① Perturbations Equations

$\rightarrow e_1 = 3M + \frac{1}{2}(l+2)(l-1)r + 4\pi r^3 p$

$\rightarrow e_2 = 8\pi r^3 e^{-\nu/2}$

$\rightarrow e_3 = -\frac{1}{2} l(l+1)(M + 4\pi r^3 p) + \omega^2 r^3 e^{-(\lambda+\nu)}$

$\rightarrow e_4 = \frac{1}{2}(l+2)(l-1)r - \omega^2 r^3 e^{-\nu} - \frac{e^{\lambda}(M + 4\pi r^3 p)(3M - r + 4\pi r^3 p)}{r}$

$\Rightarrow \boxed{e_1 H_0 = e_2 X + e_3 H_1 + e_4 K}$ → Einstein eqn

$\rightarrow a_1 = \omega^3 (p+\rho) e^{-\nu/2}$

$\rightarrow a_2 = -\frac{p' e^{(\nu-\lambda)/2}}{r}$

$\rightarrow a_3 = \frac{1}{2}(p+\rho) e^{\nu/2}$

$\Rightarrow \boxed{X = a_1 V + a_2 W + a_3 H_0}$ → Auxilary variable.

$\Rightarrow a_1 V = X - a_2 W - a_3 \frac{(e_2 X + e_3 H_1 + e_4 K)}{e_1}$

$\Rightarrow \boxed{a_1 V = X\left(1 - \frac{a_3 e_2}{e_1}\right) + \left(-\frac{a_3 e_3}{e_1}\right) H_1 + \left(-\frac{a_3 e_4}{e_1}\right) K + (-a_2) W}$

$$\boxed{\begin{bmatrix} H_0 \\ V \end{bmatrix} = \begin{bmatrix} \frac{e_3}{e_1} & \frac{e_4}{e_1} & 0 & \frac{e_2}{e_1} \\ -\frac{a_3 e_3}{a_1 e_1} & -\frac{a_3 e_4}{a_1 e_1} & -\frac{a_2}{a_1} & \left(1 - \frac{a_3 e_2}{e_1}\right)\frac{1}{a_1} \end{bmatrix} \begin{bmatrix} H_1 \\ K \\ W \\ X \end{bmatrix}}$$

$$b_1 = \frac{-\left[l+1+\frac{2M e^{\lambda}}{r} + 4\pi r^2 e^{\lambda}(p-\rho)\right]}{r}$$

$$b_2 = \frac{e^{\lambda}}{r}[1]$$

$$b_3 = -\frac{e^{\lambda}}{r}[16\pi(p+\rho)]$$

$$\boxed{H_1' = b_1 H_1 + b_2 H_0 + b_2 K + b_3 V}$$

$$b_4 = \frac{1}{r}$$

$$b_5 = \frac{1}{2}\frac{(l+1)l}{r}$$

$$b_6 = -\left[\frac{(l+1)}{r} - \frac{1}{2}\nu'\right]$$

$$b_7 = -\frac{8\pi(p+\rho)e^{\lambda/2}}{r}$$

$$\boxed{K' = b_4 H_0 + b_5 H_1 + b_6 K + b_7 W}$$

$$b_8 = -\frac{(l+1)}{r}$$

$$b_9 = \frac{r e^{\lambda/2}}{\gamma p e^{\nu/2}}$$

$$b_{10} = \frac{-r e^{\lambda/2} l(l+1)}{r^2}$$

$$b_{11} = \frac{r e^{\lambda/2}}{2}$$

$$\boxed{W' = b_8 W + b_9 X + b_{10} V + b_{11} H_0 + 2 b_{11} K}$$

$$b_{12} = -\frac{l}{r}$$

$$b_{13} = (\rho+p)e^{\nu/2}\left[\frac{1}{2}\left(\frac{1}{r} - \frac{1}{2}\nu'\right)\right]$$

$$b_{14} = (\rho+p)e^{\nu/2}\left[\frac{1}{2}(r\omega^2 e^{-\nu} + \frac{1}{2}\frac{l(l+1)}{r}\right]$$

