$$\rightarrow \quad X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn} \quad ; \; 0 \leq k < N$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\left(\frac{2\pi}{N}\right)kn} \quad ; \; 0 \leq n < N$$

Both are periodic with N, both are discrete.

* Signal cannot be both band limited and time limited. (As it violates Heisenburg's uncertanity principle - fixed time bandwidth produ
If a signal is time limited, we cannot sample it.

* **Claim:** for any finite length sequence $x[n]$ of length N, it is enough to sample $X(j\omega)$ at N uniformly spaced frequency locations.

$$\bar{x} = [x[0] \quad x[1] \quad \dots \quad x[N-1]]^T_{N\times 1}$$

$$\bar{X} = F_N \, \bar{x}$$

$$F_N = \begin{array}{c} \\ k=0 \\ k=1 \\ \vdots \\ k=N-1 \end{array} \begin{array}{c} \begin{array}{ccc} n=0 & n=1 & n=N-1 \end{array} \\ \left[ \begin{array}{c} \\ e^{-j\left(\frac{2\pi}{N}\right)kn} \\ \\ \end{array} \right]_{N\times N} \end{array}$$

We can obtain $\bar{X}$ from $\bar{x}$ if $F_n$ is invertible.

Proving that $F_N$ is orthogonal:

Let $\omega = e^{-j\left(\frac{2\pi}{N}\right)}$

$$F_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & & & & \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{bmatrix}$$

$$\Rightarrow F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Inorder to make $F_4$ a unitary matrix, some textbooks distribute $1/N$ as $1/\sqrt{N}$ and $1/\sqrt{N}$ in $x[n]$ and $X[k]$

$F_N \rightarrow$ full rank, orthogonal, unitary.

* $x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$ $\qquad 0 \leq \cancel{k}\, n < N$.

if we compute $x[N+1] = \frac{1}{N} \cdot \sum_{n=0}^{N-1} X[k]\, e^{j(\frac{2\pi}{N})k(N+1)}$

$$e^{j(\frac{2\pi}{N})kN} = 1$$

$\Rightarrow x[N+1] = x[1]$

Thus $x[N+n] = x[n]$

$\Rightarrow x[7] = x[2]$ (when $N=5$)

$\Rightarrow x[38] = x[3]$

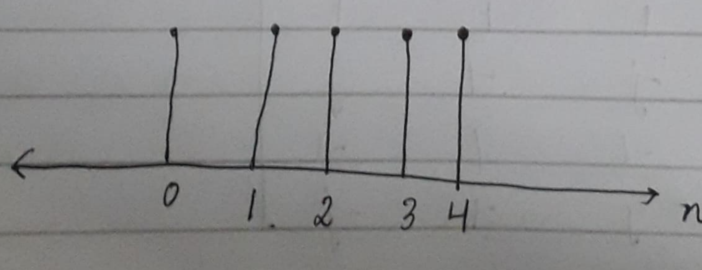
$x[-18] = x[2]$

Generalising, $x[m] = x[m \bmod N]$

Ex: 5 point DFT:

$$x[n] = \begin{cases} 1 ; & 0 \leq n \leq 4 \\ 0 ; & \text{otherwise} \end{cases}$$

Ans:



$$X[k] = \sum_{n=0}^{4} x[n] \, e^{-j\left(\frac{2\pi}{5}\right)kn}$$

$$\cancel{X[0] = \sum_{n=0}^{4} x[n] = 5} \qquad X[0] = \sum_{n=0}^{4} x[n] = 5$$

