

Separations in Proof Complexity and TFNP

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MIAO Seminar, Copenhagen

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UNDERSTANDING THE TITLE

TFNP := Total Function NP

Polytime $R(x, y)$

$\text{TFNP} := \overline{\text{Total Function}} \text{ NP}$

Polytime $R(x, y)$

Input x

Output $y : R(x, y) = 1 \wedge |y| \leq |x|^{O(1)}$

$\text{TFNP} := \overline{\text{Total Function}} \text{ NP}$

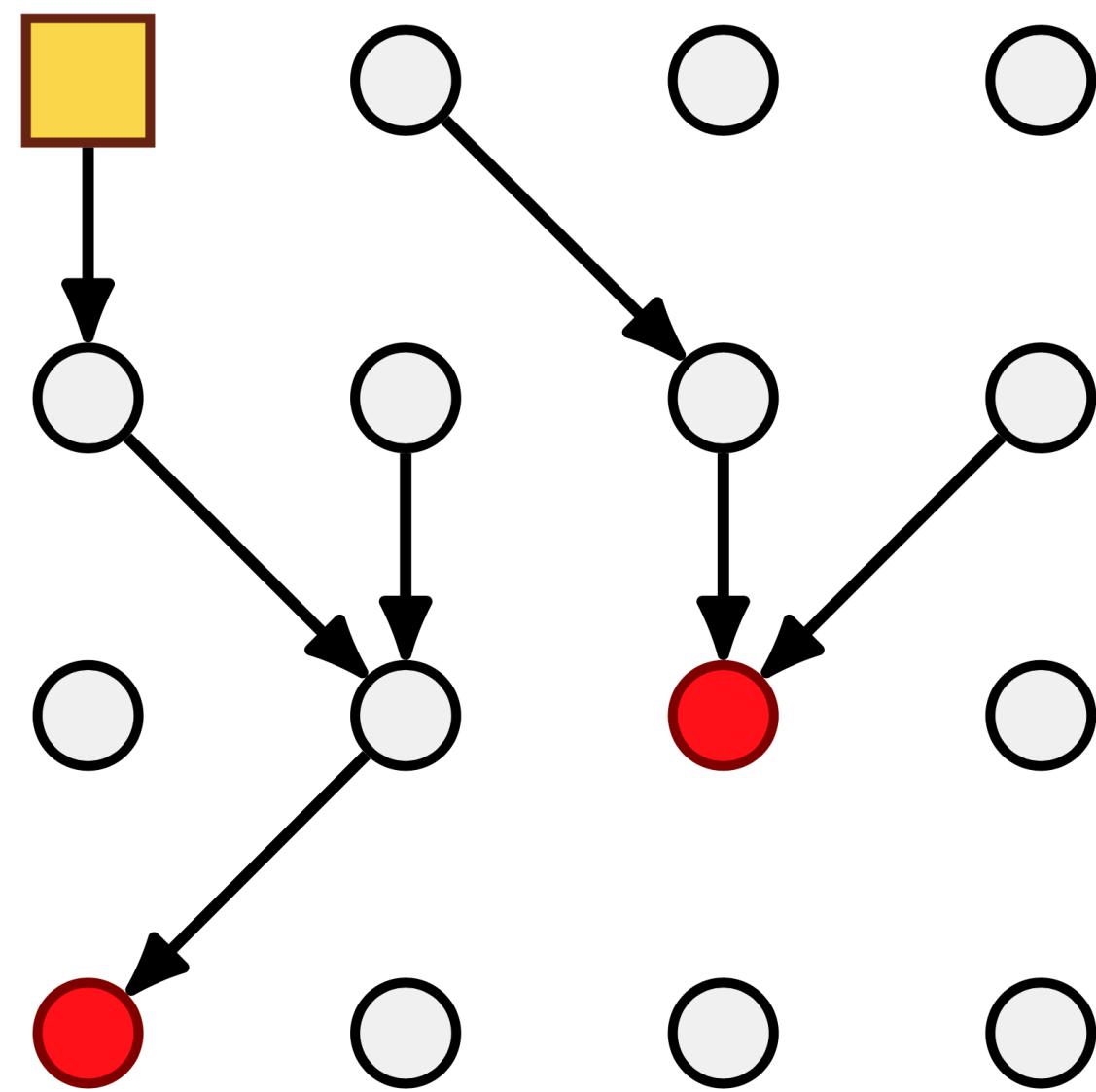
Polytime $R(n, y)$

Input x

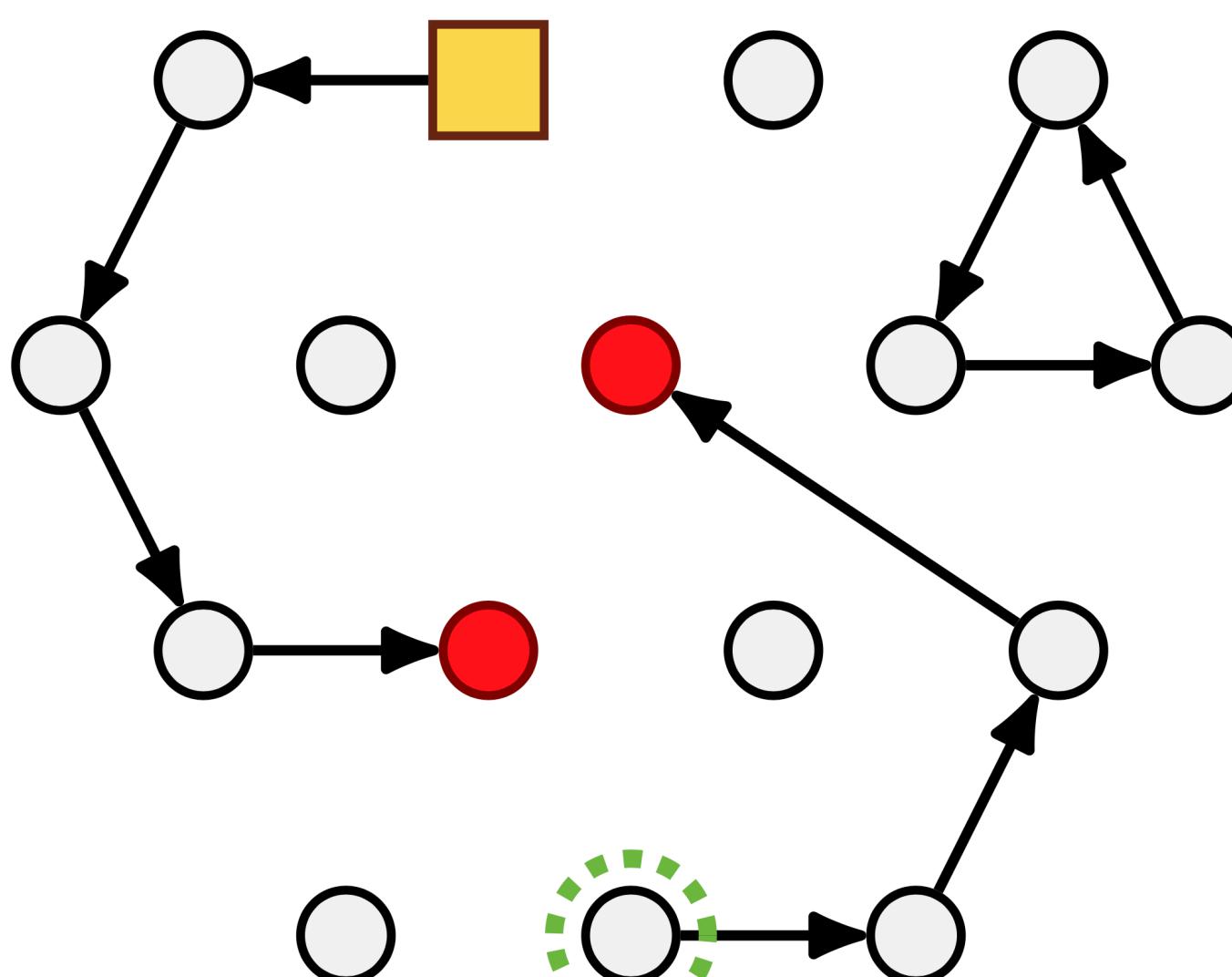
Output $y : R(n, y) = 1 \wedge |y| \leq |x|^{O(1)}$

Promise R is total: $\forall x \exists y R(n, y) = 1$

Two Problems

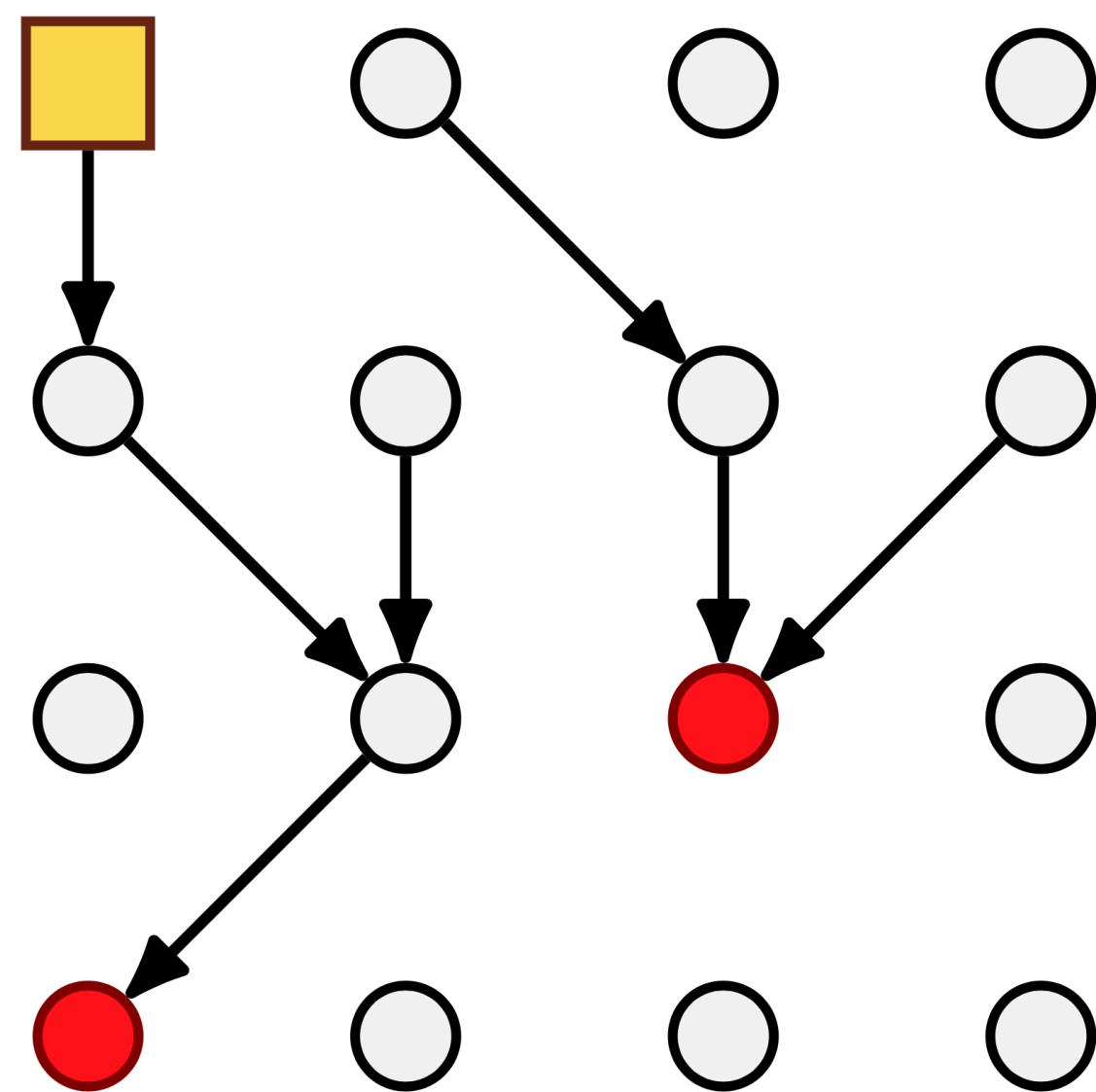


Sink-of-DAG (SoD)

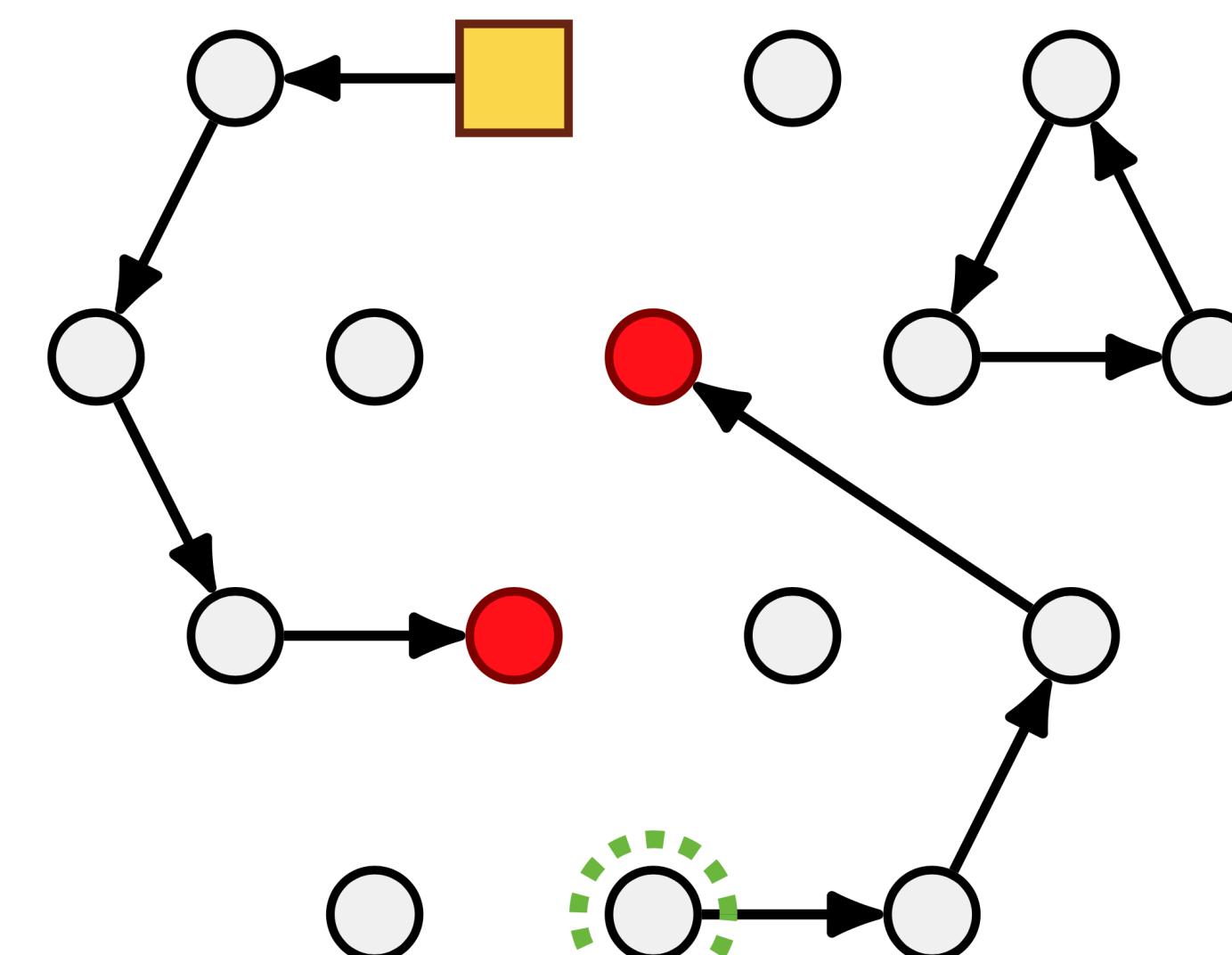


Sink-of-Line (SoL)

Two ($\& \frac{1}{2}$) Problems



Sink-of-DAG (SoD)



Sink-of-Line (SoL)
End-of-Line (EoL)

... And Three Classes

$$\text{PLS} = \{P : P \leq_{\text{SoD}}\}$$

$$\text{PPADS} = \{P : P \leq_{\text{SOL}}\}$$

$$\text{PPAD} = \{P : P \leq_{\text{EOL}}\}$$

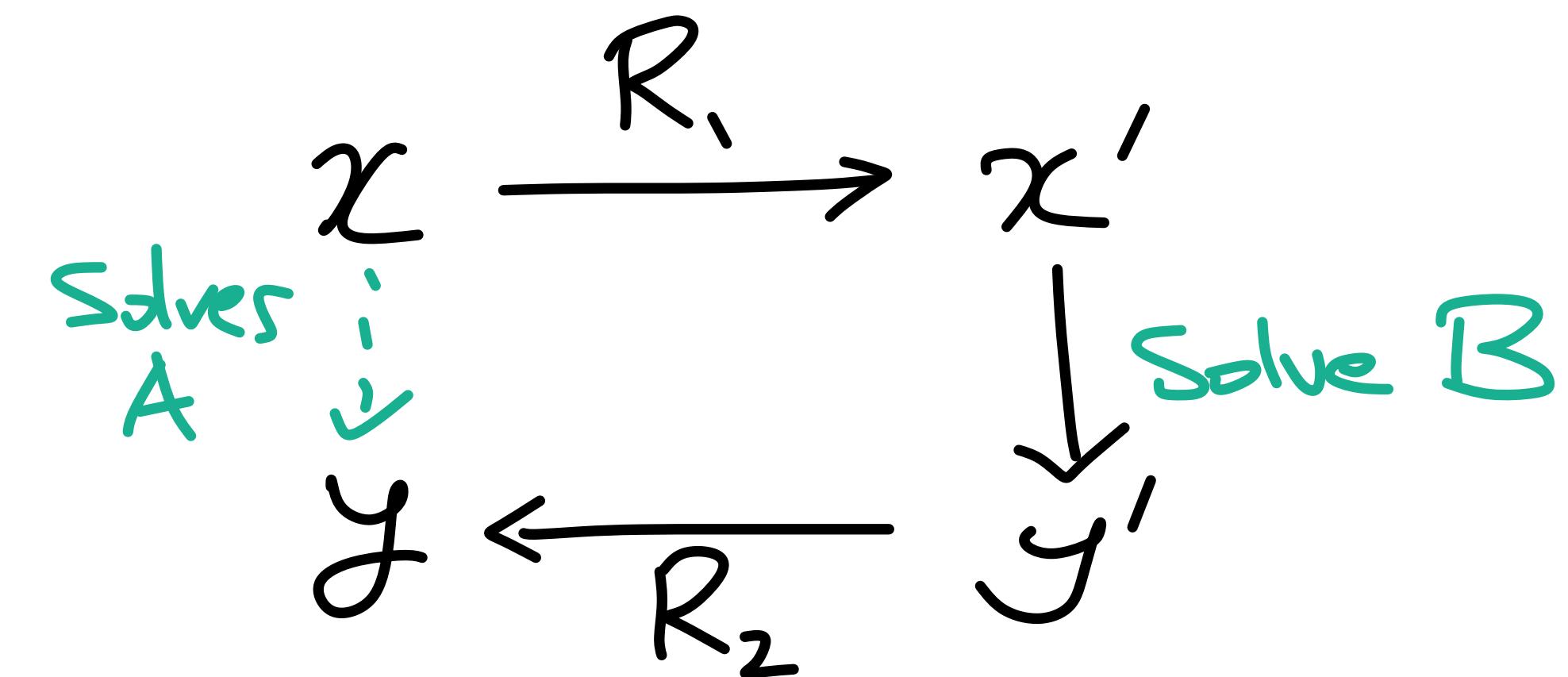
... And Three Classes

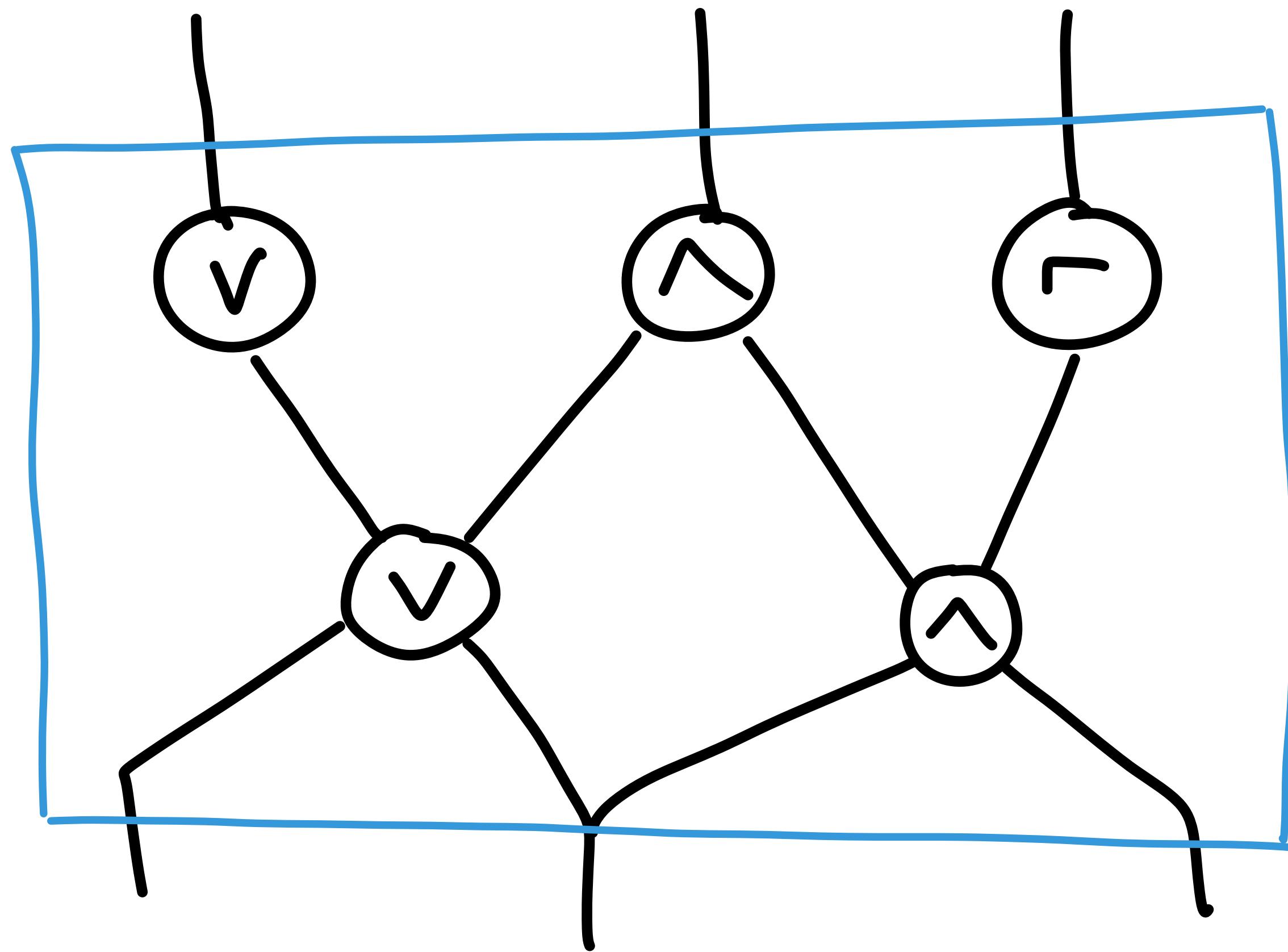
$$\text{PLS} = \{P : P \leq_{\text{SD}} S\}$$

