AI MATH Exercises

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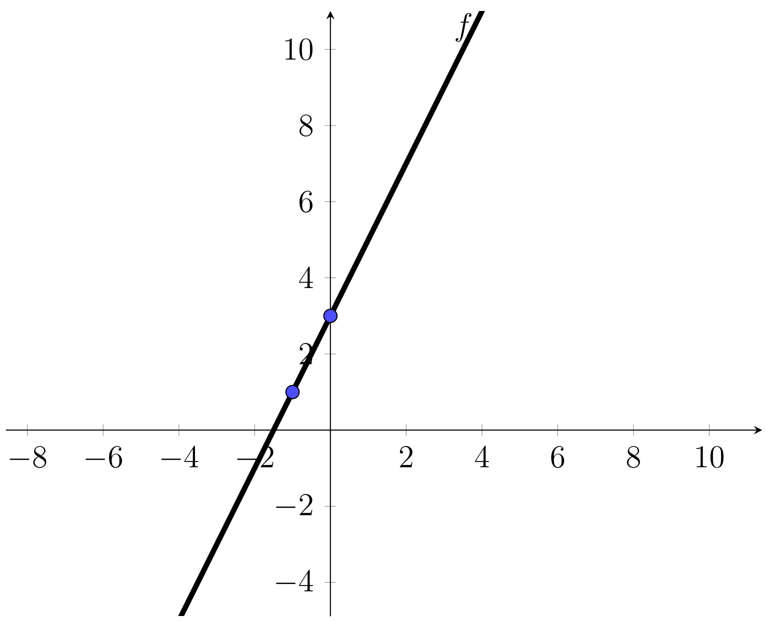
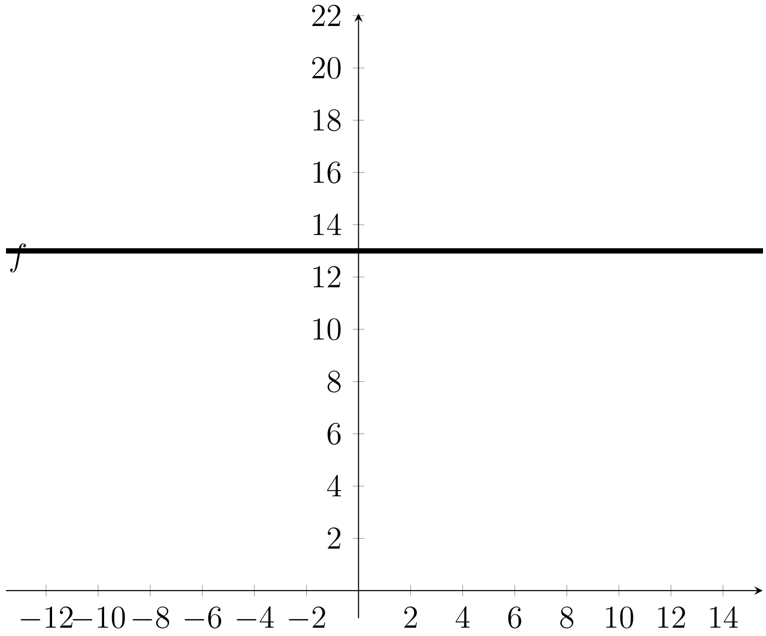
1 Functions

