Illustrate the concept of definite integral $\int_a^b f(x)$, expressing as the limit of a sum and verify it by actual integration.

Siddhanth Khera

Under Guidance of Tanu Setia Ma'am

The Shri Ram School Aravali

Signature of Examiner

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1 Introduction

1.1 Definition of an Integral

Integration is a way uniting parts to a whole, integration is the opposite of derivation. It also often called the anti-derivative.

Let f(x) be the derivative of g(x)

$$\int f(x)dx = g(x) + C$$

Definite Integrals are defined a little differently

$$\int_a^b f(x)dx = g(b) - g(a)$$

Integrals can also be represented as a limit of summation

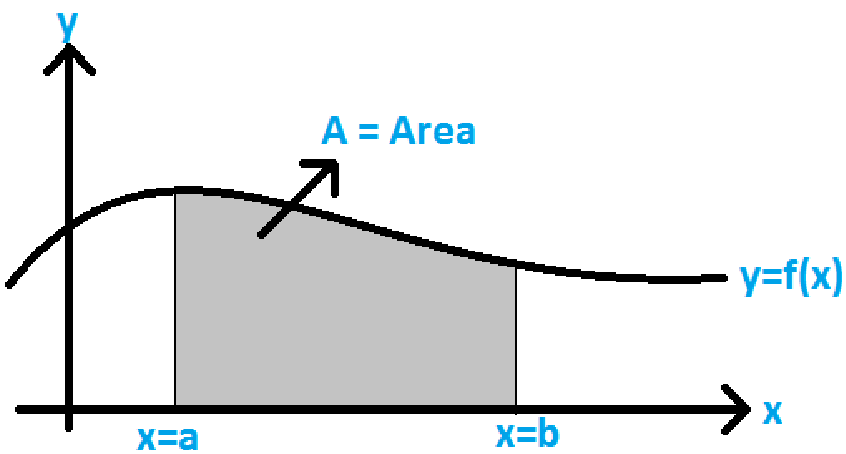
$$\int_a^b f(x) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \Delta x \cdot f(a + \Delta x \cdot i) \text{ ,where } \Delta x = \frac{b-a}{n}$$

Since integrals are the opposite of derivatives in some sense, we can derive many formulas for integration by just working backwards. It is easier to derive the formulas for derivation because we have defined it well with limits.

We will be going through other ways of deriving these formulas as well.

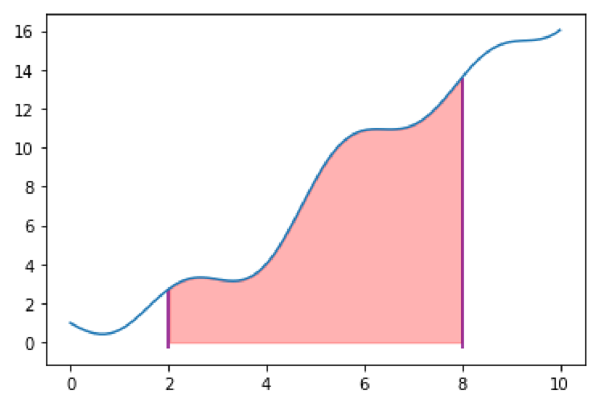
1.2 Graphical Representation of Integrals

Integrals can be used to represent the area under a curve.



For a function $f(x)$ it's integral $\int_a^b f(x)dx$ represents it's area under a graph from $x = a$ to $x = b$

An example generated in Python using Matplotlib, the area of $f(x) = cos(x) + x^{\frac{5}{4}} - sin(2x)$ from x=2 to x=8, this area can be represented as $\int_2^8 f(x)$



The integral breaks the graph into infinitesimally small segments, dx. The area of a rectangle can be given by $h \times w$ where h is the height and w is the width. We divide the graph into infinitely many rectangles with height $f(x)$ and width dx and add all their areas.

2 Application of Integrals

Integrals have applications everywhere in maths, they are used by engineers and to find out the work done, force applied etc., they are used by statisticians to find the mean value in a range and by graphic designers to find the volume of solids. Integrals have uses in almost all fields, and are used extensively in Mathematics and Physics. Here are some examples of integration in action.