$$\text{PPADS} = \{P : P \leq_{\text{SL}} S\}$$

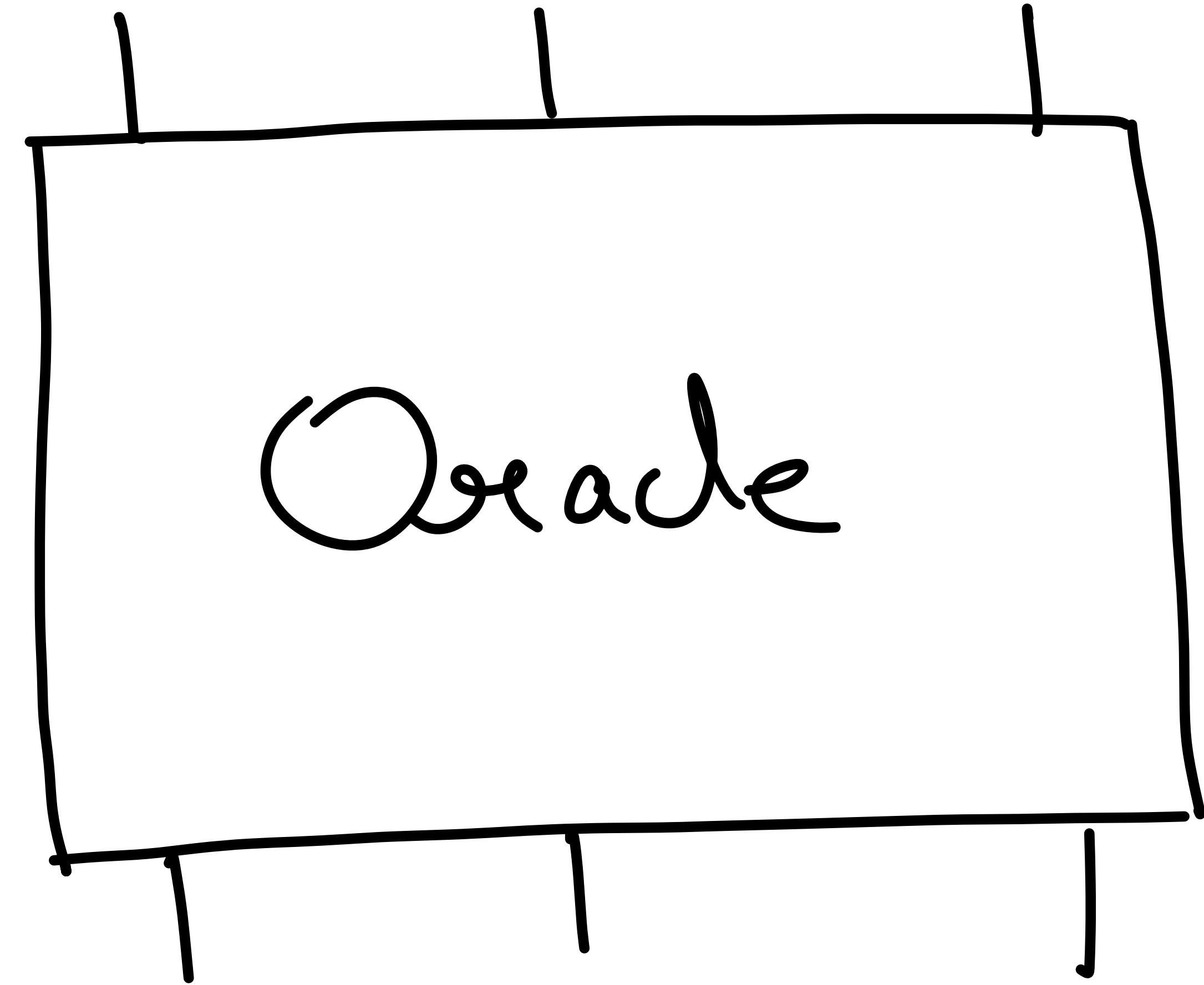
$$\text{PPAD} = \{P : P \leq_{\text{EL}} E\}$$

$A \leq B$ if $\exists R_1, R_2$



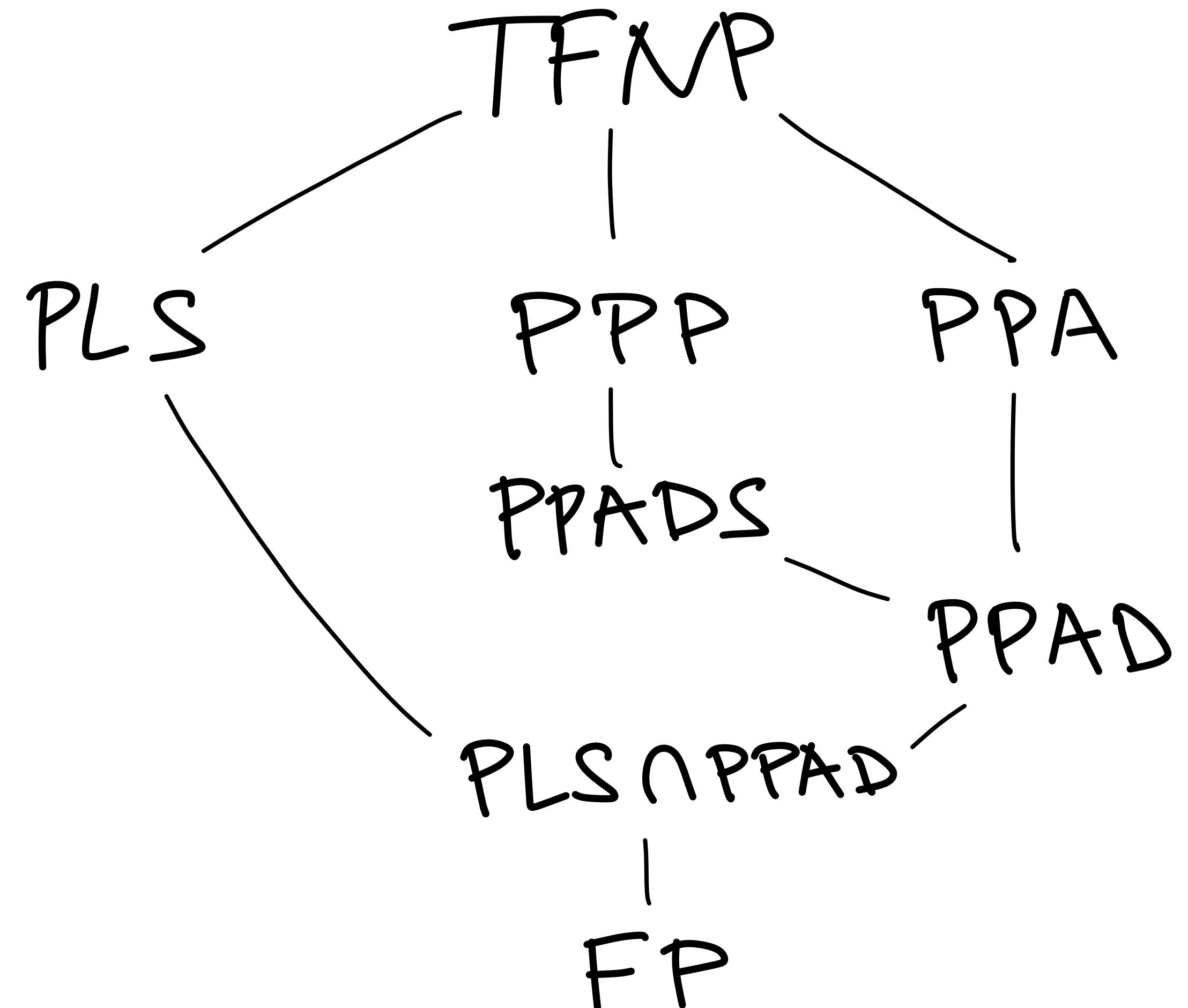


White-box



Black-box

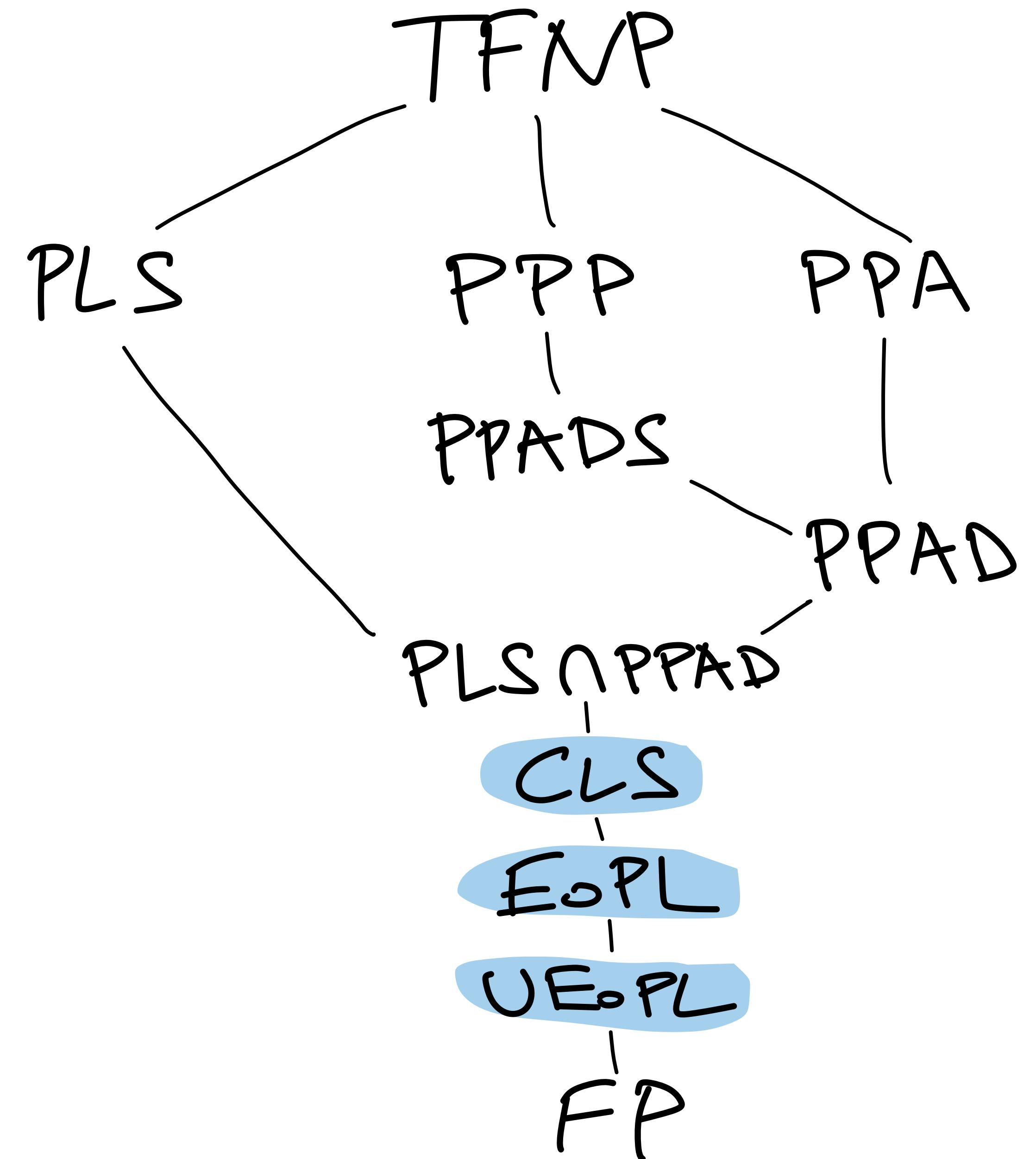
Classical hierarchy (90's and 00's)



[Pap94]

[JPY88]

New classes (10's)

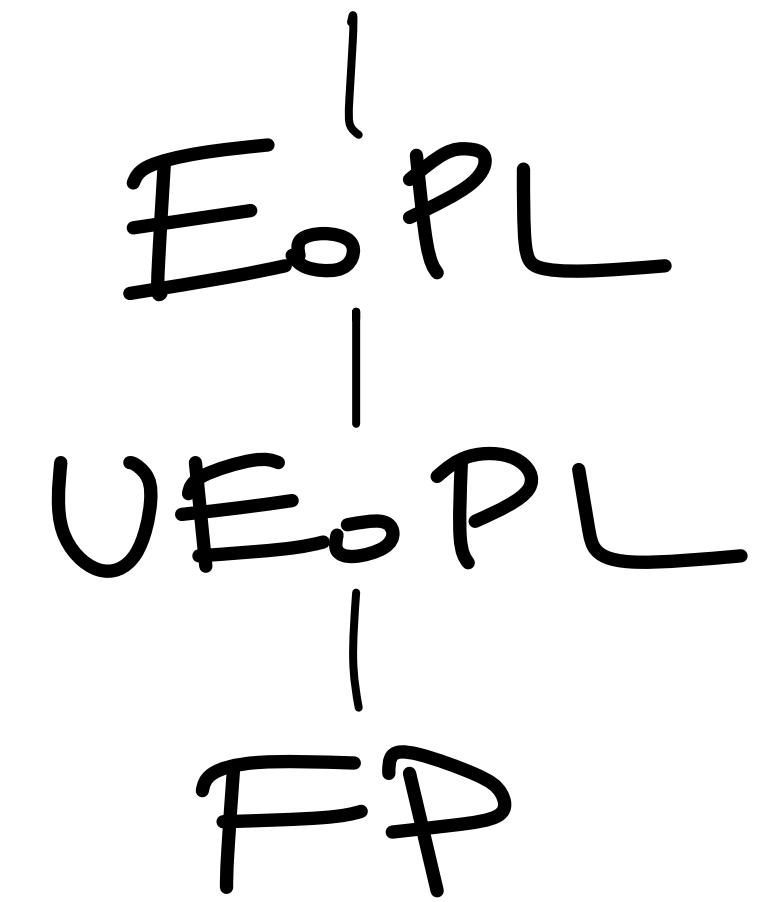
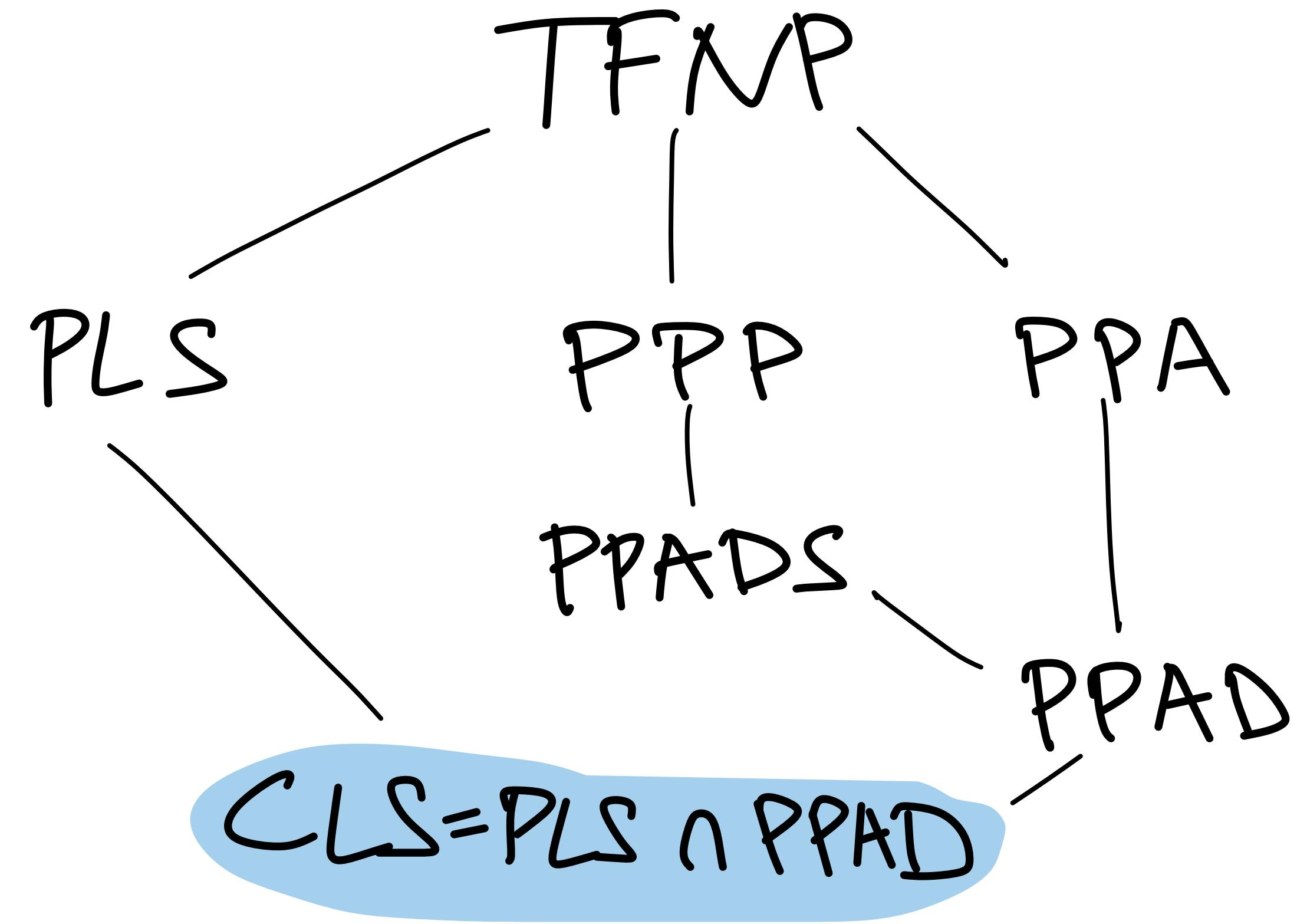


[HY20]

[FGMS20]

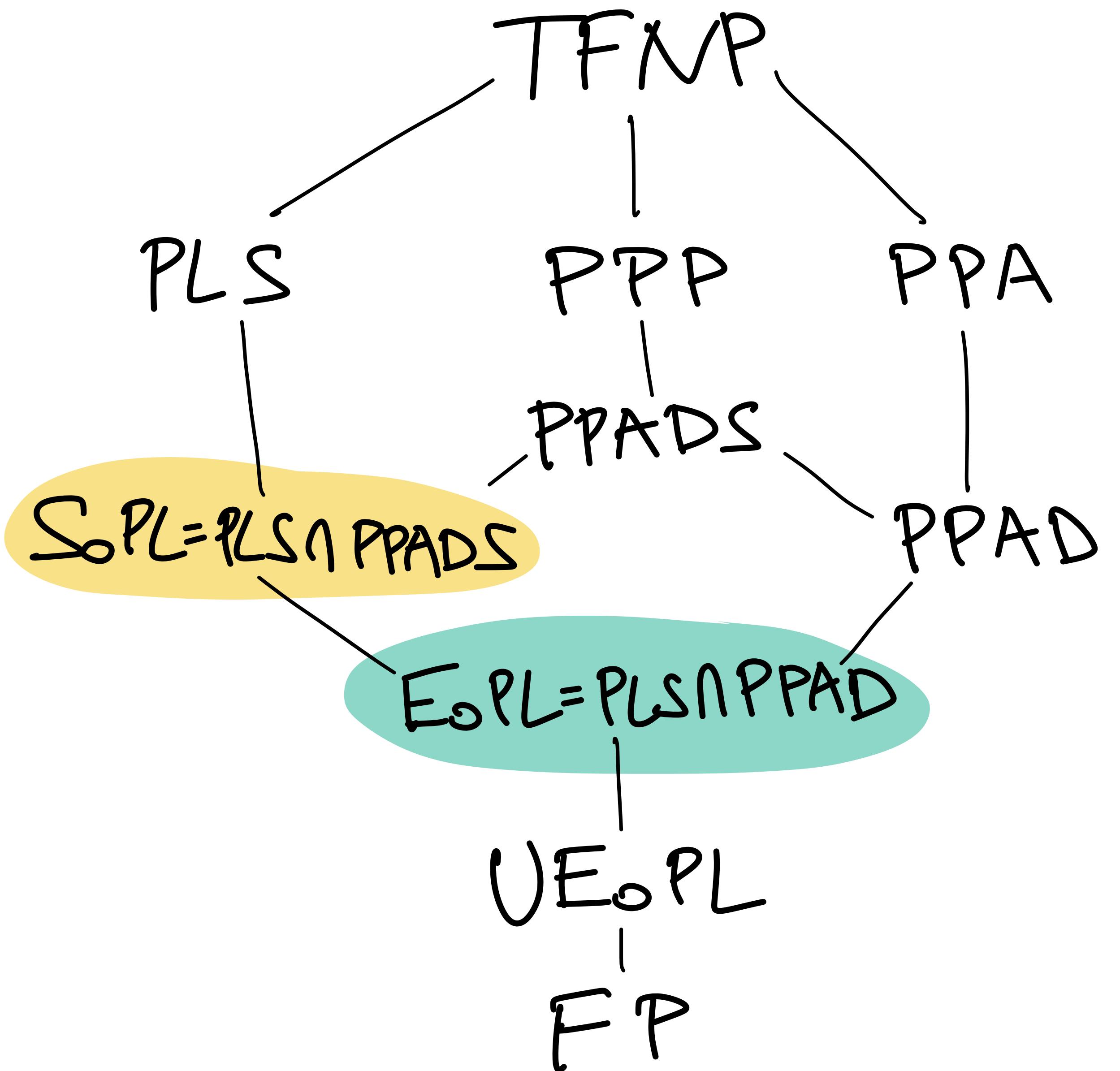
[DP11]