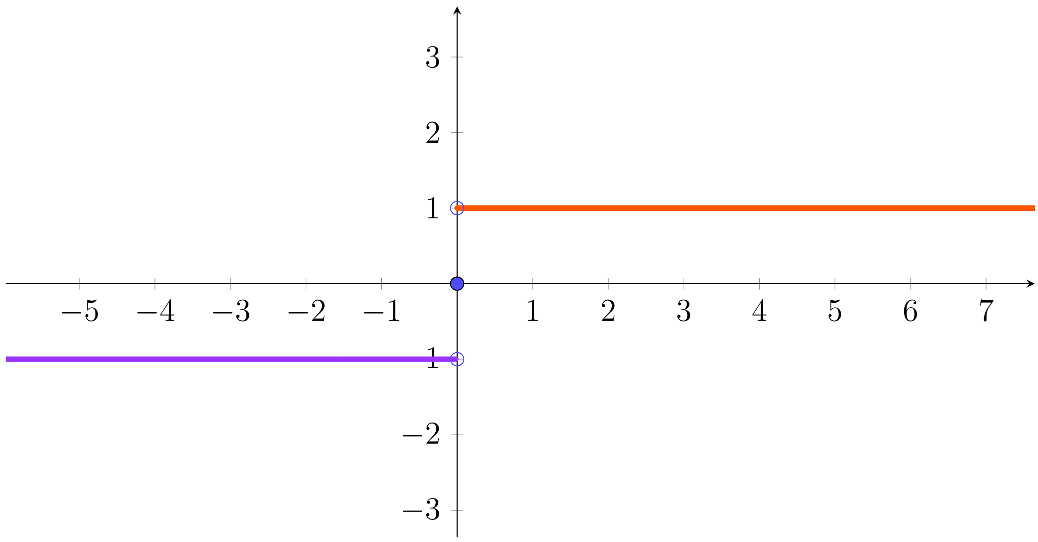
Exercise 1: The function $f(x) = 2x + 3$ is a line. TO plot it we choose two different values for x and calculate their corresponding y values. We draw the points defined by the x's and their y values. Finally we draw a line that passes through these two points. For our example we choose $x = 0, -1$. We have $y = 3, 1$. The plot is then



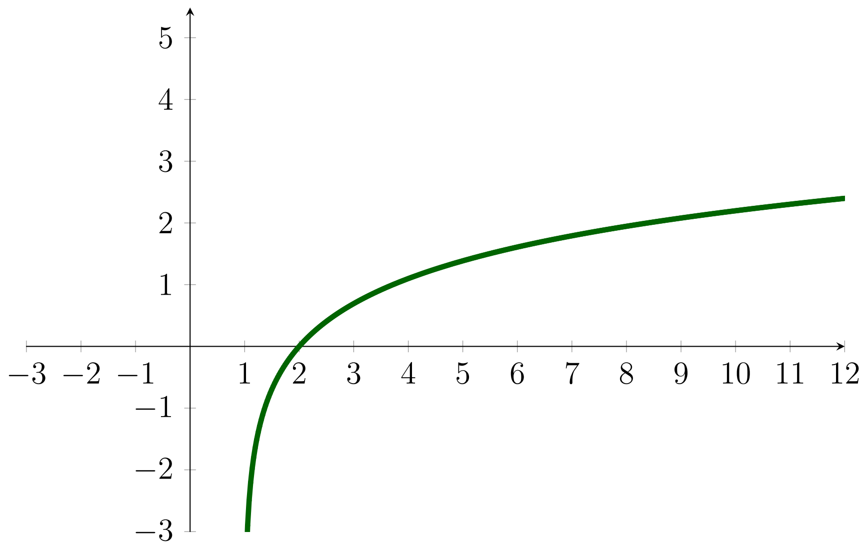
The function $f(x) = 13$ is constant hence a horizontal line.



