Separations in Proof Complexity and TFNP

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Understanding The Title

TFNP := Total Function NP

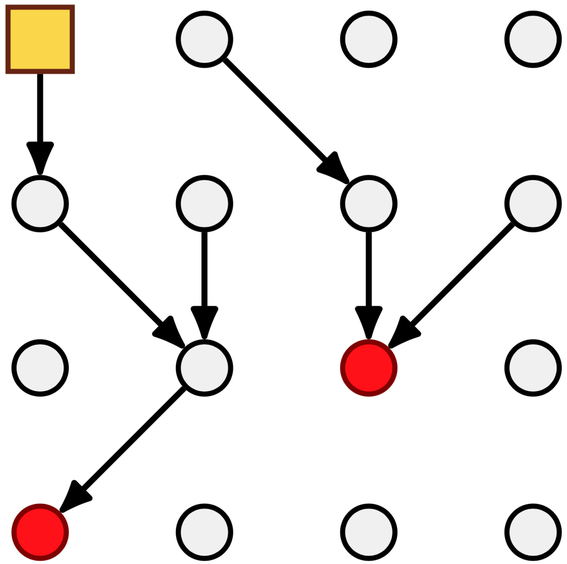
Polytime $R(x,y)$

Input x

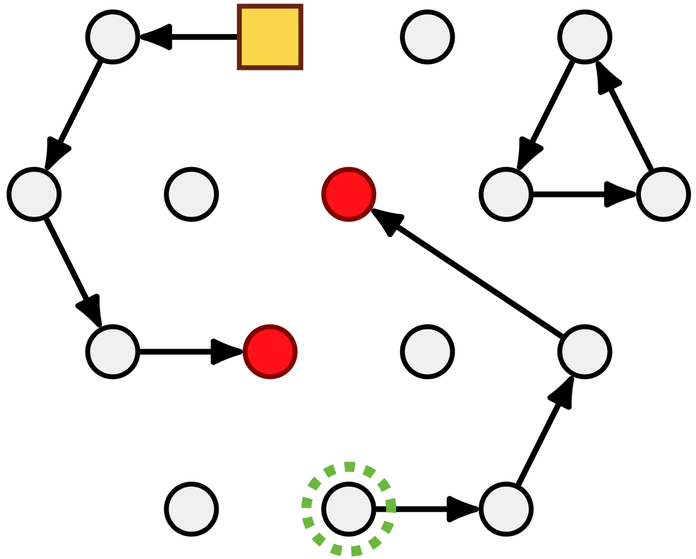
Output $y : R(x,y) = 1$ & $|y| \leq |x|^{O(1)}$

Promise R is total: $\forall x \; \exists y \; R(x,y) = 1$

Two Problems

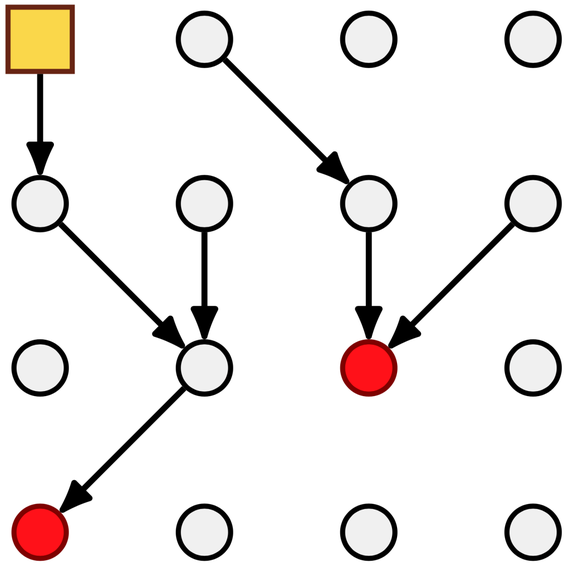


Sink-of-DAG (SoD)

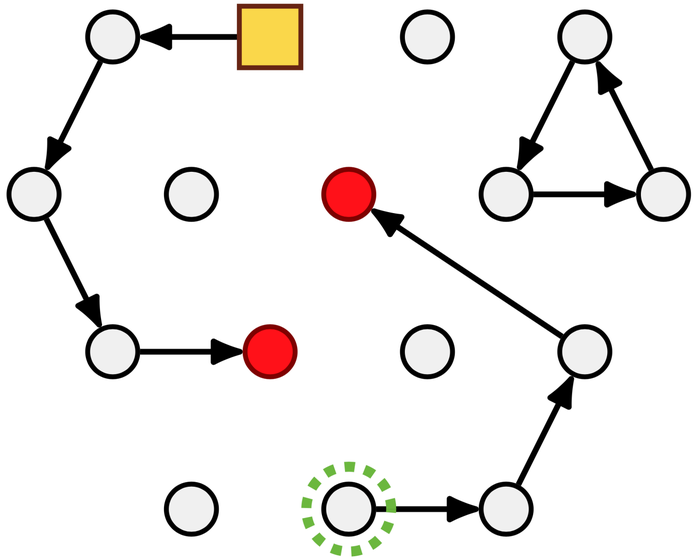


Sink-of-Line (SoL)

Two (& ½) Problems



Sink-of-DAG (SoD)



Sink-of-Line (SoL)
End-of-Line (EoL)

