 & 0 & .. & .. & 0 \\ 0 & 1 & .. & ... & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & .. & .. & 1 \end{bmatrix}_{p\times p}$$

Properties of Multivariate Normal Distribution:

1. The following term is a quadratic form (appearing inside the exponent of MVND)

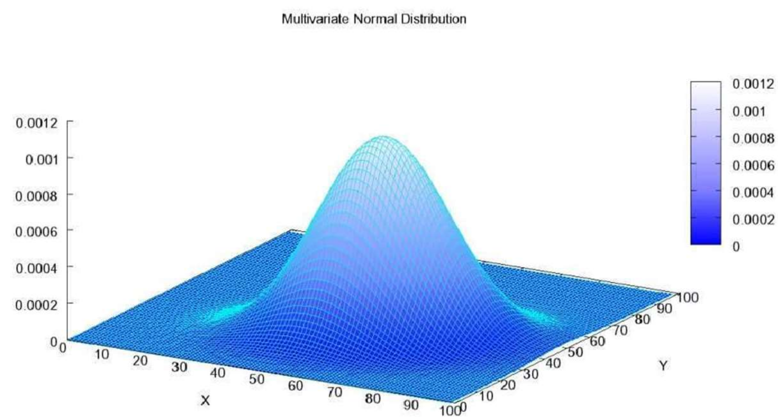
$$\left[-\frac{1}{2}(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})\right]$$

2. If variables are uncorrelated then the Σ will be like as shown below.

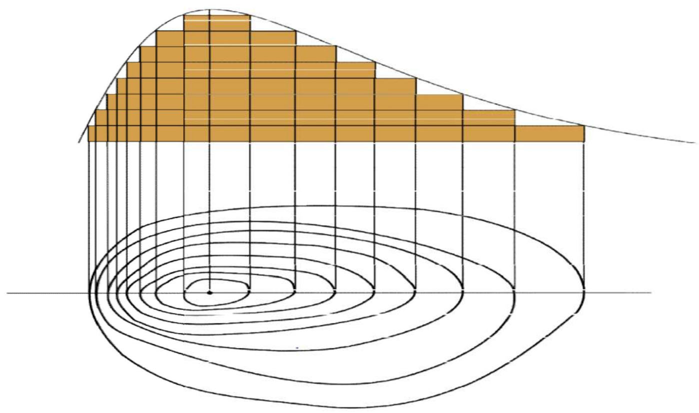
$$\Sigma = \begin{bmatrix} \sigma_1^{\,2} & 0 & .. & .. & 0 \\ 0 & \sigma_2^{\,2} & .. & ... & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & .. & .. & \sigma_p^{\,2} \end{bmatrix}$$

3. Any subset of $\underline{X}$ also has a multivariate normal distribution.

4. For the density of p- dimensional normal variables, the paths of x values yielding a constant height for the density are ellipsoids.



5. Thus multivariate normal density is constant on surfaces where the square of distance $\left[(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})\right]$ is constant. These paths are called contours.



Examples

1: Consider a bivariate normal population with $\mu_1 = 0, \mu_2 = 2, \sigma_{11} = 2, \sigma_{22} = 1, \rho_{12} = 1/2$

a) Write down the bivariate normal density.

b) Write down the squared generalized distance expression.

Solution: The multivariate normal distribution is given by,

$$f(\underline{X}) = \frac{1}{\left(\sqrt{2\pi}\right)^p \sqrt{|\Sigma|}} \exp\left[-\frac{1}{2}(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})\right] \text{........(1)}$$

To find σ_{12} *and* σ_{21}

Here p=2, $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 2 & \sigma_{12} \\ \sigma_{21} & 1 \end{bmatrix}$

$$\rho_{12} = \frac{\sigma 12}{\sqrt{\sigma 11}\sqrt{\sigma 22}} \qquad \therefore 1/2 = \frac{\sigma 12}{\sqrt{2}\sqrt{1}} \qquad \therefore \sigma_{12} = \frac{1}{\sqrt{2}}$$

$$\Sigma = \begin{bmatrix} 2 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{bmatrix} \qquad \therefore |\Sigma| = 3/2 \qquad \therefore \sqrt{|\Sigma|} = \sqrt{3/2}$$

$$\Sigma^{-1} = \frac{1}{(2-1/2)}\begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 2 \end{bmatrix} \qquad \therefore \Sigma^{-1} = \frac{2}{3}\begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 2 \end{bmatrix}$$