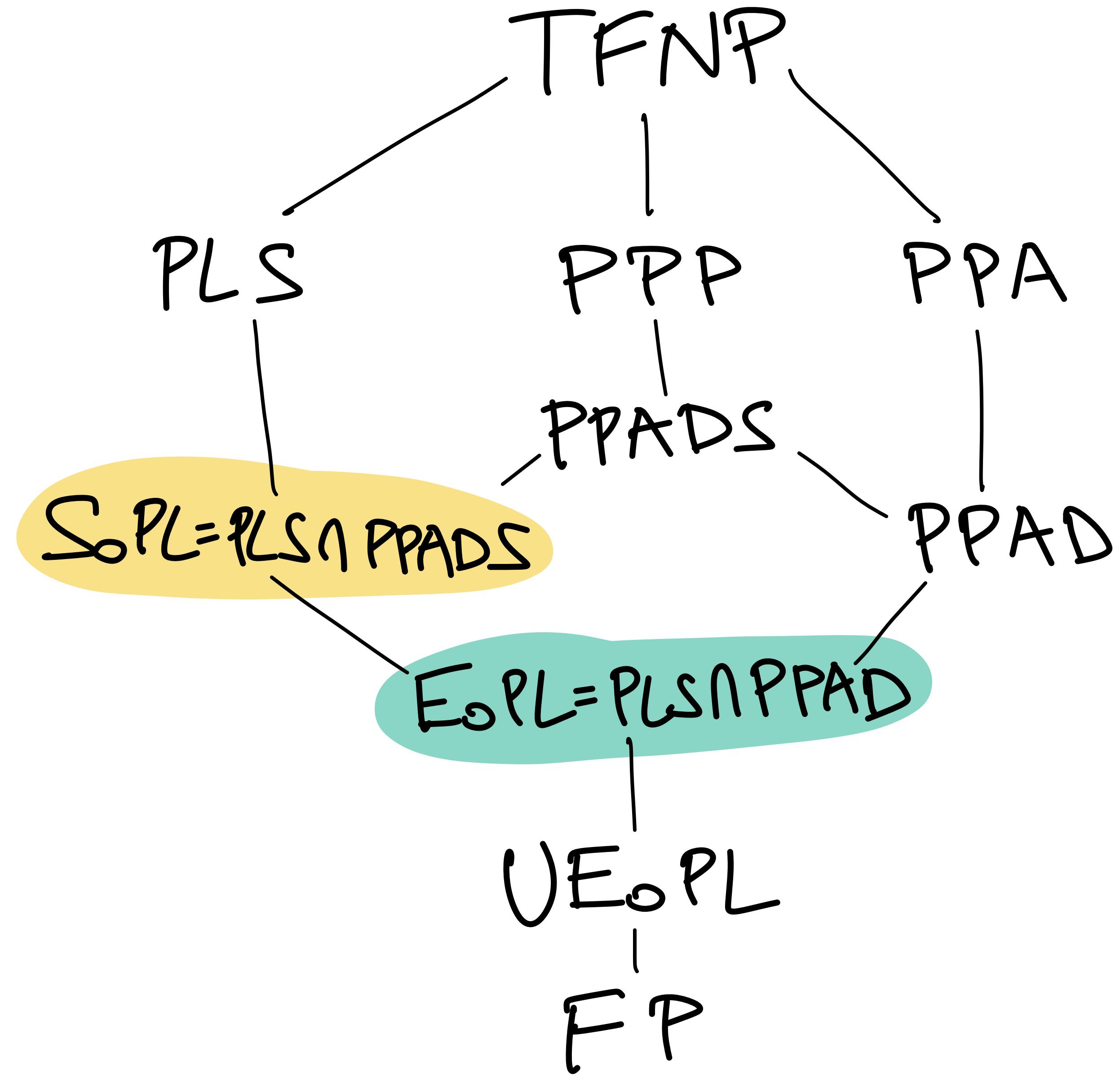
A Breakthrough Collapse (2021)



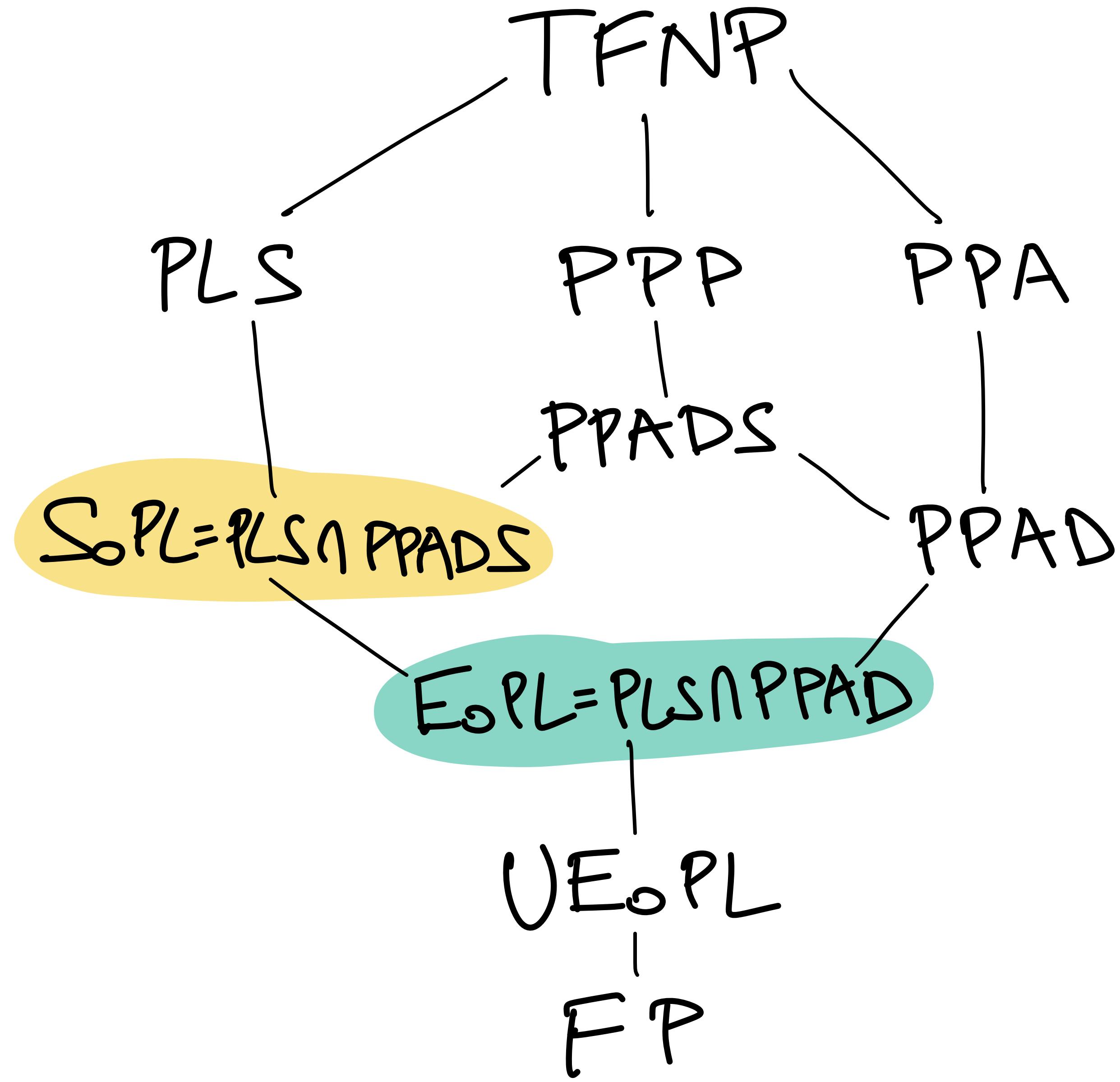
(Best paper!)
[FGHS21]

Further Collapses (2022)



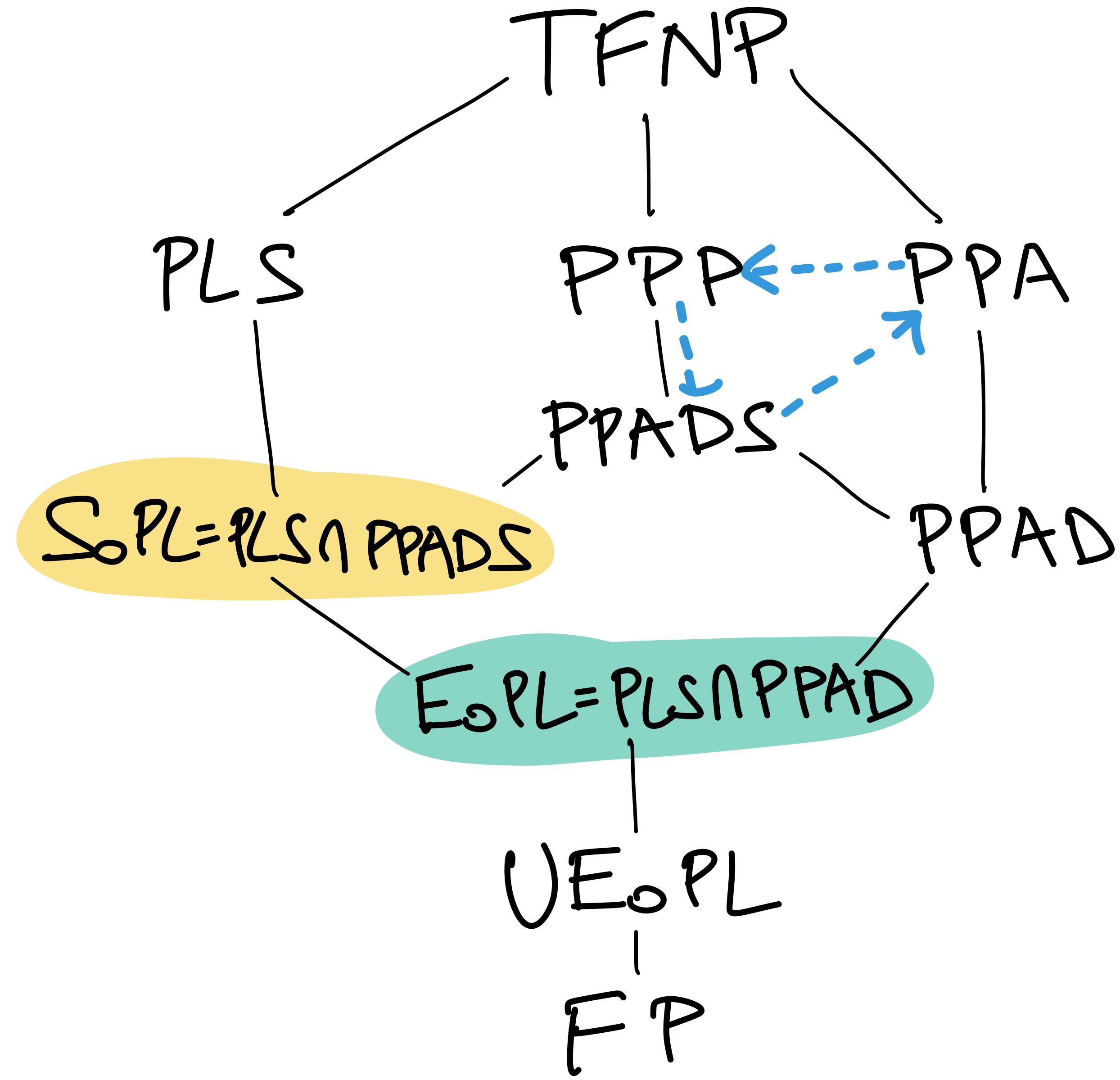


More Collapses ?



More Collapses?

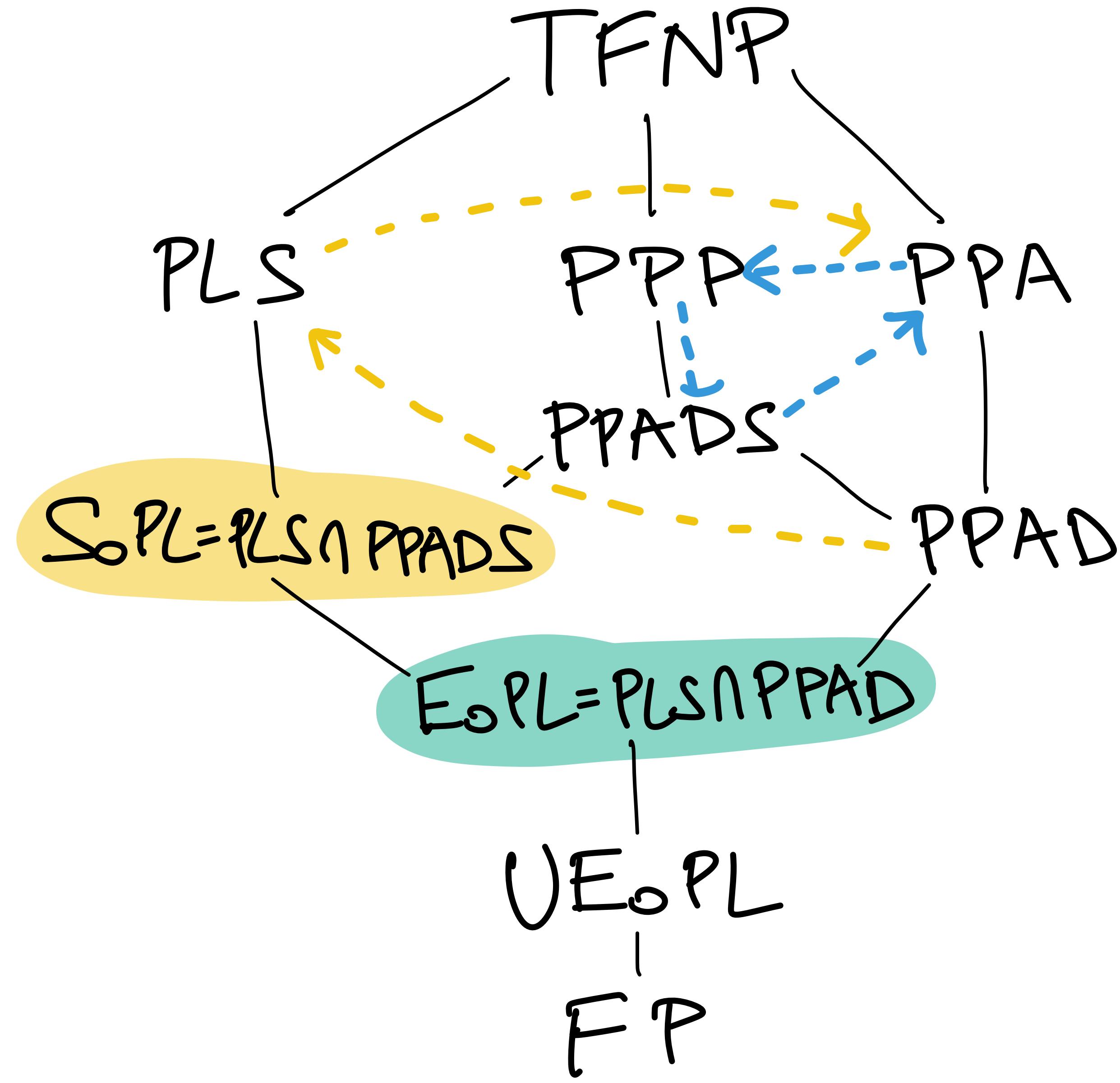
White-box sep. $\Rightarrow P \neq NP$
 Black-box sep. possible



More Collapses?

White-box sep. $\Rightarrow P \neq NP$
 Black-box sep. possible

Beame et al. 98'



More Collapses?

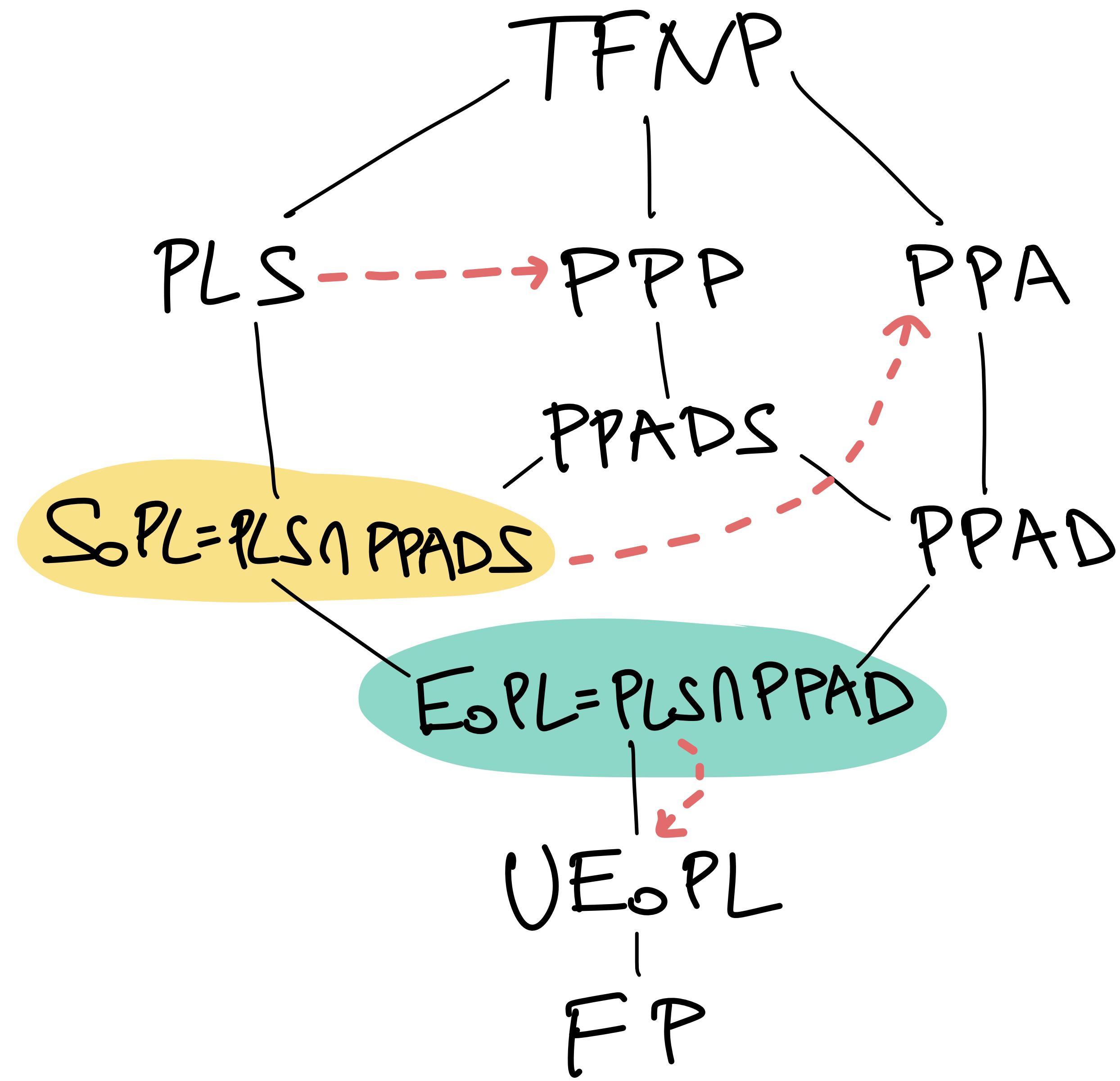
White-box sep. $\Rightarrow P \neq NP$

Black-box sep. possible

Beame et al. 98'

MarioKa 01'

Buresh-Openheim 04'



More Collapses?
 NO MORE
 (BLACK-BOX)

