ECE C247, Winter 2022
Neural Networks and Deep Learning, UCLA

Homework #1

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Question 1 : Linear Algebra refresher

1 a) Given that A is a square matrix and that $A.A^T = I$.

Let $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, so $A.A^T = \begin{bmatrix} x^2+y^2 & xz+yw \\ zx+wy & z^2+w^2 \end{bmatrix}$

Since $A.A^T = I$, we can infer that $x = -1/\sqrt{2}$

$y = z = w = 1/\sqrt{2}$ (Better to construct a matrix that is symmetric)

Hence $A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

We can now solve for its Eigenvalues & Eigenvectors :-

$$A - \lambda I = \begin{bmatrix} -1/\sqrt{2} - \lambda & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} - \lambda \end{bmatrix}$$

Now, $\det(A - \lambda I) = 0$

$$\Rightarrow \quad -\left(1/\sqrt{2}+\lambda\right)\left(1/\sqrt{2}-\lambda\right) - 1/2 = 0$$

$$\Rightarrow \quad 1/2 - \lambda^2 + 1/2 = 0$$

$$\Rightarrow \quad \lambda^2 = 1 \quad \Rightarrow \quad \lambda = \pm 1$$

Now, lets solve for the eigenvector corresponding to $\lambda = 1$.

$(A - I)x = 0$

$$\Rightarrow \begin{pmatrix} -1/\sqrt{2} - 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} - 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad -x_1\left(1/\sqrt{2} + 1\right) + x_2 = 0$$

$$x_1/\sqrt{2} + x_2\left(1/\sqrt{2} - 1\right) = 0$$

We can set $x_2 = 1$, and solve for x_1

Hence $x_1 = \sqrt{2} - 1$

Hence the eigenvector corresponding to $\lambda = 1$ is

$v_1 = \begin{bmatrix} \sqrt{2} - 1 \\ 1 \end{bmatrix}$ and we can normalize this vector to obtain

$v_1 = \begin{pmatrix} 0.383 \\ 0.924 \end{pmatrix}$ is the eigenvector corresponding to the Eigenvalue $\lambda = 1$.

Now, lets compute the eigenvector corresponding to $\lambda = -1$

$(A + I)x = 0$

$$\Rightarrow \begin{pmatrix} -1/\sqrt{2} + 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} + 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad -x_1(1 + 1/\sqrt{2}) + x_2/\sqrt{2} = 0$$

$$x_1/\sqrt{2} + x_2(1 + 1/\sqrt{2}) = 0$$

Let's set $x_2 = 1$, hence $x_1 = -(\sqrt{2} + 1)$

Hence the Eigenvector $x = \begin{bmatrix} -(\sqrt{2}+1) \\ 1 \end{bmatrix}$

Normalizing this vector to obtain :-

$$v_2 = \begin{bmatrix} -0.924 \\ 0.383 \end{bmatrix}$$

Hence the Eigenvectors corresponding to values $\lambda = 1$ and -1 are :

$$v_1 = \begin{bmatrix} 0.383 \\ 0.924 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} -0.924 \\ 0.383 \end{bmatrix}$$

* We notice that the Eigenvalues are 1
* We also notice that the Eigenvectors are Orthonormal / Orthogonal.

1 a) (ii) Let's consider the eigenvalues to be λ.

So, $Ax = \lambda x$.

$$(\lambda x)^T (\lambda x) = (Ax)^T (Ax)$$

$$|\lambda|^2 x^T x = x^T A^T A x$$

$$|\lambda|^2 = 1$$

1 a) (iii) Let's consider λ_1, λ_2 be the eigenvalues of A and let v_1, v_2 be its corresponding eigenvectors, such that $\lambda_1 \neq \lambda$

So, $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$

$$\Rightarrow \quad (Av_1)^T (Av_2) = (\lambda_1 v_1)^T (\lambda_2 v_2)$$

$$\Rightarrow v_1^T . A^T . A . v_2 = \lambda_1 \lambda_2 v_1^T v_2$$

$$\Rightarrow (\lambda_1 \lambda_2 - 1) v_1^T . v_2 = 0$$

Since we know that $\lambda_1 \neq \lambda_2$ and $|\lambda_{1,2}| = 1$, hence $(\lambda_1 \lambda_2 - 1) \neq 0$

and $v_1^T . v_2 = 0$

Hence Eigenvectors corresponding to distinct eigenvalues are Orthogonal.

1 a) (iv) under the Transformation, the length of vector will remain unchanged as the vector will either be subjected to a rotation / reflection.

For example in our case from 1 a) (i) we know that :-

$$A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Hence under transformation becomes $Ax = \begin{bmatrix} -1/\sqrt{2}\, x_1 + 1/\sqrt{2}\, x_2 \\ 1/\sqrt{2}\, x_1 + 1/\sqrt{2}\, x_2 \end{bmatrix}$

$$||Ax|| = ||x||$$

1 b)(i) We can represent the matrix A as $A = U \Sigma V^T$

$$A.A^T = U \Sigma V^T . V \Sigma^T . U^T$$

$$A.A^T = U \Sigma^T . \Sigma U^T$$

Hence the left singular vectors of A are the eigenvectors of $A.A^T$

Similarly let's compute $A^T.A$

$$A^T.A = V \Sigma^T . U^T . U \Sigma V^T = V \Sigma^T . \Sigma V^T$$

Hence the Right singular vectors of A are the Eigenvectors of $A^T A$.

1) b) ii From the previous part, $A = U \Sigma V^T$

$$A.A^T = U \Sigma^T \Sigma U^T \quad \text{and} \quad A^T A = V \Sigma^T \Sigma V^T$$

Hence the singular value of A is the square root of the Eigenvalues of AA^T and $A^T A$.

1) c) (i) FALSE

Reason: Any $n \times n$ identity matrix has all identical eigenvalues 1.

(ii) FALSE

Reason: Let's consider a 2×2 matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ where $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ are eigenvectors, but their sum is not an eigenvector of A.

(iii) TRUE

Reason: $A\bar{v} = \lambda \bar{v}$, For any matrix to be positive semi definite, $\vec{x}^T A \vec{x} \geq 0$ for all $\bar{x}$. But if $\bar{v}$ is an eigen vector of A, then $\vec{v}^T A \vec{v} = \vec{v}^T \lambda \bar{v} = \bar{v}^T v \lambda$

Since $\bar{v}^T \vec{v}$ is a positive number, in order for $\bar{v}^T A \bar{v} \geq 0$, λ must be greater than or equal to zero (non negative).

(IV) TRUE

Reason: This statement is true for any $n \times n$ identity matrix.

(V) TRUE

Reason: Let's consider $A\bar{v}_1 = \lambda \bar{v}_1$ and $A\bar{v}_2 = \lambda \bar{v}_2$

$$A\bar{v}_1 + A\bar{v}_2 = \lambda \bar{v}_1 + \lambda \bar{v}_2 \Rightarrow A(\bar{v}_1 + \bar{v}_2) = \lambda(\bar{v}_1 + \bar{v}_2)$$

Hence, from above equation, we could infer that the statement is true.