Some Thoughts on PDEs and Numerics

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Objective of this text

The goal is draw a map that motivates the topics that are encountered in functional analysis and PDEs courses for myself. It is not meant to provide proofs nor to be totally clear for the first time reader. It is meant as a map and a simple way of organizing the topics. Those who are encountering these topics for the first time can return to this map as they advance in their courses. Maybe it also can help some of those who already took the topics and wanted a way to organize and motivate them. This map is not meant to be historically accurate.

About Uniqueness and Existence

Suppose we have some fixed charge distribution and we put a small positive charge away from the distribution. The charge will either stay where it was put or begin moving in accordance with the effect of the charge distribution[1]. We expect (or do it in the laboratory) that repeating this same process with the same settings will always produce the same result. This is an example of why a differential equation modeling a physical phenomenon is expected to have a solution which is unique.
Now suppose we have a surface, we only know the potential at the surface points, the potential is the energy needed to travel to that point. In this case a charge will move such as to minimize its potential locally and hence we expect existence and uniqueness for every arbitrary potential we can impose on the surface. This also motives the variational formulation. Laplace equation which we begin the map with models this process of potential known on a surface.

Laplace Equation

Let's begin by considering the Laplace Equation on an open bounded connected subset $\Omega \subset \mathbb{R}^n$

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = g \in \mathcal{C}^0(\Omega) \end{cases}$$

We look for solutions in $\mathcal{C}^2(\Omega)$. The maximum principle guarantee the uniqueness of such a solution as well as the dependence of the solution on the boundary conditions with respect to the usual $\mathcal{C}^0$ max-norm.
To prove the existence we write the problem as a minimization problem. One can prove

[1] assuming the only force present is the electric force and with low acceleration to ignore charge radiation.

that u is a solution to the equation above iff it minimizes the function

$$\int_\Omega |\nabla u|^2$$

subject to the space $\{u \in \mathcal{C}^2 : u|_{\partial\Omega} = g\}$. We summarize the proof below since we will reference it later. The proof typically looks like this.

Proof. Define $I(u) = \int_\Omega |\nabla u|^2$. suppose u is a minimizer of I and let $v \in \mathcal{C}_0^2(\Omega)$. The function

$$\epsilon \mapsto I(u + \epsilon v)$$

must have a critical point at 0. That is

$$\int_\Omega \nabla u \cdot \nabla v = 0$$

which is equivalent, using the divergence theorem, to

$$\int_\Omega (\Delta u) v = 0$$

By the fundamental theorem of calculus of variation we have $\Delta u = 0$. Conversely if u is a solution to Laplace problem then for every function $v \in \mathcal{C}^2$ such that $v|_{\partial\Omega} = g$ and $I(v) < \infty$

$$0 = \int_\Omega (\Delta u)(u - v) = -\int_\Omega |\nabla u|^2 + \int_\Omega \nabla u \cdot \nabla v \leq -\int_\Omega |\nabla u|^2 + \int_\Omega \frac{1}{2}(|\nabla u|^2 + |\nabla v|^2)$$

Whence $I(u) \leq I(v)$. $\square$

Remark. *The critical points of the closely related functional $J(u) = \int_\Omega (\Delta u) u$ also satisfy Laplace equation, but the converse is not true. Also I is bounded below while J is not.*

Now, we pick a sequence $u_n \in \mathcal{C}^2(\Omega)$ with $u_n|_{\partial\Omega} = g$ such that $I(u_n) \to \inf_{\{u \in \mathcal{C}^2 : u|_{\partial\Omega} = g\}} I(u)$. We have

$$\begin{aligned}\int_\Omega |\nabla u_n - \nabla u_m|^2 &= \int_\Omega |\nabla u_n|^2 + \int_\Omega |\nabla u_m|^2 - 4\int_\Omega |\frac{1}{2}(\nabla u_n + \nabla u_m)|^2 \\ &\leq \int_\Omega |\nabla u_n|^2 + \int_\Omega |\nabla u_m|^2 - 4 \inf_{\{u \in \mathcal{C}^2 : u|_{\partial\Omega} = g\}} I(u) \xrightarrow{n,m\to\infty} 0\end{aligned}$$

This parallelogram identity is a technique used for convex sets as in Lax-Milgram theorem. We now know that (∇u_n) is a Cauchy sequence in L^2. Because of completeness of L^2 the sequence (∇u_n) has a limit v. Moreover using Poincare-Friedrichs Inequality we have

$$\int_\Omega |u_n - u_m|^2 \leq C \int_\Omega |\nabla u_n - \nabla u_m|^2$$

Hence (u_n) also has a limit u in L^2. Had the convergence been in $\mathcal{C}^2$ we would have concluded, but this is not the case. How are u and v related? and what does ∇u mean now for a function in L^2? In this case we simply call v the *weak derivative* of u. That is we established the existence of a minimizer for I over the set

$$H = \left\{ u \in L^2(\Omega) : \exists (u_n) \in (\mathcal{C}^2)^\infty \texttt{ such that } u_n|_\Omega = g, u_n \xrightarrow{L^2} u \texttt{ and } (\nabla u_n) \texttt{ converges in } L^2 \right\}$$