OUR WORK

**AND NOW FOR
SOMETHING
COMPLETELY
DIFFERENT**



Resolution v.s. Sherali - Adams

Resolution

$$\frac{A \vee x, B \vee \neg x}{A \vee B}$$

measure: width

Simulated by

$$\sum_i p_i(x) q_i(x) = 1 + J(x)$$

measure: degree

Sherali - Adams

Resolution v.s. Sherali - Adams

Resolution

$$\frac{A \vee x, B \vee \neg x}{A \vee B}$$

measure: width

Our RESULT: Simulation needs exp. large coefficients

Sherali - Adams

$$\sum_i p_i(x) q_i(x) = 1 + J(x)$$

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Resolution v.s. Sherali - Adams

Resolution

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measure: width

Our RESULT: Simulation needs exp. large coefficients



PLS $\not\subseteq$ PPADS

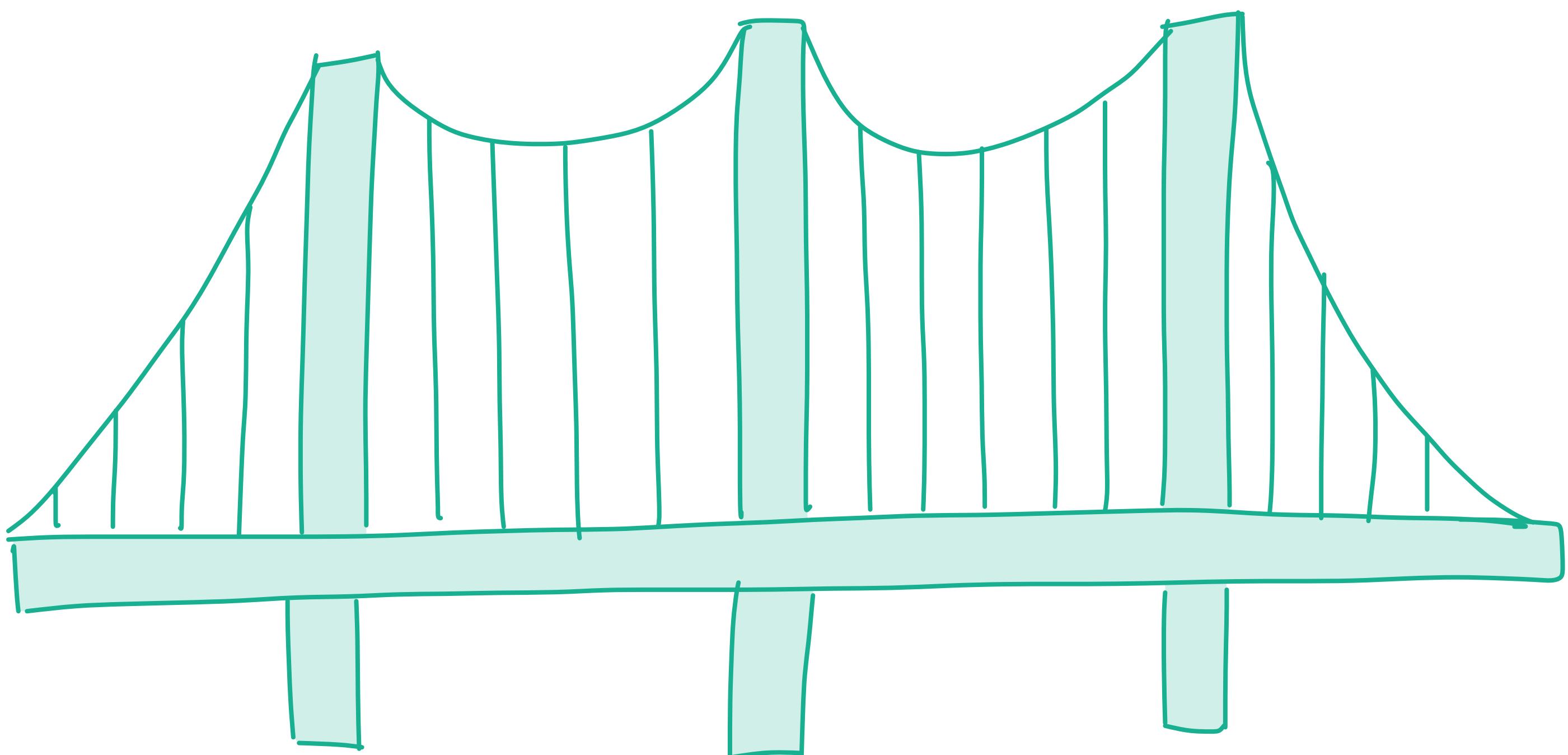
Sherali - Adams

$$\sum_i p_i(x) q_i(x) = 1 + J(x)$$

measure: degree

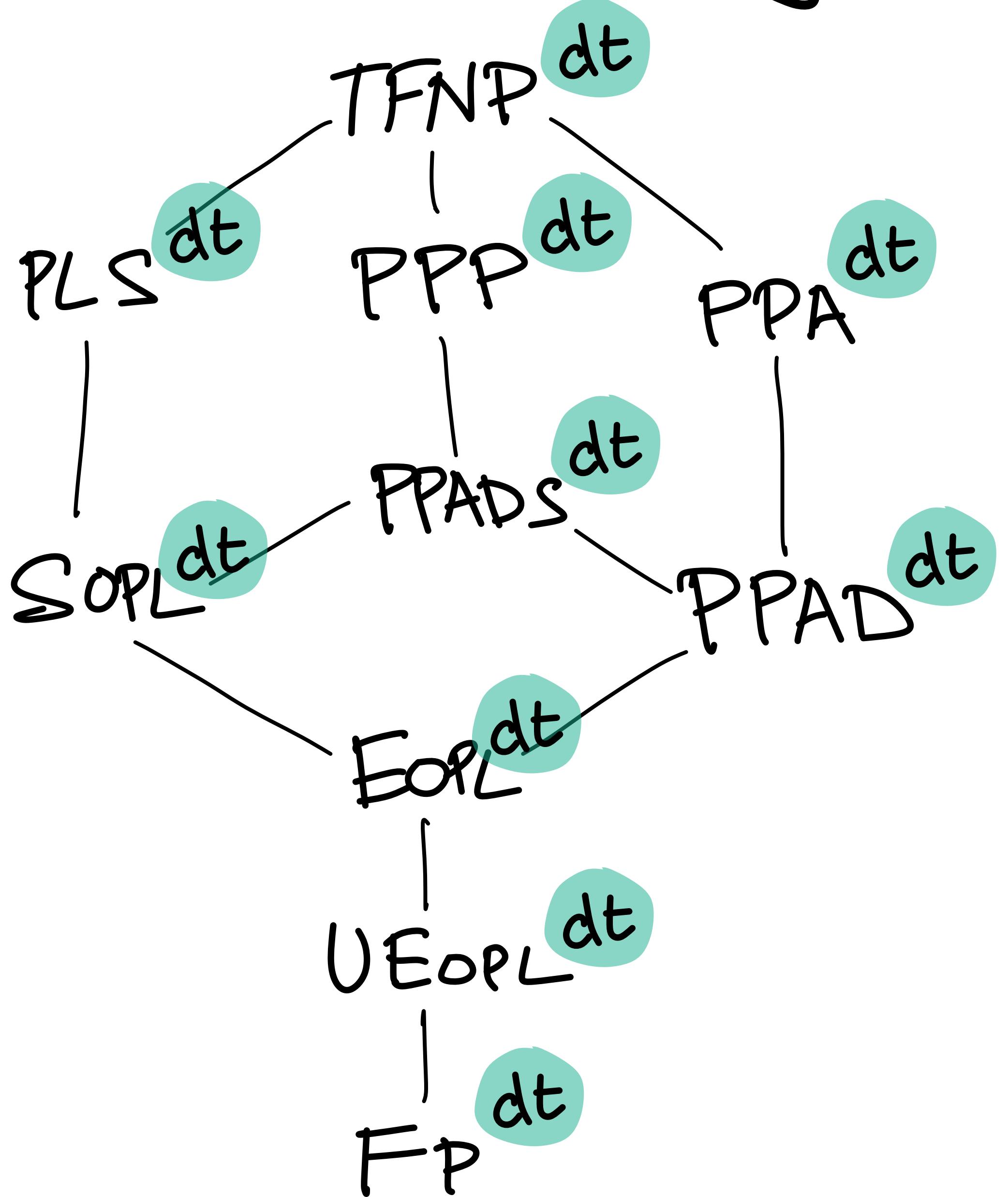
THE BRIDGE

Proof
Complexity

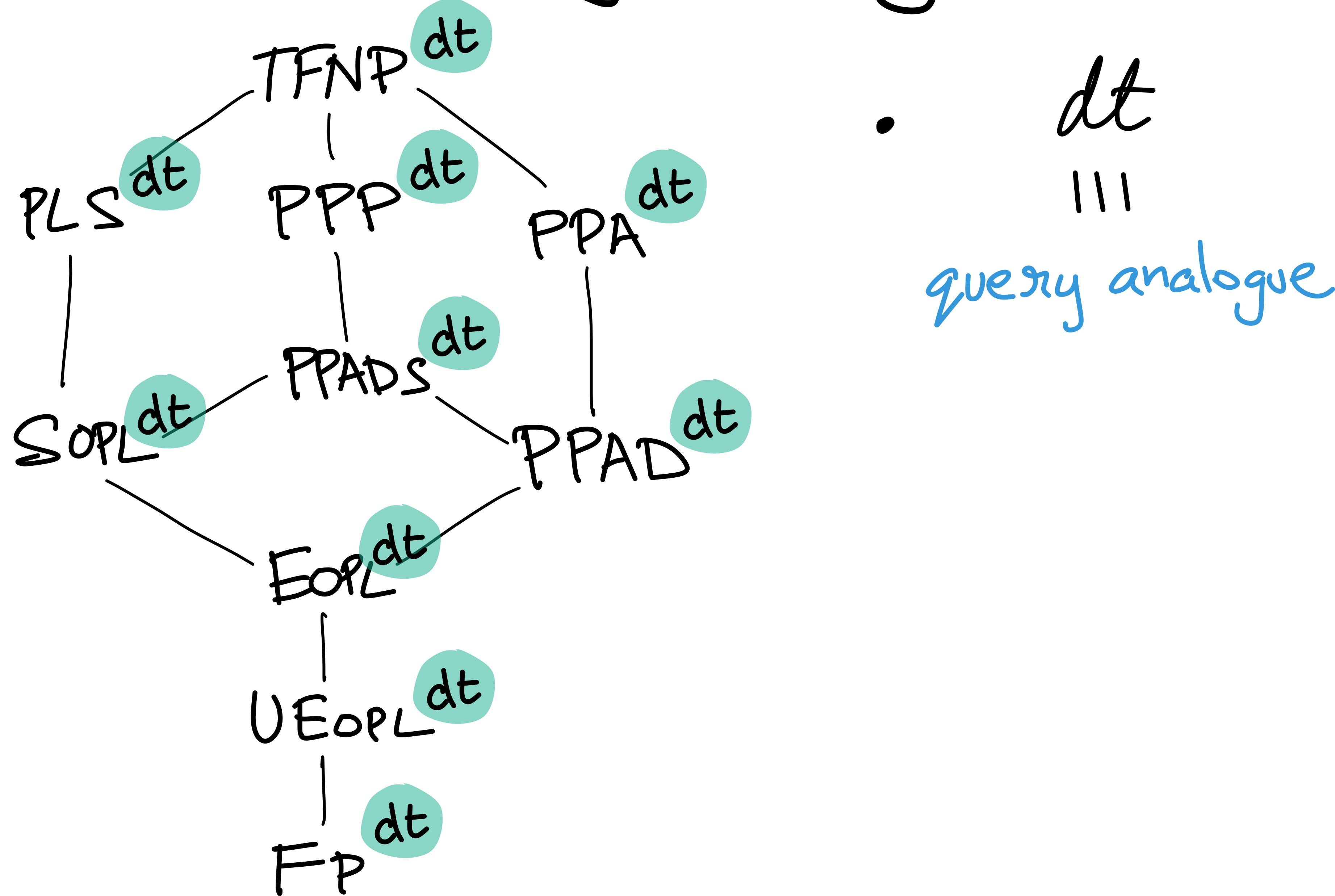


TFNP

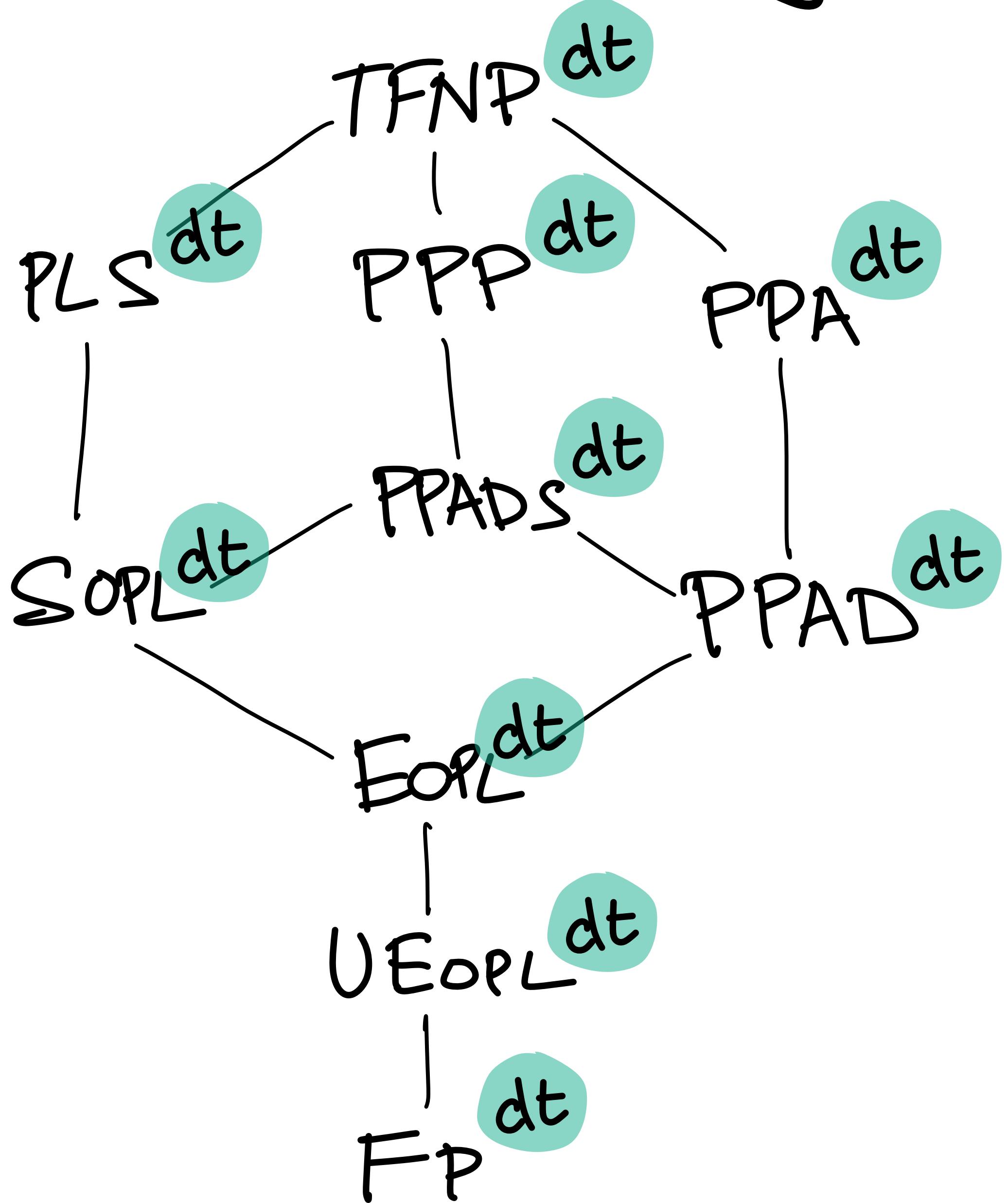
World 1: Query analogues



World 1: Query analogues



World 1: Query analogues

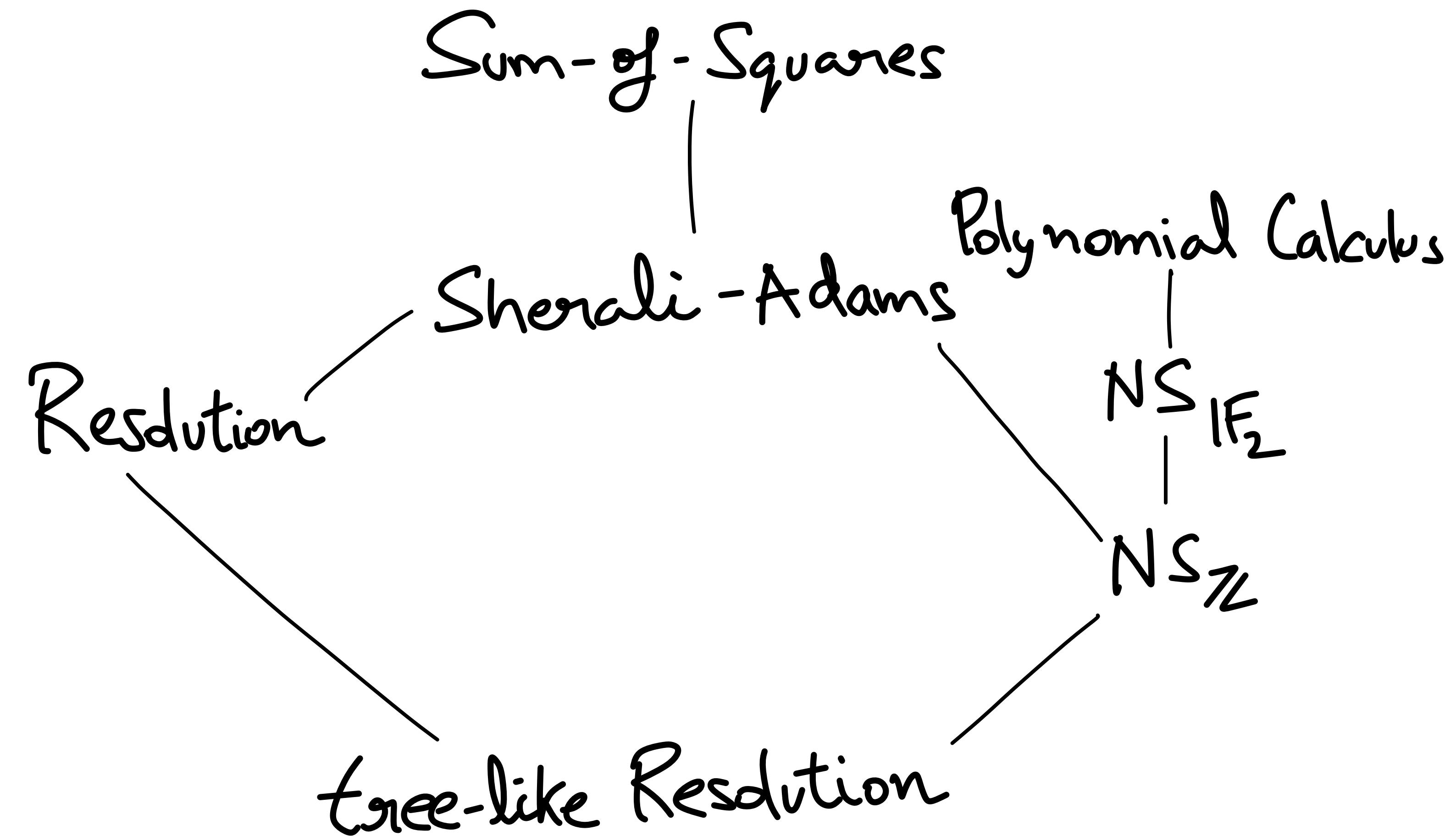


- dt
 |||
query analogue
- Reductions
 |||

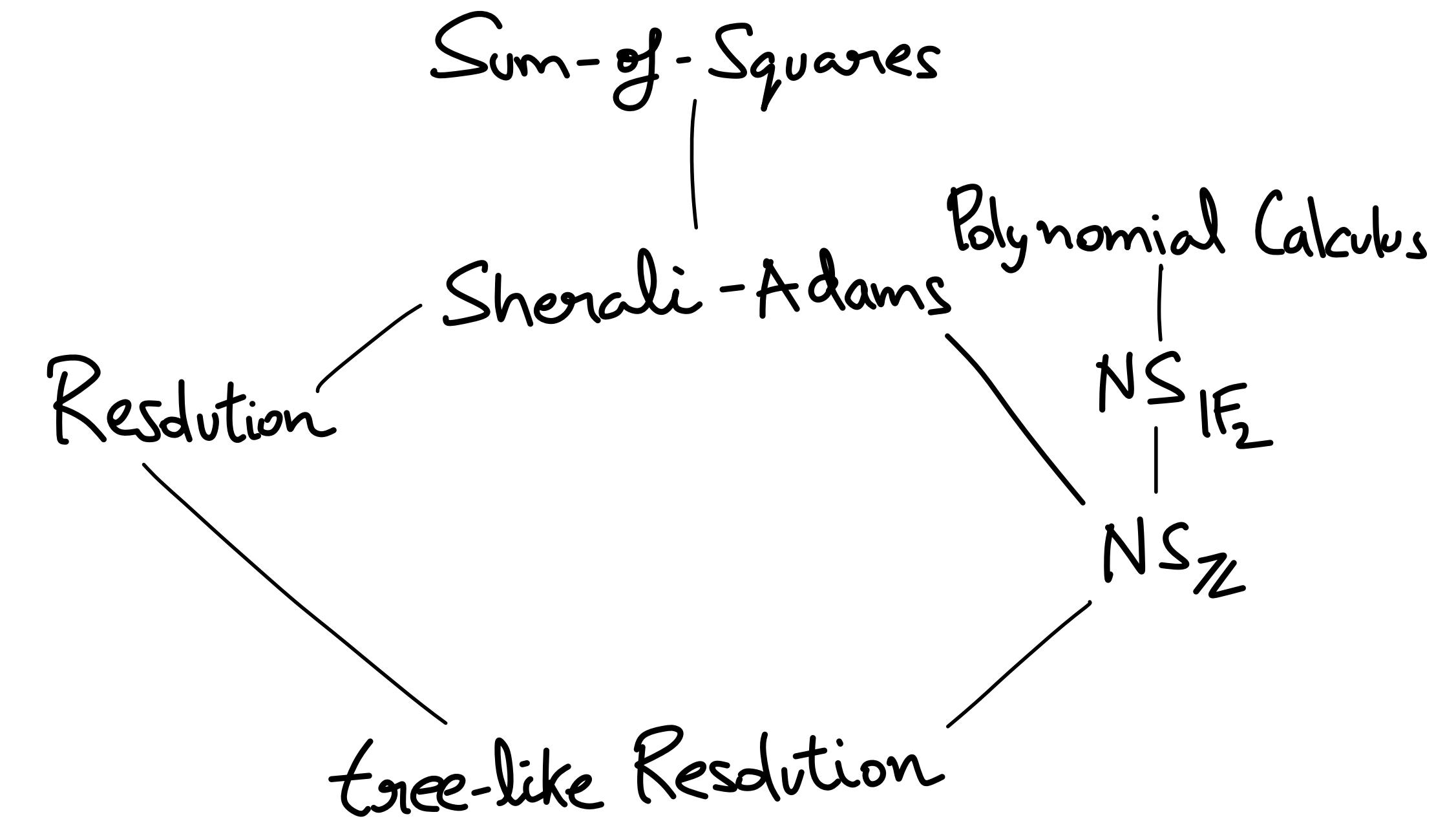
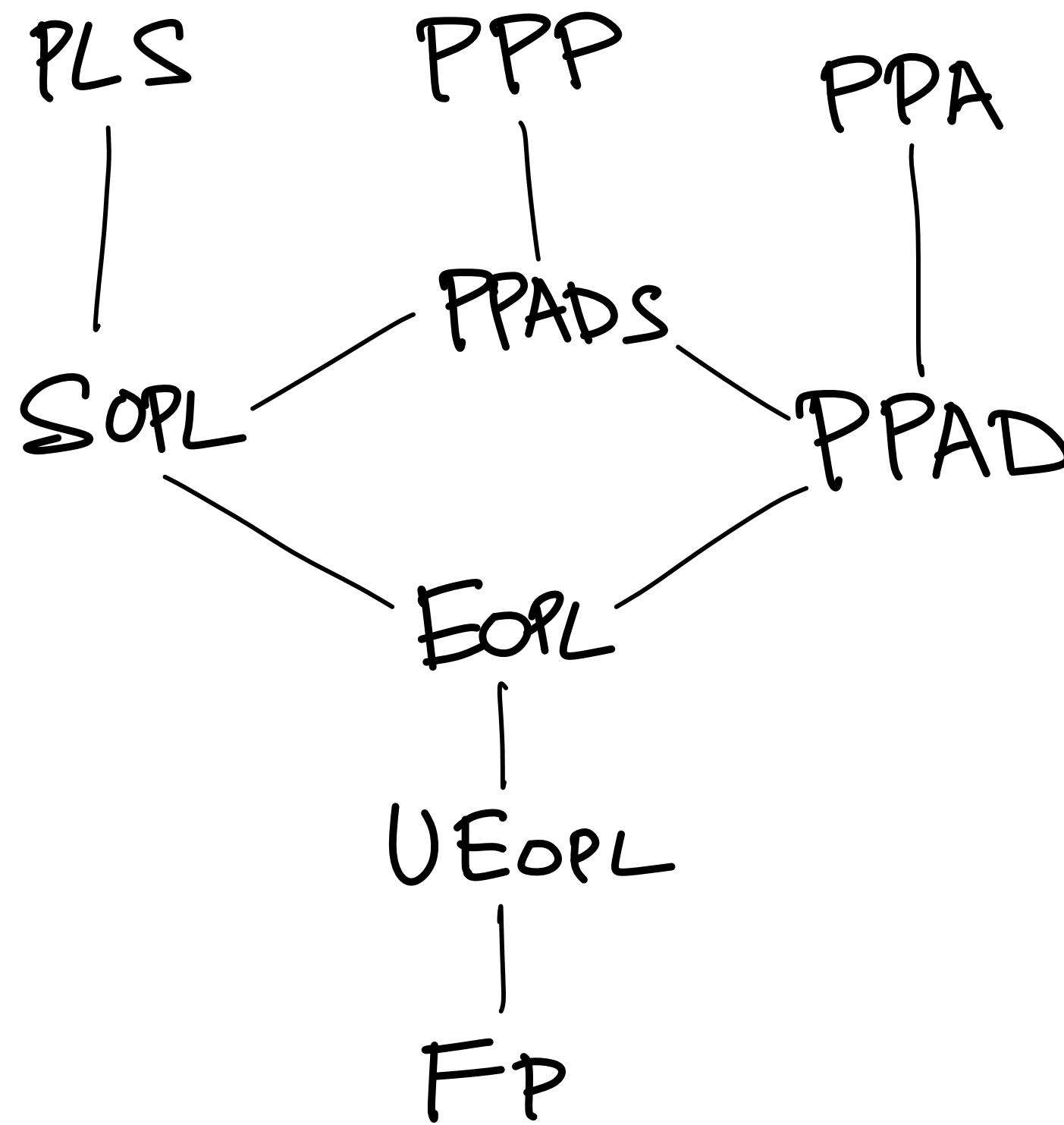
Shallow decision trees

World 2 : Proof Complexity

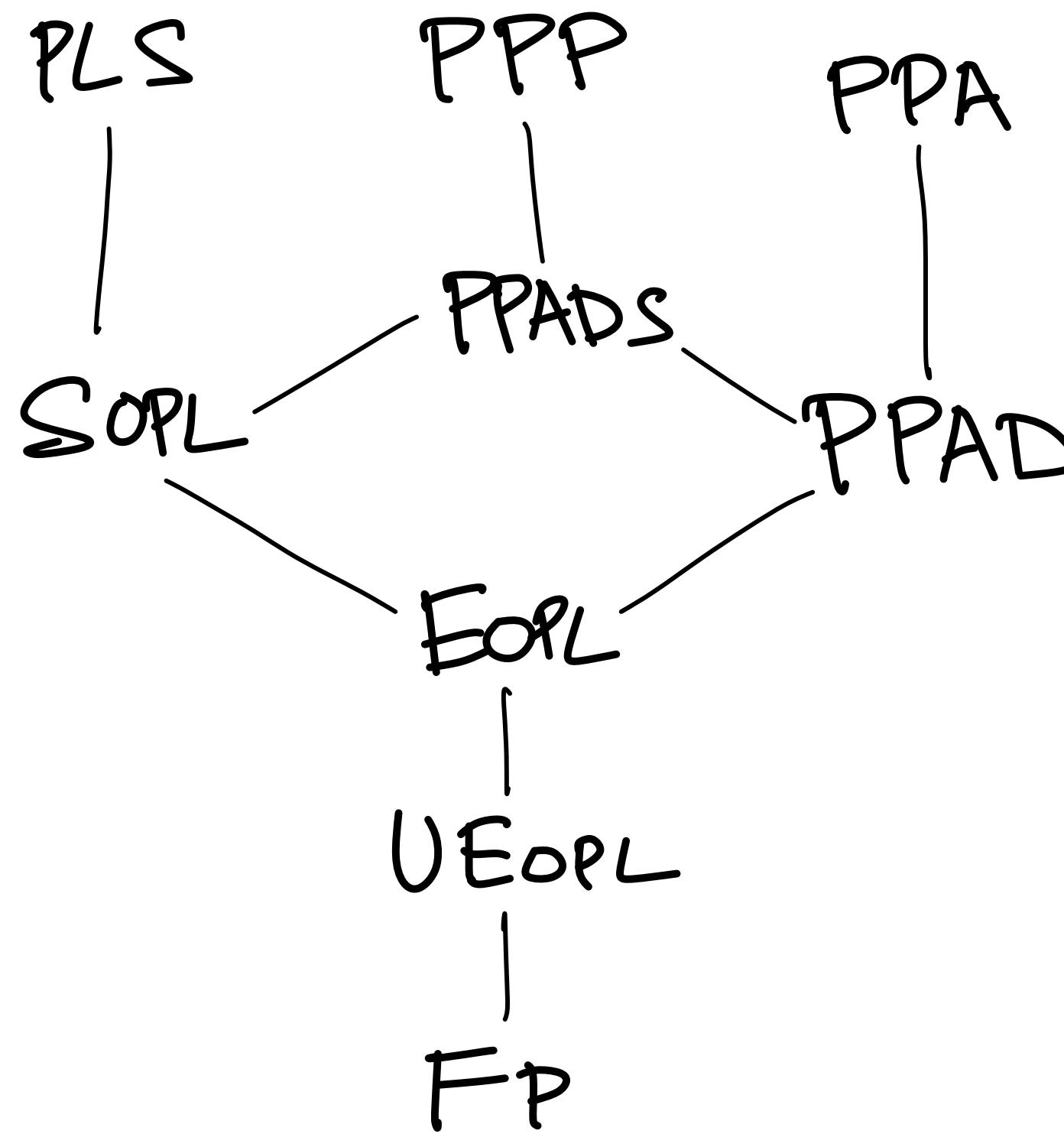
Is there a short derivation that this CNF is unsat?



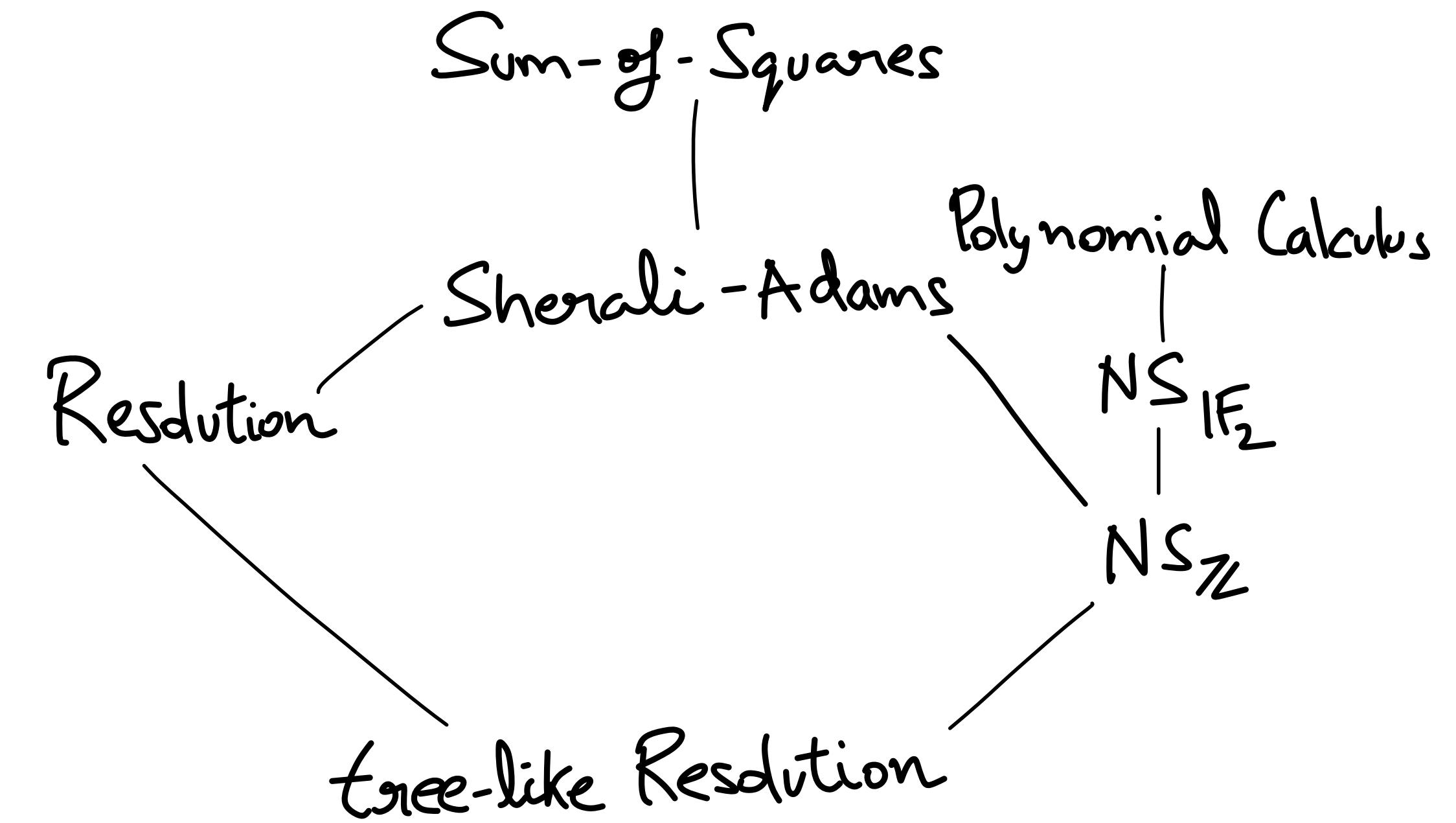
Time to Squint



Time to Squint



???



The Bridge : Characterizations

- TFNP^{dt} ^{Search Problems} can be translated into CNF fallacies

SINK-OF-DAG \mapsto "this dag has
no sinks"

The Bridge : Characterizations

- TFNP^{dt} **Search Problems** can be translated into **CNF fallacies**
- **CNF fallacies** define **search problems**

$$\varphi = x_1 \wedge (\bar{x}_1 \vee \bar{x}_2) \wedge x_2 \mapsto \begin{array}{l} \text{find}(x_1, x_2) \\ \text{falsified clause} \end{array}$$

More Explicitly



