AI MATH Exercises

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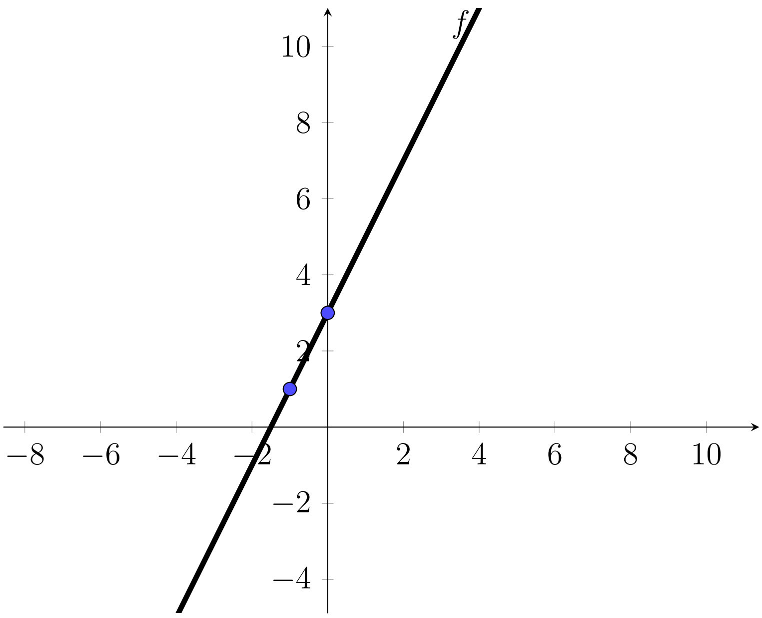
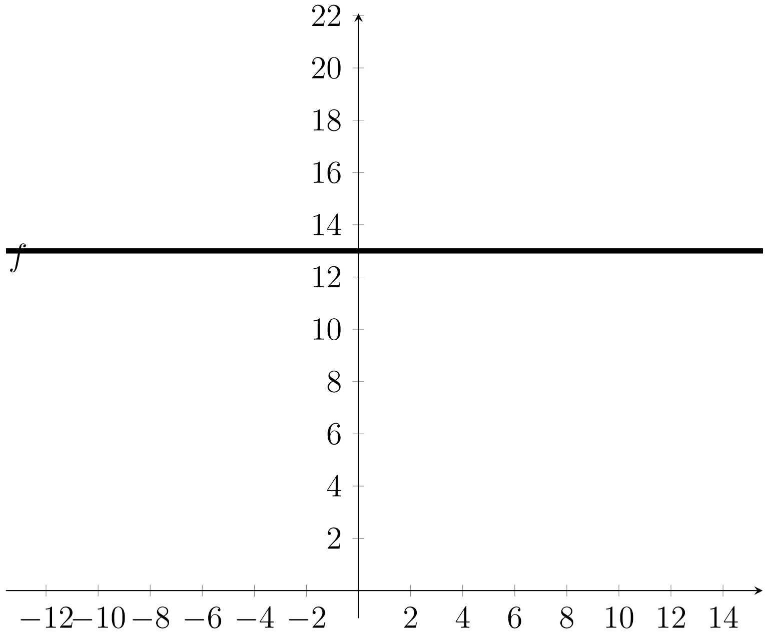
1 Functions