This set is similar to what is called Sobolev Space $H^1(\Omega)$. We will denote the function that ∇u_n converges towards ∇u and call it the weak derivative of u. Using Cauchy-Schwartz inequality one can see that for every $w \in L^2$

$$\int_\Omega u_n w \to \int_\Omega uw$$
$$\int_\Omega \nabla u_n w \to \int_\Omega \nabla uw$$

If moreover w was $\mathcal{C}^1$ with $w|_\Omega = 0$ and $I(w) < \infty$ then by integration by parts

$$\int_\Omega u \nabla w = \lim \int_\Omega u_n \nabla w = -\lim \int_\Omega \nabla uw = -\int_\Omega \nabla uw$$

This kind of convergence is called *weak convergence.* This line also provide us with the most used definition of weak derivatives which is equivalent to our definition of weak derivative above. A function denoted $\nabla u \in L^2(\Omega)$ is the weak derivative of $u \in L^2(\Omega)$ if it satisfies

$$\int_\Omega u \nabla w = -\int_\Omega \nabla uw$$

for every $w \in \mathcal{C}_c^\infty(\Omega)$. The choice $\mathcal{C}_c^\infty(\Omega)$ seems to be a convention but we lose nothing as $\mathcal{C}_c^\infty(\Omega)$ is dense in $\left\{ w \in \mathcal{C}^1 : w|_\Omega = 0 \texttt{ and } \int_\Omega |\nabla w|^2 < \infty \right\}$. We now have a solution but in a different sense than we started with and even with this new way of differentiating we only got a function that is differentiable once. Reforming the original in this new way is called *weak formulation* while the formulation of the problem as a minimization of the functional I is called the *variational formulation.* The problem we altered through the previous discussion and proved the existence of solution is stated as

$$\begin{cases} \int_\Omega \nabla u \cdot \nabla v = 0 \quad \forall v \in \mathcal{C}_c^\infty(\Omega) \\ u|_{\partial\Omega} = g \end{cases}$$

A solution for the weak formulation is called a weak solution. Luckily for us Weyl's lemma states that such a solution to the weak formulation above is in fact in $\mathcal{C}^\infty(\Omega)$ (keep in mind a weak solution is the equivalence class up to almost everywhere equality equivalence). Thus we have the maximum principle we stated at the beginning and consequently uniqueness and dependence on boundary conditions.

Before we move on there are two points we mention quickly. First, we must clarify what we mean by $u|_{\partial\Omega}$ in the weak formulation since we are dealing with almost equality now and $\partial\Omega$ could be of measure zero. If $\partial\Omega$ is $\mathcal{C}^1$ (locally can be written as a $\mathcal{C}^1$ function) the way we defined H using converging sequences makes it possible find a sequence in $\mathcal{C}^\infty(\bar{\Omega})$ which will allow us to make sense of $u|_{\partial\Omega}$ (of course independent of the sequence). We call this new definition of $u|_{\partial\Omega}$ the *trace* of u.

The second point is that in our analysis of the variational formulation we implicitly assumed that $I(u_n) \to \inf_{\{u\in\mathcal{C}^2 : u|_{\partial\Omega}=g\}} I(u) < \infty$. That is $H \neq \emptyset$. This is not necessarily the case, that is there exist a function g for which the corresponding space H is empty. We want g to be in the range of the trace operator we just discussed, the range turns out to be isomorphic to what is denoted $H^{1/2}(\partial\Omega)$.

Remark. *The classical sense of solution of Laplace equation we started with must be satisfied point-wise not almost everywhere, as one can see that for $g = 1$ that uniqueness is not satisfied. The weak sense agree with the classical one if a classical solution existed.*

Weak and Weak* Topology

Similar to what we said earlier a sequence (f_n) in L^2 converges weakly to f if

$$\int f_n g \xrightarrow{n\to\infty} \int fg$$

Or in usual inner product notations $< f_n, g > \to < f, g >$. This is a form of "component-wise" convergence, and recall that the product topology is defined using this component-wise convergence. L^2 is isomorphic to the space of sequence l^2. l^2 is homeomorphic to a subset of the product topological space $\mathbb{R}^{\mathbb{N}}$. If we focus on the unit ball of l^2 we see that it is a closed subspace of $[-1, 1]^{\mathbb{N}}$. Tychonoff theorem states that $[-1, 1]^{\mathbb{N}}$ is compact and hence any closed subspace of it is compact. This result is known as *Banach-Alaoglu theorem.* This result can be stated in a way similar fashion to Heine–Borel for finite dimentional spaces as any bounded sequence of functions in L^2 has a subsequence which has a weak limit. These concepts in this section are abstracted to general settings named *weak topology* and *weak* topology.*

Are Weak Solutions Physical? [To Be Written]

We discussed that uniqueness and existence should be the case for a PDE that models a physical phenomenon. Weak solutions as we saw came from the variational formulation of PDE subject to a larger appropriate space since it is not guaranteed that the minimum can be achieved within the space of differentiable functions.
Topics To Check:

- Wave Equation with triangular initial conditions.
- Shock Waves. Check Evan.
- solutions with approximate initial conditions do approximate desired solutions?
- tent erection center point and minimal surfaces.
- examples of PDEs with only weak solutions, better be radical xD.