Electronic Colloquium on Computational Complexity
Under the auspices of the Computational Complexity Foundation (CCF)

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REPORTS > DETAIL:

Revision(s):

[Revision #2 to TR22-141 | 30th November 2022 03:22](#)

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TFNP Characterizations of Proof Systems and Monotone Circuits



Authors: Sam Buss, Noah Fleming, Russell Impagliazzo

Accepted on: 30th November 2022 03:22

Downloads: 81

Revision #2 Keywords:

Abstract:

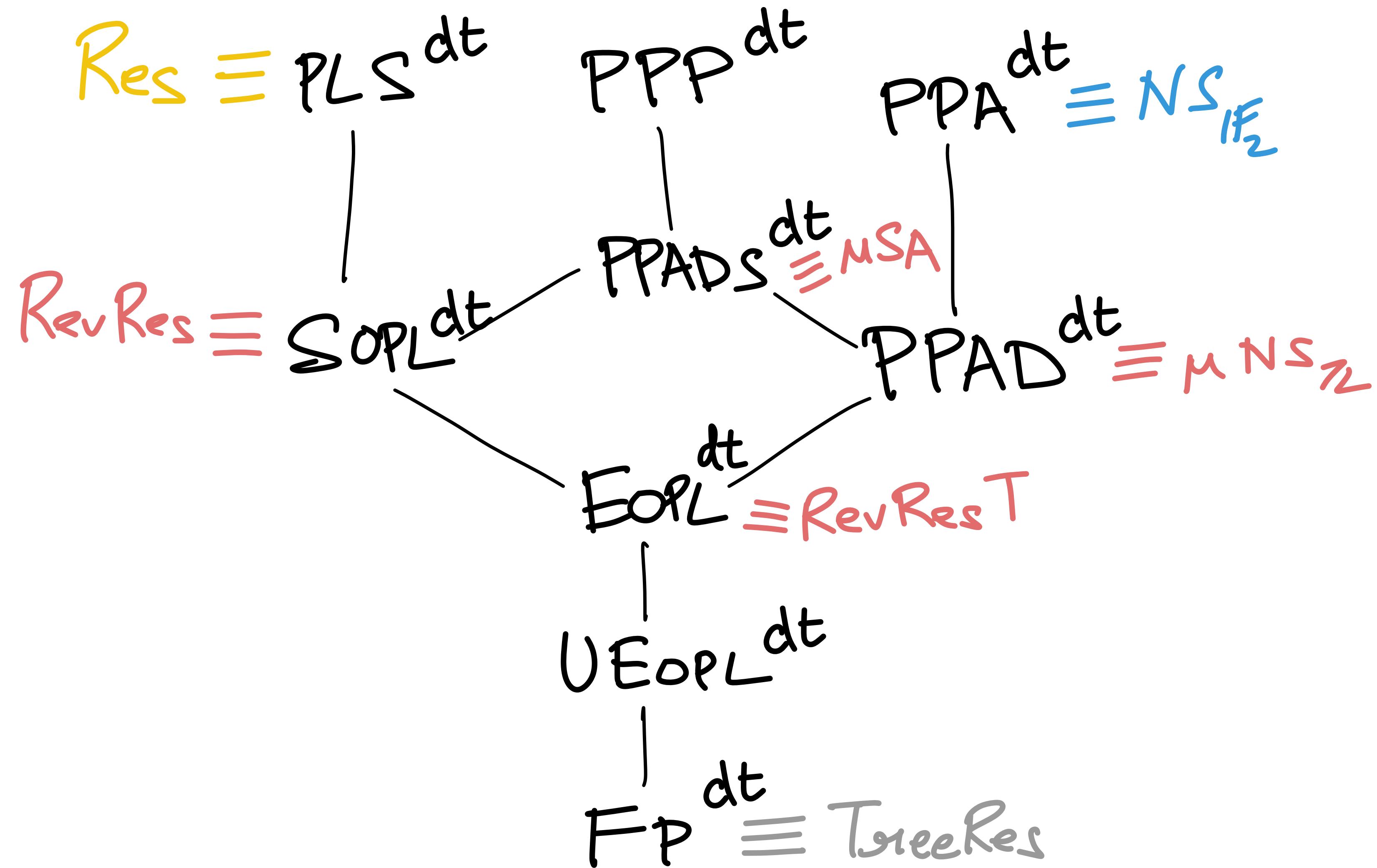
Connections between proof complexity and circuit complexity have become major tools for obtaining lower bounds in both areas. These connections -- which take the form of interpolation theorems and query-to-communication lifting theorems -- translate efficient proofs into small circuits, and vice versa, allowing tools from one area to be applied to the other. Recently, the theory of TFNP has emerged as a unifying framework underlying these connections. For many of the proof systems which admit such a connection there is a TFNP problem which characterizes it: the class of problems which are reducible to this TFNP problem via query-efficient reductions is equivalent to the tautologies that can be efficiently proven in the system. Through this, proof complexity has become a major tool for proving separations in black-box TFNP. Similarly, for certain monotone circuit models, the class of functions that it can compute efficiently is equivalent to what can be reduced to a certain TFNP problem in low communication. When a TFNP problem has both a proof and circuit characterization, one can prove an interpolation theorem. Conversely, many lifting theorems can be viewed as relating the communication and query reductions to TFNP problems. This is exciting, as it suggests that TFNP provides a roadmap for the development of further interpolation theorems and lifting theorems.