$$b_{15} = (\rho+p)e^{\nu/2}\left[\frac{1}{2}\left(\frac{3}{2}\nu' - \frac{1}{r}\right)\right]$$

$$b_{16} = -(\rho+p)e^{\nu/2}\left[\frac{1}{2}\frac{l(l+1)}{r^2}\nu'\right]$$

$$b_{17} = -\frac{(p+\rho)e^{\nu/2}}{r}\left[4\pi(p+\rho)e^{\lambda/2} + \omega^2 e^{\lambda/2-\nu} - \frac{1}{2}r^2\left[r^{-2}e^{-\lambda/2}\nu'\right]'\right]$$

$$\boxed{X' = b_{12}X + b_{13}H_0 + b_{14}H_1 + b_{15}K + b_{16}V + b_{17}W}$$

② →

Boundary conditions → $\boxed{Y(r) = Y(0) + \frac{r^2}{2} Y''(0)}$

$$Y(0) \begin{cases} W(0) = 1. \\ K(0) = \pm(\rho_0 + P_0). \\ X(0) = (P_0+\rho_0) e^{\nu_0 l/2} \left\{ \left[\frac{4\pi}{3}(3P_0+\rho_0) - \frac{\omega^2 e^{-\nu_0}}{l} \right] W(0) + \frac{K(0)}{2} \right\} \\ H_1(0) = \dfrac{2l K(0) + 16\pi(\rho_0+P_0) W(0)}{l(l+1)}. \end{cases}$$

State variables.

$$\rho(r) = \rho_0 + \frac{\rho_2}{2} r^2 \qquad \qquad \gamma_0, P_0, \rho_0 \to ?$$

$$P(r) = P_0 + \frac{1}{2}P_2 r^2 + \frac{1}{4}P_4 r^4$$

$$\nu(r) = \nu_0 + \frac{1}{2}\nu_2 r^2 + \frac{1}{4}\nu_4 r^4.$$

$$P_2 = -\frac{4\pi}{3}(\rho_0+P_0)(\rho_0+3P_0)$$

$$\rho_2 = \frac{P_2(\rho_0+P_0)}{\gamma_0 P_0}$$

$$\nu_2 = \frac{8\pi}{3}(\rho_0+3P_0)$$

$$P_4 = -\frac{2\pi}{5}(\rho_0+P_0)(\rho_2+5P_2) - \frac{2\pi}{3}(\rho_2+P_2)(\rho_0+3P_0) - \frac{32\pi^2}{9}\rho_0(\rho_0+P_0)(\rho_0+3P_0).$$

$$\nu_4 = \frac{4\pi}{5}(\rho_2+5P_2) + \frac{64\pi^2}{9}\rho_0(\rho_0+3P_0).$$

③ second-order perturbations values at singularity.

$$\boxed{\begin{array}{l} Y(0) \rightarrow Y \\ Y''(0) \rightarrow Y'' \end{array}}$$

→ $C_1 = \frac{4}{(l+2)(l-1)} 8\pi e^{-\nu_0/2}$

→ $C_2 = \frac{-4}{(l+2)(l-1)} \left(\frac{8\pi}{3} \rho_0 + \omega^2 e^{-\nu_0} \right)$

→ $C_3 = \frac{-4}{(l+2)(l-1)} \left[\frac{2\pi}{3} l(l+1)(\rho_0 + 3p_0) - \omega^2 e^{-\nu_0} \right]$

$$\boxed{Q_0 = C_1 X + C_2 K + C_3 H_1}$$

→ $C_4 = \frac{2}{l(l+1)} \frac{e^{-\nu_0/2}}{\gamma_0 P_0}$

→ $C_5 = \frac{3}{l(l+1)}$

→ $C_6 = \frac{8\pi}{3(l(l+1))} (l+1) \rho_0$

$$\boxed{Q_1 = C_4 X + C_5 K + C_6 W}$$

$\rightarrow d_1 = -\frac{1}{4}(\rho_0 + P_0)$

$d_2 = \frac{1}{2}\left[P_2 + (\rho_0 + P_0)\frac{\omega^2(l+3)}{l(l+1)} e^{-\nu_0}\right]$