$$\cancel{X[1] =}$$

$$X[k] = \sum_{n=0}^{4} e^{-j\left(\frac{2\pi}{5}\right)kn}$$

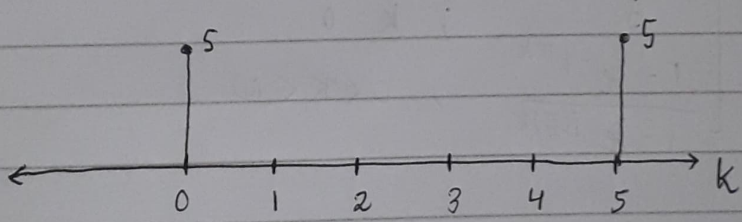
$$= \frac{1 - e^{-j\left(\frac{2\pi}{5}\right)kn \times 5}}{1 - e^{-j\left(\frac{2\pi}{5}\right)kn}} = \frac{1 - e^{-j(2\pi)kn}}{1 - e^{-j\left(\frac{2\pi}{5}kn\right)}}$$

$$= \underline{\underline{0}} \quad \text{(when } k \neq 0\text{)}$$

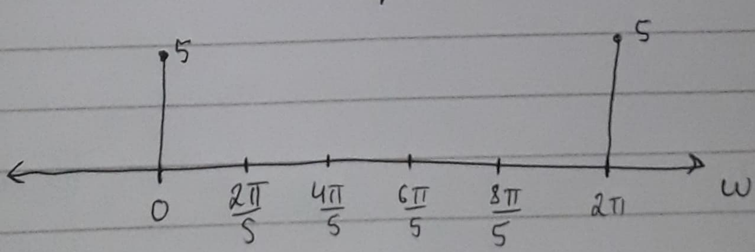
$$\Rightarrow X[k] = \begin{cases} 5 & ; \ k = 0 \\ 0 & ; \ \cancel{\text{otherwise}} \ 0 < k < 5 \end{cases}$$

↳ periodic with 5

$X[k]$:




$X_q(j\omega)$ :




$\underline{\underline{\text{Ex2}}}$ 10 point DFT for the above example

Ans 2

$$X[k] = \sum_{n=0}^{9} x[n]\, e^{-j\left(\frac{2\pi}{10}\right)kn}$$

$$= \sum_{n=0}^{4} x[n]\, e^{-j\left(\frac{2\pi}{10}\right)kn}$$

$x[n] = 0 \;\forall\; n > 4$

$$\cancel{X[0]} = \sum_{n=0}^{4} e^{-j\left(\frac{2\pi}{10}\right)kn}$$

$$X[k] = \frac{1 - e^{-j\left(\frac{2\pi}{10}\right)k \times 5}}{1 - e^{-j\left(\frac{2\pi}{10}\right)k}}$$

$$= \frac{1 - e^{-j\left(\frac{10\pi}{10}\right)k}}{1 - e^{-j\left(\frac{2\pi}{10}k\right)}} \qquad \cancel{= \frac{1 + e^{-j(\pi/9)k}}{1 - e^{-j\left(\frac{2\pi}{9}\right)k}}}$$

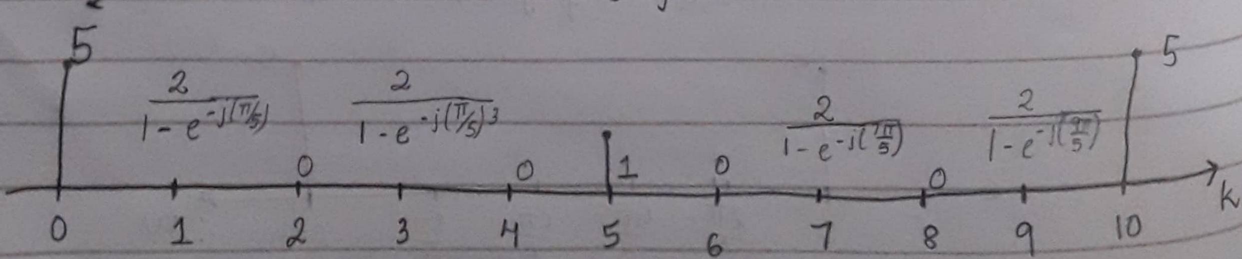
When $k = 0$, $X[k] = \sum_{n=0}^{4} (1)$