TFNP Problems

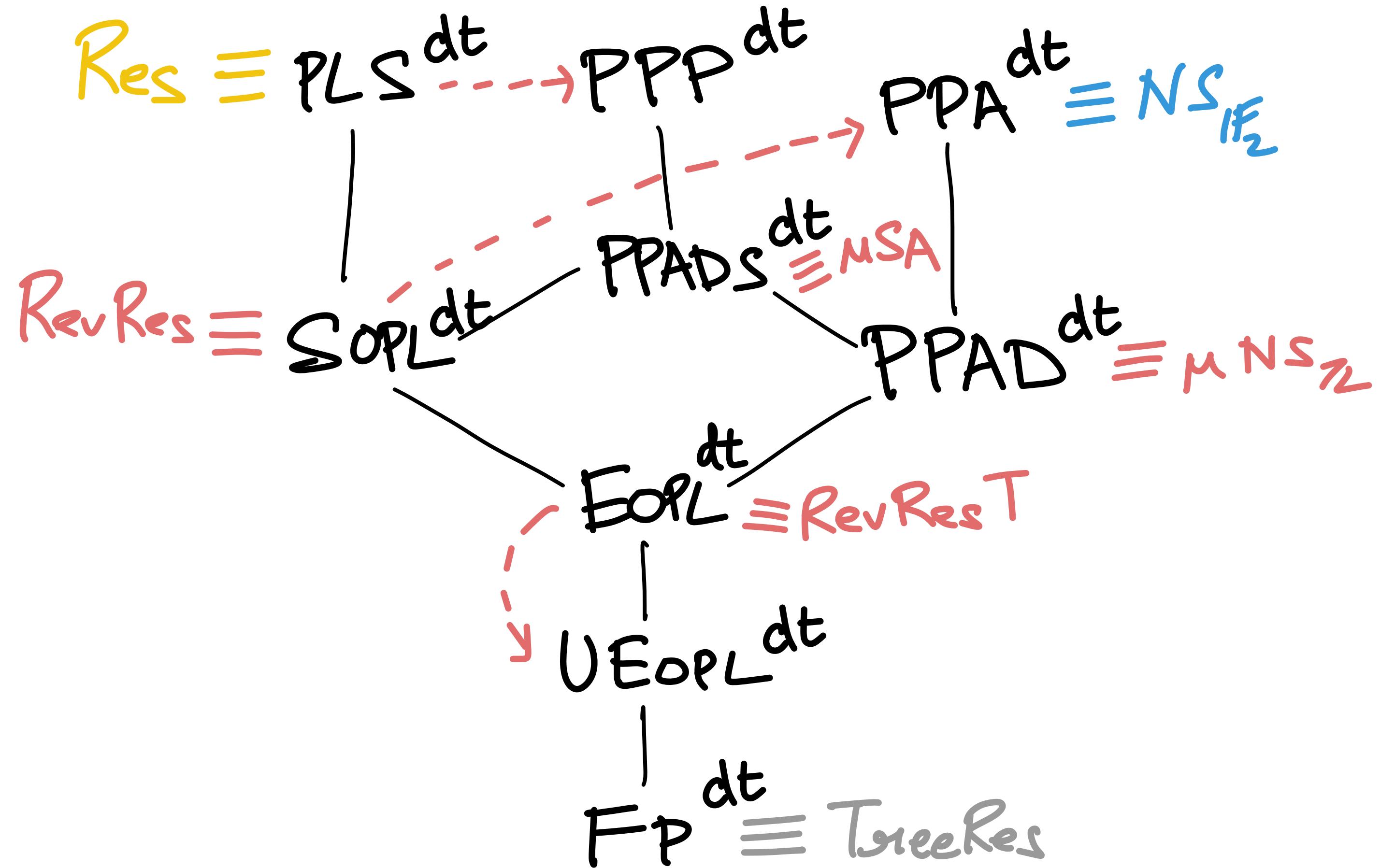
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Proof Systems
with
Reflection Principle

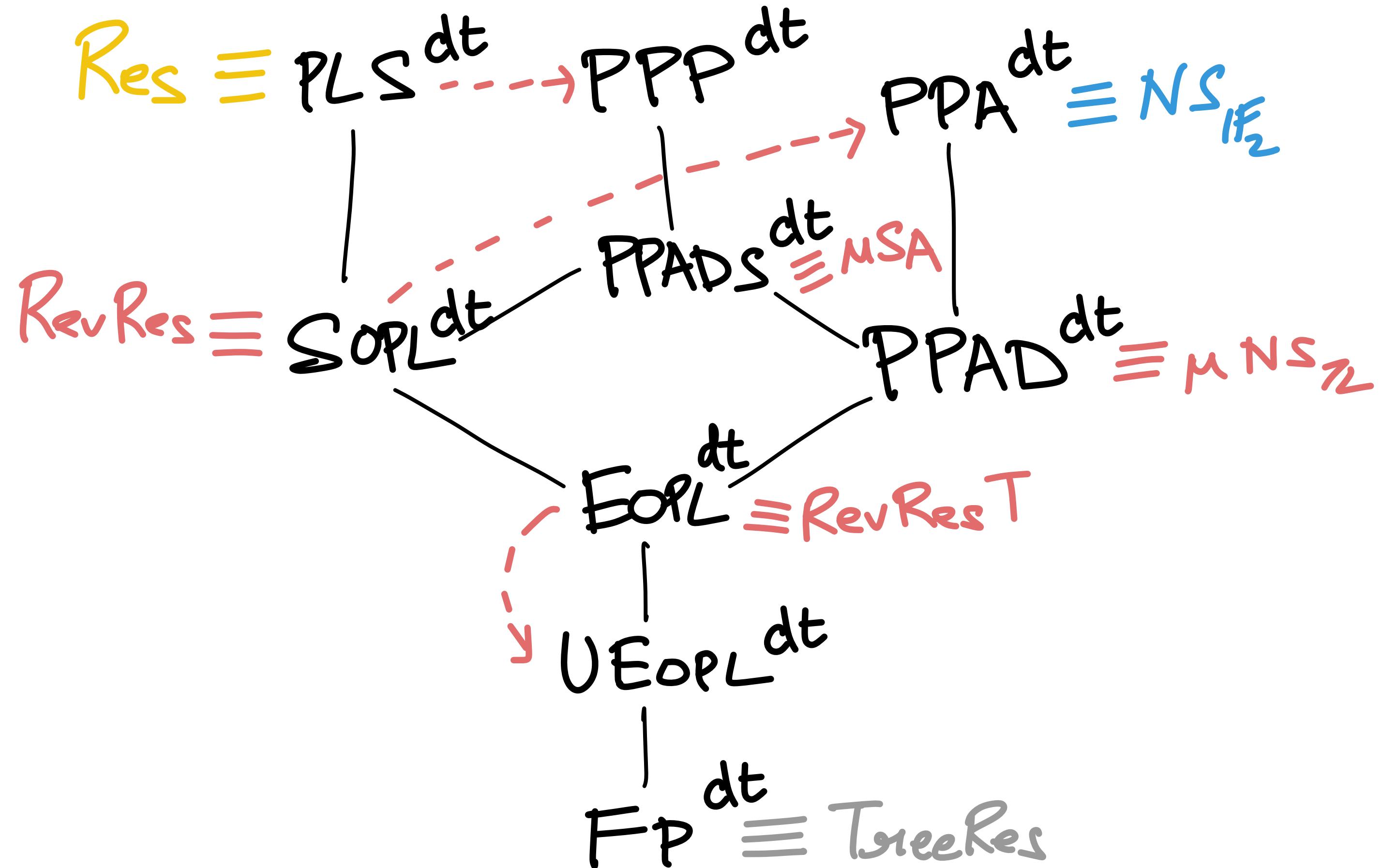
The Bridge : Characterizations



The Bridge : Characterizations

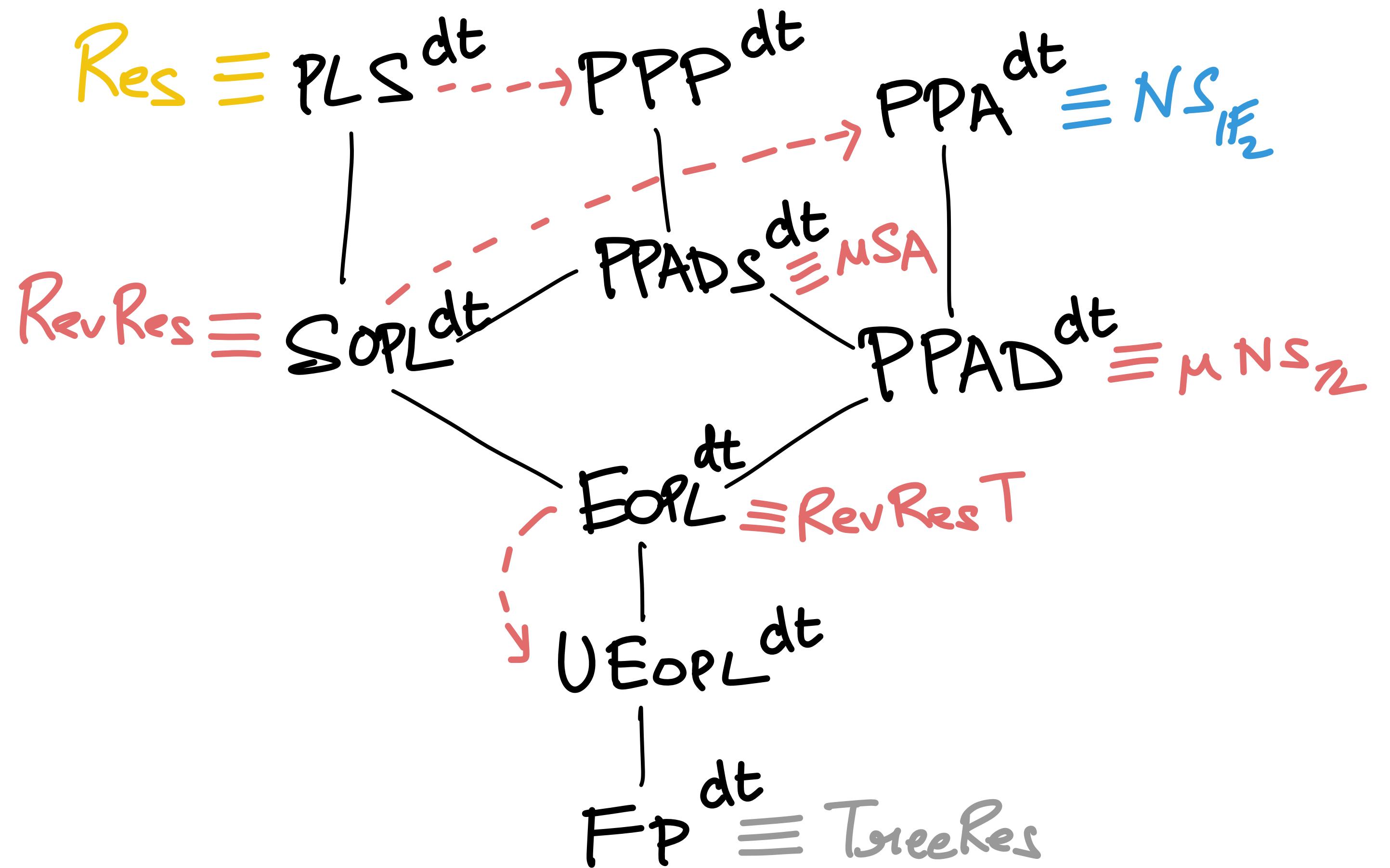


The Bridge : Characterizations



Results rephrased :

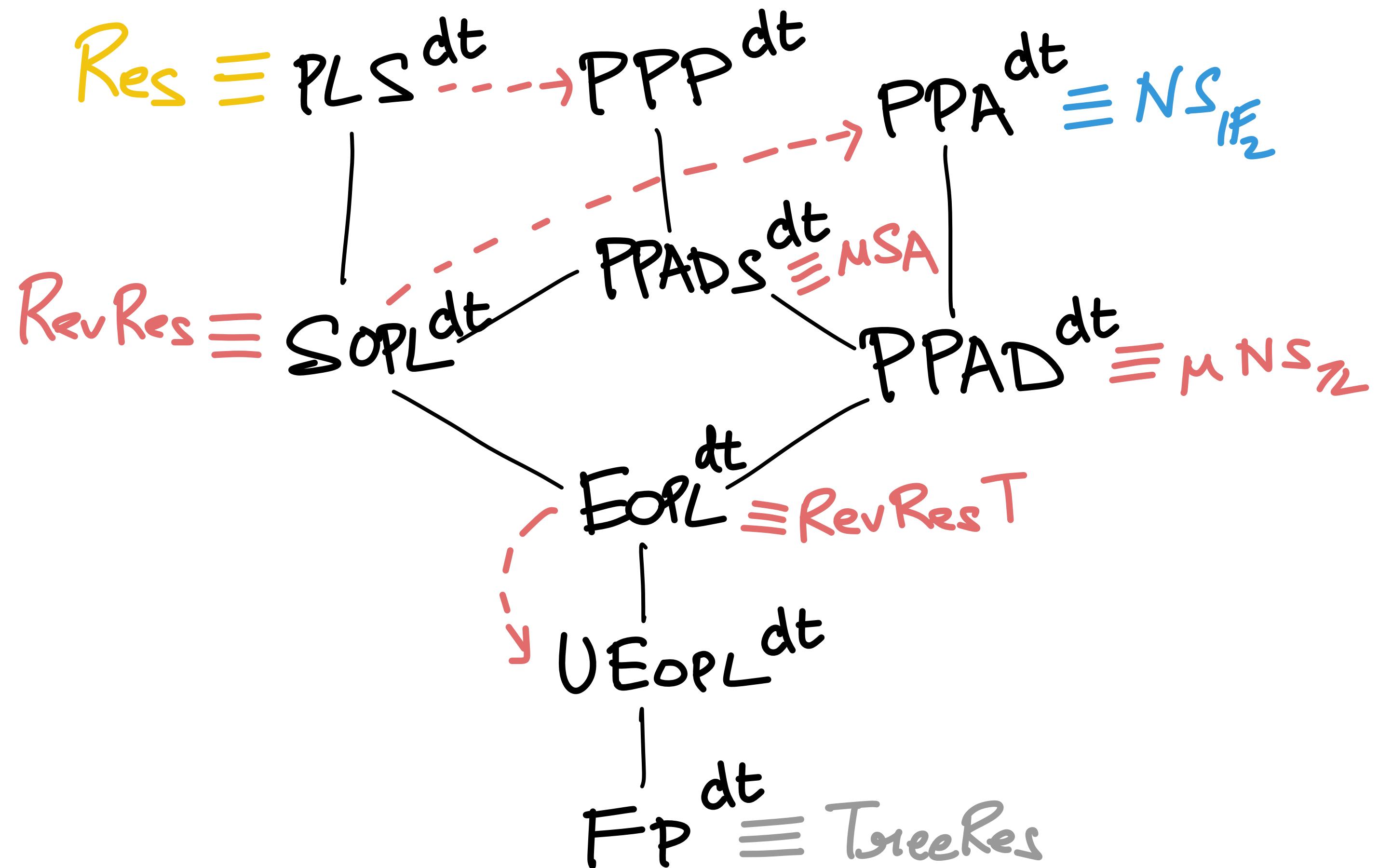
The Bridge : Characterizations



Results rephrased :

- **Res** ~~is~~ uSA

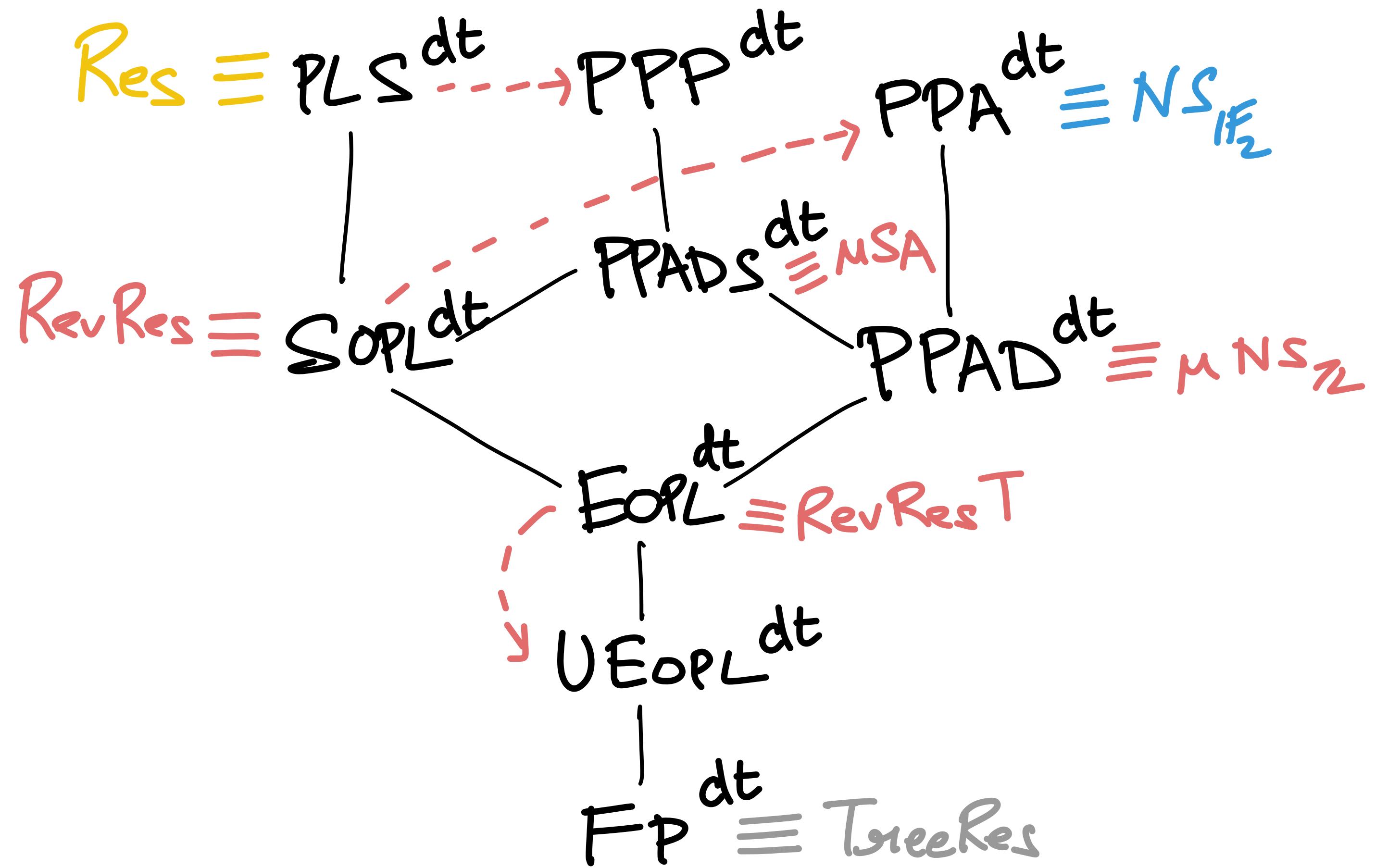
The Bridge : Characterizations



Results rephrased:

- $\text{Res} \not\leq \text{uSA}$
- $\text{RevRes} \not\leq \text{NS}$

The Bridge : Characterizations



Results rephrased:

- $\text{Res} \not\leq \text{uSA}$
- $\text{RevRes} \not\leq \text{NS}$

Independent work: $\text{PLS}^\circ \nleq \text{PPADS}^\circ \Rightarrow \text{PLS}^\circ \nleq \text{PPP}^\circ$ by [BT22]

Some Characterizations

let's see why:

- ① Resolution width \approx PLS^{dt} depth

Some Characterizations

let's see why:

i Resolution width $\approx \text{PLS}^{\text{dt}}$ depth

Formally, given $R(x, y) \in \text{PLS}^{\text{dt}}$; PLS^{dt} depth
= depth of reduction to SoD

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For every $v \in V$, we have decision trees

$$\Pi_v(x) = s_v$$

$$O_v(x) = y \text{ s.t. } (x, y) \in R \text{ if } v \text{ is a sink*}$$

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$$\text{PLS}^{\text{dt}} \text{ depth} = \log |V| + \max_{v \in V} |\Pi_v|$$

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Some Characterizations

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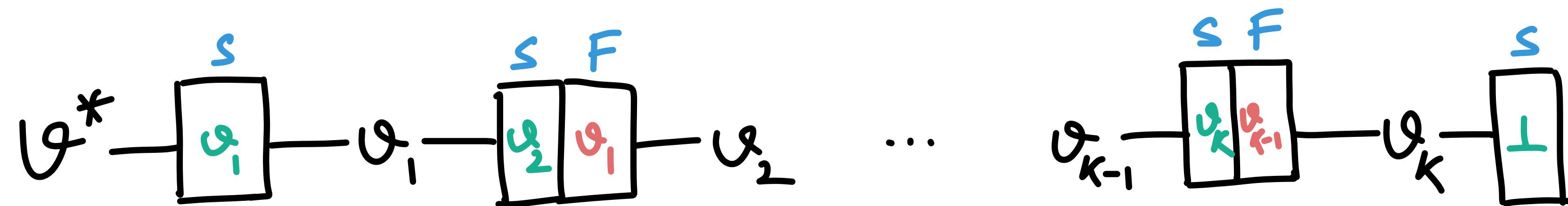
We will use Prover-Delayer characterization

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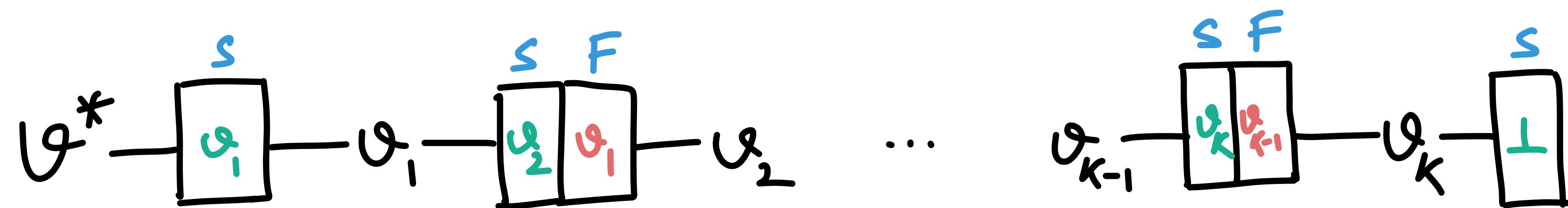


Some Characterizations

let's see why:

i a Resolution width $\leq \text{PLS}^{\text{dt}}$ depth

We will use Prover-Delayer characterization



$$\text{Res Width} \leq 2 \log |V| + \max_{v \in V} |\Pi_v|$$

Some Characterizations

let's see why:

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Some Characterizations

let's see why:

i \square Resolution width $\geq \text{PLS}^{\text{dt}}$ depth

Dag-like Res proof is already kinda a SoD reduction

Some Characterizations

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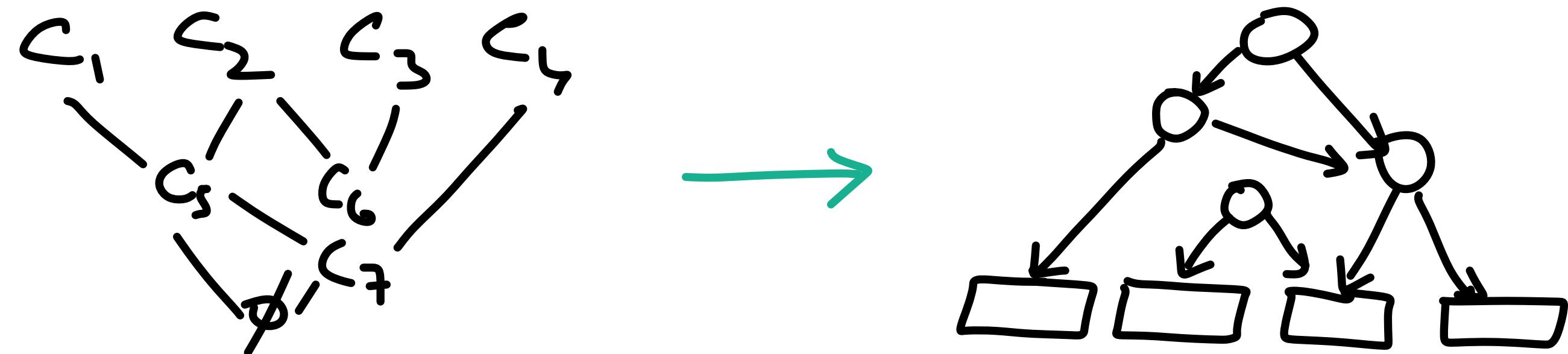
Dag-like Res proof is already kinda a SoD reduction
Just have to Flip The Proof!