$d_3 = \frac{1}{2} e^{-\nu_0/2}$

$d_4 = \frac{1}{4}\nu_2 e^{-\nu_0/2}$

$d_5 = -\frac{1}{4}(\rho_2 + P_2)$

$d_6 = \frac{1}{4}(\rho_0 + P_0)$

$d_7 = \frac{1}{2}\omega^2(P_0 + \rho_0) e^{-\nu_0}$

$d_8 = -\{P_4 - \frac{4\pi}{3}\rho_0 P_2 + \frac{\omega^2}{2l}[\rho_2 + P_2 - (\rho_0 + P_0)\nu_2] e^{-\nu_0}\}.$

$d_9 = \frac{1}{2}(l+2)$

$d_{10} = -\frac{1}{4} l(l+1)$

$d_{11} = 4\pi(\rho_0 + P_0).$

$d_{12} = \frac{4\pi}{3}(\rho_0 + 3P_0)$

$d_{13} = \frac{1}{2}$

$d_{14} = -4\pi[\rho_2 + P_2 + \frac{8\pi}{3}\rho_0(\rho_0 + P_0)]$

$d_{15} = \frac{1}{2}(l+3)$

$d_{30} = -1.$

$d_{16} = -8\pi(\rho_0 + P_0)\frac{(l+3)}{l(l+1)}.$

$d_{17} = 4\pi\left[\frac{1}{3}(2l+3)\rho_0 - P_0\right]$

$d_{18} = \frac{8\pi}{l}(\rho_2 + P_2)$

$d_{19} = -8\pi(\rho_0 + P_0).$

$d_{20} = \frac{1}{2}.$

$$d_{21} = \frac{1}{2}(l+2)$$

$$d_{22} = -\frac{1}{8} l(l+1)(\rho_0+p_0)\, e^{\nu_0/2}$$

$$d_{23} = -(\rho_0+p_0)\, e^{\nu_0/2} \left[\frac{1}{4}(l+2)\nu_2 - 2\pi(\rho_0+p_0) - \frac{1}{2}\omega^2 e^{-\nu_0} \right]$$

$$d_{24} = \frac{\frac{l}{2}\left[\rho_2+p_2 + \frac{1}{2}(\rho_0+p_0)\nu_2\right]}{\rho_0+p_0}$$

$$d_{25} = (\rho_0+p_0)\, e^{\nu_0/2} \left\{ \frac{1}{2}\nu_2 \right\}$$

$$d_{26} = \frac{(\rho_0+p_0)\, e^{\nu_0/2}}{4}$$

$$d_{27} = \frac{(\rho_0+p_0)\, e^{\nu_0/2}\, \omega^2 e^{-\nu_0}}{2}$$

$$d_{28} = -\frac{(\rho_0+p_0)\, e^{\nu_0/2}\, l(l+1)\nu_2}{4}$$

$$d_{29} = \left[\frac{1}{2}(l+1)\nu_4 - 2\pi(\rho_2+p_2) - \frac{16\pi^2}{3}\rho_0(\rho_0+p_0) + \frac{1}{2}\left(\nu_4 - \frac{4\pi}{3}\rho_0\nu_2\right) + \frac{1}{2}\omega^2 e^{-\nu_0}\left(\nu_2 - \frac{8\pi}{3}\rho_0\right) \right] (p_0+\rho_0)\, e^{\nu_0/2}$$

→ $d_1 K'' + d_2 W'' + d_3 X'' = d_4 X + d_5 K + d_6 Q_0 + d_7 Q_1 + d_8 W.$

→ $d_9 K'' + d_{10} H_1'' + d_{11} W'' = d_{12} K + d_{13} Q_0 + d_{14} W$

→ $d_{15} H_1'' + d_{30} K'' + d_{16} W'' = d_{17} H_1 + d_{18} W + d_{19} Q_1 + d_{20} Q_0$