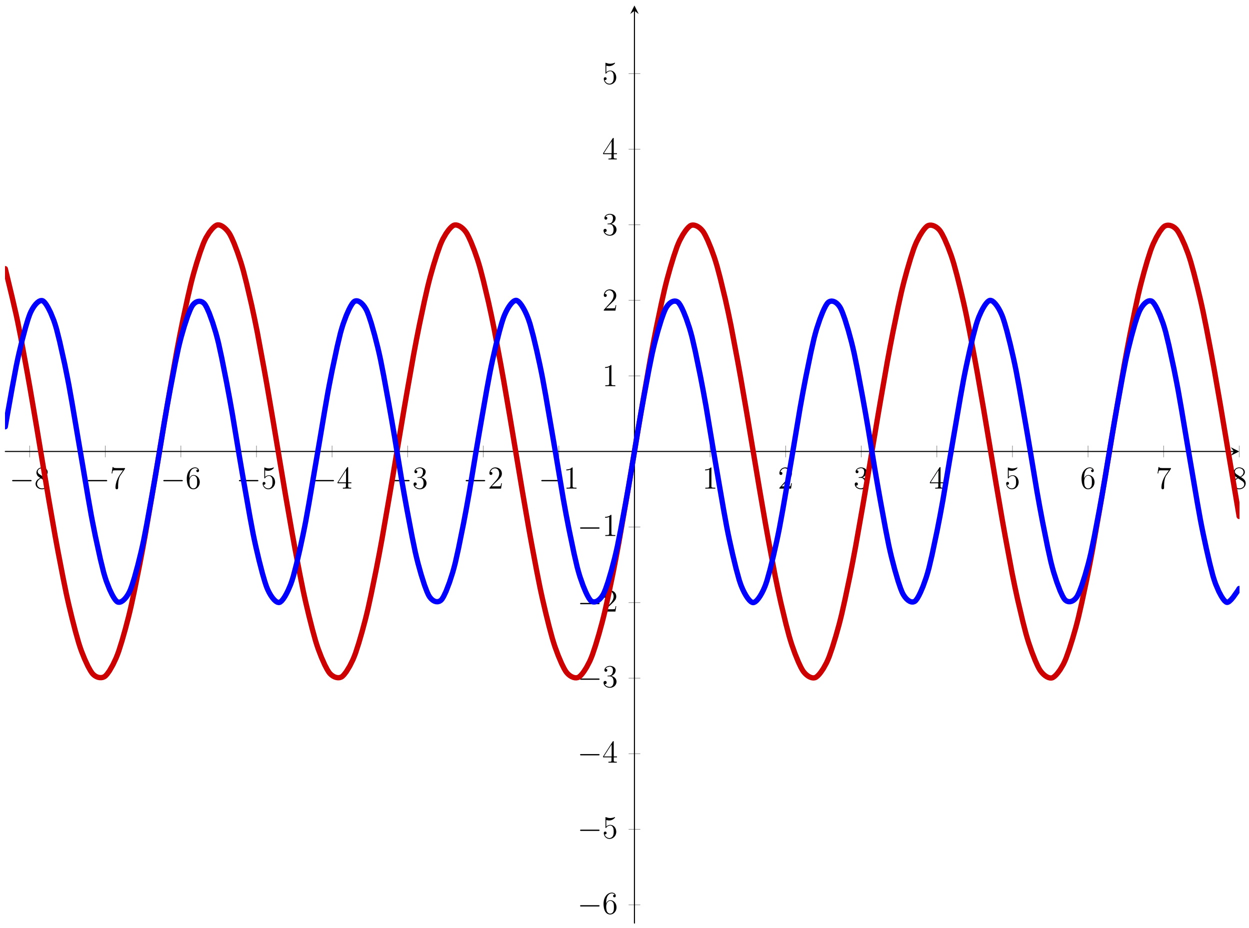
The function $sgn(x)$ is constant on three different regions. When x is larger than 0 is constant and equals 1, when x is less than 0 it is constant and equals -1, and when $x = 0$ its value is 0. Thus we plot is as