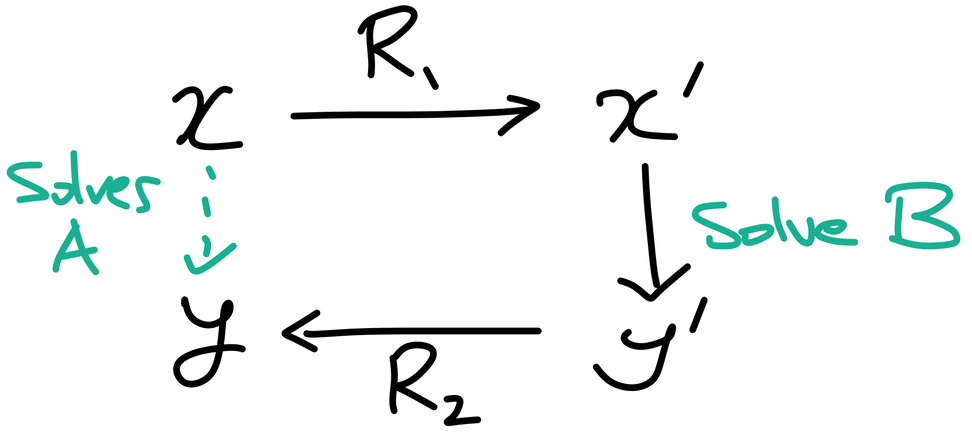
... And Three Classes

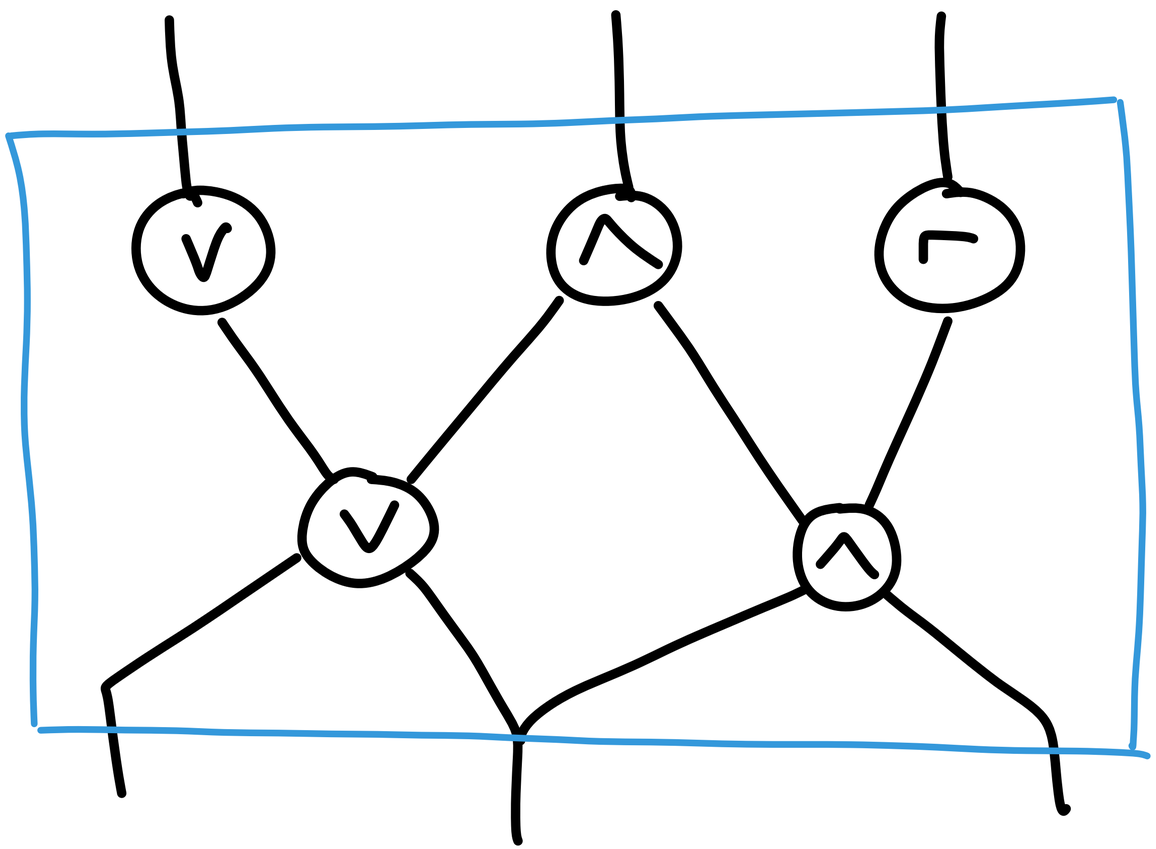
$$PLS = \{P : P \subseteq S \circ D\}$$

$$PPADS = \{P : P \subseteq S \circ L\}$$

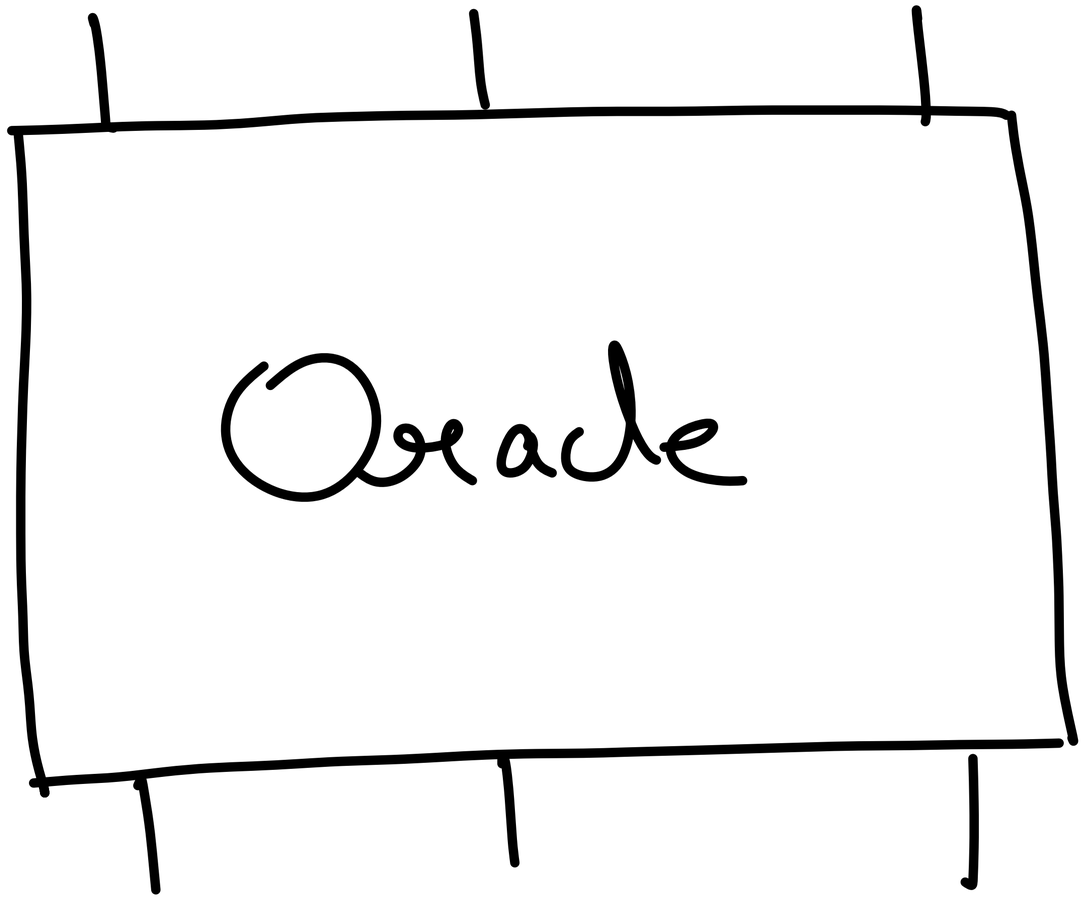
$$PPAD = \{P : P \subseteq E \circ L\}$$

$A \subseteq B$ if $\exists R_1, R_2$



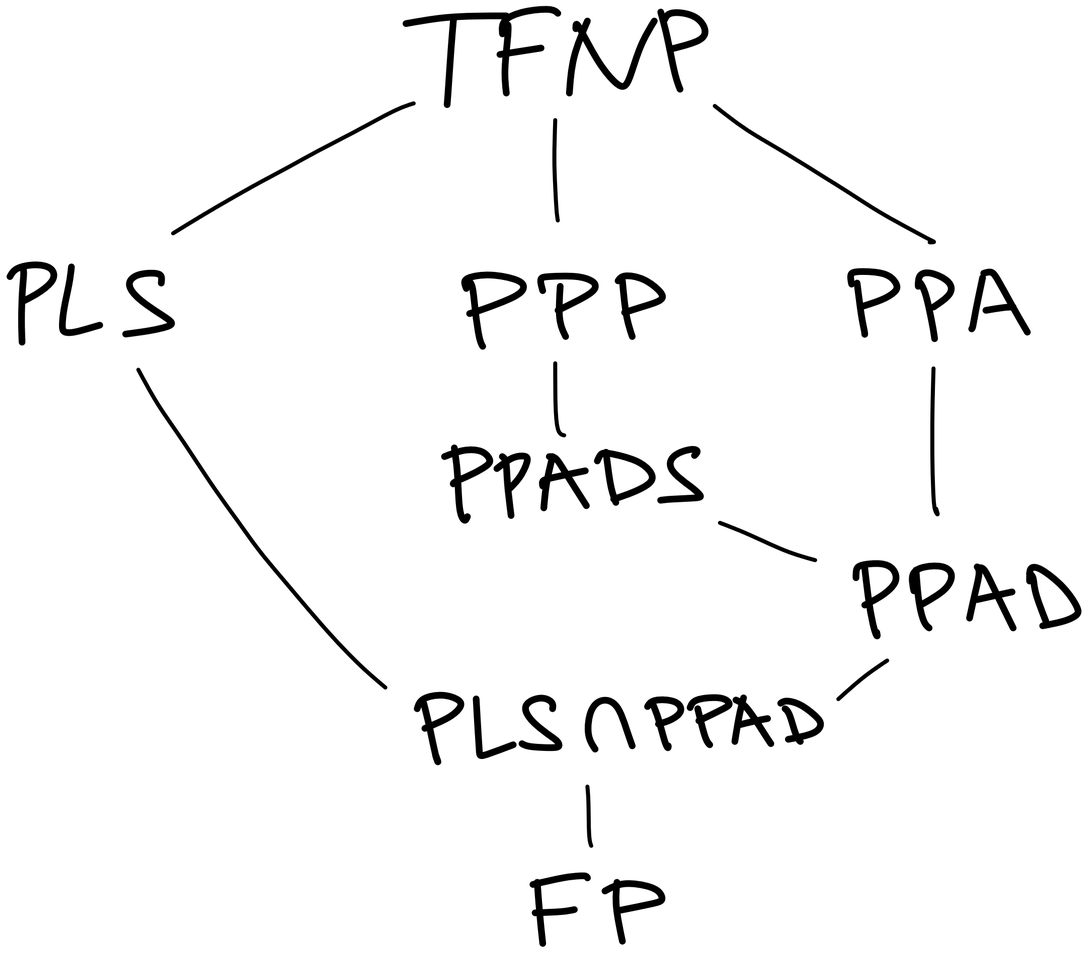


White-box



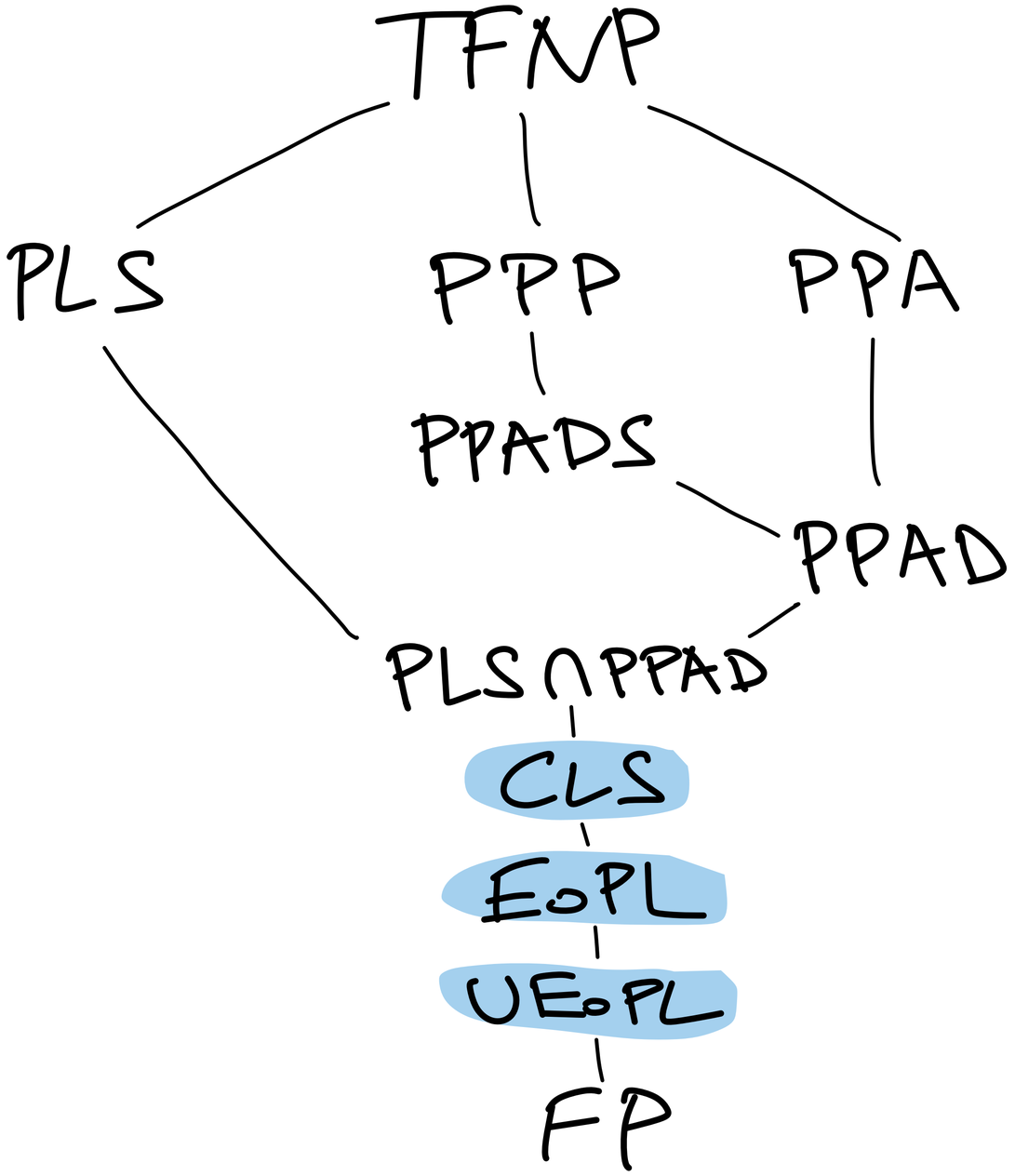
Black-box

Classical hierarchy (90's and 00's)