→ $d_{21} X'' + d_{22} H_1'' + d_{23} W'' = d_{24} X + d_{25} K + d_{26} Q_0 + d_{27} H_1 + d_{28} Q_1 + d_{29} W.$

$$\begin{bmatrix} 0 & d_1 & d_2 & d_3 \\ d_{10} & d_9 & d_{11} & 0 \\ d_{15} & d_{30} & d_{16} & 0 \\ d_{22} & 0 & d_{23} & d_{21} \end{bmatrix} \begin{bmatrix} H_1''(0) \\ K''(0) \\ W''(0) \\ X''(0) \end{bmatrix} = \begin{bmatrix} 0 & d_5 & d_8 & d_4 & d_6 & d_7 \\ 0 & d_{12} & d_{14} & 0 & d_{13} & 0 \\ d_{17} & 0 & d_{18} & 0 & d_{19} & d_{20} \\ d_{27} & d_{25} & d_{29} & d_{24} & d_{26} & d_{28} \end{bmatrix} \begin{bmatrix} H_1(0) \\ K(0) \\ W(0) \\ X(0) \\ Q_0(0) \\ Q_1(0) \end{bmatrix}$$

$$\begin{bmatrix} H_1' \\ K' \\ W' \\ X' \end{bmatrix} = \begin{bmatrix} b_2 & b_1 & b_2 & 0 & b_3 & 0 \\ b_4 & b_5 & b_6 & b_7 & 0 & 0 \\ b_{11} & 0 & 2b_{11} & b_8 & b_{10} & b_9 \\ b_{13} & b_{14} & b_{15} & b_{17} & b_{16} & b_{12} \end{bmatrix} \begin{bmatrix} H_0 \\ H_1 \\ K \\ W \\ V \\ X \end{bmatrix}$$

$$\begin{bmatrix} H_0 \\ V \end{bmatrix} = \begin{bmatrix} \frac{e_3}{e_1} & \frac{e_4}{e_1} & 0 & \frac{e_2}{e_1} \\ \frac{-a_3 e_3}{a_1 e_1} & \frac{-a_3 e_4}{a_1 e_1} & \frac{-a_2}{a_1} & \frac{1}{a_1}\left(1 - \frac{a_3 e_2}{e_1}\right) \end{bmatrix} \begin{bmatrix} H_1 \\ K \\ W \\ X \end{bmatrix}$$

L&D eqns.

$$\begin{bmatrix} Q_0(0) \\ Q_1(0) \end{bmatrix} = \begin{bmatrix} c_3 & c_2 & 0 & c_1 \\ 0 & c_5 & c_6 & c_4 \end{bmatrix} \begin{bmatrix} H_1(0) \\ K(0) \\ W(0) \\ X(0) \end{bmatrix}$$

Power series analysis at singularity.

→ Power Series Analysis

$r = 0$ to $r = \frac{R}{25}$

① $T = \begin{bmatrix} 0 & d_1 & d_2 & d_3 \\ d_{10} & d_9 & d_{11} & 0 \\ d_{15} & d_{30} & d_{16} & 0 \\ d_{22} & 0 & d_{23} & d_{21} \end{bmatrix}$

② $U = \begin{bmatrix} 0 & d_5 & d_8 & d_4 & d_6 & d_7 \\ 0 & d_{12} & d_{14} & 0 & d_{13} & 0 \\ d_{17} & 0 & d_{18} & 0 & d_{19} & d_{20} \\ d_{27} & d_{25} & d_{29} & d_{24} & d_{26} & d_{28} \end{bmatrix}$

③ $C = \begin{bmatrix} C_3 & C_2 & 0 & C_1 \\ 0 & C_5 & C_6 & C_4 \end{bmatrix}$

④ $Y = \begin{bmatrix} H_1 \\ K \\ W \\ X \end{bmatrix}$

⑤ $Q = \begin{bmatrix} Q_0 \\ Q_1 \end{bmatrix}$

Eqn ① :-

$$\boxed{T \cdot Y''(0) = U \cdot \begin{bmatrix} Y(0) \\ Q(0) \end{bmatrix}} \Rightarrow \boxed{Y''(0) = T^{-1} \cdot U \cdot \begin{bmatrix} Y(0) \\ Q(0) \end{bmatrix}}$$

$$\boxed{Q(0) = C \cdot Y(0)}$$

$Y(0)$:

→ $X(0) = (\rho_0 + P_0)\, e^{\nu_0/2} \left\{ \left[\frac{4\pi}{3}(\rho_0 + 3P_0) - \frac{\omega^2 e^{-\nu_0}}{l} \right] W(0) + \frac{1}{2} K(0) \right\}$

→ $H_1(0) = \dfrac{2l\,K(0) + 16\pi(\rho_0 + P_0)\,W(0)}{l(l+1)}$

→ $W(0) = 1$

→ $K(0) = \pm(\rho_0 + P_0)$

→ for $0 \le r \le \frac{R}{25.0}$; $\boxed{Y(r) = Y(0) + \frac{r^2}{2} Y''(0)}$

Notes.

→ This region fetches approx Y near singularity.

→ Lindblom and Detweiler eqn.

$r = \frac{R}{25}$ to $r = R$.

① $$E = \begin{bmatrix} \frac{e_3}{e_1} & \frac{e_4}{e_1} & 0 & \frac{e_2}{e_1} \\ \frac{-a_3 e_3}{a_1 e_1} & \frac{-a_3 e_4}{a_1 e_1} & \frac{-a_2}{a_1} & \frac{e_1 - a_3 e_2}{a_1 e_1} \end{bmatrix}$$

② $$B = \begin{bmatrix} b_2 & b_1 & b_2 & 0 & b_3 & 0 \\ b_4 & b_5 & b_6 & b_7 & 0 & 0 \\ b_{11} & 0 & 2b_{11} & b_8 & b_{10} & b_9 \\ b_{13} & b_{14} & b_{15} & b_{17} & b_{16} & b_{12} \end{bmatrix}$$

③ $$Y = \begin{bmatrix} H_1 \\ K \\ W \\ X \end{bmatrix}$$

④ $$P = \begin{bmatrix} H_0 \\ H_1 \\ K \\ W \\ V \\ X \end{bmatrix}$$

⑤ $$F = \begin{bmatrix} H_0 \\ V \end{bmatrix}$$

Eqn ②:-

$$\boxed{F = E \cdot Y}$$

$$\boxed{Y' = B \cdot P} \Rightarrow \boxed{Y = \int_{R/25}^{r} B \cdot P \, dr}$$

Initial value of Y comes from Eqn ①.

→ Exactly at the surface $X(R) = 0$. (and beyond too as there's no fluid element present).

$$\Rightarrow \boxed{H_0(R) = \frac{e_3 H_1(R) + e_4 K(R)}{e_1}}$$

Notes:-

→ This region shows the behaviour inside the star and fetches boundary value condition for the next region using the fact that $X(R) = 0$.

→ Zerilli eqn :-

from $r = R$ to $r = 25\omega^{-1}$

① $a(r) = \dfrac{-(nr+3M)}{\omega^2 r^2 - \frac{(n+1)M}{r}}$

② $b(r) = \dfrac{nr(r-2M) - \omega^2 r^4 + M(r-3M)}{(r-2M)\left[\omega^2 r^2 - \frac{(n+1)M}{r}\right]}$

③ $g(r) = \dfrac{n(n+1)r^2 + 3nMr + 6M^2}{r^2(nr+3M)}$

④ $h(r) = \dfrac{-nr^2 + 3nMr + 3M^2}{(r-2M)(nr+3M)}$

⑤ $K(r) = \dfrac{-r^2}{r-2M}$

⑥ $n = \dfrac{(l-1)(l+2)}{2}$

⑦ $r_* = r + 2M\log\left(\frac{r}{2M} - 1\right)$

Eqn ⑧

$$\boxed{V_Z(r^*) = \frac{(1-2M/r)}{r^3(nr+3M)^2}\left[2n^2(n+1)r^3 + 6n^2Mr^2 + 18nM^2r + 18M^3\right]}$$

$$\boxed{\frac{d^2Z}{dr_*^2} = (V_Z - \omega^2)Z \quad (OR) \quad \frac{d^2Z}{dr^2} = \left\{\frac{[V-\omega^2]r^3}{(r^2+2M)(r-2M)}\right\}Z} \rightarrow (i)$$

$$\begin{bmatrix} 0 & 1 \\ a(r) & b(r) \end{bmatrix}\begin{bmatrix} H_0(r) \\ K(r) \end{bmatrix} = \begin{bmatrix} g(r) & 1 \\ h(r) & K(r) \end{bmatrix}\begin{bmatrix} Z \\ \frac{dZ}{dr_*} \end{bmatrix}$$