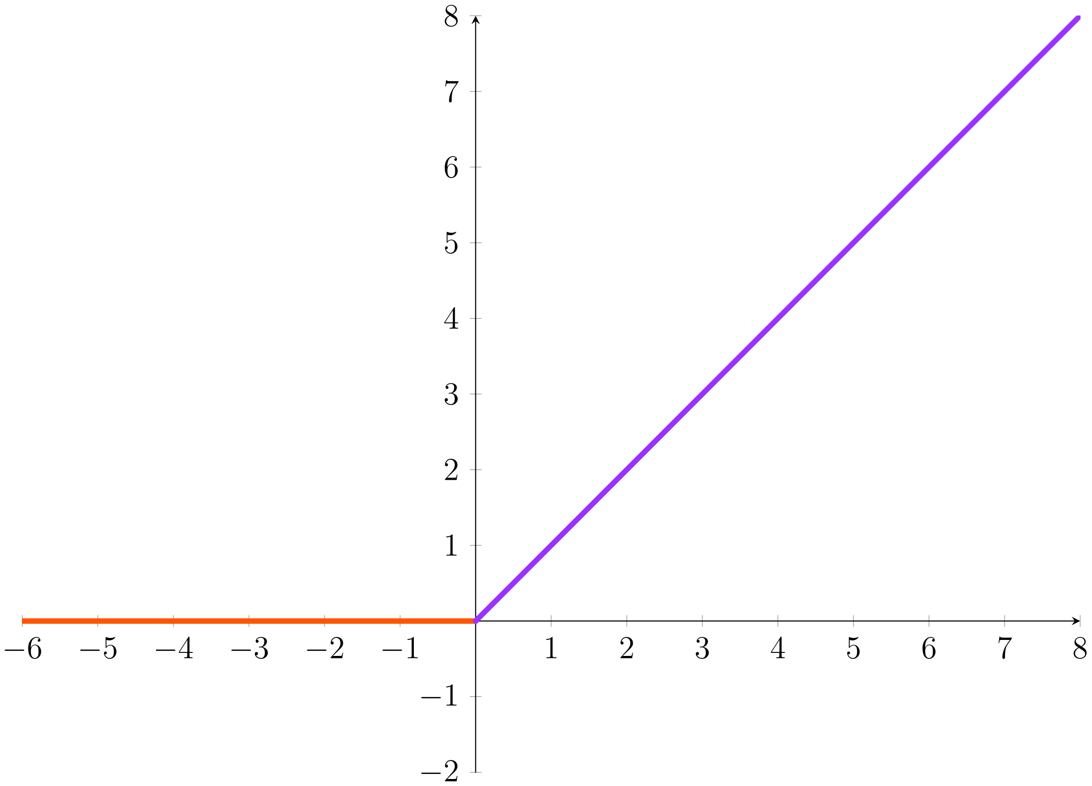
To plot the function $f(x) = log(x - 1)$ we first remember plot $log(x)$ and translate it 1 unit to the right. The log function is increasing to infinity but slowly and very close to 0 from right it become too small and approaches $-\infty$. log is undefined for negative x. Putting all of this together we get the plot.



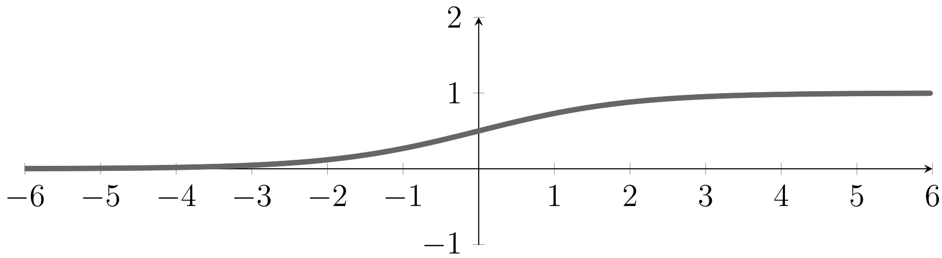
To plot the functions $3\sin(2x)$ and $2\sin(3x)$ we first look at $\sin(x)$. sin is between -1 and 1. Multiply sin by a number changes this bound. We also know that sin is periodic with period 2π. Multiplying the argument (x) of sin by a number changes the period. Therefore $3\sin(2x)$ has is between -3 and 3 and has period $2\pi/2 = \pi$ while the function $2\sin(3x)$ is between -2 and 2 and has period $2\pi/3$.