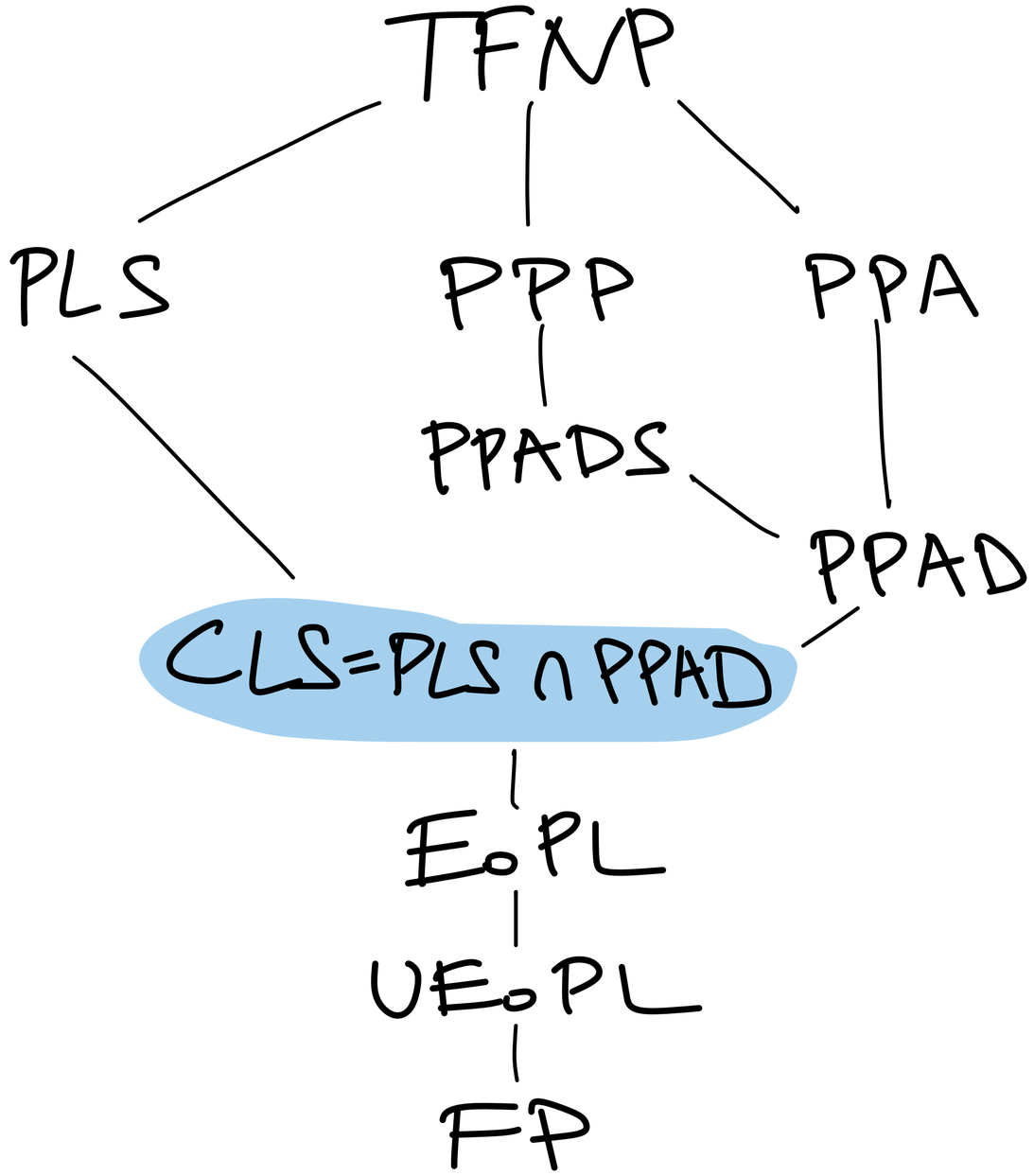
[Pap94]
[JPY88]

New classes (10's)



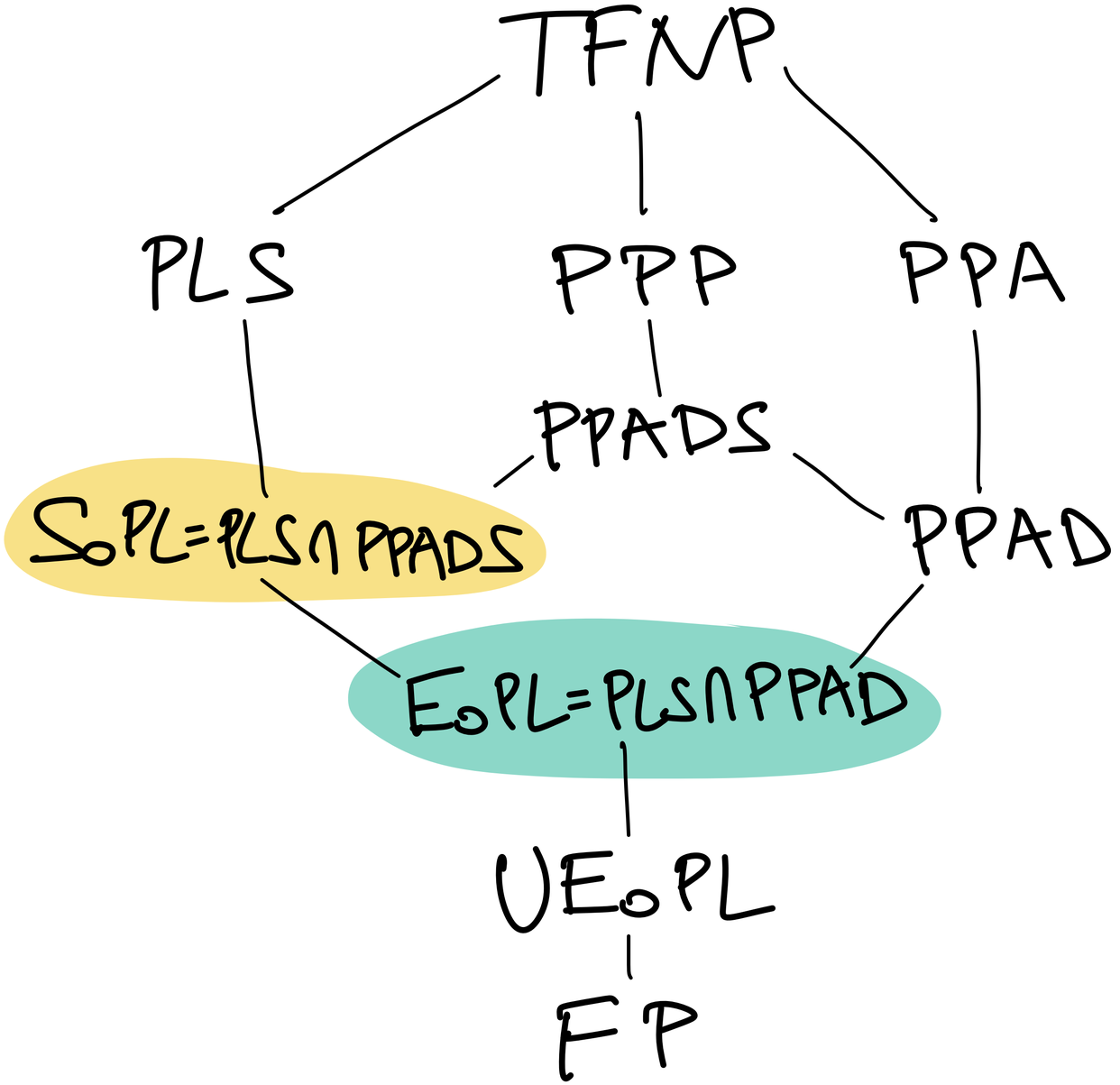
[HY20]
[FGMS20]
[DP11]

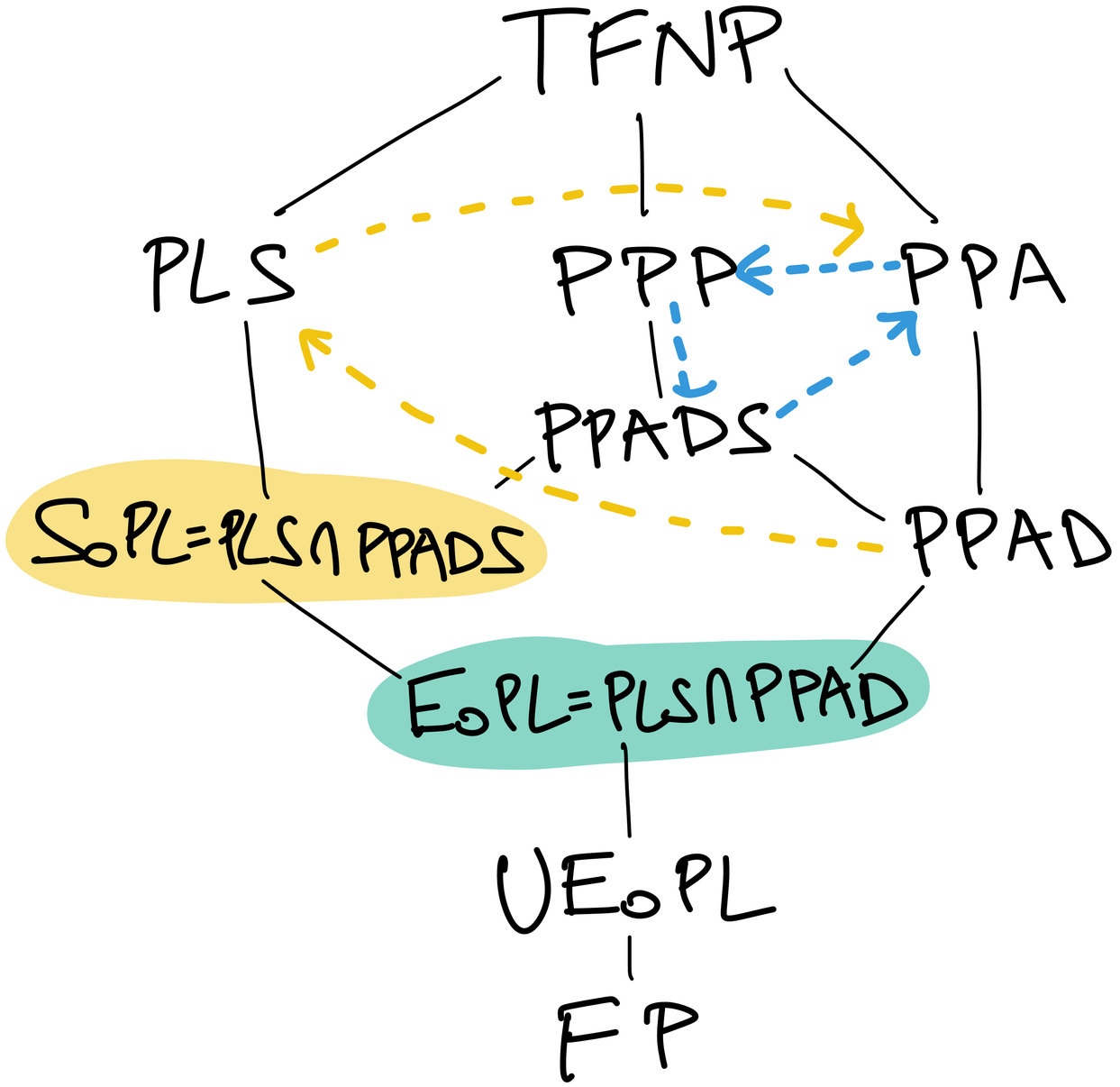
A Breakthrough Collapse (2021)