Now, $(\underline{x}-\underline{\mu}) = \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} = \begin{bmatrix} x_1 - 0 \\ x_2 - 2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - 2 \end{bmatrix} \qquad (\underline{x}-\mu)' = \begin{bmatrix} x_1 & x_2 - 2 \end{bmatrix}$

Now substituting all above the values in equation (1)

$$f(\underline{X}) = \frac{1}{\left(\sqrt{2\pi}\right)^2 \sqrt{3/2}} \exp\left[-\frac{1}{2}\begin{bmatrix} x_1 & x_2 - 2 \end{bmatrix} \frac{2}{3}\begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 - 2 \end{bmatrix}\right]$$

$$f(\underline{X}) = \frac{1}{\pi\sqrt{6}} \exp\left[-\frac{1}{3}(x_1^2 - \sqrt{2}x_1(x_2-2) + 2(x_2-2)^2\right]$$

This is required bivariate normal density function.

b) The squared generalized distance expression is $\left[\frac{2}{3}(x_1^2 - \sqrt{2}x_1(x_2-2) + 2(x_2-2)^2\right]$

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Examples:

1. Consider a bivariate normal population with $\mu_1 = 0, \mu_2 = 2$
$\sigma_{11} = 2, \sigma_{22} = 1, \rho_{12} = 1/2$.

Find ① write down the bivariate normal density.

② write down the squared generalised distance expression $(\underline{x}-\underline{\mu})' \Sigma^{-1} (\underline{x}-\underline{\mu})$ as a function of x_1, x_2.

→ We know that the multivariate normal distribution function

$$f(x) = \frac{1}{(\sqrt{2\pi})^p \sqrt{|\Sigma|}} \exp\left\{-\frac{1}{2} (\underline{x}-\underline{\mu})' \Sigma^{-1} (\underline{x}-\underline{\mu})\right\} \quad \text{— ①}$$

Here $p = 2$, $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 2 & \sigma_{12} \\ \sigma_{21} & 1 \end{bmatrix}$$

To find σ_{12} & σ_{21},

$$\rho_{12} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} \Rightarrow \frac{1}{2} = \frac{\sigma_{12}}{\sqrt{2}\sqrt{1}}$$

$$\therefore \sigma_{12} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \sigma_{12} = \sigma_{21} = \frac{1}{\sqrt{2}}$$

Hence $\Sigma = \begin{bmatrix} 2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{bmatrix}$

$$|\Sigma| = 2 - \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) = 2 - \frac{1}{2} = \frac{3}{2}$$

$$\sqrt{|\Sigma|} = \sqrt{3/2}$$

and $\Sigma^{-1} = \frac{2}{3} \begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 2 \end{bmatrix}$

Now, $(\underline{x}-\underline{\mu}) = \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} = \begin{bmatrix} x_1 - 0 \\ x_2 - 2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - 2 \end{bmatrix}$

$$[\underline{x}-\underline{\mu}]' = [x_1 \quad x_2 - 2]$$

From eq^n ①

$$f(x) = \frac{1}{(\sqrt{2\pi})^2 \sqrt{3/2}} \cdot \exp\left\{-\frac{1}{2}\,[x_1 \quad x_2-2]\,\frac{2}{3}\begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2-2 \end{bmatrix}\right\}$$

$$= \frac{\sqrt{2}}{2\pi\sqrt{3}} \exp\left\{-\frac{1}{3}\left[x_1 - \frac{1}{\sqrt{2}}(x_2-2) \quad -\frac{x_1}{\sqrt{2}} + 2(x_2-2)\right]\begin{bmatrix} x_1 \\ x_2-2 \end{bmatrix}\right\}$$

