AI MATH Exercises

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1 Functions

Example: Find the domain of the function

$$f(x) = x + \frac{\sqrt{1-x^2}}{\log(1+x)}$$

Solution: We need to consider the following restrictions

- Square root argument must be non-negative.
- log argument must be positive.
- $\log(1+x)$ must be non-zero since it is in the denominator.

The first condition is $1 - x^2 \geq 0$ which can be manipulated further as

$$\begin{aligned} & 1 - x^2 \geq 0 \\ \Leftrightarrow\ & 1 \geq x^2 \\ \Leftrightarrow\ & 1 \geq |x| \\ \Leftrightarrow\ & 1 \geq x \geq -1 \end{aligned}$$

The second condition is simply $1 + x > 0$ which is equivalent to $x > -1$.

The third condition is $\log(1+x) \neq 0$. Since log equals 0 only when its input is 1 the condition becomes $1 + x \neq 1$ which is simplified to $x \neq 0$.

Putting all these conditions together the domain is $(-1, 0) \cap (0, 1]$. This domain can be represented on the real line as shown in figure 1.

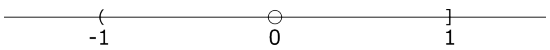

Figure 1: real line representation

Exercise 1: Graph the following functions.

- $f(x) = 2x + 3$
- $f(x) = 13$
- $sgn(x) = \begin{cases} 1 & ; x > 0 \\ 0 & ; x = 0 \\ -1 & ; x < -1 \end{cases}$
- $f(x) = \log(x - 1)$
- $f(x) = 3\sin(2x)$
- $f(x) = 2\sin(3x)$
- $\text{ReLU}(x) = \max(x, 0)$
- $\sigma(x) = \frac{1}{1+e^{-x}}$

Exercise 2: Find the domain of the following functions.

- $f(x) = \frac{x}{x-1}$
- $f(x) = \frac{x-2}{x^2-3x+2} + \frac{1}{x+5}$
- $f(x) = \log\left(\frac{1}{x-7}\right)$

Exercise 3: Let $f(x + 1) = x^2 - x + 1$. Express $f(x)$ and $f(x - 1)$.

Exercise 4: Let $h \neq 0$. Express and simplify the expression $\frac{f(x+h)-f(x)}{h}$ for the following functions

- $f(x) = 2x + 3$.
- $f(x) = 5x^2$.
- $f(x) = 5x^2 - 2x - 3$.

Exercise 5: Write the equation of the line that passes through the points $(3, 4)$ and $(0, 5)$.

Exercise 6: Let

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \text{ defined by } f(x, y) = (x^2, y^2)$$
$$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ defined by } g(x, y) = x + y$$
$$h : \mathbb{R} \to \mathbb{R} \times \mathbb{R} \text{ defined by } h(x) = (\sin(x), \cos(x))$$

Express $g \circ f \circ h$

Exercise 7: Let $f(x) = 2^x - 1$ and $g(x) = \log_2(x + 1)$. What is $g \circ f$ and $f \circ g$?

Exercise 8: Differentiate the following functions

- $f(x) = \sin(3x)$
- $f(x) = x\ln(x)$
- $f(x) = x\sqrt{x} + \sqrt[3]{x}$
- $\tan(x^2 + 1)$

- $(e^x + x^3 - 1)^6$

Challenge: Solve $x = \sin(x)$

Exercise 9: Study the monotonic behaviour of the following functions and find their maximum and minimum:

- $x^2 - 5x + 1$
- $x^4 - 4x^2 + 1$
- $(x^4 - 4x^2 + 1)^4$
- $(x^4 - 4x^2 + 1)^3$
- $\sin(x) + \cos(x)$
- $x \ln(x)$
- $\sqrt{x} \ln(x)$
- $x^3 + x + 1$
- $(2x - 1)^2 + (3x - 5)^2$. Where have we seen this before?

Exercise 10: Check the the functions $x \ln(x)$ and $\sqrt{x} \ln(x)$ have the same tangent line at $x = 1$. What is the equation of the line?

Challenge: How many critical points does $2^x - x^2$ have? Why we only asked about their number, can you find them?

Exercise 11: Apply the following operations if applicable

- $\begin{bmatrix} 1 & 3 & 6 \end{bmatrix} + \begin{bmatrix} -2 & 1 & 0 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \\ -0.5 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \\ -0.5 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 4 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 4 \end{bmatrix}^T$