(Best paper!)
[FGHS21]

Further Collapses (2022)





More Collapses?

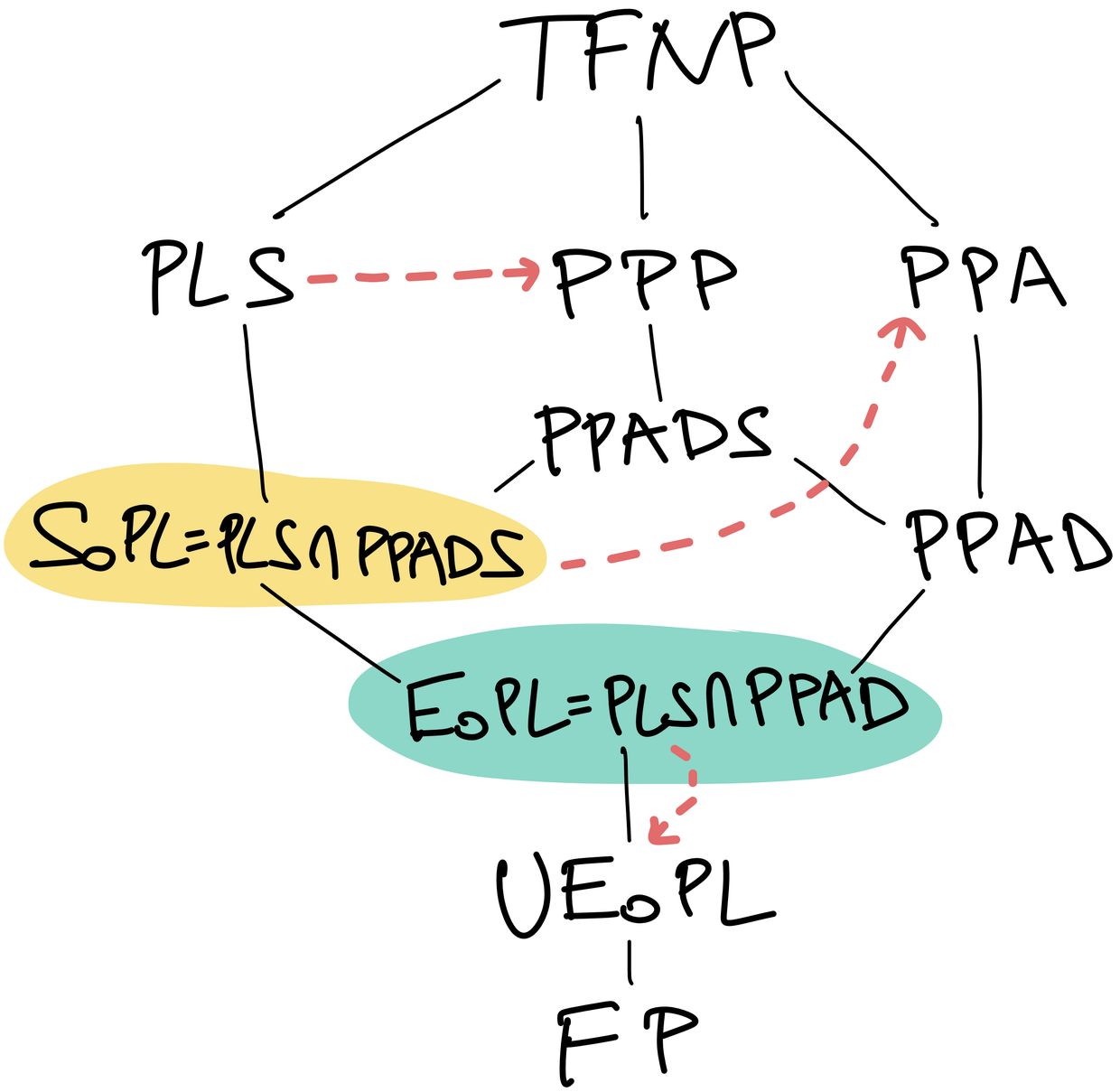
White-box sep. $\Rightarrow$ $P \neq NP$

Black-box sep. possible

Beame et al. 98'

Morioka 01'

Buresh-Oppenheim 04'



More Collapses?

NO MORE (BLACK-BOX)

OUR WORK

Image from https://fanart.tv/movie/9267/and-now-for-something-completely-different/

Resolution v.s. Sherali-Adams

Resolution

$$\frac{A \vee x \quad , \quad B \vee \neg x}{A \vee B}$$

measure: width

Simulated by

Sherali-Adams

$$\sum_i p_i(x) q_i(x) = 1 + J(x)$$

measure: degree

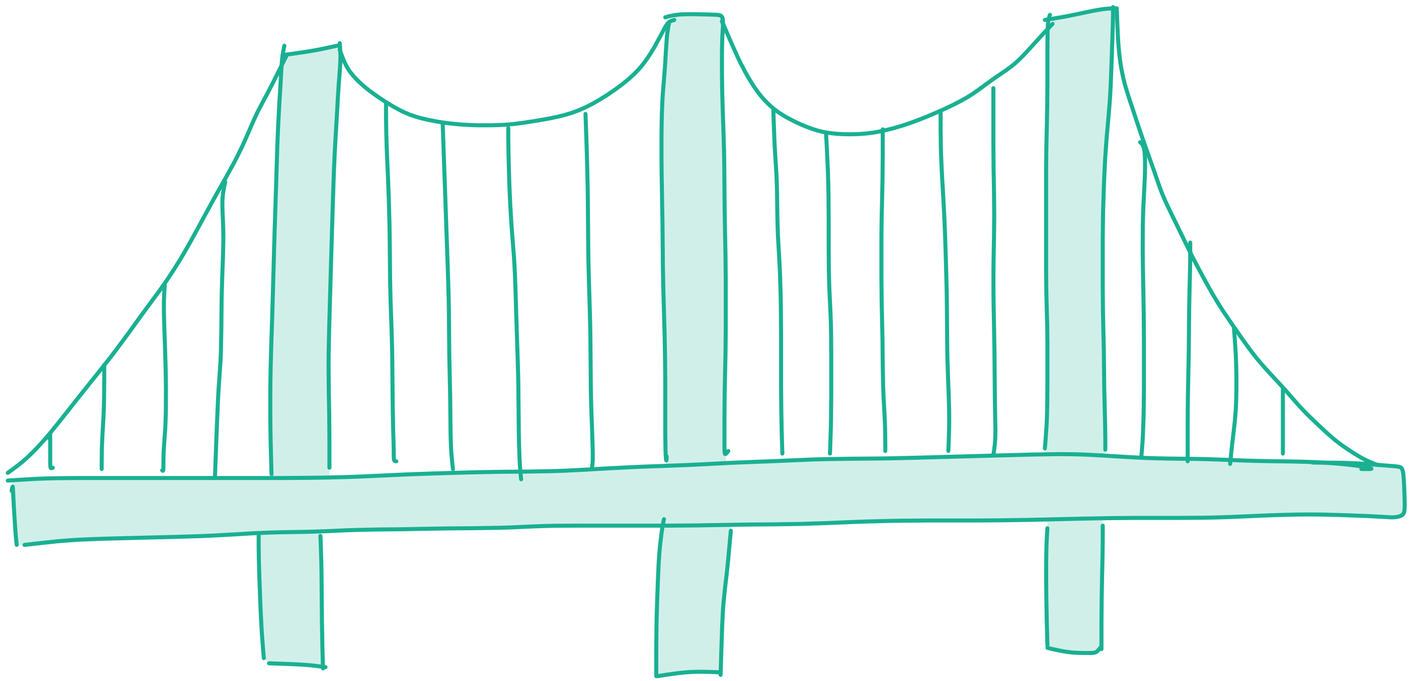
OUR RESULT: Simulation needs exp. large coefficients