$$= \frac{1}{\pi\sqrt{6}} \exp\left\{-\frac{1}{3}\left(x_1\left(x_1 - \frac{1}{\sqrt{2}}(x_2-2)\right) + (x_2-2)\left(-\frac{x_1}{\sqrt{2}} + 2(x_2-2)\right)\right)\right\}$$

$$= \frac{1}{\pi\sqrt{6}} \exp\left\{-\frac{1}{3}\left(x_1^2 - \frac{1}{\sqrt{2}}x_1(x_2-2) - \frac{x_1}{\sqrt{2}}(x_2-2) + 2(x_2-2)^2\right)\right\}$$

$$= \frac{1}{\pi\sqrt{6}} \exp\left\{-\frac{1}{3}\left(x_1^2 - \sqrt{2}\,x_1(x_2-2) + 2(x_2-2)^2\right)\right\}$$

which is a required bivariate normal density function.

② Required squared generalised distance expression is

$$-\frac{1}{2}(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu}) = -\frac{1}{3}\left(x_1^2 - \sqrt{2}\,x_1(x_2-2) + 2(x_2-2)^2\right)$$

$$\therefore (\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\mu) = \frac{2}{3}\left(x_1^2 - \sqrt{2}\,x_1(x_2-2) + 2(x_2-2)^2\right)$$

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② Find μ & Σ^{-1} so that the following pdf can be written in the form of normal pdf.

$$\frac{1}{2\pi} \exp\left[-\frac{1}{2}(x^2+y^2+4x-6y+13)\right]$$

→ We know that the multivariate normal distribution pdf

$$f(x) = \frac{1}{(\sqrt{2\pi})^p \sqrt{|\Sigma|}} \exp\left[-\frac{1}{2}(\underline{x}-\underline{\mu})' \Sigma^{-1} (\underline{x}-\underline{\mu})\right]$$

Here $p=2$ ie x & y

consider,

$$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \text{ie } [\underline{x}-\underline{\mu}] = \begin{bmatrix} x-\mu_1 \\ y-\mu_2 \end{bmatrix}$$

$$\Sigma^{-1} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$\therefore (\underline{x}-\underline{\mu})' \Sigma^{-1} (\underline{x}-\underline{\mu}) = \underset{1\times 2}{\begin{bmatrix} x-\mu_1 & y-\mu_2 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} a & b \\ b & c \end{bmatrix}} \begin{bmatrix} x-\mu_1 \\ y-\mu_2 \end{bmatrix}$$

$$= \begin{bmatrix} a(x-\mu_1)+b(y-\mu_2) & b(x-\mu_1)+c(y-\mu_2) \end{bmatrix} \begin{bmatrix} x-\mu_1 \\ y-\mu_2 \end{bmatrix}$$

$$= \begin{bmatrix} ax-a\mu_1+by-b\mu_2 & bx-b\mu_1+cy-c\mu_2 \end{bmatrix} \begin{bmatrix} x-\mu_1 \\ y-\mu_2 \end{bmatrix}$$

$$= \left[(x-\mu_1)(ax-a\mu_1+by-b\mu_2) + (y-\mu_2)(bx-b\mu_1+cy-c\mu_2)\right]$$

$$= \left[ax^2 - ax\mu_1 + bxy - b\mu_2 x - \mu_1 ax + a\mu_1^2 - b\mu_1 y + b\mu_1\mu_2 + bxy - b\mu_1 y + cy^2 - c\mu_2 y - bx\mu_2 + b\mu_1\mu_2 - c\mu_2 y + c\mu_2^2\right]$$

$$= ax^2 + cy^2 + x(-a\mu_1 - b\mu_2 - \mu_1 a - b\mu_2) + y(-b\mu_1 - b\mu_1 - c\mu_2 - c\mu_2) + xy(b+b) + (a\mu_1^2 + b\mu_1\mu_2 + b\mu_1\mu_2 + c\mu_2^2)$$

$$= ax^2 + cy^2 + x(-2a\mu_1 - 2b\mu_2) + y(-2b\mu_1 - 2c\mu_2) + 2bxy + (a\mu_1^2 + c\mu_2^2 + 2b\mu_1\mu_2)$$