$$= 5$$

$$\cancel{X[k] = \begin{cases} 5 & ;\; k = 0 \\ \dfrac{1 - e^{-j\left(\frac{10\pi}{9}\right)k}}{1 - e^{-j\left(\frac{2\pi}{9}\right)k}} & ;\; 0 < k < 10 \end{cases}}$$

$$X[k] = \begin{cases} 5 & ;\; k = 0 \\ \dfrac{1 - e^{-j\pi k}}{1 - e^{-j\left(\frac{\pi}{5}\right)k}} & ;\; 0 < k < 10 \end{cases}$$

X[k] :



* $F_N$ is full rank, orthogonal, unitary

orthogonality:

$$F_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & & \omega^{N-1} \\ 1 & \omega^2 & & \omega^{2(N-1)} \\ 1 & & & \\ 1 & & & \\ 1 & \omega^{N-1} & & \omega^{(N-1)(N-1)} \end{bmatrix} \qquad \omega = e^{-j\left(\frac{2\pi}{N}\right)}$$

Clearly $i^{th}$ column vector is :
where $1 \leq i \leq N$

$$\begin{bmatrix} 1 \\ \omega^{(i-1)} \\ \omega^{2(i-1)} \\ \\ \\ \omega^{(N-1)(i-1)} \end{bmatrix}$$

For complex exponentials, orthogonality condition

$\Rightarrow$ $c_i (c_j)^* = 0 \qquad (i \neq j)$

$\Rightarrow$ $1 + \omega^{(i-1)}\omega^{(1-j)} + \omega^{2(i-1)}\omega^{2(1-j)} + \cdots + \omega^{(N-1)(i-1+1-j)}$

$\Rightarrow$ $1 + \omega^{i-j} + \omega^{2(i-j)} + \cdots + \omega^{(N-1)(i-j)}$

$\Rightarrow$ $\dfrac{\left(1 - \omega^{(i-j)N}\right)}{1 - \omega^{(i-j)}}$

$$e^{-j\left(\frac{2\pi}{N}\right) \times (i-j)N} = e^{-j(i-j)2\pi} = 1$$

$\Rightarrow$ $c_i (c_j)^* = 0 \qquad (\text{as } i \neq j)$

Hence it is orthogonal.

## Full rank matrix

We can say that $F_N$ is full rank as:

It is an orthogonal matrix. (Thus the columns are independent of each other). Thus, the matrix is invertible.

## Unitary matrix

Inverse of matrix = complex conjugate transpose

$\Rightarrow A(A^*)^T = I$

(A, $A^*$ are symmetric, $\Rightarrow (A^*)^T = A^*$)

$$\Rightarrow \frac{1}{\sqrt{N}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & & \omega^{N-1} \\ 1 & \omega^2 & & & \\ 1 & & & & \\ 1 & \omega^{N-1} & & & \omega^{(N-1)^2} \end{bmatrix} \frac{1}{\sqrt{N}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & & \omega^{-(N-1)} \\ 1 & & & & \\ 1 & & & & \\ 1 & \omega^{-(N-1)} & & & \omega^{-(N-1)^2} \end{bmatrix}$$

$$\Rightarrow \frac{1}{N}\times N\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & & & \\ 0 & & & & 1 \end{bmatrix} = I$$

Pf

$$1 + \omega^{-1} + \dots + \omega^{-(N-1)}$$

$$= \frac{1 - \omega^{-N}}{1 - \omega^{-1}} = \frac{1 - 1}{1 - \omega^{-1}} = 0$$

similarly,

$$1 + \omega^2 + \dots + \omega^{2(N-1)}$$

$$= \frac{1 - \omega^{2N}}{1 - \omega^2} = \frac{1 - 1}{1 - \omega^2} = 0$$

Similarly, all the other 0's were obtained.

As the inverse of $F_N$ is equal to its conjugate transpose, $F_N$ is an unitary matrix.