⇕

PLS & PPADS

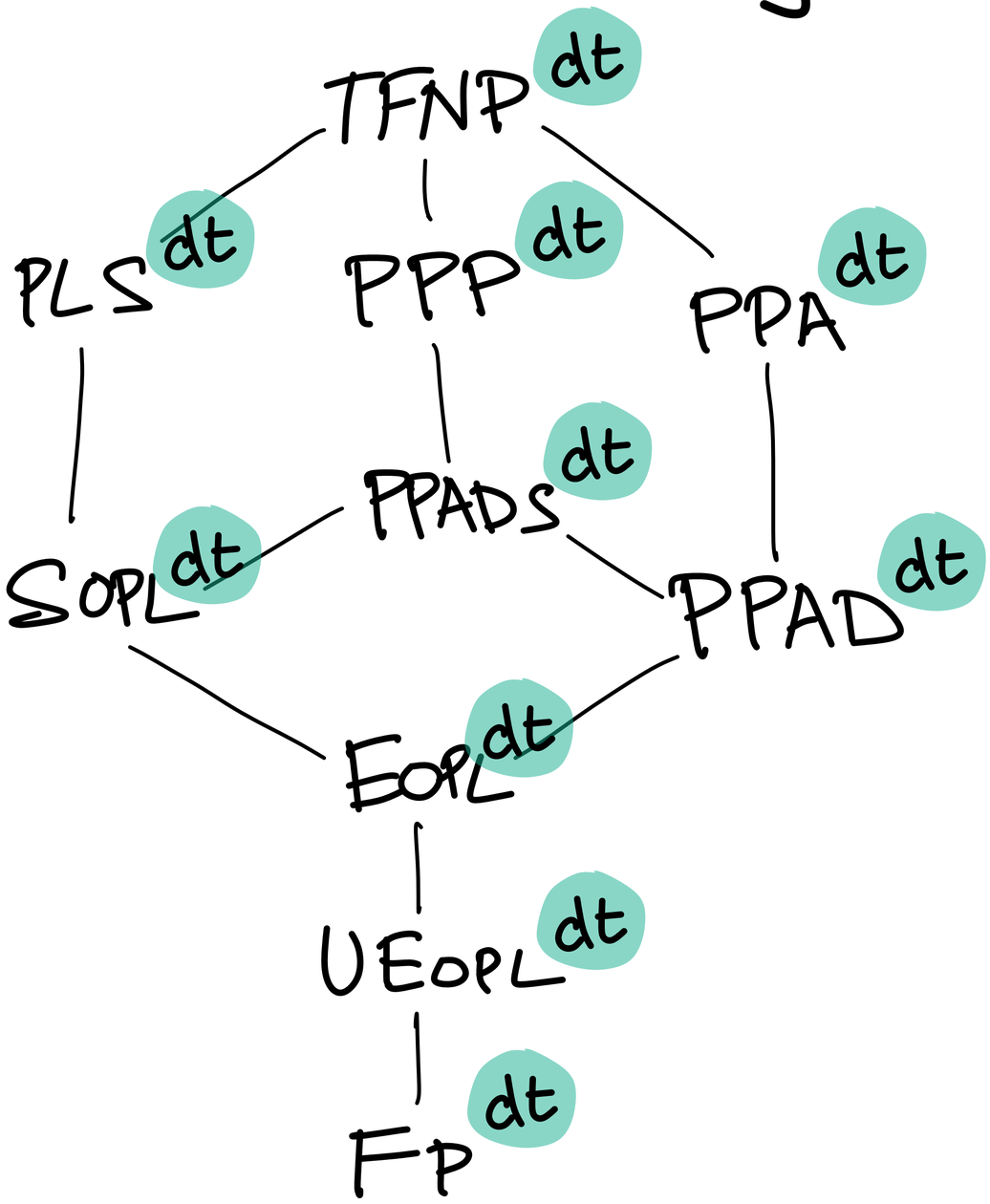
THE BRIDGE

Proof Complexity



TFNP

World 1 : Query analogues



- dt

|||

query analogue

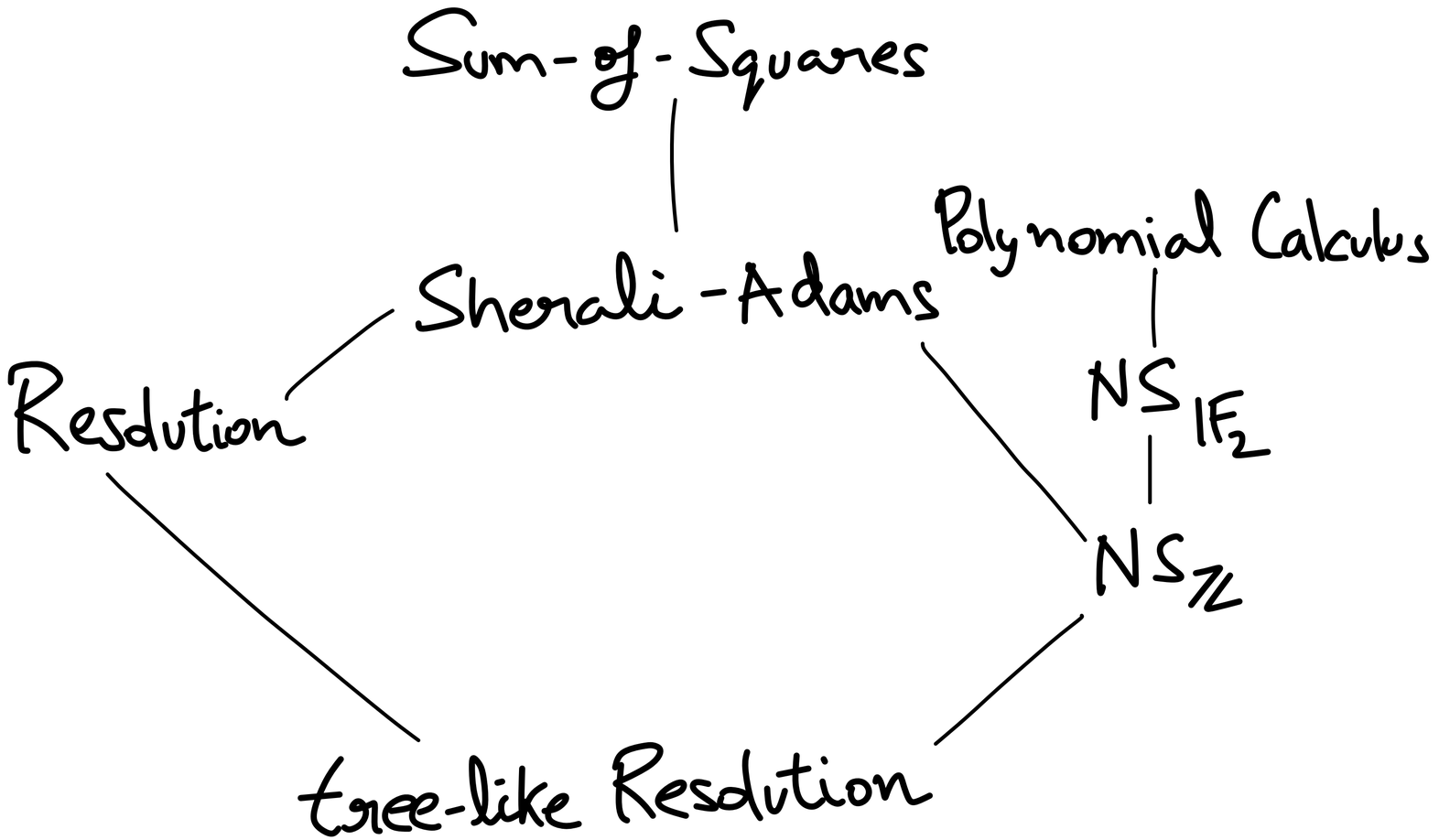
- Reductions

|||

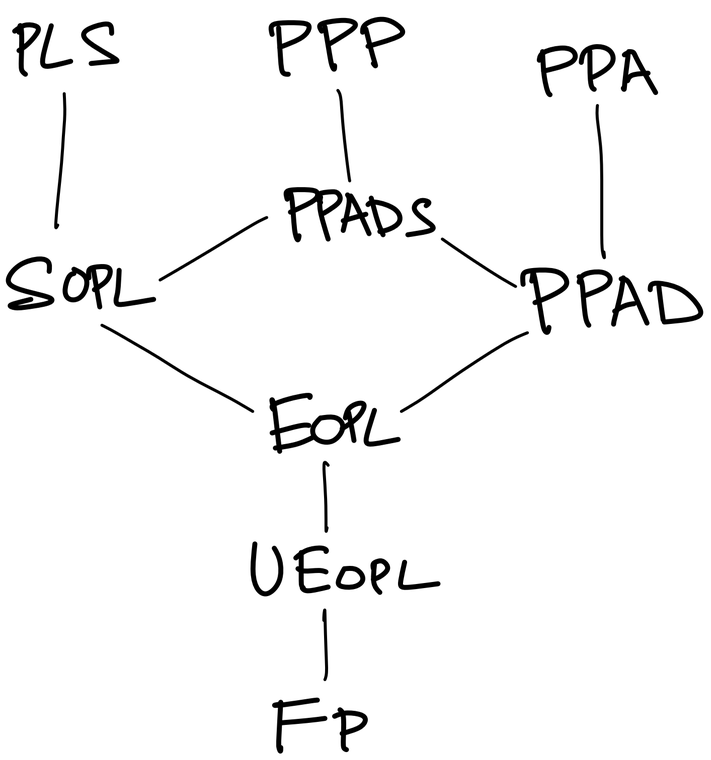
Shallow decision trees

World 2 : Proof Complexity

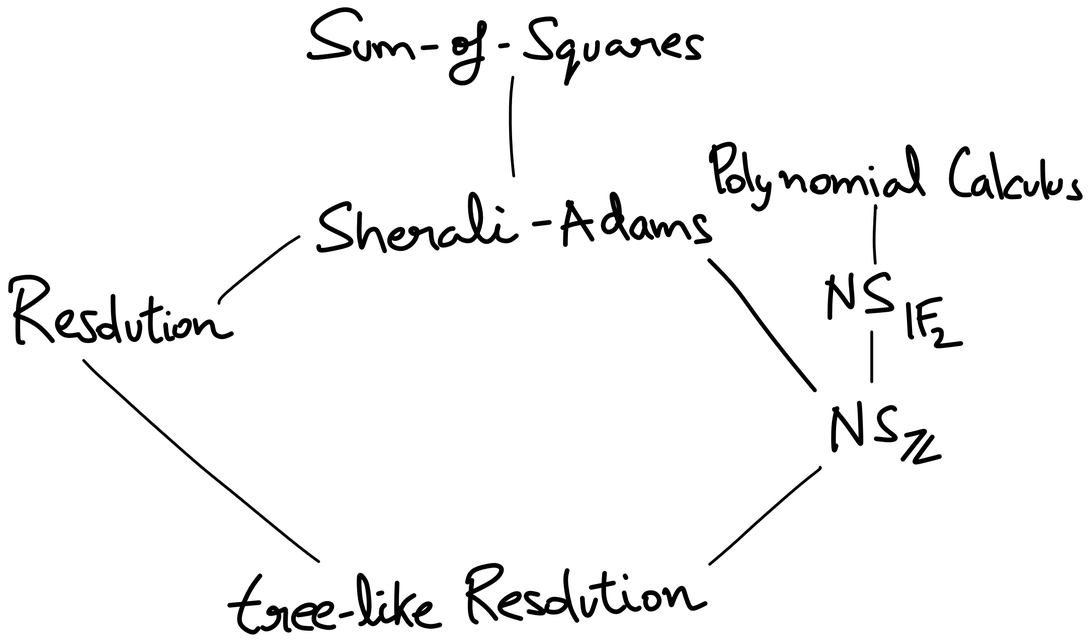
Is there a short derivation that this CNF is unsat?