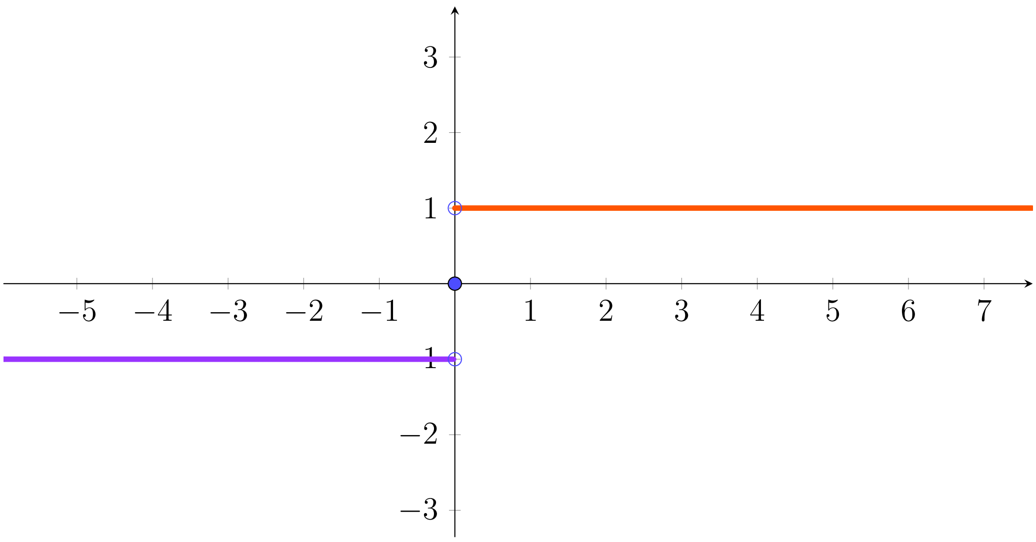
Exercise 1: The function $f(x) = 2x + 3$ is a line. TO plot it we choose two different values for x and calculate their corresponding y values. We draw the points defined by the x's and their y values. Finally we draw a line that passes through these two points. For our example we choose $x = 0, -1$. We have $y = 3, 1$. The plot is then



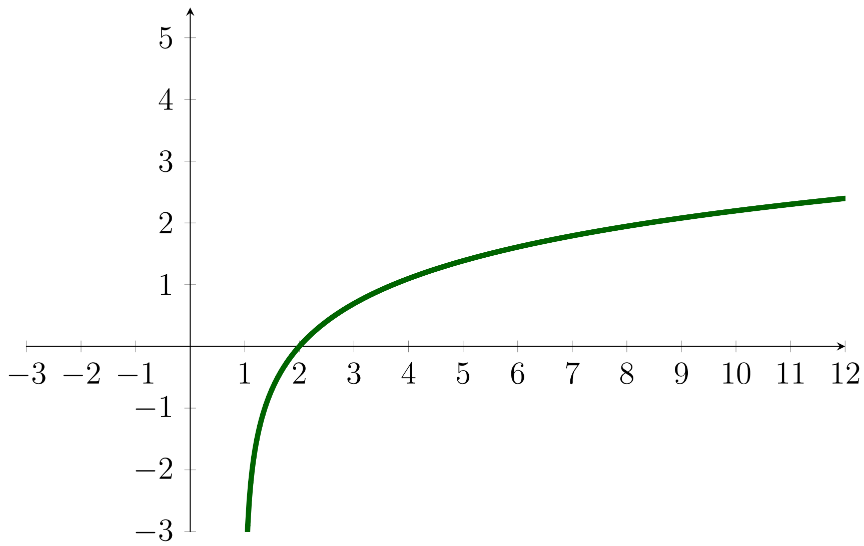
The function $f(x) = 13$ is constant hence a horizontal line.



