Q5.1.

Q1) A pair of jointly continuous random variables X and Y, have a joint probability density function given by.

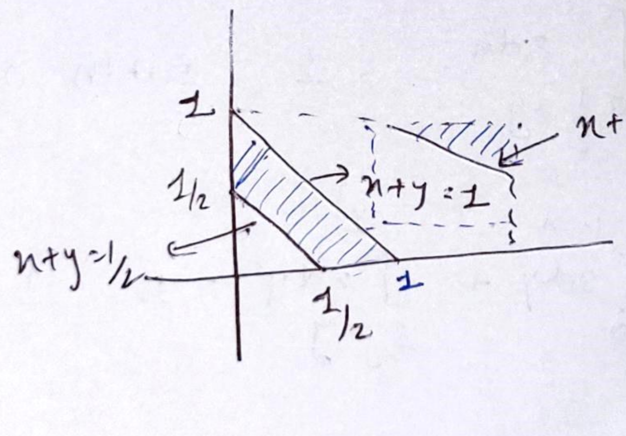
$$F_{x,y}(x,y) = \begin{cases} c, & \text{shaded} \\ 0, & \text{otherwise} \end{cases}.$$

1. find c.

Solution :- we know that

$$\iint f_x(x,y) = 1.$$

$$c = \text{Area of shaded region.}$$



$$c = \frac{1}{\text{Area of shaded region}}$$

$$c = \frac{1}{4 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}}$$

Q.5.1 $c = 2$

(b) Find the marginal PDF of X and Y

Marginal of x: $f_x(x) = \int f_x(x,y)\,dy$.

$$= \int_{\frac{1}{2}-x}^{1-x} 2\,dy = 1 \quad \text{when. } 0 \le x < \tfrac{1}{2}$$

$$= \int_0^{1-x} 2\,dy + \int_{\frac{3}{2}-x}^{1} 2\,dy = 1 \quad \text{when } \tfrac{1}{2} \le x < 1$$

Hence $X \sim U(0,1)$

marginal of y: $f_y(y)$. =

$$f_y(y) = \int_{\frac{1}{2}-y}^{1-y} 2\,dx = 1 \quad \text{when } 0 \le y < \tfrac{1}{2}$$

$$\int_0^{1-x} 2\,dy + \int_{3/2-y}^{1} 2\,dy = 1 \quad \tfrac{1}{2} \le y < 1$$

Hence $Y \sim \text{unif}(0,1)$

Q 5.1.

Q3. find $E(x|y=1/4)$ and $\text{var}(x|y=\frac{1}{4})$

Solution: $E(x/y=\frac{1}{4}) = \int_{-\infty}^{\infty} x f_{x/y}(x|y)\,dx$

Now,

$$f_{x|y}(x|y) = \frac{f_{xy}(x,y)}{f_y(y)}$$

from graph $= \begin{cases} 2 & \frac{1}{4} \le x \le \frac{3}{4} \\ 0 & \text{otherwise} \end{cases}$.

so,

$$E(x/y=\frac{1}{4}) = \int_{1/4}^{3/4} x.2\,dx.$$

$$= \left[\frac{9}{16} - \frac{1}{16}\right] = \frac{8}{16} = \frac{1}{2}$$

$$\text{var}(x/y=\frac{1}{4}) = \int_{-\infty}^{\infty} x^2 f_{x/y}(x/y)\,dx$$

$$= \int_{1/4}^{3/4} x^2.2\,dx.$$

$$= \frac{2}{3}\left[\frac{27-1}{64}\right].$$

$$= \frac{2}{3} \times \frac{26}{64}.$$

$$= \frac{13}{48}.$$

Q4.) Find the conditional PDF for x given that $y = 3/4$.

$$f_{x/y}(x/y) = \frac{f_{xy}(x,y)}{f(y)}$$

$$f_{x/y}\left(\frac{x}{y}=3/4\right) = f_{xy}(x,y).$$ $\left\{ y = \frac{3}{4},\ f_{xy}(x,y)\right.$ is in shaded area y.

when

$$x+y=1 \Rightarrow x = 1-\frac{3}{4} = \frac{1}{4}.$$

$$x+y=\frac{3}{2} \Rightarrow x = \frac{3}{2}-\frac{3}{4} = \frac{3}{4}.$$

So,

$$f_{x/y}\left(\frac{x}{y=3/4}\right) = \begin{cases} 2, & 0 \le x \le \frac{1}{4}, \frac{3}{4} \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(Q5.2.