Time to Squint



≈≈



The Bridge: Characterizations

- $\mathrm{TFNP}^{\mathrm{dt}}$ search problems can be translated into CNF fallacies

SINK-OF-DAG $\mapsto$ "this dag has no sinks"

The Bridge: Characterizations

- $TFNP^{dt}$ search problems can be translated into CNF fallacies

- CNF fallacies define search problems

$$\varphi = x_1 \wedge (\overline{x}_1 \vee \overline{x}_2) \wedge x_2 \longmapsto \begin{array}{c} \text{find}(x_1, x_2) \\ \text{falsified clause} \end{array}$$

More Explicitly

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Revision(s):

Revision #2 to TR22-141 | 30th November 2022 03:22

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TFNP Characterizations of Proof Systems and Monotone Circuits

Authors: Sam Buss, Noah Fleming, Russell Impagliazzo
Accepted on: 30th November 2022 03:22
Downloads: 81
Revision #2 Keywords:

Abstract:

Connections between proof complexity and circuit complexity have become major tools for obtaining lower bounds in both areas. These connections -- which take the form of interpolation theorems and query-to-communication lifting theorems -- translate efficient proofs into small circuits, and vice versa, allowing tools from one area to be applied to the other. Recently, the theory of TFNP has emerged as a unifying framework underlying these connections. For many of the proof systems which admit such a connection there is a TFNP problem which characterizes it: the class of problems which are reducible to this TFNP problem via query-efficient reductions is equivalent to the tautologies that can be efficiently proven in the system. Through this, proof complexity has become a major tool for proving separations in black-box TFNP. Similarly, for certain monotone circuit models, the class of functions that it can compute efficiently is equivalent to what can be reduced to a certain TFNP problem in low communication. When a TFNP problem has both a proof and circuit characterization, one can prove an interpolation theorem. Conversely, many lifting theorems can be viewed as relating the communication and query reductions to TFNP problems. This is exciting, as it suggests that TFNP provides a roadmap for the development of further interpolation theorems and lifting theorems.

TFNP Problems

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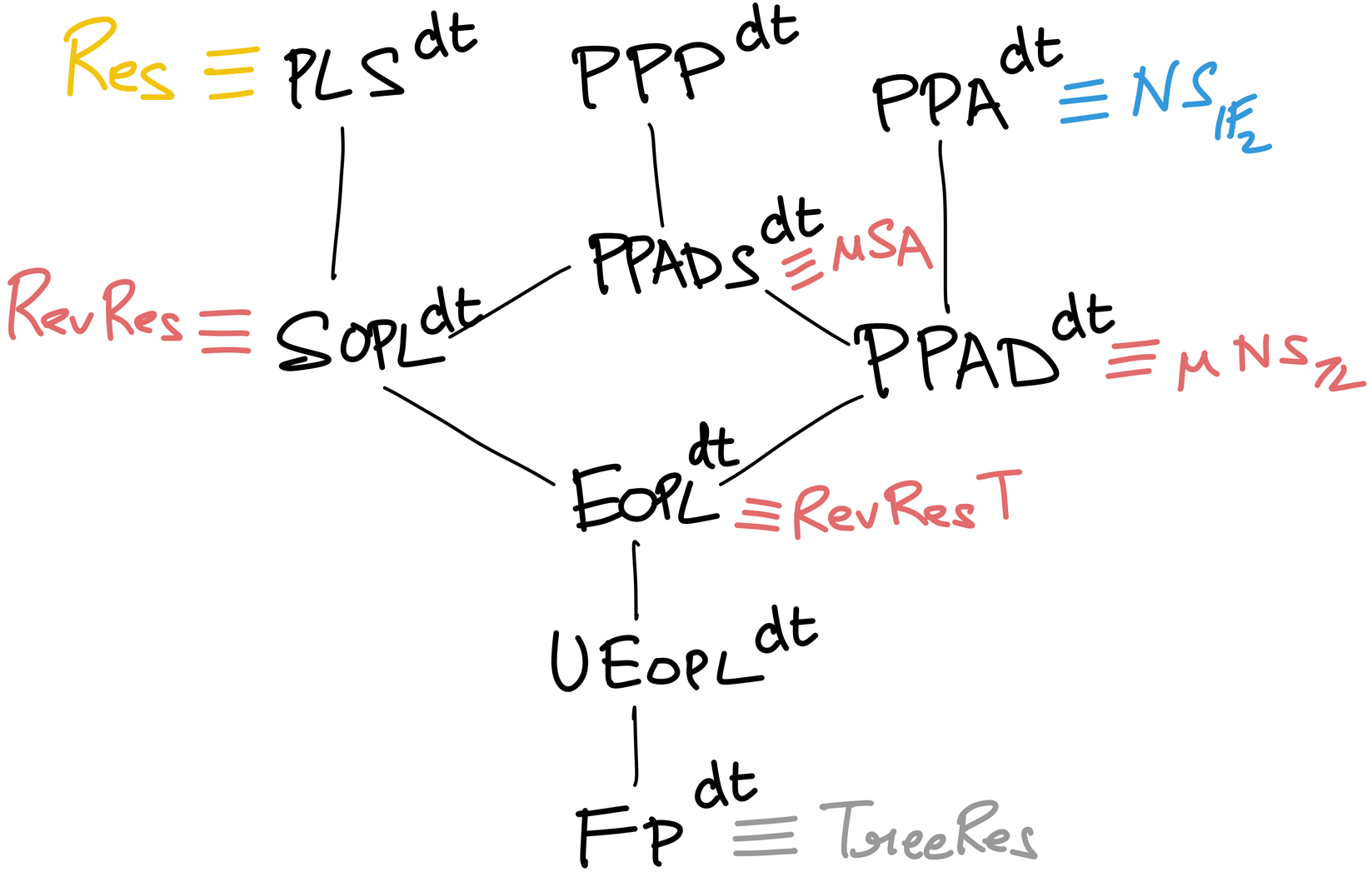
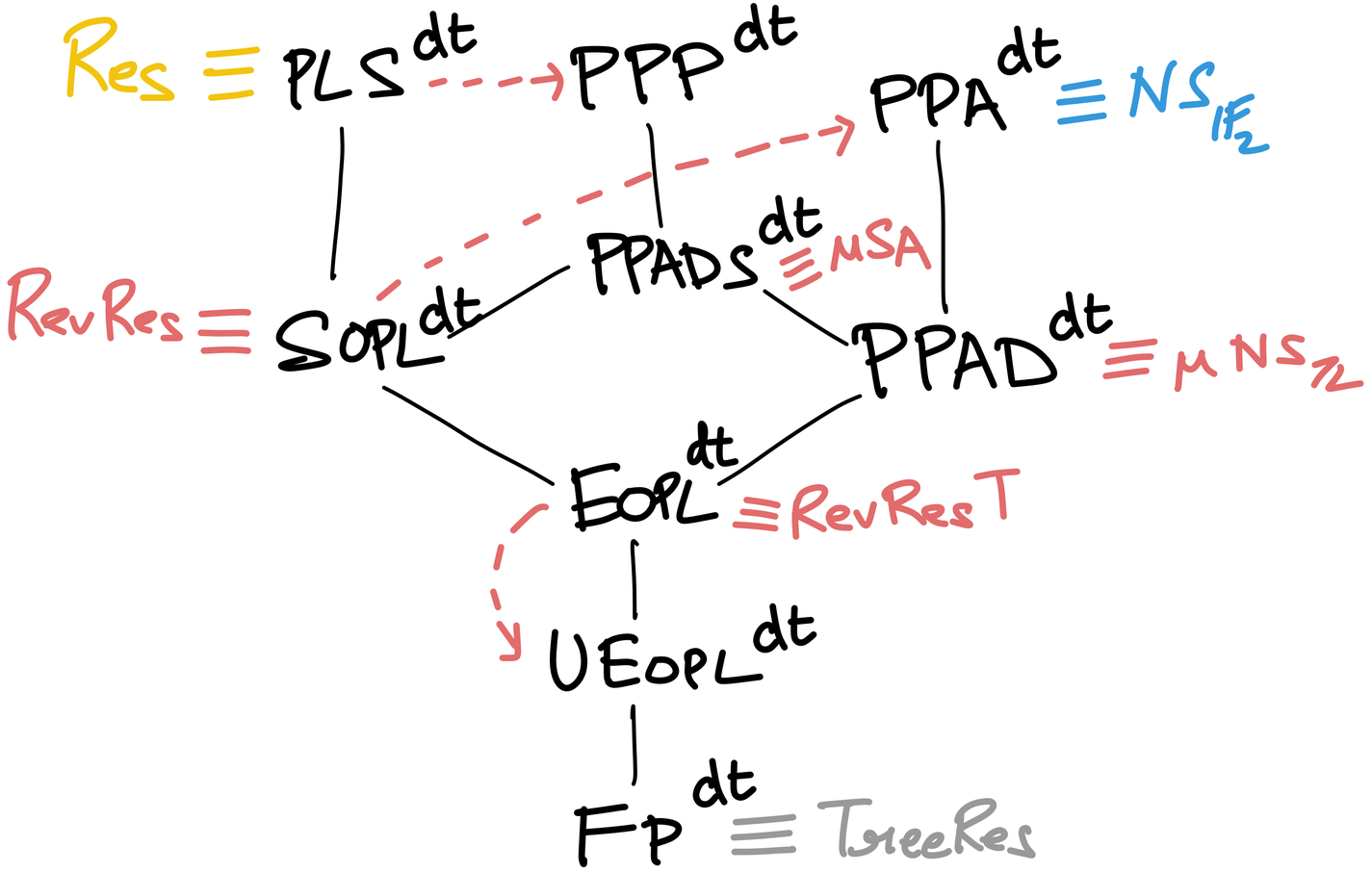
Proof Systems with Reflection Principle