Some Characterizations

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Just have to *Flip The Proof!*

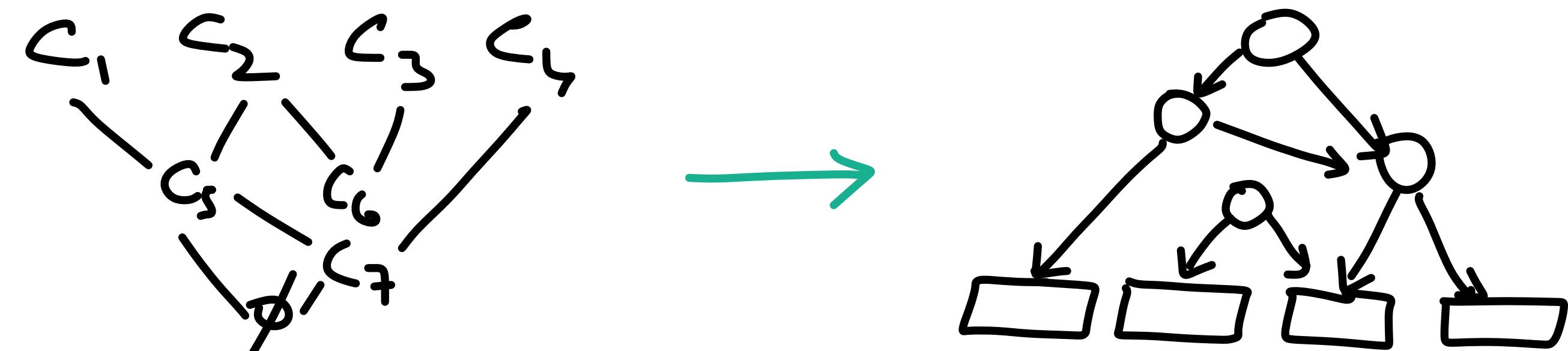


Some Characterizations

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Dag-like Res proof is already kinda a SoD reduction
Just have to *Flip The Proof!*



$$\text{PLS}^{\text{dt}} \text{ depth} \leq \log(\text{ResSize}) + \text{ResWidth}$$

Some Characterizations

let's see why:

- i Resolution width $\approx \text{PLS}^{\text{dt}}$ depth
- ii Unary NS deg $\approx \text{PPAD}^{\text{dt}}$ depth

Lemma : If \exists depth d EoL-formulation of F
then \exists uNS refutation of F
with degree $O(d)$ and size $L 2^{O(d)}$.

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Proof : EoL formulation : $(V = [L], \{s_\alpha, p_\alpha, o_\alpha\})$

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Define $S_\alpha(x) = \begin{cases} -1 & \text{if } v \neq v^* \text{ is a source in } G_x \\ 1 & \text{if } v \neq v^* \text{ is a proper sink in } G_x \\ 0 & \text{otherwise} \end{cases}$

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S_v can be computed
in depth $5d$.

Lemma : If \exists depth d EoL-formulation of F
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Proof : EoL formulation : $(V = [L], \{S_\vartheta, P_\vartheta, O_\vartheta\})$

Define $S_\vartheta(x) = \begin{cases} -1 & \text{if } \vartheta \neq \vartheta^* \text{ is a source in } G_x \\ 1 & \text{if } \vartheta \neq \vartheta^* \text{ is a proper sink in } G_x \\ 0 & \text{otherwise} \end{cases}$

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leaves
axe
solutions

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leaves
axe
solutions

$$\Rightarrow \sum_{\vartheta \in V} S_\vartheta = \sum_i p_i \bar{C}_i$$

for some $\{p_i\}$

Lemma : If \exists depth d EoL-formulation of F
then \exists uNS refutation of F
 with degree $O(d)$ and size $L 2^{O(d)}$.

Proof : EoL formulation : $(V = [L], \{S_\vartheta, P_\vartheta, O_\vartheta\})$

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leaves
axe
solutions

$$\Rightarrow \sum_{v \in V} S_\vartheta = \sum_i p_i \bar{C}_i = \#\text{sinks in } G_x - \#\text{non-}\vartheta^*\text{ sources in } G_x = 1$$

for some $\{p_i\}$

•

•

Lemma : If \exists uNS refutation of F with $\deg d$ and size L
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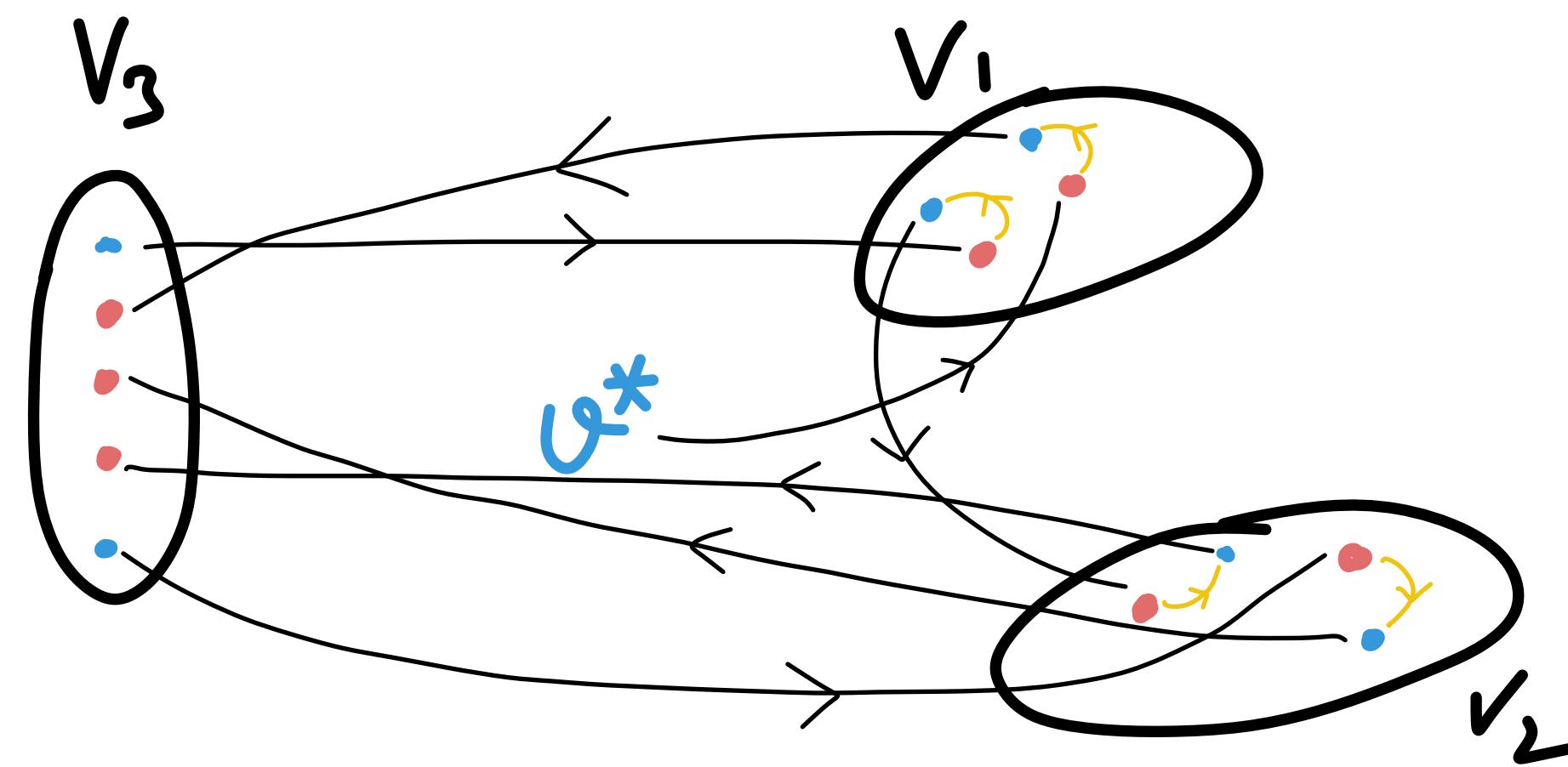
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NOTE: Not a Cook-Reckhow proof system!
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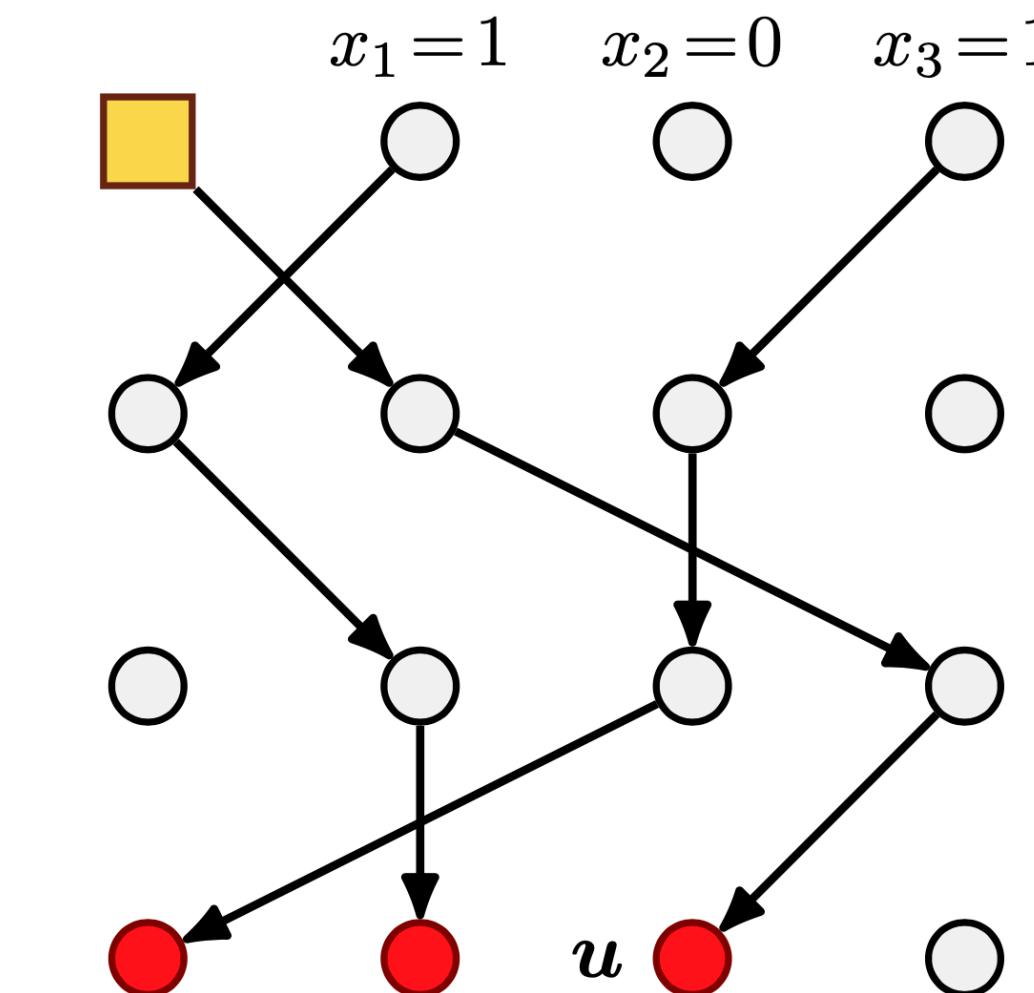
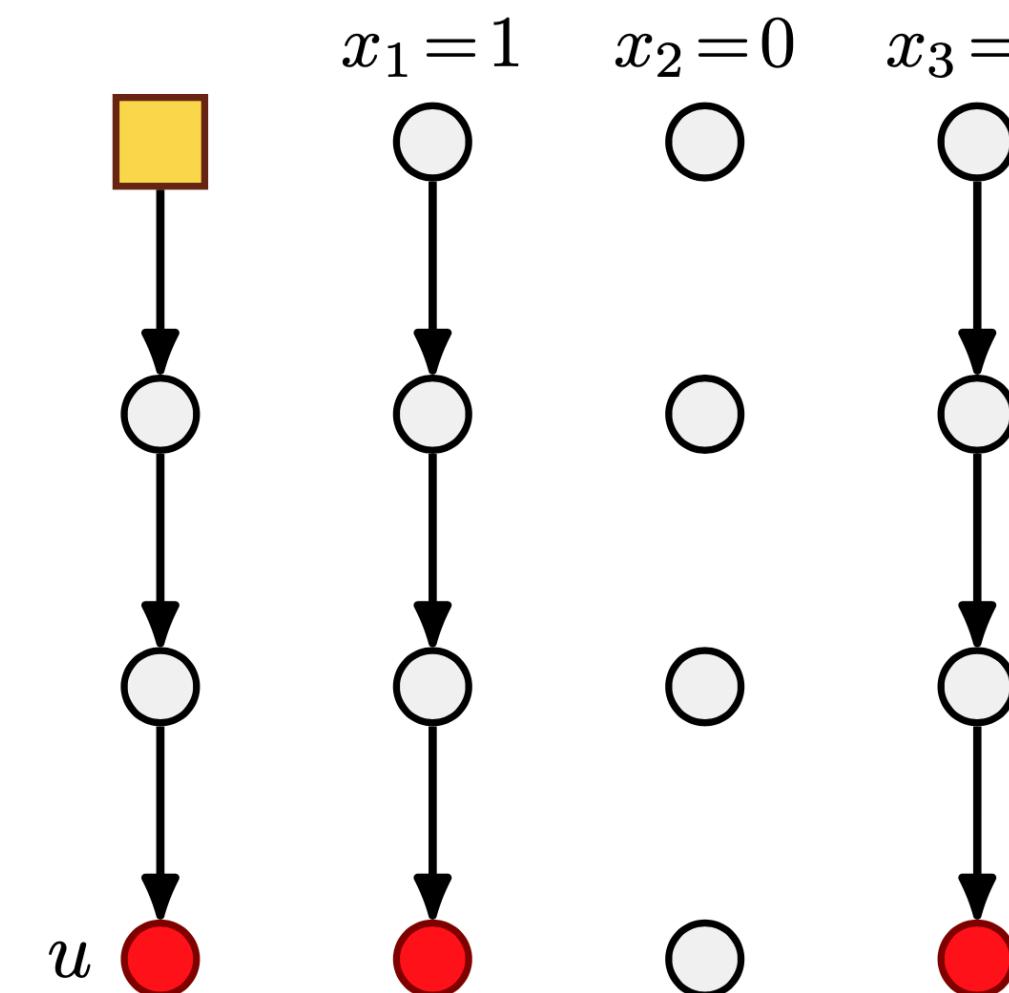
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↳ Randomised reduction is a distribution over reductions

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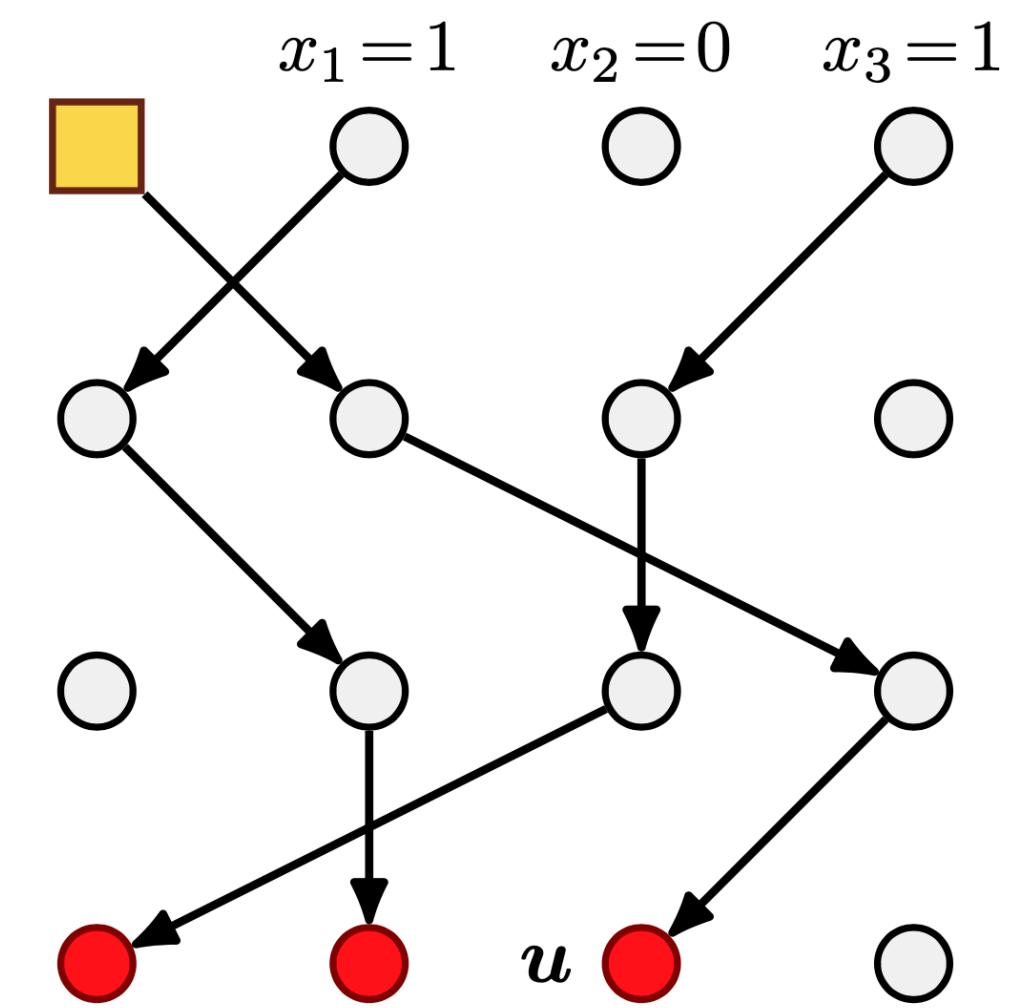
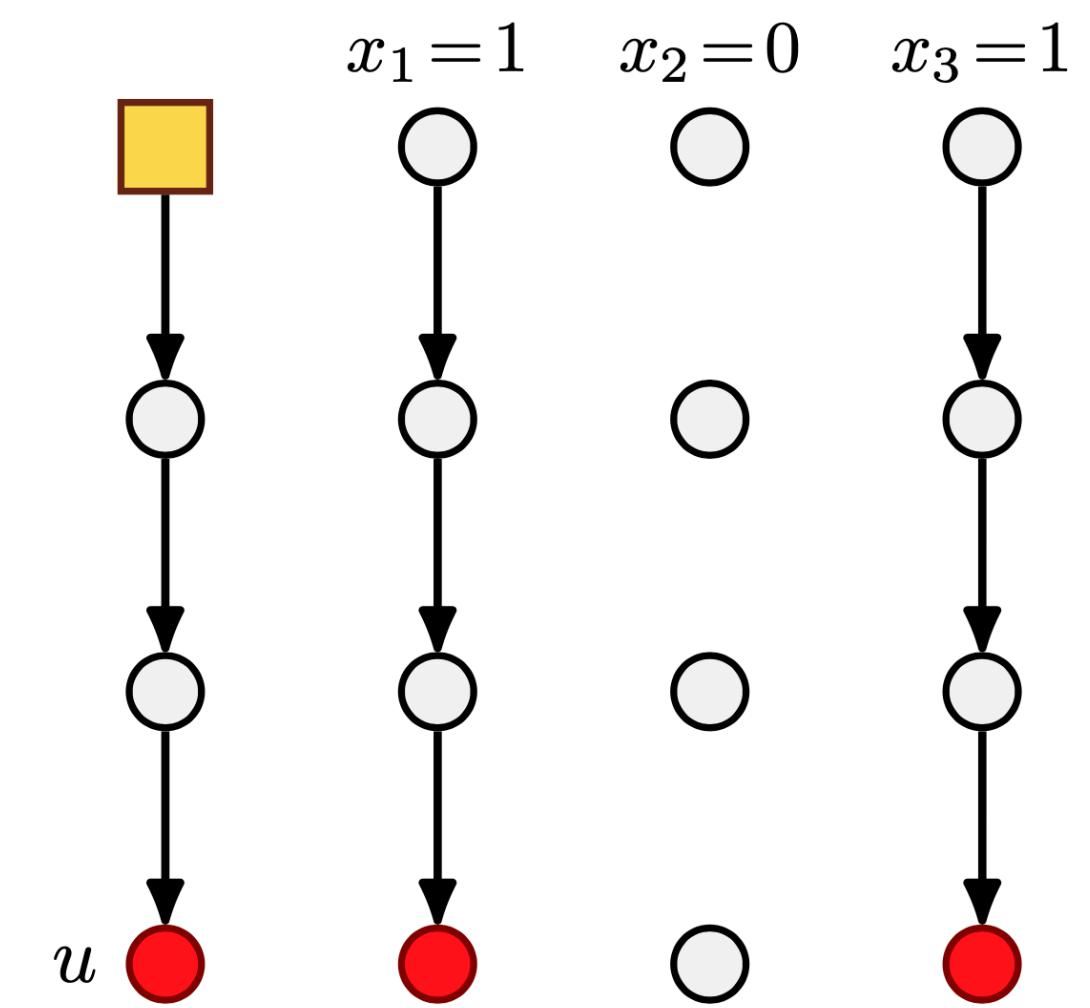
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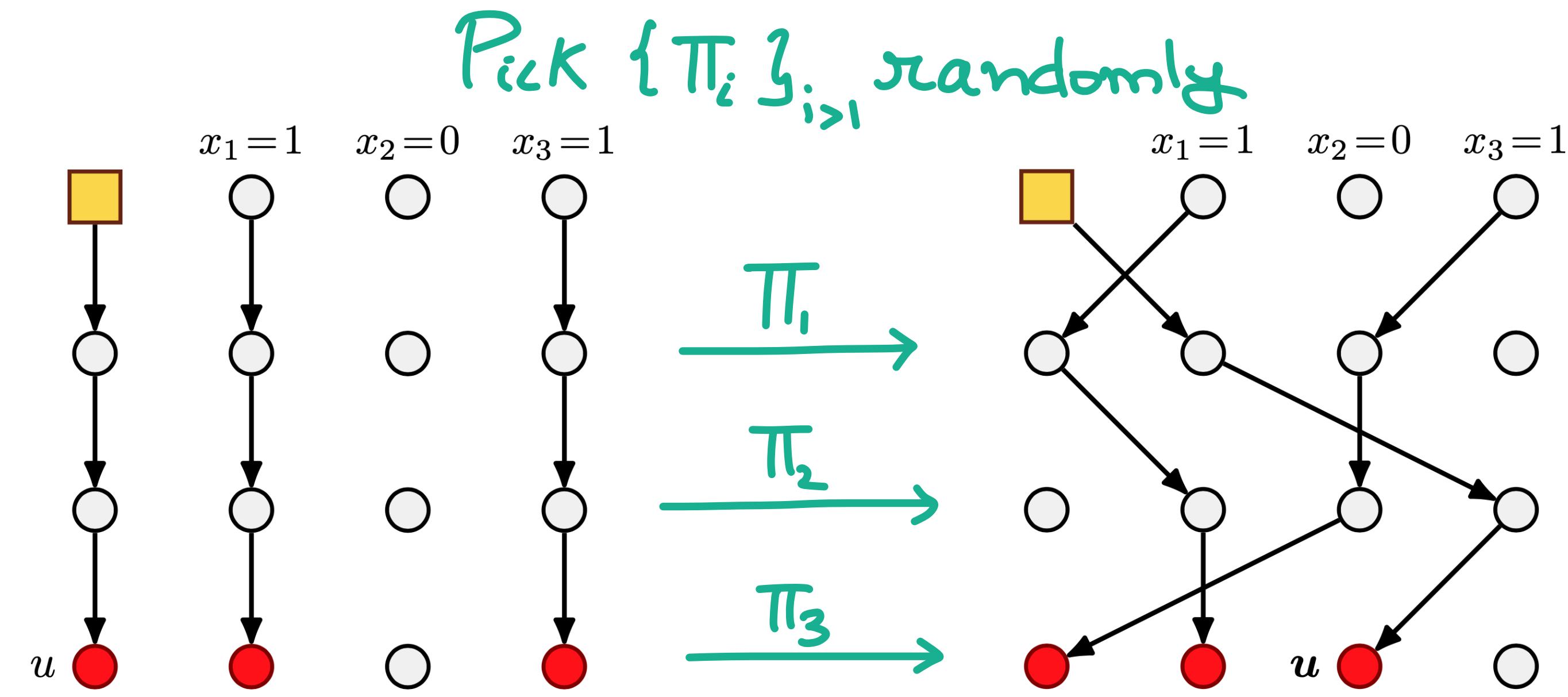
cause it's
IDEAL