2.1 Finding the Average Function Value

We usually take Average to mean Arithmetic Mean. The Arithmetic Mean is defined as $\frac{\text{Sum of the terms}}{Numberofterms}$. The problem is that the Arithmetic Mean is only defined for data sets, there are infinite points in a range, so we have to change the definition of average for these.

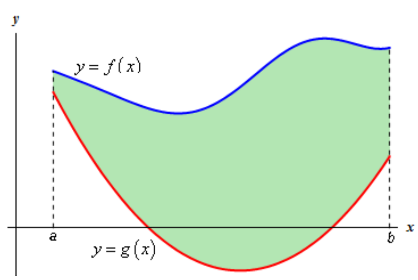
$$Average = \frac{\int_a^b f(x)}{b-a}$$

This is the Average Function Value and gives us the Average Value a function holds over a range.

2.2 Area Between Curves

We have said that integration helps us find the area under a curve, but it also helps us find the area between curves. If we have two curves $f(x)$ and $g(x)$, the area between their curves can be defined as

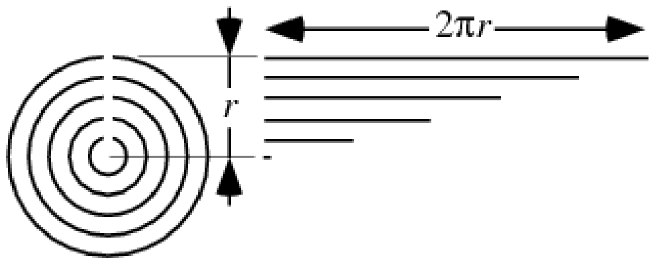
$$\int_a^b f(x) - g(x)$$



Where a and b are the points of intersection of the curves, or the range under which you want to find the area between the curves

2.3 Derivation of the Area of Shapes

We all know that the area of a circle is πr^2, there are many ways to derive this area, but one of the involves integration, which I will now show you.

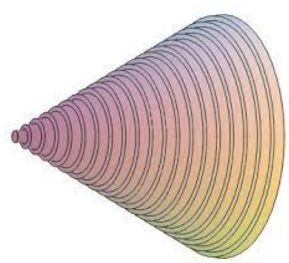


We can find the area of the circle by breaking it into infinitely many circular boundaries with circumference $2\pi r$ and integrate it with dr

$$\int_0^r 2\pi r dr = 2\pi \int_0^r r dr$$
$$= 2\pi|_0^r \frac{r^2}{2} = 2\pi(\frac{r^2}{2} - \frac{0^2}{2})$$
$$= \pi r^2$$

2.4 Finding the Volume using the Method of Disks

The Method of Disks or the Solids of Revolution method is a method used for finding the volume of solids using integration.



When we take a rectangle and rotate it with 2π degrees of rotation, we get a cylinder. To find the area under the graph, we divide the area into infinitesimally small rectangles. To find the volume, we divide the 3D shape into cylinders.

The volume of a cylinder is given by multiplying the area of the circular base with it's height. We have already derived the formula for the area of a circle, so the volume of a cylinder can be given by $\pi r^2 h$

The radius of the cylinders we divide the shape into is given by the function we are rotating around the axis. The height of the cylinder is given by dx. Therefore the volume can be given by