The Bridge: Characterizations



The Bridge: Characterizations



Results rephrased:

- $\mathrm{Res} \not\leq \mathrm{uSA}$
- $\mathrm{RevRes} \not\leq \mathrm{NS}$

Independent work: $\mathrm{PLS}^\circ \& \mathrm{PPADS}^\circ \Rightarrow \mathrm{PLS}^\circ \& \mathrm{PPP}^\circ$ by [BT22]

Some Characterizations

let's see why:

① Resolution width $\approx$ PLS^{dt} depth

Formally, given $R(x, y) \in PLS^{dt}$; PLS^{dt} depth = depth of reduction to SoD

Reduction: construct vertex set V

For every $v \in V$, we have decision trees

$\pi_v(x) = s_v$

$O_v(x) = y$ s.t. $(x, y) \in R$ if v is a sink*

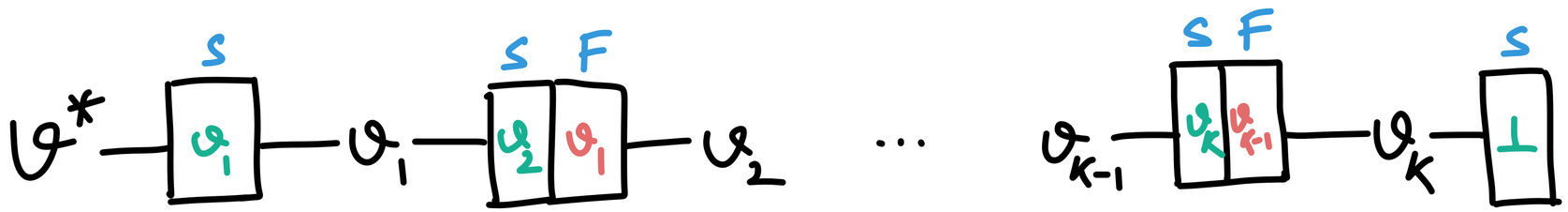
$$PLS^{dt} \text{ depth} = \log |V| + \max_{v \in V} |\pi_v|$$

Some Characterizations

Let's see why:

① [a] Resolution width $\leq$ PLS^{dt} depth

We will use Prover-Delayer characterization



$$\text{Res Width} \leq 2\log|V| + \max_{v \in V} |\pi_v|$$

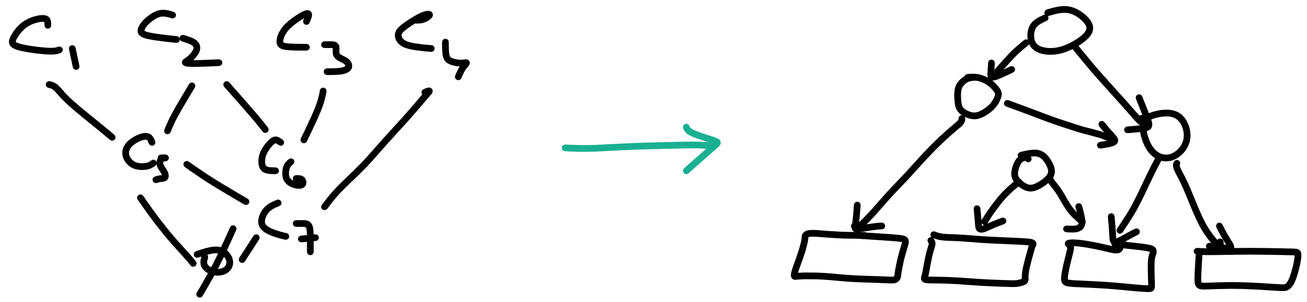
Some Characterizations

let's see why:

① [b] Resolution width $\gtrsim$ PLS^{dt} depth

Dag-like Res proof is already kinda a SoD reduction

Just have to Flip The Proof!



PLS^{dt} depth $\leq$ log(Res Size) + ResWidth

Some Characterizations

Let's see why:

(i) Resolution width $\approx$ PLS^{dt} depth

(ii) Unary NS deg $\approx$ $PPAD^{dt}$ depth

Lemma: If $\exists$ depth d EoL-formulation of F
then $\exists$ uNS refutation of F
with degree $O(d)$ and size $L2^{O(d)}$.

Proof: EoL formulation: $(V=[L], \{S_v, P_v, O_v\})$

Define $S_v(x) = \begin{cases} -1 & \text{if } v \neq v^* \text{ is a source in } G_x \\ 1 & \text{if } v \neq v^* \text{ is a proper sink in } G_x \\ 0 & \text{otherwise} \end{cases}$

S_v can be computed in depth $5d$.

$$S_v = \sum_{(-1)\text{-leaf } l} -D_l + \sum_{1\text{-leaf } l} D_l = \sum_{(-1)\text{-leaf } l} -D'_l \bar{C}_l + \sum_{1\text{-leaf } l} D'_l \bar{C}_l$$

leaves are Solutions

$$\Rightarrow \sum_{v \in V} S_v = \sum_i p_i \bar{C}_i = \#\text{sinks in } G_x - \#\text{non-}v^*\text{ sources in } G_x = 1$$

for some $\{p_i\}$

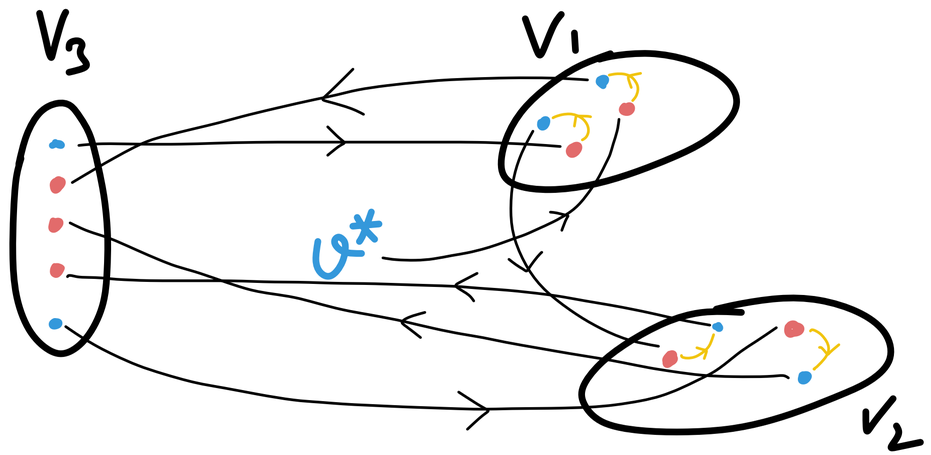
Lemma: If $\exists$ uNS refutation of F with deg d and size L
then $\exists$ deg $O(d)$ $EoL_{O(L)}$-formulation of F.

Proof: The refutation: $\sum_{i=1}^{m} p_i \bar{C}_i = 1 = \sum_i \sum_j c_{ij} q_{ij}$

In EoL formulation,

Nodes $\equiv$ q_{ij} with multiplicity $|c_{ij}|$ $\rightarrow$ in +, − set based on $\text{sgn}(c_{ij})$
$+ V^* = \{v^*\} \subseteq +$

Edges $\equiv$ Outer Matching + Inner Matching



On Separations

Key Lemma: Robust separation of SoPL from NS

SoPL ≈ SoD without merging of paths

Robust? We modify NS to refute approximately

$$\varepsilon\text{-NS} := \sum_{i \in [m]} p_i(x) \cdot q_i(x) = 1 \pm \varepsilon \quad \forall\, x \in \{0,1\}^n$$

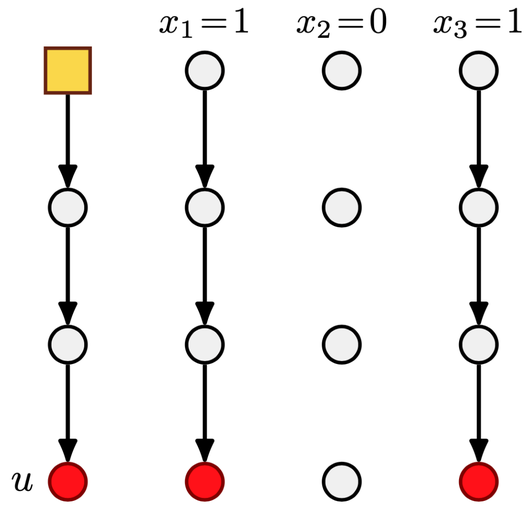
NOTE: Not a Cook-Reckhow proof system!
Verification is coNP-complete.

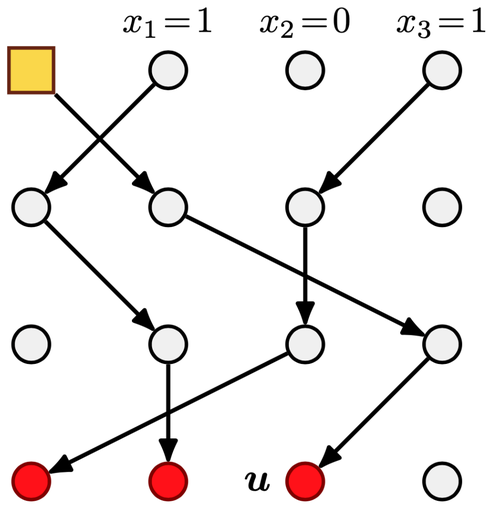
Lemma: Every $\frac{1}{2}$-NS refutation of SoPL_n requires deg $n^{\Omega(1)}$.

