Matrix Differenciation Assigment

Chain Rule Assigment:

1) Given $f(z) = \log_e(1+z)$ where $z = X^T X$, $x \in \mathbb{R}^{n}$

if $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ then $X^T = [x_1 \quad x_2 \quad x_3 \ldots x_n]$

$$X^T X = [x_1^2 + x_2^2 + x_3^2 \ldots + x_n^2]$$

Applying chain rule,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{d}{dz}(\log(1+z)) \cdot \frac{d}{dx}(x^T \cdot x)$$

$$= \frac{1}{1+z} \cdot \frac{d}{dz}(1+z) \cdot \frac{d}{dx}(x_1^2 + x_2^2 + x_3^2 \ldots + x_n^2)$$

$$= \frac{1}{1+z} \cdot (2x_1 + 2x_2 + 2x_3 \ldots + 2x_n)$$

$$= \frac{2}{1+z}(x_1 + x_2 + x_3 \ldots + x_n)$$

$$= \frac{2}{1+z} \sum_{i=1}^{n} x_i$$

(Ans.)

2) $F(x) = e^{-z/2}$, where $z = g(z)$, $g(y) = y^T \cdot S^{-1} y$,
$y = h(x)$, $h(x) = x - \mu$

using chain rule

$$\frac{d}{dx} f = \frac{d}{dz} f \cdot \frac{d}{dz} y \cdot \frac{d}{dx} y$$

here, $\frac{df}{dz} = \frac{d}{dz}\left(e^{-z/2}\right) = -\frac{e^{-z/2}}{2}$

$$\frac{dz}{dy} = \frac{d}{dy}\left(Y^T \cdot S^{-1} \cdot y\right)$$

$$= \lim_{h \to 0} \frac{g(y+h) - g(y)}{h}$$

$$= \lim_{h \to 0} \frac{(Y^T + h) S^{-1} (y+h) - Y^T \cdot S^{-1} Y}{h}$$

$$= \lim_{h \to 0} \frac{(y^T S^{-1} + h S^{-1})(y+h) - y^T \cdot S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{y^T S^{-1} y + y^T S^{-1} h + h S^{-1} y + h^2 S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{h(y^T S^{-1} + S^{-1} y + h S^{-1})}{h}$$

$$= \lim_{h \to 0} \left(y^T S^{-1} + S^{-1} y + h S^{-1}\right)$$

$$= y^T S^{-1} + S^{-1} y$$

$$\frac{dy}{dx} = \frac{d(x-\mu)}{dx} = 1$$

There force, $\dfrac{df}{dx}$

$$= \frac{df}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$= -\frac{e^{-z/2}}{2} \left(y^T S^{-1} + S^{-1} y\right) \cdot 1$$

$$= -\frac{e^{-z/2}\left(y^T + y\right)}{2S} \text{ (Ans.)}$$