Q1.) Now Let T_i be a random variable which counts no of shuffle . . . . . ?.

Solution

1) let T_i be a random variable denoting number of shuffles to get i cards under card n.

The intial given state is $1, 2, 3 \ldots n$ with 1 at top position.

T_1 = no of shuffle to get 1st card on top.

T_i = Geometric variable as each shuffle is a failure when i cards are not under n and last event is succes. with $p = \frac{i}{n}$

PMF =

$$P(T_i) = \left(1 - \frac{i}{n}\right)^{n-1} \frac{i}{n}.$$

$$E(T_i) = \frac{1}{P_{succes}}.$$

$E(T_i) = \frac{n}{i}$

give T_1 i.e. 1 card below n^{th} card., no of shuffles until 2 card is below n.

$T_2 - T_1 = Y_2$

similarly.

$T_i - T_{i-1} = Y_i$ (No of shuffles require to put i card under n, given i-1 cards are already ~~belo~~ under it.

$Y_i = \text{Geometric}\left(\frac{i}{n}\right)$. ; $P_{success} = \frac{i}{n}$.

$E(Y_i) = \frac{n}{i}$.

$\text{PDF}(T_i - T_{i-1}) =$ ~~i~~ $\text{Geometric}\left(\frac{i}{n}\right)$.

The T_ith state is in fact independent of T_{i-1} state as no of shuffles. is dependent on only "i" value

Hence only current state matter

Q 5.2.

(2) Prove that after $T_{n-1}+1$ shuffles, all possible $n!$ arrangement are equally likely

Solution Consider a card x. at bottom position.

i.e: no card is below x.

Total cards $= n$.

Let T_{n-1} = no of shuffles such that there are $n-1$ cards below x.

as we observe

any combination of $(n-1)!$ ~~cards~~ which is of $n-1$ cards arranged is equally likely.

$$P(T_{n-1}) = \frac{1}{(n-1)!}$$

~~for at T_n time~~

→ $T_{n-1}+1$ shuffle i.e shuffle after $n-1$ cards are below x, there would $n!$ ways. as x can go to any position.

$\therefore$ $n!$ ways equally likely. after $T_{n-1}+1$ shuffle.

Q3.) find expectation of $T_{n-1}+1$

solution:- T is the first time x become top card.

i.e.

~~$T = T_{n-1}$~~

$$\Rightarrow T = (T_{n-1} - T_{n-2}) + (T_{n-2} + T_{n-3}) \cdots (T_1 - T_0)$$

$(T_{n-1} - T_{n-2})$ = First time there are $n-1$ cards below x given that $n-2$ cards are already below x.

As we prove that

$T_i - T_{i-1}$ is a geometric Random variable with $P_s = \frac{i}{n}$.

$$E(T_i - T_{i-1}) = \frac{n}{i}$$

$$E(T) = \frac{n}{n-1} + \frac{n}{n-2} + \frac{n}{n-3} + \cdots 1.$$

$E(T) = n\left(1 + \frac{1}{2} + \frac{1}{3} \cdots \frac{1}{n}\right).$

$E(T) = n\,H(n)$ { $H(n)$ = nth harmonic mean }

Answer :- $E(T) = n\left(1 + \frac{1}{2} + \frac{1}{3} \cdots \frac{1}{n}\right).$

Q4. show that if you shuffle $100 \times n \log n$ bing

- - - - - - - - - - ?

solution :- T be first time - x become top card of deck.

$$E(T) = n\,H(n).$$

Then so. to shuffle k times such that so that with 99%. chance, the deck is completely shuffled - 1% to not shuffled.

so, $$P(T \geq k) \leq \frac{1}{100}$$

using Markov's inequality

$$P(x \geq a) \leq \frac{E(x)}{a}.$$

$$P(T \geq k) \leq \frac{n\,H(n)}{k}.$$

we know that

$$n H(n) \approx n \log n.$$

$$P(T \geq k) \leq \frac{n \log n}{k}.$$

Based on comparison.