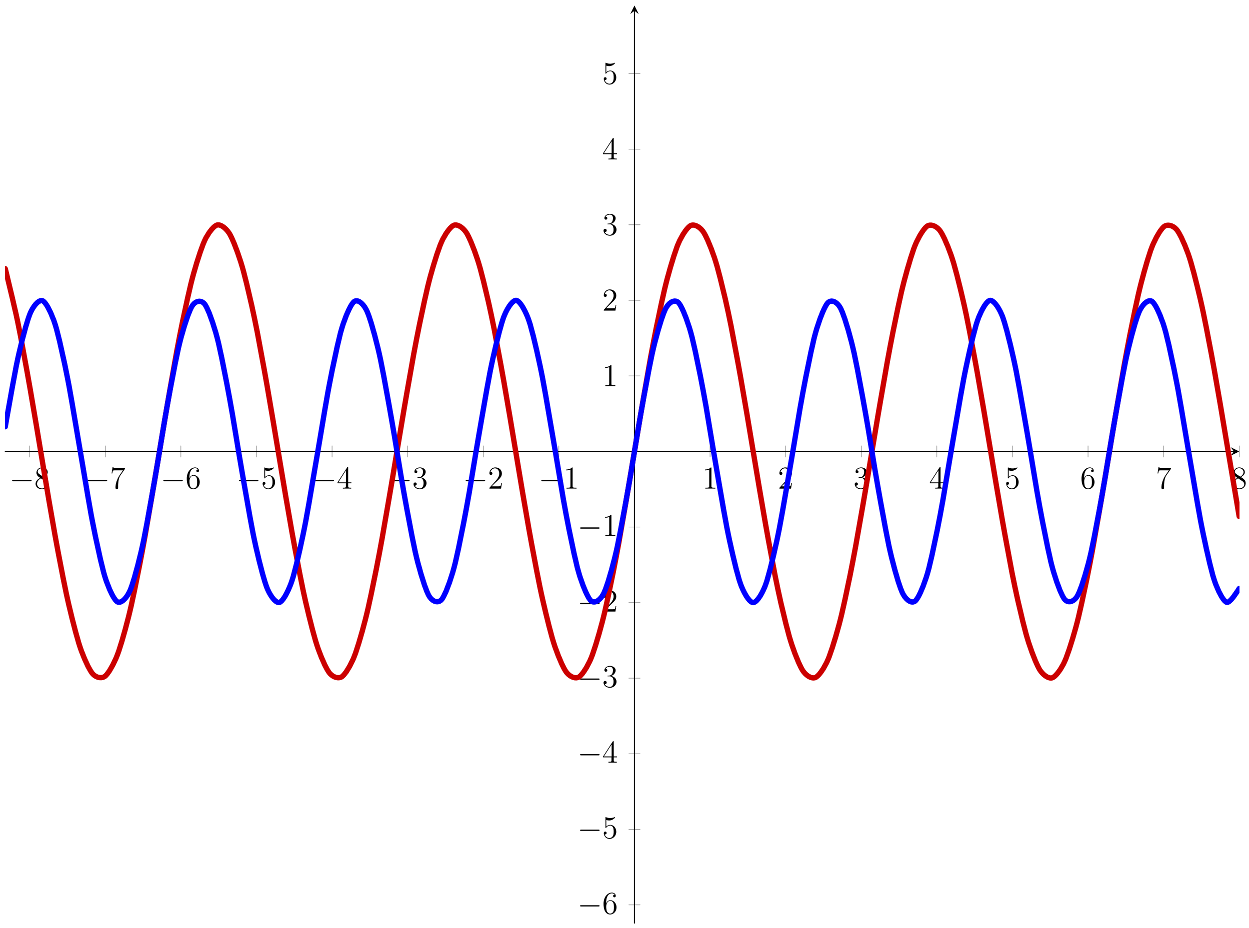
The function $sgn(x)$ is constant on three different regions. When x is larger than 0 is constant and equals 1, when x is less than 0 it is constant and equals -1, and when $x = 0$ its value is 0. Thus we plot is as