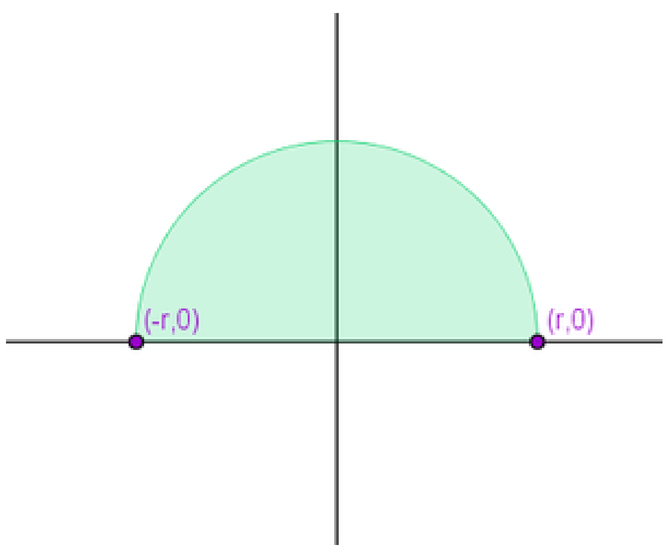
$$\int_a^b \pi f(x)^2 dx$$

2.4.1 Volume of a Sphere

Like in 2D where we used integration to find the volume of a circle, here in 3D we will use integration to find the volume of a sphere.

The formula for a semicircle is given by $\sqrt{r^2 - x^2}$, where r is the radius. We can use the formula we derived for the Method of Disks to find the area of a circle.

$$\pi \int_{-r}^{r} f(x)^2 dx = \pi \int_{-r}^{r} (\sqrt{r^2 - x^2})^2 dx = \pi \int_{-r}^{r} (r^2 - x^2) dx$$
$$|\pi r^2 x - \frac{\pi x^3}{3}|_{-r}^{r} = 2\pi r^3 - \frac{2}{3}\pi r^3 = \frac{4}{3}\pi r^3$$



3 Estimation of Integral or Riemann Sums

For simple functions, we can find the exact value of their integrals. Unfortunately not all functions can be integrated. Most of the time, rough estimates will work. We use calculators and estimate the Integrals. Luckily we have Riemann sums!

A Riemann sum is a certain kind of approximation of an integral by a finite sum. It is named after nineteenth century German mathematician Bernhard Riemann. One very common application is approximating the area of functions or lines on a graph, but also the length of curves and other approximations. We will cover 4 kinds of Riemann Sums, along with their graphs for x^3.

In Riemann Sums, we divide the range into -usually equal- partitions. If we divide it into n sub intervals, each of length.

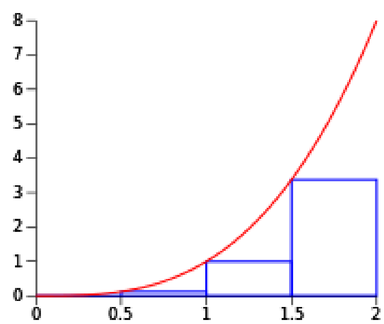
$$\Delta x = \frac{b-a}{n}$$

The partitions therefore will be $a, a+\Delta x, a+2\Delta x, a+3\Delta x \ldots$

We estimate the integral based on these partitions. As $n \to \infty$, we get the exact value.

3.1 Left Riemann Sum

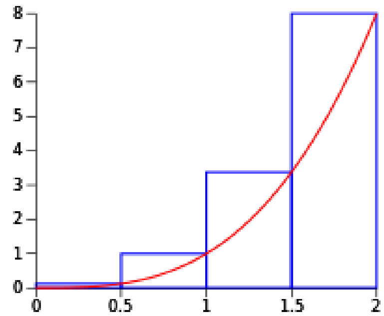
In the Left Riemann Sum, we approximate the area by taking it's value at the left point and multiplying it with it's width.



$$A_{left} = \Delta x(f(x) + f(a+\Delta x) + f(a+2\Delta x) \cdots + f(b-\Delta x))$$

3.2 Right Riemann Sum

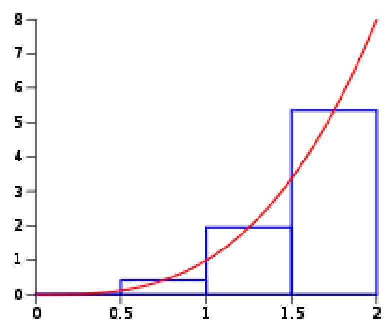
In the Right Riemann Sum, we approximate the area by taking it's value at the right point and multiplying it with it's width.



$$A_{right} = \Delta x(f(a + \Delta x) + f(a + 2\Delta x) + f(a + 3\Delta x) \cdots + f(b))$$