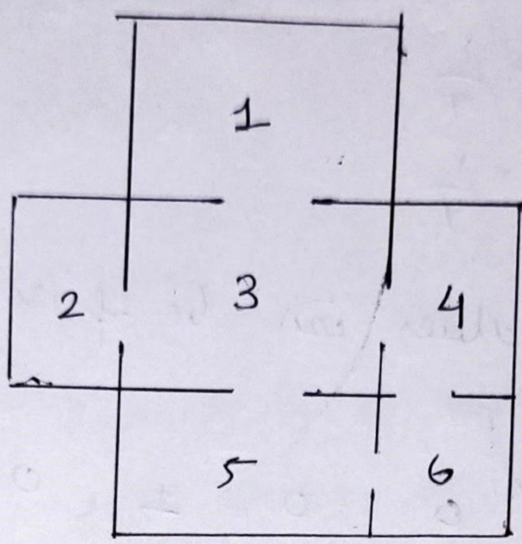
$$\frac{n \log n}{k} \leq \frac{1}{100}$$

$$\boxed{k \geq 100 n \log n}$$

Hence, After shuffling $100 n \log n$ times there will be $\frac{99}{100}$ chance, that the deck is random.

Q1. A rat runs through the maze shown below. At each step it leaves the room it is in by choosing at random one of the doors out of room.

Solution:-



(a) Give the transition matrix P for this markov chain

Solution :- $P[i][j]$ = Represents probability of moving i to j.

$P[1][2]$ = No path is possible

$\therefore P[i][2] = 0$.

$P[1][3] = 1$, since only 1 path is possible from location 1

Similarly

$P[2][3] = 1$.

$P[3][1] = \frac{1}{4}$

Four paths are possible from 3, with equal probability

$$P[3][1] = \frac{1}{4}$$

$$P[3][2] = \frac{1}{4}.$$

$$P[3][4] = \frac{1}{4}$$

$$P[3][5] = \frac{1}{4}.$$

Similarly all values can be filled in transition matrix

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

(b) Show that it is irreducible but not aperiodic.

Solution :- Irreducible :- A chain is irreducible when it has single communicating state. i.e. all states can be reached from every state.

or,

$$\begin{array}{ccccccccc} 1 & \rightleftarrows & 3 & \rightleftarrows & 4 & \rightleftarrows & 6 & \rightleftarrows & 5 \\ & & \updownarrow & & & & & & \\ & & 2 & & & & & & \end{array}$$

∴ the chain is irreducible

(~~c~~) Aperiodic :- A chain is said to be aperiodic when for some value of n $p^n > 0$, i.e all values are greater than 0 of transition matrix

$p^{(n)}$ = n step probability matrix, where n = 1, 2, 3, 4 ..

$p^n[6][3]$: Probability of going to location 3 from location 6 in n steps, n = 1, 2, 3 ---

$p^n[6][3] = 0$, when, n = odd.

$p^n[6][5] = 0$, when n = even.

∴ $p^n[6][3]$ and $p^n[6][5]$ cannot be both positive at one value of n.

∴ chain is aperiodic.

Q3) Find the stationary distribution of the chain

Solution.. To find stationary distribution we have to calculate following values $\pi_1, \pi_2 \ldots \pi_n$.

$$\pi_j = \lim_{n \to \infty} P(X_n = j \mid X_0 = i) \quad \forall\ i, j \in S.$$

Also,

$$\sum_{j \in S} \pi_j = 1 \quad \text{i.e.} \quad \pi_1 + \pi_2 + \pi_3 \ldots \pi_n = 1.$$

Given.

$$\pi_1 = \frac{1}{4}\pi_3$$

$$\pi_2 = \frac{1}{4}\pi_3.$$

$$\pi_3 = \pi_1 + \pi_2 + \frac{1}{2}\pi_4 + \frac{1}{2}\pi_5$$

$$\pi_4 = \frac{1}{4}\pi_3 + \frac{1}{2}\pi_6$$

$$\pi_5 = \frac{1}{4}\pi_3 + \frac{1}{2}\pi_6$$

$$\pi_6 = \frac{1}{2}\pi_4 + \frac{1}{2}\pi_5$$

from here. $z_1 = z_2$, z_3 , $z_4 = z_5$

let ~~to~~ let $z_1 = z_2 = x$

$z_4 = z_5 = y$.