From the definition the function ReLU is constant and equals 0 when $x < 0$ and it coincides with the line $y = x$ when $x \geq 0$. Thus the plot is



We first need to look at the behaviour of the function $\sigma = \frac{1}{1+e^{-x}}$. The function e^x increases to infinity quickly as x increases and decreases to 0 quickly as x decreases (try to plot it and compare it with a plotting program). The function e^{-x} is decreasing and behaves the other way. Since the reciprocal of a positive decreasing function is increasing the function σ is increasing. Now we look at the asymptotic behaviour of σ. When x is very large, the value of e^{-x} is too small and hence $1 + e^{-x} \approx 1$ that is $\sigma(x) \approx 1$ when x is large. When x is very small (negative) the value of e^{-x} is extremely large thus the value $\frac{1}{1+e^{-x}} \approx 0$. From this analysis we can plot the following



Exercise 2:

The function $\frac{x}{1-x}$ is defined everywhere except when the denominator is zero which happens only when $x = 1$. Therefore the domain is $\mathbb{R} - \{1\}$ which can also be written as $(-\infty, 1) \cup (1, \infty)$.

The function $\frac{x-2}{x^2-3x+2}+\frac{1}{x+5}$ is defined everywhere except when at least one of the denominators is zero. The roots of $x^2-3x+2=(x-2)(x-1)$ are $x=1,2$ and second denominator equals zero exactly when $x=-5$. Therefore the domain is $\mathbb{R}-\{-5,1,2\}=(-\infty,-5)\cup(-5,1)\cup(1,2)\cup(2,\infty)$

The function $\log\left(\frac{1}{x-7}\right)$ is defined when the input of log is positive and when the denominator is not zero. This happens when $x>7$. The domain is $(7,\infty)$

Exercise 3: We are given $f(x+1)=x^2-x+1$. To find $f(x)$ we simply substitute $x-1$ in place of x and expand everything

$$f(x)=(x-1)^2-(x-1)+1=(x^2-2x+1)-(x-1)+1=x^2-3x+3$$

To find $f(x-1)$ we substitute $x-2$

$$f(x-1)=(x-2)^2-(x-2)+1=(x^2-4x+4)-(x-1)+1=x^2-5x+6$$

Exercise 4: This exercise is closely related to calculating the derivative (recall the definition of derivative).
For $f(x)=2x+3$ we have

$$\frac{f(x+h)-f(x)}{h}=\frac{(2(x+h)+3)-(2x+3)}{h}=\frac{2h}{h}=2$$

For the function $f(x)=5x^2$ we have

$$\frac{f(x+h)-f(x)}{h}=\frac{5(x+h)^2-5x^2}{h}=\frac{5(x^2+2hx+h^2)-5x^2}{h}=\frac{10hx+5h^2}{h}=10x+5h$$

Notice that the function $f(x)=5x^2-2x-3$ is the subtraction of the previous two functions, therefore we can quickly we that the answer is $10x+5h-2$

Exercise 5: There are several ways to do this task. Here is one: First thing to notice is that it is not a vertical line since the x components of the points are not the same. The equation of a line has the formula $y=mx+b$. Substituting the pair $x=3$ and $y=4$ and the pair $x=0$ and $y=5$ we have the linear system

$$\begin{cases} 4=3m+b \\ 5=b \end{cases}$$

The solution of this system is $m=-1/3$ and $b=5$. The equation of the line is $y=-\frac{1}{3}x+5$.

Exercise 6: To compute $g\circ f\circ h$ we calculate $h(x)$, $f(h(x))$ then $g(f(h(x)))$. $h(x)=(\sin(x),\cos(x))$, $f(h(x))=(\sin^2(x),\cos^2(x))$ and finally $g(f(h(x)))=\sin^2(x)+\cos^2(x)=1$.