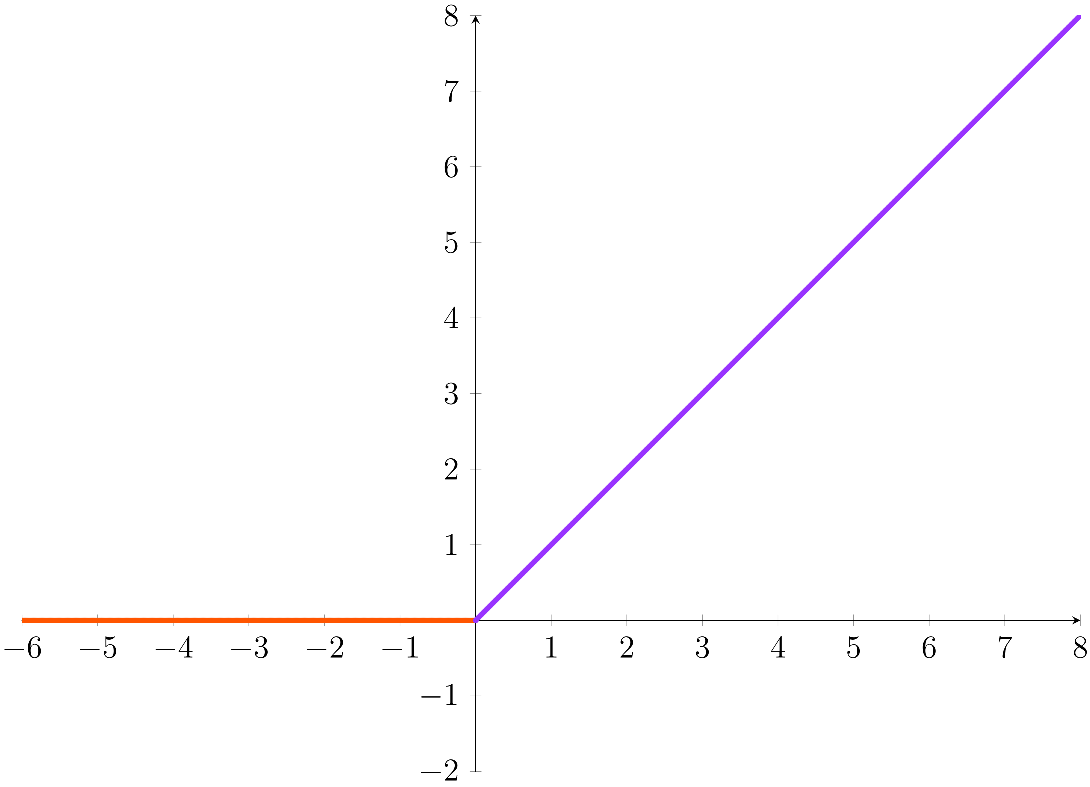
To plot the function $f(x) = log(x - 1)$ we first remember plot $log(x)$ and translate it 1 unit to the right. The *log* function is increasing to infinity but slowly and very close to 0 from right it become too small and approaches $-\infty$. log is undefined for negative x. Putting all of this together we get the plot.



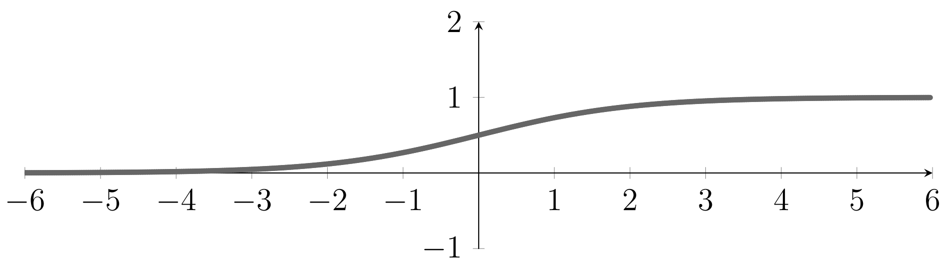
To plot the functions $3\sin(2x)$ and $2\sin(3x)$ we first look at $\sin(x)$. sin is between -1 and 1. Multiply sin by a number changes this bound. We also know that sin is periodic with period 2π. Multiplying the argument (x) of sin by a number changes the period. Therefore $3\sin(2x)$ has is between -3 and 3 and has period $2\pi/2 = \pi$ while the function $2\sin(3x)$ is between -2 and 2 and has period $2\pi/3$.



From the definition the function ReLU is constant and equals 0 when $x < 0$ and it coincides with the line $y = x$ when $x \geq 0$. Thus the plot is



We first need to look at the behaviour of the function $\sigma = \frac{1}{1+e^{-x}}$. The function e^x increases to infinity quickly as x increases and decreases to 0 quickly as x decreases (try to plot it and compare it with a plotting program). The function e^{-x} is decreasing and behaves the other way. Since the reciprocal of a positive decreasing function is increasing the function σ is increasing. Now we look at the asymptotic behaviour of σ. When x is very large, the value of e^{-x} is too small and hence $1 + e^{-x} \approx 1$ that is $\sigma(x) \approx 1$ when x is large. When x is very small (negative) the value of e^{-x} is extremely large thus the value $\frac{1}{1+e^{-x}} \approx 0$. From this analysis we can plot the following



Exercise 2:

The function $\frac{x}{1-x}$ is defined everywhere except when the denominator is zero which happens only when $x = 1$. Therefore the domain is $\mathbb{R} - \{1\}$ which can also be written as $(-\infty, 1) \cup (1, \infty)$.