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by comparing with given pdf

$$= x^2 + y^2 + 4x - 6y + 3$$

Here $a = 1$, $c = 1$ $\quad 2bxy = 0$

$\Rightarrow 2b = 0 \Rightarrow b = 0$

coeff of x

$$-2a\mu_1 - 2b\mu_2 = 4$$
$$-2a\mu_1 = 4$$
$$-2\mu_1 = 4$$
$$\mu_1 = -2$$

coeff of y.

$$-2b\mu_1 - 2c\mu_2 = -6$$
$$-2c\mu_2 = -6$$
$$-2\mu_2 = -6$$
$$\mu_2 = 3.$$

Hence,

$$\mu = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \quad \& \quad \Sigma^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a\mu_1^2 + c\mu_2^2 + 2b\mu_1\mu_2 = 13$$
$$1(-2)^2 + 1(3)^2 + 2(0) = 13.$$
$$4 + 9 = 13$$

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One variable Jacobian Transformation Method :-

→ Let variable x & it's function $f(x)$ convert to variable y and it's function $f(y)$.

$\therefore \quad f(y) = f(x) \cdot |J|$ Here $J = \frac{dx}{dy}$

Ex. Given $f(x) = 2x$, $0 < x < 1$ find function of $y = 3x+1$

Here, $y = 3x+1$

i.e $x = \frac{y-1}{3}$ and $\frac{dx}{dy} = \frac{1}{3}$ $\therefore \left|\frac{dx}{dy}\right| = \left|\frac{1}{3}\right| = \frac{1}{3}$

$$f(y) = f\left(\frac{y-1}{3}\right)\left(\frac{1}{3}\right) = 2\left(\frac{y-1}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{9}(y-1)$$

limits, if $x = 0$, $y = 1$ & if $x = 1$, $y = 4$

$$\therefore \quad f(y) = \frac{2}{9}(y-1) \quad , \quad 1 < y < 4$$

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Note: The random variables $X_1, X_2, \ldots, X_p$, $p \geq 2$ are said to be jointly normal or multivariate normal distribution if there exist p independent random variable $Z_i \sim N(0,1)$ constants μ_i and $p \times p$ non singular matrix $A = (a_{ij})$ of real number such that $\boxed{X = \mu + AZ}$

Property: Let X_i, $i = 1, 2, \ldots, p$ be multivariate normal distributed random variables then find it's mean μ & covariance matrix Σ

→ By definition of multi-normality we know that,

$$X = \mu + AZ$$

where μ is const, A is $p \times p$ nonsingular matrix.

and $Z \sim N_p(0, I)$

since $Z \sim N_p(0, I)$ then $E(Z) = 0$, $V(Z) = I$

$$E(X) = E(\mu + AZ) = E(\mu) + A\,E(Z)$$
$$= \mu + A(0) = \mu.$$

$$\text{covariance matrix} = \Sigma = \text{cov}(X) \cancel{= \text{cov}(}$$
$$= E[(X-\mu)(X-\mu)']$$
$$= E[(AZ)(AZ)']$$
$$= E[AZZ'A']$$
$$= \cancel{E\,E\,A}\; A\,E(ZZ')A' = AIA'$$
$$\Sigma = AA'$$

$(AZ)' = Z'A'$

$E(ZZ') = I$

$V(X) = E(X^2) - [E(X)]^2$

$V(Z) = E(Z^2) - E(Z)$

$I = E(ZZ') - 0$

$I = E(ZZ')$.

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Property | If $X \sim N_p(\mu, \Sigma)$ & B is a nonsingular $(p \times p)$ matrix then $BX \sim N_p(B\mu, B\Sigma B')$.

→ We know that,

$$X \sim N_p(\mu, \Sigma)$$

by definition of multi-normality $X = \mu + AZ$
where $Z \sim N_p(0, I)$ & A is nonsingular matrix

Also $\Sigma = AA'$

$$\therefore \quad X = \mu + AZ$$

$$BX = \underline{B\mu} + \underline{BAZ}$$

$\therefore$ BA is non singular matrix.