using that it's
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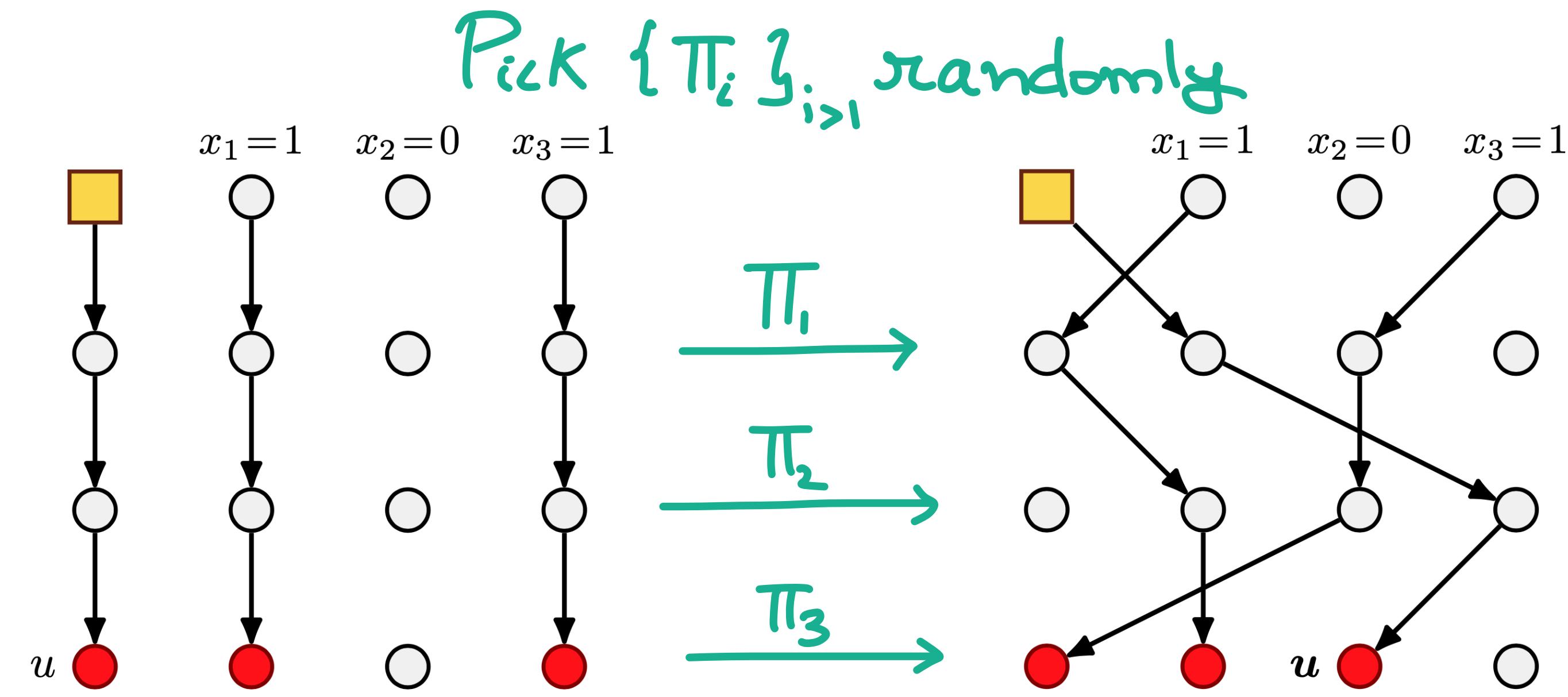
Our Reduction



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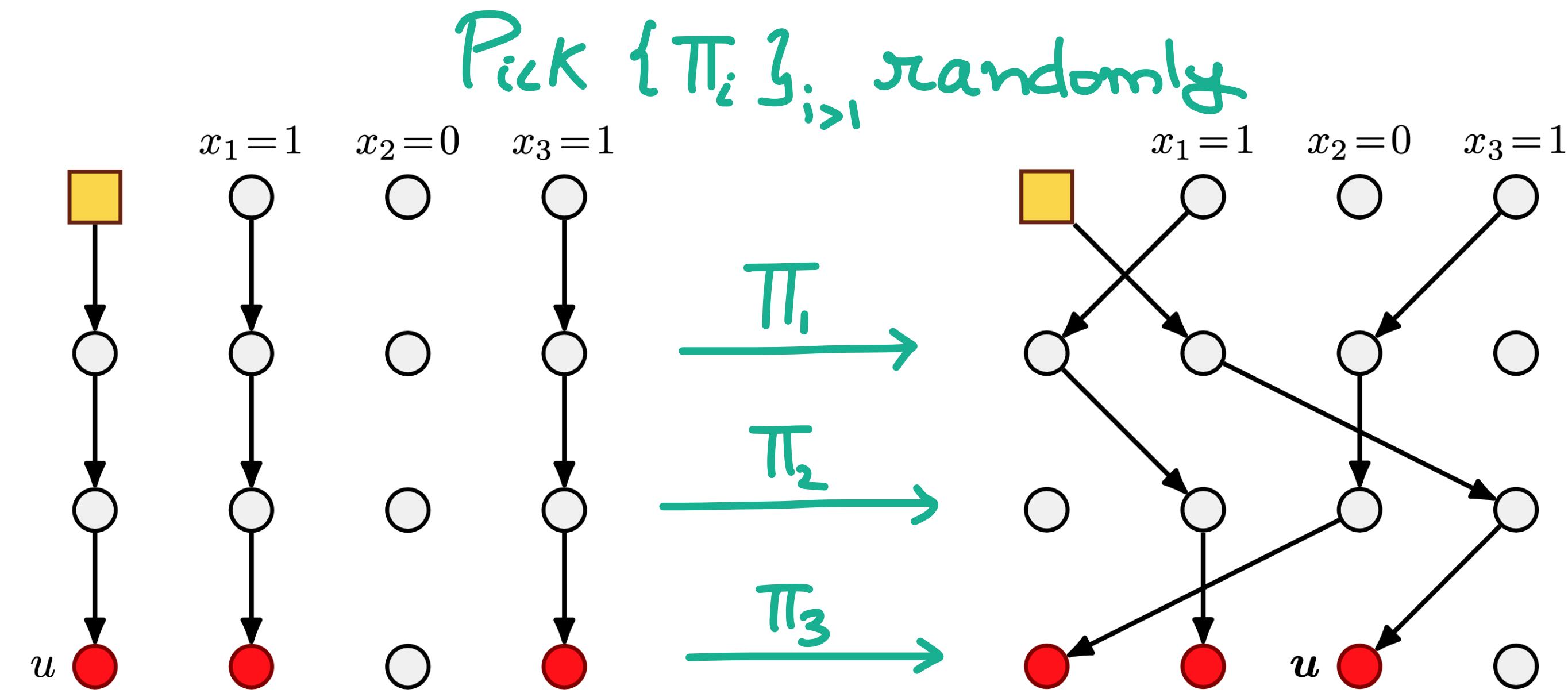


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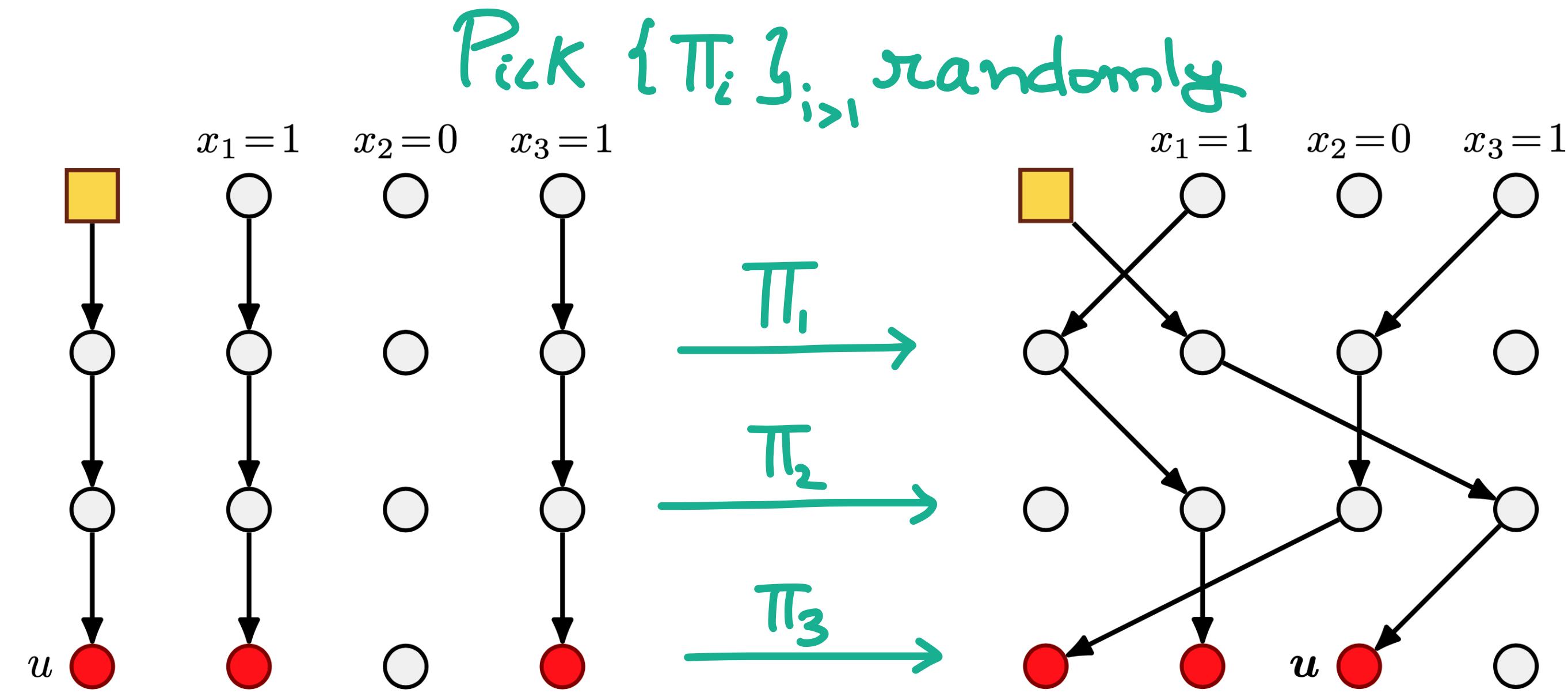
Is this Ideal?

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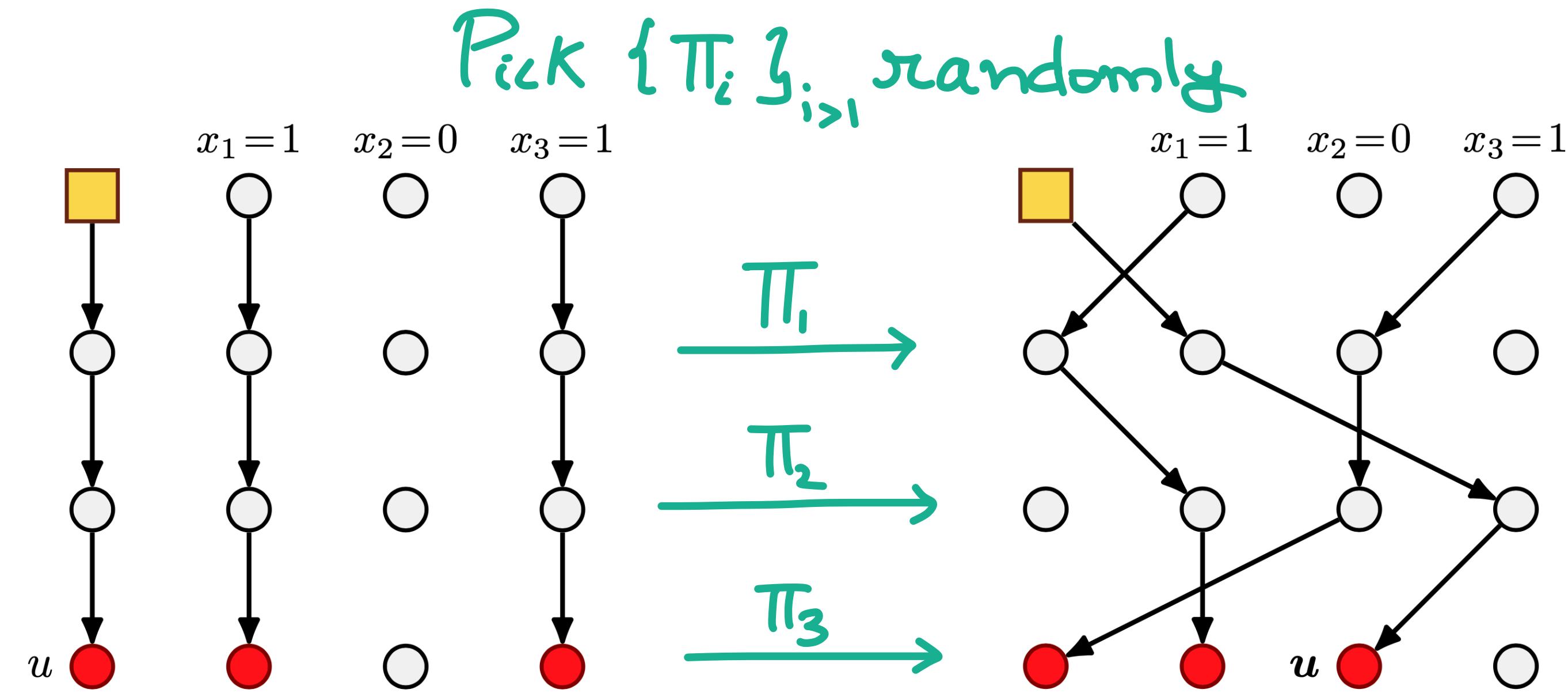


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Note

$$|\text{Sol}(y)| = 1 + |x| \quad \text{w. p. 1}$$

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\Rightarrow Given $(y, u) \sim R_x$ let $A \subseteq \{i\} \times [n+1]$, $B \subseteq \{i+1\} \times [n+1]$
be the active nodes. We can apply a random bijection
 $A \rightarrow B$ and get an Ideal reduction.

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Hard instance for $\epsilon\text{-NS} \rightarrow \text{l.b. on } J(x)$ in SA

$$\sum_i p_i(x) a_i(x) = 1 + J(x)$$

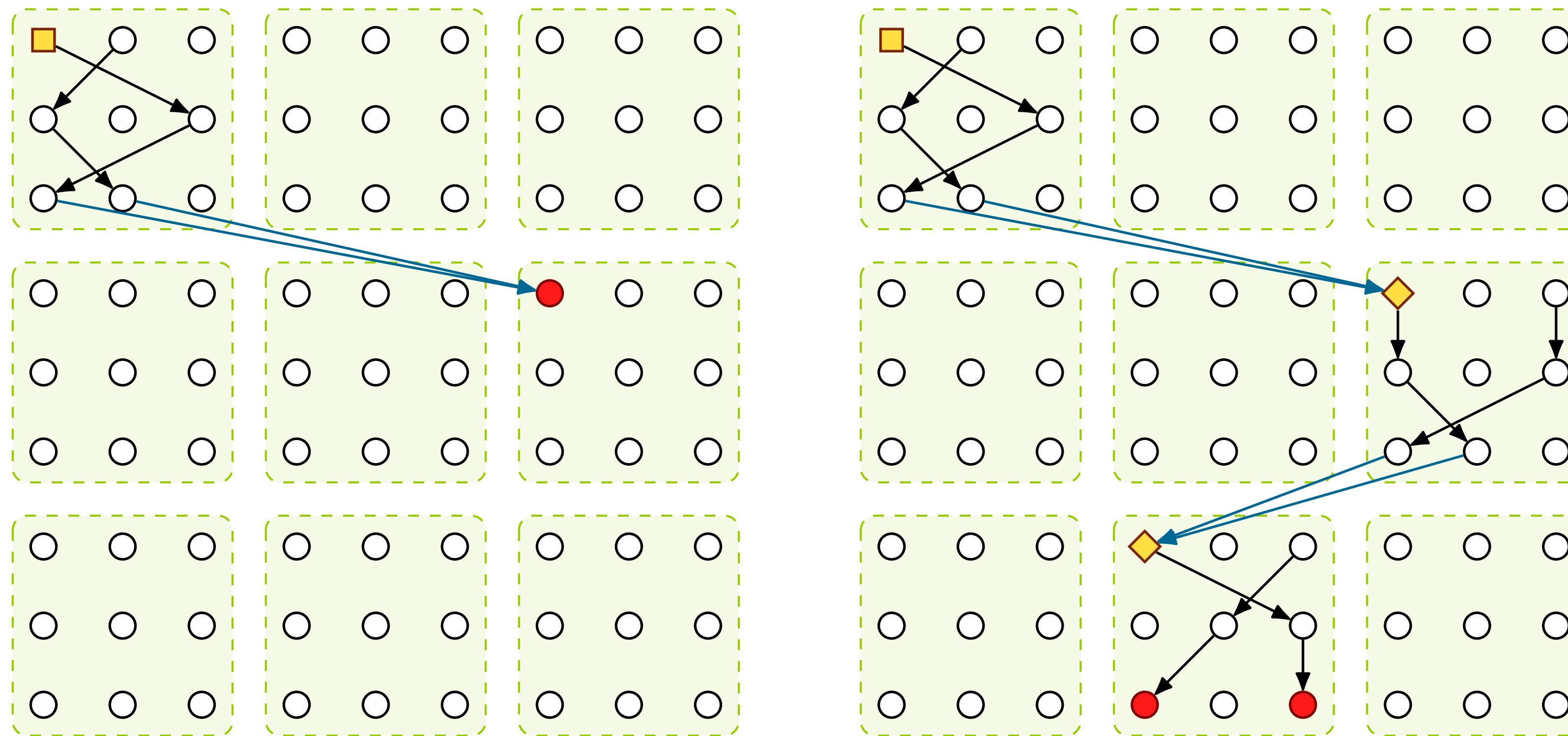
$$\sum_i \frac{p_i(x) a_i(x)}{\deg < n^{o(1)}} = \frac{1 + J(x)}{2^{\Omega(n)}}$$

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Thanks!
for your attention!