Exercise 7:

$$g(f(x))=\log_2(f(x)+1)=\log_2(2^x-1+1)=x$$

$$f(g(x)) = 2^{g(x)} - 1 = 2^{\log_2(x+1)} - 1 = (x+1) - 1 = x$$

Exercise 8:

- We use chain rule. The derivative of $\sin(x)$ is $\cos(x)$ and the derivative of $3x$ is 3. Thus
$$(\sin(3x))' = 3\cos(3x)$$
- We use the product rule
$$(x\ln(x))' = (1)\ln(x) + x\frac{1}{x} = \ln(x) + 1$$
-
$$f(x) = x\sqrt{x} + \sqrt[3]{x} = x^{3/2} + x^{1/3}$$
Therefore
$$f'(x) = \frac{3}{2}x^{3/2-1} + \frac{1}{3}x^{1/3-1} = \frac{3}{2}\sqrt{x} + \frac{1}{3}\frac{1}{\sqrt[3]{x^2}}$$
- We use the chain rule. The derivatives of $\tan(x)$ is $\sec^2(x)$ and the derivative of x^2+1 is $2x$. Thus
$$(\tan(x^2+1))' = 2x\sec^2(x^2+1)$$
- We use the chain rule. The derivative of x^6 is $6x^5$ and the derivative of $e^x + x^3 - 1$ is $e^x + 3x^2$. Thus
$$\left((e^x + x^3 - 1)^6\right)' = 6(e^x + 3x^2)(e^x + x^3 - 1)^5$$

Exercise 9:

- $f(x) = x^2 - 5x + 1$. The derivative is $f'(x) = 2x - 5$. The solution of $f'(x) = 0$ is $x = 5/2 = 2.5$. When $x > 2.5$ the derivative is positive (take a value and substitute) and when $x < 2.5$ the derivative is negative. Therefore the function decreases on $(-\infty, 2.5)$ and increases on $(2.5, \infty)$. This means that the critical point $x = 2.5$ is a local minimum and is actually the global minimum. Therefore the minimum value of f is $f(2.5) = (2.5)^2 - 5(2.5) + 1 = -5.25$.
- $f(x) = \sin(x) + \cos(x)$. Since sin and cos are periodic with on $[0, 2\pi]$ it is enough to study f on this interval. We differentiate $f'(x) = \cos(x) - \sin(x)$. We solve $f'(x) = 0$. The solutions of this equation satisfy $\cos(x) = \sin(x)$ which are $x = \pi/4$ or $x = 5\pi/4$. We have $f(\pi/4) = 2/\sqrt{2}$ and $f(5\pi/4) = -2/\sqrt{2}$ and these values are the maximum and minimum respectively. For the monotonic behaviour notice that the function $\cos(x) - \sin(x)$ is positive on $[0, \pi/4)$ (check it by considering a value and substituting it) and negative on $(\pi/4, 5\pi/4)$ then again positive on $(5\pi/4, 2\pi]$. This translates to f is increasing on $[0, \pi/4) \cup (5\pi/4, 2\pi]$ and decreasing on $(\pi/4, 5\pi/4)$.
- $f(x) = (2x-1)^2 + (3x-5)^2$. We have $f'(x) = (2)(2)(2x-1) + (3)(2)(3x-5) = 26x - 34$. Solving $f'(x) = 0$ we get $x = 17/13 \approx 1.3$. Similarly to the first function the derivative in positive after $17/13$ and negative before it. Hence it is the minimum. The minimum value of f is $f(17/13) \approx 3.77$. The function f increases on $(17/13, \infty)$ and decreases on $(-\infty, 17/13)$.

Exercise 12: We form the matrix A and the vector $\mathbf{b}$ as follows

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 5 \end{bmatrix}$$

The parameters (slope and intercept) of the line of best fit are given by

$$\begin{aligned}
& \left(A^T A\right)^{-1} A^T \mathbf{b} \\
= & \left(\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 3 \\ 5 \end{bmatrix} \\
= & \left(\begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 3 \\ 5 \end{bmatrix} \\
= & \frac{1}{20} \begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 3 \\ 5 \end{bmatrix} \\
= & \frac{1}{20} \begin{bmatrix} -6 & -2 & 2 & 6 \\ 13 & 8 & 2 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 3 \\ 5 \end{bmatrix} \\
= & \frac{1}{20} \left[30//10 \right] \\
= & \begin{bmatrix} 3/2 \\ 1/3 \end{bmatrix}
\end{aligned}$$

Therefore the slope of the line of best fit is $3/2 = 1.5$ and its intercept is $1/3 \approx 0.333$.