Here $E(BX) = B\mu$ and $C = (BA)(BA)'$

$$= (BAA'B')$$

$$= B\Sigma B'$$

Hence, $BX \sim N_p(B\mu, B\Sigma B')$

Example: Given $X \sim N_3(0, I_3)$ & $Y_1 = X_2 + X_3$, $Y_2 = 2X_3 - X_1$ & $Y_3 = X_2 - X_1$ then find the mean & variance covariance matrix of Y,

→ Given, $X \sim N_3(0, I_3)$

i.e $E(X) = 0$, $V(X) = I_3 = \Sigma$

$p = 3$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mu = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

~~W.k.T. $X \sim N_p(\mu, \Sigma)$ then~~

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\therefore \quad Y = BX$$

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WKT $X \sim N(\mu, \Sigma)$ then $BX \sim N(B\mu, B\Sigma B')$

Here, $y = BX \sim N(B\mu, B\Sigma B')$ & B is nonsingular

$$|B| = \begin{vmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ -1 & 1 & 0 \end{vmatrix} = 0(0-2) - 1(0+2) + 1(-1-0)$$

$$= 0 - 2 - 1 = -3 \neq 0$$

$\Rightarrow B$ is nonsingular matrix.

$$B\mu = \underset{3\times 3}{\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

& covariance matrix of $Y = B\Sigma B'$

$$\therefore B\Sigma B' = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$ - it is symmetric matrix.

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Example: Let $X \sim N_3(\mu, \Sigma)$ with $\mu = (-3, 1, 4)'$ & $\Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

obtain the distribution of $\frac{x_1 + x_2}{2}$ & x_3.

→ Given,

$X \sim N_3(\mu, \Sigma)$

$$\mu = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} \quad \& \quad \Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- Take,