IDEA: Randomized decision-to-search reduction in the style of Raz-Wigderson '92 (matching)

We show ε-NS proofs for SoPL $\Rightarrow$ apx poly for OR

$\deg_{\varepsilon\text{-NS}}(\mathrm{SoPL}_n) \leq \deg_{\varepsilon}(\mathrm{OR}_n)$





What's a Reduction?

A pair (f, u) s.t.

(i) $f : \{0,1\}^{n-1} \to \{0,1\}^{O(n^2 \log n)}$ mapping inputs of OR_{n-1} to $SOPL_n$ s.t. $f_i(x)$ is depth d decision tree

(ii) For any x, the only solutions of $y = f(x)$ are active sinks on the last row.
Moreover, u is a planted solution

(iii) $OR(x) = \begin{cases} 0 \\ 1 \end{cases}$ then $\begin{matrix} Sol(y) = \{u\} \\ |Sol(y)| \geq 2 \end{matrix}$

↳ Randomised reduction is a distribution over reductions

Ideal Reduction ⇒ Apx to OR

↳ Ideal (y,u): for every y, $(u \mid y=y)$ is uniform over $\mathrm{Sol}(y)$

$$\mathbb{E}[q(y,u)] = \mathop{\mathbb{E}}_{y\sim y}\left[\mathop{\mathbb{E}}_{u'\sim(u\mid y=y)}\left[\phi_{i_{u'}}(y)\, a_{i_{u'}}(y)\right]\right]$$

$$= \mathop{\mathbb{E}}_{y\sim y}\left[\mathop{\mathbb{E}}_{u'\sim \mathrm{Sol}(y)}\left[\phi_{i_{u'}}(y)\, a_{i_{u'}}(y)\right]\right]$$

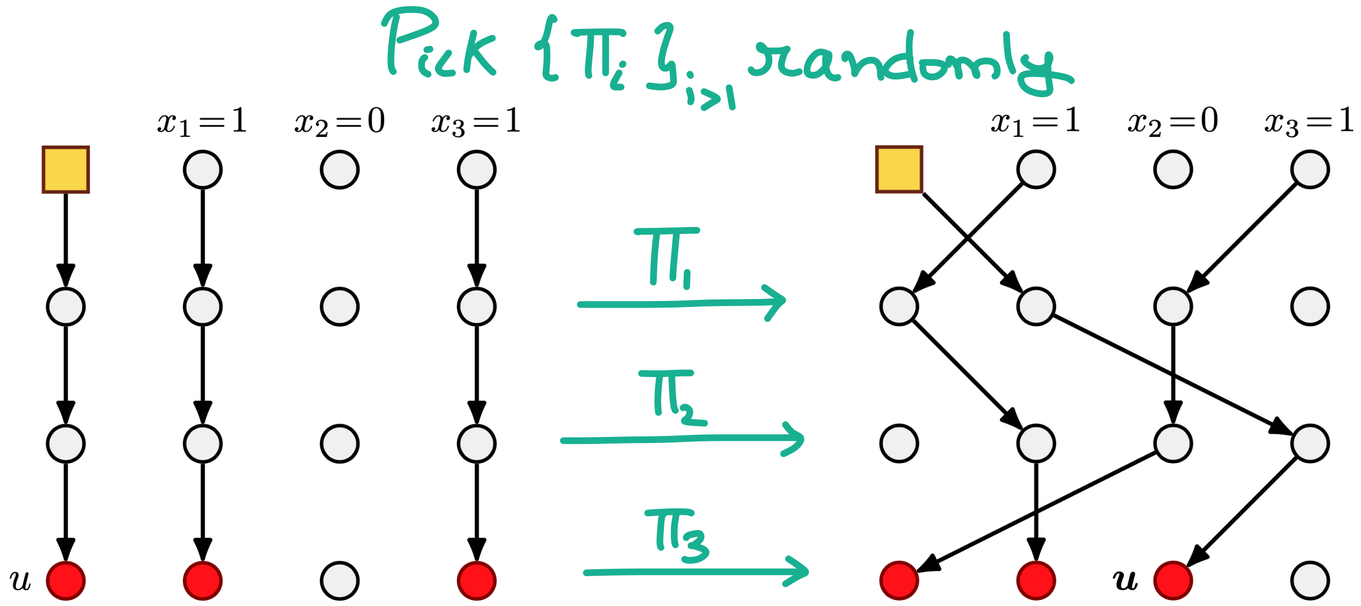
$$= \mathop{\mathbb{E}}_{y\sim y}\left[\frac{1}{|\mathrm{Sol}(y)|}\sum_{u'}\phi_{i_{u'}}(y)\, a_{i_{u'}}(y)\right]$$

$$= \frac{1\pm\varepsilon}{|\mathrm{Sol}(y)|}$$

↙ cause it's IDEAL

↙ using that it's an ε-NS proof

Our Reduction



Is this Ideal? NO.

Claim This reduction is Locally Ideal, which suffices.

Note $|Sol(y)| = 1 + |x|$ w.p. 1

$r(x) = \mathbb{E}_{R_x}[q(y,u)]$ R_x: "Our Reduction"

$r'(x) = \mathbb{E}_{I_x}[q(y,u)]$ I_x: An Ideal Reduction

Claim $r(x) = r'(x)$ for all $x \in \{0,1\}^{n-1}$

Proof By Linearity of Expectation, enough to show

$\mathbb{E}_{R_x}[m(y,u)] = \mathbb{E}_{I_x}[m(y,u)]$ for any monomial.

We assume $\deg(m) = o(\text{poly}(n))$ ← else nothing to prove

$\Rightarrow \exists\ i \in [\frac{n}{3}, \frac{2n}{3}]$ s.t. m reads none of the vars in rows $i, i+1$

$\Rightarrow$ Given $(y,u) \sim R_x$ let $A \subseteq \{i\} \times [n+1]$, $B \subseteq \{i+1\} \times [n+1]$ be the active nodes. We can apply a random bijection $A \to B$ and get an Ideal reduction.

On Separations

Lemma: Every $\frac{1}{2}$-NS refutation of SOPL_n requires deg $n^{\Omega(1)}$.

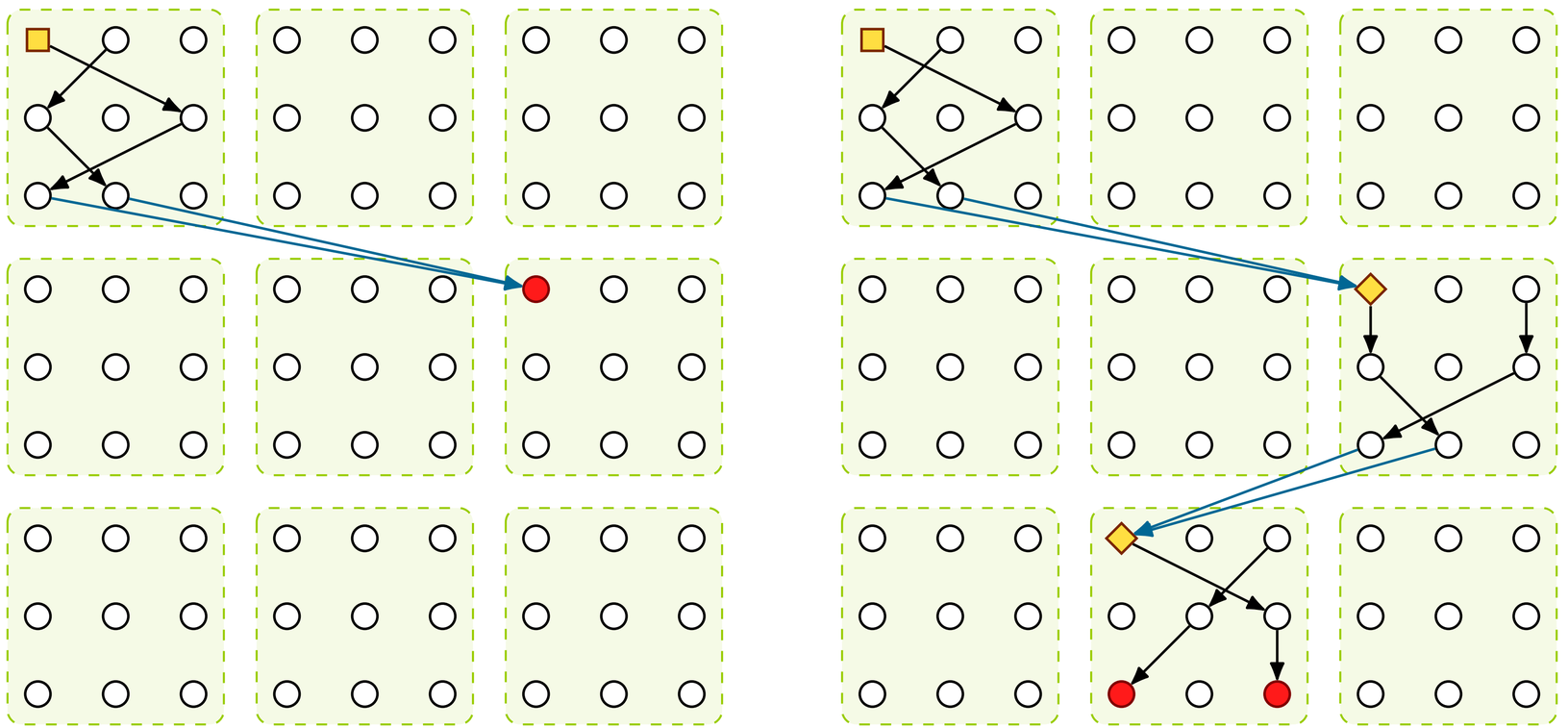
How do we amplify coefficients? Repeat instance!

Lemma: Any degree $n^{o(1)}$ SA proof of SoD_{n^2} requires coefficients of magnitude $\exp(\Omega(n))$.

Hard instance for ε-NS $\rightarrow$ l.b. on $J(x)$ in SA

$$\sum_i \underbrace{p_i(x)a_i(x)}_{\deg < n^{o(1)}} = \underbrace{1 + J(x)}_{\geq 2^{\Omega(n)}}$$

$\Rightarrow$ Some monomial with coefficient $\geq \dfrac{2^{\Omega(n)}}{n^{n^{o(1)}}} = 2^{\text{poly}(n)}$



Thanks
for your attention!