$$z_6 = \frac{xy}{2} + \frac{y}{2}$$

$$\boxed{z_6 = y}$$

$$\boxed{z_3 = 2x + y}$$

$$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 1$$

$$2x + 3y + 2x + y = 1$$

$$\boxed{x + y = \frac{1}{4}} \quad \text{---- (i)}$$

Also,
$$z_5 = \frac{1}{4} z_3 + \frac{1}{2} z_6 .$$

$$y = \frac{1}{4}(2x + y) + \frac{y}{2}$$

$$\frac{y}{2} = \frac{2x + y}{4}$$

$$2y = 2x + y$$

$$\boxed{y = 2x} \quad \text{.... (ii)}$$

putting eq (ii) in eq (i).

$$x + y = \frac{1}{4}$$

$$3x = \frac{1}{4}$$

$$x = \frac{1}{12}.$$

$$y = \frac{2}{12}$$

$$\pi_3 = 2x + y$$

$$\pi_3 = 2 \times \frac{1}{12} + \frac{2}{12}$$

$$\boxed{\pi_3 = \frac{4}{12}}$$

$$\boxed{\pi_1 = \pi_2 = x = \frac{1}{12}}$$

$$\boxed{\pi_4 = \pi_5 = \pi_6 = \frac{2}{12}}$$

Answer:- Stationary Distribution $\Rightarrow \pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6)$

$$= \left(\frac{1}{12}, \frac{1}{12}, \frac{4}{12}, \frac{2}{12}, \frac{2}{12}, \frac{2}{12}\right)$$

(d) Now suppose there is trap in room 5 and we start in room 1. Find expected no. of steps before reaching room 5 for first time, starting in room 1.

Solution:- let

$r(i) = E(\text{No of steps to reach state } 5 \mid X_0 = i)$

we know

$$r_i = 1 + \sum_k r_i P_{ik} \quad \text{for } i \in S.$$

$S = \{1, 2, 3, 4, 5, 6\}$ (Total states).

$r(5) = 0$ (same state).

$$r(6) = 1 + \frac{1}{2} r(5) + \frac{1}{2} (r(4)).$$

$$r(4) = 1 + \frac{1}{2} r(6) + \frac{1}{2} r(3)$$

$$r(3) = 1 + \frac{1}{4} r(1) + \frac{1}{4} r(2) + \frac{1}{2} r(4) + \frac{1}{4} r(5)$$

$r(1) = 1 + r(3)$.

$r(2) = 1 + r(3)$

converting everything to r_3.

$$r(4) = 1 - \frac{1}{2} r(6) = \frac{1 r_3}{2}.$$

(e)

$$r(6) = 1 + \frac{1}{2} r_3$$

$$r(4) - 1 - \frac{1}{2}\left(1 + \frac{1}{2} r_4\right) = \frac{1}{2} r_3.$$

$$r_4 - \frac{1}{2}\left(1 + \frac{1}{2} r_4\right) - 1 = \frac{1}{2} r_3.$$

$$r_4 - \frac{1}{2} - \frac{1}{4} r_4 - 1 = \frac{1}{2} r_3.$$

$$\frac{3}{4} r_4 - \frac{3}{2} = \frac{1}{2} r_3$$

$$3r_4 - 6 = 2r_3, \quad r_4 = \frac{2r_3 + 6}{3}.$$

$$r_3 = 1 + \frac{1}{4} r_1 + \frac{1}{4} r_2 + \frac{1}{2} r_4 + 0.$$

$$r_3 = 1 + \frac{2r_3 + 6}{\cancel{6}} + \frac{1}{4}(r_3 + 1) + \frac{1}{4}(r_3 + 1)$$

$$12r_3 = 12 + 4r_3 + \overset{12}{\cancel{24}} + 3r_3 + 3 + 3r_3 + 3$$

$$2r_3 = 12$$

$$r_3 = 6$$

$$r_i = 1 + r_3, \quad r_1 = 1 + 6, \quad r = 7$$

$$\boxed{r_1 = 7}$$

Answer = 7

(e) Find the expected time to return to room 1.

Solution:- r_y = Mean or expected time to start 1.

$$r_y = \frac{1}{\lambda_y}$$

$\cancel{\lambda_y}$

$$r_1 = \frac{1}{\lambda_1} \quad (\lambda_1 = \frac{1}{12})$$

(From question - 3)

$$r_1 = \frac{1}{1/12}$$

$$\boxed{r_1 = 12}$$