$$y_1 = \frac{x_1 + x_2}{2}, \quad y_2 = x_3$$

$$Y = BX = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Here,

$$B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B \text{ is not a square matrix.}$$

- ~~~~ it is singular matrix.

$$\rho(B) = 2 < 3$$

$$\therefore Y_2 \sim N_2(B\mu, B\Sigma B')$$

$$\therefore B\mu = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$B\Sigma B' = \underset{2\times 3}{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$$

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Example: Let $X \sim N(\mu, \Sigma)$, where $X = (X_1, X_2, X_3)'$ & $\mu = [0, 0, 0]'$

$$\Sigma = \begin{bmatrix} 7 & 3 & 2 \\ 3 & 4 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$ obtain the distribution of $y = X_1 + 2X_2 - 3X_3$.

→ Given, $X \sim N(\mu, \Sigma)$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \quad \mu = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 7 & 3 & 2 \\ 3 & 4 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$y = X_1 + 2X_2 - 3X_3 = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$= BX$$

$\therefore B = [1 \quad 2 \quad -3]$ $\quad \rho(B) = 1 < 3$

ie B is singular.

$\therefore Y = N_1(B\mu, B\Sigma B')$

$$\therefore B\mu = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$B\Sigma B' = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 7 & 3 & 2 \\ 3 & 4 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= 29$$

$\therefore Y \sim N_1(0, 29)$

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* Independence of subvector property:

If $x_1, x_2, \ldots, x_p$ have a joint normal distributions a necessary + sufficient condition for one subset of the random variable and the subset consisting of the remaining variables to be independent is that each covariance of a variable from one set and a variable from other set is zero.

Theorem:

Let ~~$X \sim N_{p+q}$~~ $X \sim N_{p+q}(\mu, \Sigma)$ where $X = \begin{bmatrix} X_{(1)} \\ X_{(2)} \end{bmatrix}$, $\mu = \begin{bmatrix} \mu_{(1)} \\ \mu_{(2)} \end{bmatrix}$

and $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$ where $X_{(1)}$ + $\mu_{(1)}$ are p-dimensional

" $X_{(2)}$ + $\mu_{(2)}$ " "

Σ is $p \times p$ matrix.

A necessary + sufficient condition that $X_{(1)}$ + $X_{(2)}$ are independent is $\operatorname{cov}(X_{(1)}, X_{(2)}) = 0$

Examples:- Let $X \sim N_3(\mu, \Sigma)$ with $\mu = [-3, 1, 4]$ + $\Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ which of the following random variables are independent & Justify yours answers

1. X_1 + X_2.
2. X_2 + X_3
3. (X_1, X_2) + X_3
4. $\frac{X_1 + X_2}{2}$ + X_3
5. X_2 & $X_2 - \frac{5}{2}X_1 - X_3$.

→ Given $X \sim N_3(\mu, \Sigma)$

$\mu = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

① X_1 + X_2

$\operatorname{cov}(X_1, X_2) = -2 \neq 0$

$\therefore$ X_1 + X_2 ~~are~~ not independent.

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② X_2 & X_3

$\therefore \text{cov}(X_2, X_3) = 0 \Rightarrow X_2$ & X_3 are independent.

③ (X_1, X_2) & X_3.

$\cancel{X^{(1)} + X^{(2)}}$ Let $X^{(1)} = (X_1, X_2)$, $X^{(2)} = X_3$

$$\Sigma = \left[\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 5 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]$$

$\text{cov}(X^{(1)}, X^{(2)}) = 0$

$\therefore (X_1, X_2)$ & X_3 are independent.

④ $\frac{X_1+X_2}{2}$, X_3

Let $y_1 = \frac{X_1+X_2}{2}$, $y_2 = X_3$.

$$y = BX = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$Y = BX \sim N[B\mu, B\Sigma B']$$

$$\therefore B\Sigma B' = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} \frac{1}{2} & 0 \\ 0 & 2 \end{array}\right] \qquad \therefore \text{Cov}(y_1, y_2) = 0$$

$\therefore \frac{X_1+X_2}{3}$ & X_3 are independent

(5) X_2 + $X_2 - \frac{5}{2}X_1 - X_3$

let $y_1 = X_2$, $y_2 = X_2 - \frac{5}{3}X_1 - X_3$.

$$\therefore\ y = BX = \begin{bmatrix} 0 & 1 & 0 \\ -5/2 & 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$\therefore\ Y \sim N\left[B\mu,\ B\Sigma B'\right]$$

$$\therefore\ B\Sigma B' = \underset{2\times3}{\begin{bmatrix} 0 & 1 & 0 \\ -5/2 & 1 & -1 \end{bmatrix}} \underset{3\times3}{\begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} 0 & -5/2 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}}$$

$$= \begin{bmatrix} 5 & 10 \\ 10 & 93/4 \end{bmatrix}$$