(OR)

$$\begin{bmatrix} 0. & 1 \\ a(R) & b(R) \end{bmatrix}\begin{bmatrix} \frac{e_3 H_1(R) + e_4 K(R)}{e_1} \\ K(R) \end{bmatrix} = \begin{bmatrix} g(R) & 1 \\ h(R) & K(R) \end{bmatrix}\begin{bmatrix} Z|_{r=R} \\ \left(1-\frac{2M}{R}\right)\frac{dZ}{dr}\Big|_{r=R} \end{bmatrix} \rightarrow (ii)$$

→ Eqn (ii) is boundary value condition for Eqn (i).

Notes :-

→ This region connects with the previous region using Zerilli transformation and is used for comparing values at asymptotes.

Asymptotic power series analysis.

At $r = 25\omega^{-1}$.

① $$\boxed{Z_{\pm} = e^{\pm i\omega r_*} \sum_j \begin{pmatrix} + \to \bar{\alpha}_j \\ - \to \alpha_j \end{pmatrix} r^{-j}.} \to \boxed{Z = \beta Z_- + r Z_+}$$

② $$\alpha_1 = \frac{-i(n+1)\alpha_0}{\omega} \; ; \; \alpha_2 = \frac{-1}{2\omega^2}\left[n(n+1) - 3iM\omega\left(1+\frac{2}{n}\right)\right]\alpha_0 .$$

Eqn ④ :-

$$\boxed{\begin{bmatrix} Z_- & Z_+ \\ Z_-' & Z_+' \end{bmatrix}_{r=25\omega^{-1}} \begin{bmatrix} \beta \\ r \end{bmatrix} = \begin{bmatrix} Z \\ Z' \end{bmatrix}_{r=25\omega^{-1}}}$$

→ This value is fetched after numerical integration of eqn ③.

$$\to Z_- = \frac{\alpha_0}{625}\left[1250 - (n+50i)(n+1) + 3iM\left(1+\frac{2}{n}\right)\omega\right]$$

$$\to Z_-' = \frac{\alpha_0 \omega}{15625}\left[(n+25i)(n+1) - 3iM\left(1+\frac{2}{n}\right)\omega\right]$$

$$\to Z_+ = \frac{\bar{\alpha}_0}{625}\left[1250 - (n-50i)(n+1) - 3iM\left(1+\frac{2}{n}\right)\omega\right]$$

$$\to Z_+' = \frac{\bar{\alpha}_0 \omega}{15625}\left[(n-25i)(n+1) + 3iM\left(1+\frac{2}{n}\right)\omega\right].$$

Notes

→ After comparing values at asymptotes, value of r is estimated using the above eqn.

Pseudo-code for QNM.

```
→  y0 = region_singularity()
E0 = Energy_eigen(y0, region_inside). //→ Finding first eigenvalue
E1 = inf.
while ||E0 - E1|| > 1e-8 :  //→ Iterating till the real part of ω
                                 doesn't change much
    E2 = [E0]
    Γ = []
    for(0, some_no_of_iterations) :  //→ Estimating Γ values by sampling
                                          ω values.
        y = runge_kutta(y0, E2[-1], region_inside).
        y = rungekutta(y[-1], E2[-1], region_outside).
        Γnew = compute_Γ(y[-1], E2[-1]).
        Γ.append(Γnew)
        Enew = normal_dist(E2[0], 0.01*E2[0]).
        E2.append(Enew).

    E1 = E0   //→ Storing past value of ω.
    ~~Eoeeeat~~
    gamma_poly = fit(Γ, E[:-2]).
    root_poly = zero_of_quadratic_eqn(gamma_poly).
    E0 = Real_part_of(root_poly)  //→ Finding new value
                                       of real part of
                                       ω
    E2.clear()
    Γ.clear().
```

———— x ————