The function $\frac{x-2}{x^2-3x+2} + \frac{1}{x+5}$ is defined everywhere except when at least one of the denominators is zero. The roots of $x^2-3x+2 = (x-2)(x-1)$ are $x = 1, 2$ and second denominator equals zero exactly when $x = -5$. Therefore the domain is $\mathbb{R}-\{-5, 1, 2\} = (-\infty, -5) \cup (-5, 1) \cup (1, 2) \cup (2, \infty)$

The function $\log\left(\frac{1}{x-7}\right)$ is defined when the input of log is positive and when the denominator is not zero. This happens when $x > 7$. The domain is $(7, \infty)$

Exercise 3: We are given $f(x + 1) = x^2 - x + 1$. To find $f(x)$ we simply substitute $x - 1$ in place of x and expand everything

$$f(x) = (x-1)^2 - (x-1) + 1 = (x^2 - 2x + 1) - (x-1) + 1 = x^2 - 3x + 3$$

To find $f(x - 1)$ we substitute $x - 2$

$$f(x-1) = (x-2)^2 - (x-2) + 1 = (x^2 - 4x + 4) - (x-1) + 1 = x^2 - 5x + 6$$

Exercise 4: This exercise is closely related to calculating the derivative (recall the definition of derivative).
For $f(x) = 2x + 3$ we have

$$\frac{f(x+h) - f(x)}{h} = \frac{(2(x+h)+3) - (2x+3)}{h} = \frac{2h}{h} = 2$$

For the function $f(x) = 5x^2$ we have

$$\frac{f(x+h) - f(x)}{h} = \frac{5(x+h)^2 - 5x^2}{h} = \frac{5(x^2 + 2hx + h^2) - 5x^2}{h} = \frac{10hx + 5h^2}{h} = 10x + 5h$$

Notice that the function $f(x) = 5x^2 - 2x - 3$ is the subtraction of the previous two functions, therefore we can quickly we that the answer is $10x + 5h - 2$

Exercise 5: There are several ways to do this task. Here is one: First thing to notice is that it is not a vertical line since the x components of the points are not the same. The equation of a line has the formula $y = mx + b$. Substituting the pair $x = 3$ and $y = 4$ and the pair $x = 0$ and $y = 5$ we have the linear system

$$\begin{cases} 4 = 3m + b \\ 5 = b \end{cases}$$

The solution of this system is $m = -1/3$ and $b = 5$. The equation of the line is $y = -\frac{1}{3}x + 5$.

Exercise 6: To compute $g \circ f \circ h$ we calculate $h(x)$, $f(h(x))$ then $g(f(h(x)))$. $h(x) = (\sin(x), \cos(x))$, $f(h(x)) = (\sin^2(x), \cos^2(x))$ and finally $g(f(h(x))) = \sin^2(x) + \cos^2(x) = 1$.

Exercise 7:

$$g(f(x)) = \log_2(f(x) + 1) = \log_2(2^x - 1 + 1) = x$$

$$f(g(x)) = 2^{g(x)} - 1 = 2^{\log_2(x+1)} - 1 = (x+1) - 1 = x$$

Exercise 8:

- We use chain rule. The derivative of $\sin(x)$ is $\cos(x)$ and the derivative of $3x$ is 3. Thus

$$(\sin(3x))' = 3\cos(3x)$$

- We use the product rule

$$(x\ln(x))' = (1)\ln(x) + x\frac{1}{x} = \ln(x) + 1$$

- $$f(x) = x\sqrt{x} + \sqrt[3]{x} = x^{3/2} + x^{1/3}$$

  Therefore

$$f'(x) = \frac{3}{2}x^{3/2-1} + \frac{1}{3}x^{1/3-1} = \frac{3}{2}\sqrt{x} + \frac{1}{3}\frac{1}{\sqrt[3]{x^2}}$$

- We use the chain rule. The derivatives of $\tan(x)$ is $\sec^2(x)$ and the derivative of x^2+1 is $2x$. Thus

$$(\tan(x^2+1))' = 2x\sec^2(x^2+1)$$

- We use the chain rule. The derivative of x^6 is $6x^5$ and the derivative of $e^x + x^3 - 1$ is $e^x + 3x^2$. Thus

$$\left((e^x + x^3 - 1)^6\right)' = 6(e^x + 3x^2)(e^x + x^3 - 1)^5$$