Here $\mathrm{cov}(y_1, y_2) = 10 \neq 0$

$\therefore$ X_2 + $X_2 - \frac{5}{2}X_1 - X_3$ are not independent.

Conditional Distribution :-

Let $X = \begin{bmatrix} X_{(1)} \\ X_{(2)} \end{bmatrix}$ be the partition of X & $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$

then,

$$X_{(1)} / X_{(2)} \sim N_p\left(\mu_{1/2}, \Sigma_{1/2}\right)$$

where

$$\mu_{1/2} = \mu_{(1)} + \Sigma_{12}\Sigma_{22}^{-1}\left(X_{(2)} - \mu_{(2)}\right)$$

$$\Sigma_{1/2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

Example: Let $X \sim N_3(\mu, \Sigma)$ with $\mu = [2 \ -3 \ 1]'$ & $\Sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$

Find conditional distribution of X_3 given $X_1 = x_1$ & $X_2 = x_2$

→ Given,

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \quad \mu = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

$E(X_1) = 2$, $V(X_1) = 1$, $cov(X_1, X_2) = 1$

$E(X_2) = -3$, $V(X_2) = 3$, $cov(X_2, X_3) = 2$

$E(X_3) = 1$, $V(X_3) = 2$, $cov(X_1, X_3) = 1$.

To find X_3 given $X_1 = x_1$ + $X_2 = x_2$

Let $X_3 = X_{(1)}$ & $X_{(2)} = (X_1, X_2)$

$$\therefore \mu = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \quad \Sigma = \begin{array}{cc} & \begin{array}{ccc} X_3 & X_1 & X_2 \end{array} \\ \begin{array}{c} X_3 \\ X_1 \\ X_2 \end{array} & \left[\begin{array}{c|cc} 2 & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 1 & 3 \end{array}\right] \end{array}$$

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We know that,

$X_{(1)}/X_{(2)} \sim N(\mu_{1/2}, \Sigma_{1/2})$

$\mu_{1/2} = \mu_{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(X_{(2)} - \mu_{(2)})$

$\Sigma_{1/2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

$\Sigma_{21} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\Sigma_{11} = 2$

~~$\mu_{1/2} = 1 +$~~ Here $X_{(2)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mu_{(2)} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $\Sigma_{12} = [1 \;\; 2]$

$\Sigma_{22} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

$\Sigma_{22}^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$

- $\mu_{1/2} = \mu_{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(X_{(2)} - \mu_{(2)})$

$= 1 + \underset{1\times 2}{(1 \;\; 2)}\underset{2\times 2}{\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}}\begin{bmatrix} x_1 - 2 \\ x_2 + 3 \end{bmatrix}$

$= 1 + \left[\frac{1}{2} \;\; \frac{1}{2}\right]\begin{bmatrix} x_1 - 2 \\ x_2 + 3 \end{bmatrix} = 1 + \frac{1}{2}(x_1 - 2) + \frac{1}{2}(x_2 + 3)$

$= 1 + \frac{x_1}{2} - 1 + \frac{1}{2}x_2 + 3/2$

$\mu_{1/2} = 1.5 + 0.5x_1 + 0.5x_2$

- $\Sigma_{1/2} = 2 - [1 \;\; 2]\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= 2 - \left[\frac{1}{2} \;\; \frac{1}{2}\right]\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= 2 - \left[\frac{1}{2} + 1\right] = 2 - \frac{3}{2} = \frac{1}{2}$

$\therefore X_3/(X_1, X_2) \sim N\left(1.5 + 0.5x_1 + 0.5x_2, \frac{1}{2}\right)$

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Remark: M.L.E of Multivariate Normal Distribution.

M.L.E of μ is $\hat{\mu} = \bar{x}$.

M.L.E of Σ is $\hat{\Sigma} = \frac{1}{n} \Sigma (x_i - \bar{x})(x_i - \bar{x})'$

Example: Find MLE of the mean vector μ of (2×1) + ~~variance~~ covariance matrix Σ of order 2×2 on the random sample from $N_2(\mu, \Sigma)$

$$X = \begin{bmatrix} 3 & 6 \\ 4 & 4 \\ 5 & 7 \\ 4 & 7 \end{bmatrix}$$

→ let X_1, X_2, X_3, X_4 are from normal popⁿ.

~~with mea~~

To find μ + Σ of order (2×1) + (2×2).

$$\mu = \begin{bmatrix} (3+4+5+4)/4 \\ (6+4+7+7)/4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}_{2\times1}$$

for Σ, calculate Σ_{11}, $\Sigma_{12} = \Sigma_{21}$, Σ_{22}.

$$\Sigma_{11} = \frac{1}{n} \Sigma (x_1 - \bar{x})^2 = \frac{1}{4}\left[(3-4)^2 + (4-4)^2 + (5-4)^2 + (4-4)^2\right]$$

$$= 2/4 = 1/2$$

$$\Sigma_{12} = \frac{1}{n} \Sigma (x_1 - \bar{x})(x_2 - \bar{x}) = \frac{1}{4}\left[(-1)(0) + (0)(-2) + (1)(1) + (0)(1)\right] = 1/4.$$

$$\Sigma_{21} = \Sigma_{12} = 1/4.$$

$$\Sigma_{22} = \frac{1}{n} \Sigma (x_2 - \bar{x})^2 = \frac{1}{4}\left[(6-6)^2 + (4-6)^2 + (7-6)^2 + (7-6)^2\right]$$

$$= \frac{1}{4}\left[0+4+1+1\right] = 6/4 = 3/2$$

MLE of Σ is,

$$\Sigma = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 3/2 \end{bmatrix}$$