AI Summer Camp: Linear Algebra

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A **vector** is a quantity that has a magnitude and direction.

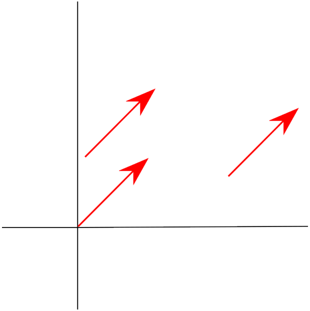
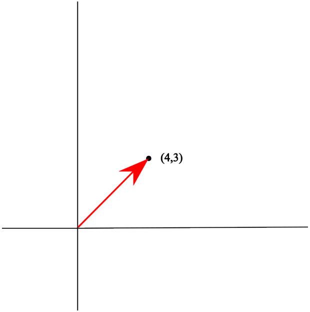


Figure: Representations of the same vector

In the plane we can represent a vector by a pair of real numbers (x, y). The set of such pairs are denoted $\mathbb{R}^2$.



It is staight forward to generalize to $\mathbb{R}^n$ for any integer $n \geq 1$. A **vector x** in $\mathbb{R}^n$ is a sequence of n real numbers $\mathbf{x} = (x_1, ..., x_n)$. n is called the dimension of the space or of the vector. The numbers x_i's are called the components of the vector $\mathbf{x}$.

Although a vector is independent from its starting point is we can sill represent a vector by a starting point and an end point. The vector in this case will just be the vector represented by the subtraction of the components of the points.

Example: The vector that starts at $(3, 1)$ and ends at $(0, 5)$ is represented by

$$(0 - 3, 5 - 1) = (-2, 4)$$

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Vector Operations

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
By definition we can write $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$.

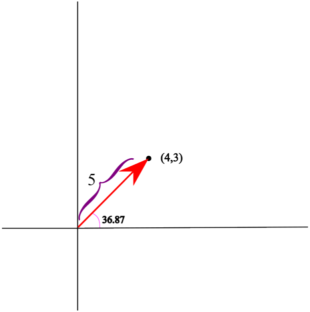
- We say that $\mathbf{x} = \mathbf{y}$ if and only if $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.
- Addition: $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n)$.
- Scalar Multiplication: $c\mathbf{x} = (cx_1, \ldots, cx_n)$.
- Length (Norm): $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$.
- Dot (Scalar) Product: $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$.

Notice that some operations return a vector while some return a real number.

The vector $(4, 3)$ has length

$$\|(4,3)\| = \sqrt{4^2 + 3^2} = 5$$

and its angle $\theta \approx 36.87$ which is the angle that satisfies $\cos(\theta) = 4/5$ and $\sin(\theta) = 3/5$.



Angle between two vectors

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The angle θ between the vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as the unique angle in the interval $[0, \pi]$ that satisfies

$$\cos(\theta) = \frac{\mathbf{x} \cdot \mathbf{y}}{\|x\|\|y\|}$$

Why do only define the angle on $[0, \pi]$?

Unit Vectors

A vector $\mathbf{x} \in \mathbb{R}^n$ that satisfies $\|x\| = 1$ is called a unit vector. Writing this using the vector components

$$x_1^2 + x_2^2 + ... + x_n^2 = 1$$

This is called the n-dimensional unit sphere equation.

Matrix

An $n \times m$ matrix is a sequence of length$n$ of m-vectors written as rows. A matrix can be viewed as a sequence of length m of n-vectors written as columns.

$$\begin{bmatrix} 2 & 5 & 0.3 \\ 5 & 6 & 10 \\ -100 & \frac{5}{3} & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

Addition of matrices and scalar multiplication are defined analogously to way they were defined for vectors.

Matrix Operations

Matrix Multiplication:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{x} \begin{bmatrix} 10 & 11 \\ 20 & 21 \\ 30 & 31 \end{bmatrix}$$

$$= \begin{bmatrix} 1\text{x}10 + 2\text{x}20 + 3\text{x}30 & 1\text{x}11 + 2\text{x}21 + 3\text{x}31 \\ 4\text{x}10 + 5\text{x}20 + 6\text{x}30 & 4\text{x}11 + 5\text{x}21 + 6\text{x}31 \end{bmatrix}$$

$$= \begin{bmatrix} 10+40+90 & 11+42+93 \\ 40+100+180 & 44+105+186 \end{bmatrix} = \begin{bmatrix} 140 & 146 \\ 320 & 335 \end{bmatrix}$$

Transpose:

$$\begin{bmatrix} 2 & 5 & 0.3 \\ 5 & 6 & 10 \\ -100 & \frac{5}{3} & 0 \\ -2 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 5 & -100 & -2 \\ 5 & 6 & \frac{5}{3} & 1 \\ 0.3 & 10 & 0 & 1 \end{bmatrix}$$