Problem 1

Part a

Base Case, we take two positive real numbers x_1, x_2

$$(x_1 - x_2)^2 \geq 0 \implies x_1^2 + x_2^2 - 2x_1x_2 \geq 0$$
$$x_1^2 + x_2^2 \geq 4x_1x_2 \implies \frac{(x_1+x_2)^2}{4} \geq x_1x_2$$
$$\implies \frac{x_1+x_2}{2} \geq \sqrt{x_1x_2}$$

We have proved it for any two positive real numbers x_1, x_2

We take 2 sets of n positive real numbers

$x_1, x_2, x_3, x_4 \ldots x_n$ and $x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4} \ldots n_{2n}$

such that

$\frac{x_1,x_2,x_3,x_4\ldots x_n}{n} \geq \sqrt[n]{x_1x_2x_3x_4\ldots x_n}$

and

$\frac{x_{n+1}+x_{n+2}+x_{n+3}+x_{n+4}+\ldots x_{2n}}{n} \geq \sqrt[n]{x_{n+1}x_{n+2}x_{n+3}x_{n+4}\ldots x_{2n}}$

Adding these two inequalities we get

$$\frac{x_1+x_2+x_3+x_4+\ldots x_{2n}}{n} \geq \sqrt[n]{x_1x_2x_3\ldots x_n} + \sqrt[n]{x_{n+1}x_{n+2}x_{n+3}\ldots x_{2n}}$$

$$\frac{x_1+x_2+x_3+x_4+\ldots x_{2n}}{2n} \geq \frac{\sqrt[n]{x_1x_2x_3\ldots x_n} + \sqrt[n]{x_{n+1}x_{n+2}x_{n+3}\ldots x_{2n}}}{2} \tag{1}$$

Let $\sqrt[n]{x_1x_2x_3\ldots x_n}, \sqrt[n]{x_{n+1}x_{n+2}x_{n+3}\ldots x_{2n}}$ be positive real numbers α, β respectively

$$(\alpha - \beta)^2 \geq 0 \implies \alpha^2 + \beta^2 - 2\alpha\beta \geq 0$$
$$\alpha^2 + \beta^2 \geq 4\alpha\beta \implies \frac{(\alpha+\beta)^2}{4} \geq \alpha\beta$$
$$\implies \frac{\alpha+\beta}{2} \geq \sqrt{\alpha\beta}$$

Therefore

$$\frac{\sqrt[n]{x_1x_2x_3\ldots x_n} + \sqrt[n]{x_{n+1}x_{n+2}x_{n+3}\ldots x_{2n}}}{2} \geq \sqrt[2n]{x_1x_2x_3\ldots x_{2n}} \tag{2}$$

$$\implies \frac{x_1+x_2+x_3+x_4+\ldots x_{2n}}{2n} \geq \sqrt[2n]{x_1x_2x_3\ldots x_{2n}} \tag{3}$$

Therefore using induction we can prove the AM GM inequality for $n = 2^k$ since we have proved base case of $n = 2$

Part b

We have proven that the AM GM inequality holds true for $n = 2^k$ numbers. If we can prove that the inequality being true for n numbers implies that it is true for $n - 1$ numbers, we can fill in the gaps between exponents of two and show that it holds true for all numbers.

I worked on this backwards on paper and that's how I will be proving it here by using bi-implicative statements.

We need to prove that for any real numbers $x_1, x_2, x_3 \ldots x_{n-1}$

$$\frac{x_1 + x_2 + x_3 + \ldots x_{n-1}}{n-1} \geq \sqrt[n-1]{x_1 x_2 x_3 \ldots x_{n-1}} \tag{4}$$

$$\iff \left(\frac{x_1 + x_2 + x_3 + \ldots x_{n-1}}{n-1}\right)^{n-1} \geq x_1 x_2 x_3 \ldots x_{n-1} \tag{5}$$

If we multiply both sides of an inequality by a positive number they inequality holds and visa versa.
We multiply it by the positive number α where $\alpha = \frac{x_1+x_2+x_3+\ldots x_{n-1}}{n-1}$

$$\iff \left(\frac{x_1 + x_2 + x_3 + \ldots x_{n-1}}{n-1}\right)^{n-1} \cdot \alpha \geq x_1 x_2 x_3 \ldots x_{n-1} \alpha \tag{7}$$

$$\iff \frac{x_1 + x_2 + x_3 + \ldots x_{n-1}}{n-1} \geq \sqrt[n]{a_1 a_2 a_3 a_4 \ldots a_{n-1} \alpha} \tag{8}$$

We observe that (expansion has been left as exercise for TA Hint: Multiple numerator and denominator by n)
$\frac{x_1+x_2+x_3+\ldots x_{n-1}}{n-1} = \frac{x_1+x_2+x_3+\ldots x_{n-1}+\alpha}{n}$

$$\iff \frac{x_1 + x_2 + x_3 + \ldots x_{n-1} + \alpha}{n} \geq \sqrt[n]{a_1 a_2 a_3 a_4 \ldots a_{n-1} \alpha} \tag{9}$$

Since α is a positive real number, we can take a set of numbers $a_1, a_2, a_3 \ldots a_n$ where a_n is α. We know that the AM GM inequality holds for it since $n = 2^k$ and therefore the last statement is true. Since these statements are bi-implicative, it must also hold for the first $n-1$ numbers of this set or all numbers in this set other than $a_n(\alpha)$ confirming our initial statement:

$$\frac{x_1 + x_2 + x_3 + \ldots x_{n-1}}{n-1} \geq \sqrt[n-1]{x_1 x_2 x_3 \ldots x_{n-1}} \tag{10}$$

to be true

Part c

Let's take a set of n positive real numbers $a_1, a_2, a_3 \ldots a_n$ where $a_i = i$

Because of the AM GM inequality we know that

$$\sqrt[n]{\prod a_i} \leq \frac{\sum a_i}{n} \tag{11}$$

In this case since $a_i = i$ we know that

$$\sqrt[n]{\prod_{i=1}^{n} i} \leq \frac{\sum_{i=1}^{n} i}{n} \tag{12}$$

$n! = \prod_{i=1}^{n} i$ (This is the definition of factorials)
$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (This is trivial)

Therefore

$$\sqrt[n]{n!} \leq \frac{\frac{n(n+1)}{2}}{n} \implies \sqrt[n]{n!} \leq \frac{n+1}{2} \implies n! \leq \left(\frac{n+1}{2}\right)^n \tag{13}$$