3.3 Midpoint Riemann Sum

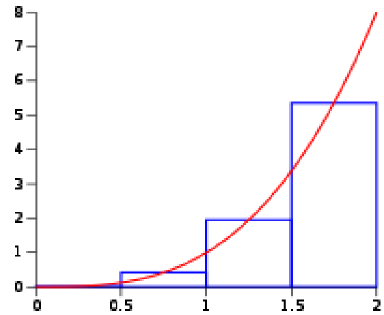
In the Midpoint Riemann Sum, we estimate the areas by making rectangles who's midpoint touches the line of the function.



$$A_{mid} = \Delta x \left[f(a + \frac{\Delta x}{2}) + f(a + \frac{3\Delta x}{2}) + f(a + \frac{5\Delta x}{2}) \cdots + f(b - \frac{\Delta x}{2}) \right]$$

3.4 Trapezoidal Riemann Sum

In the Trapezoidal Riemann Sum, we estimate the value by creating trapezoids. We essentially find the average value between the right and left points and multiply it by the width. This is usually the most accurate of these estimations.



$$A_{trap} = \Delta x \left[\frac{f(a) + f(a + \Delta x)}{2} + \frac{f(a\Delta x) + f(a + 2\Delta x)}{2} + \frac{f(a2\Delta x) + f(a + 3\Delta x)}{2} \cdots \right]$$

$$A_{trap} = \frac{1}{2}\Delta x(f(a + \Delta x) + 2f(a + 2\Delta x) + 2f(a + 3\Delta x) \cdots + f(b))$$

4 Deriving Formulas for Integration

As I mentioned earlier, integration is in some sense 'anti-derivation'. We can derive formulas for Integration by reversing the formulas for derivatives. I will proof a few formulas that, but will also use another method to prove the integral of x^n.

$$\text{Let f(x) be the derivative of g(x)}$$
$$\int f(x)dx = g(x) + C$$

4.1 Proving Integral of x^n

4.1.1 Anti-Derivative

We know the derivative of x^n is $nx^n - 1$. Integration or the anti-derivative is the opposite of the derivatives. If $f(x)$ is the derivative of $g(x)$, then the integration/anti derivative of $f(x)$ is $g(x)$. We have to transform $nx^n - 1$ to x^n when we integrate it. Therefore we can derive the formula to be $\frac{x^{n+1}}{n+1}$.

To derive the initial formula for the derivative of x^n, we can use the formal definition of derivatives.

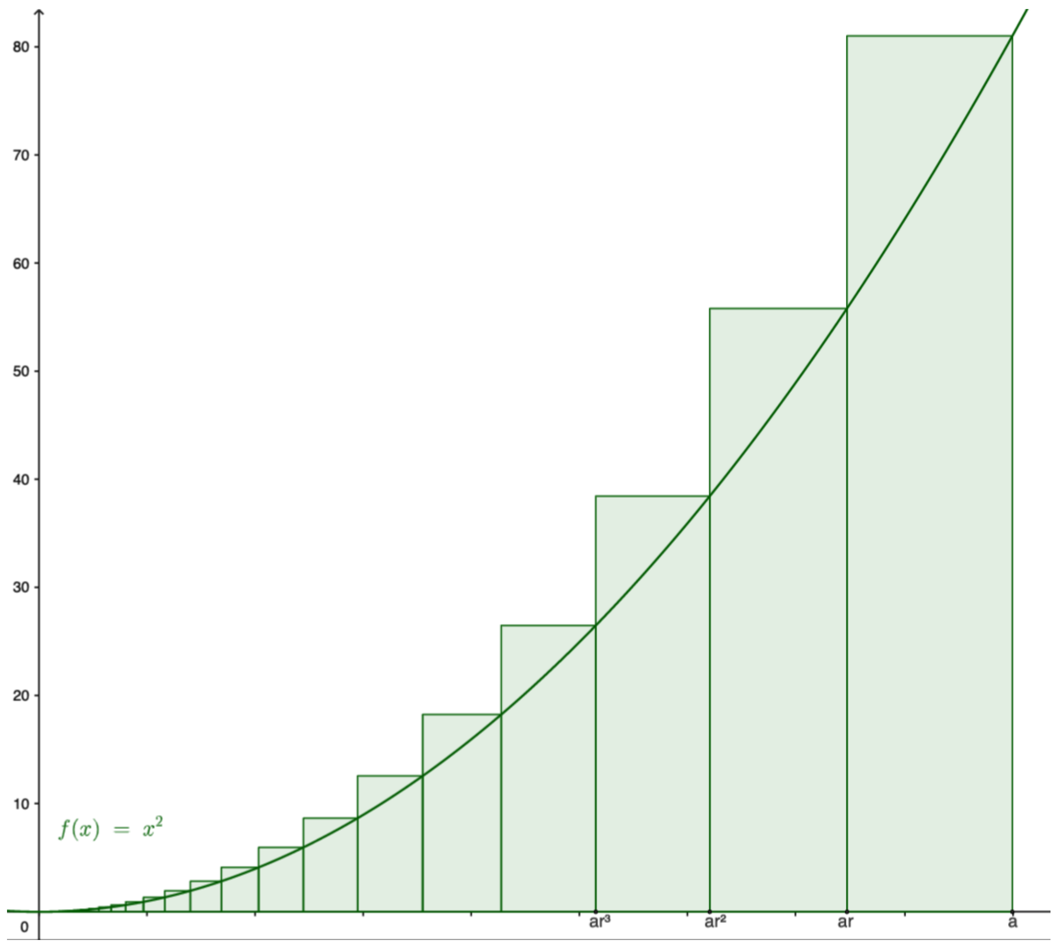
$$\lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{x^n + nx^{n-1}h \cdots - x^n}{h} = nx^{n-1}$$

On expansion, all the other terms can be ignored since they have a factor of h higher than 1. On cancelling with the denominator they all go to 0 as $h \to 0$.

4.1.2 Proving using Summation of Geometric Series

The formulas for summation of a geometric series are

$$1 + r + r^2 + r^3 + r^4 \cdots = \frac{1 - r^{n+1}}{1 - r}$$
$$\text{When there are infinite terms, and } r < 1 \lim_{n \to \infty} r^{n+1} = 0.$$
$$\therefore n \to \infty, r < 1 \text{The formula for Summation of this series is } \frac{1}{1-r}$$



We will take unequal Right Riemann sums in this instance, and take them to infinity. This is Fermat's Proof.

$$\int_0^a x^n = a^n(a - ar) + (ar)^n(ar - ar^2) + (ar^2)^n(ar^2 - ar^3)\ldots \text{as } r \to 1$$

$$= a^{n+1}(1-r) + a^{n+1}r^{r+1}(1-r) + a^{n+1}r^{2(r+1)}(1-r)\ldots$$

$$= a^{n+1}(1-r)\left[1 + r^{n+1} + r^{2(r+1)} + r^{3(r+1)}\ldots\right] = a^{n+1}(1+r)\frac{1}{1-r^{n+1}}$$

$$= a^{n+1}\frac{1-r}{1-r^{n+1}} = \frac{a^{n+1}}{1+r+r^2+r^3\ldots r^n} = \frac{a^{n+1}}{n+1} \text{ as } r \to 1$$

4.2 Deriving Substitution Rule

We may derive other integration formulas, such as those in trigonometry from reversing the rules of derivatives. The functions will in some sense be the inverse of those from derivatives.

We also may think of the substitution rule in integration as an extension of the Chain Rule.

$$\text{By the Chain Rule: } f(g(x))' = f'(g(x))g'(x)$$
$$f(g(b)) - f(g(b)) = \int_a^b f(g(x))'dx = \int_a^b f'(g(x))g'(x)$$
$$\text{Also } \int_{g(a)}^{g(b)} f'(u)du = \int_a^b f'(g(x))g'(x)$$
$$\therefore \int_a^b f(g(x))'dx = \int_{g(a)}^{g(b)} f'(u)du$$

5 Computing Riemann Sums

I wrote a python program to Estimate integrals using trapezoid Riemann Sums.