Ex The change of variable $x = uv$ and $y = \frac{1}{2}(u^2 - v^2)$ transforms $f(x,y)$ to $g(u,v)$.

(a) Calculate $\frac{\partial g}{\partial u}$, $\frac{\partial g}{\partial v}$, $\frac{\partial^2 g}{\partial u \partial v}$ in terms of partial derivative of f.

(b) If $\|\nabla f(x,y)\|^2 = 2$ for all x and y determine constants a and b s.t.

$$a\left(\frac{\partial g}{\partial u}\right)^2 + b\left(\frac{\partial g}{\partial v}\right)^2 = u^2 + v^2.$$

(a) We have $$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$
$$= v\frac{\partial f}{\partial x} + u\frac{\partial f}{\partial y}.$$

$$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$
$$= u\frac{\partial f}{\partial x} + (-v)\frac{\partial f}{\partial y}.$$

$$\frac{\partial^2 g}{\partial v \partial x} = \frac{\partial f}{\partial x} + v\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial v}\right)$$
$$= \frac{\partial f}{\partial x} + vu\frac{\partial^2 f}{\partial x^2} - v^2\frac{\partial^2 f}{\partial y \partial x}.$$

(b) $\|\nabla f(x,y)\|^2 = 2$

$$\frac{\partial f}{\partial x}(x,y)^2 + \frac{\partial f}{\partial y}(x,y)^2 = 2$$

Now $a\left(\frac{\partial g}{\partial u}\right)^2 + b\left(\frac{\partial g}{\partial v}\right)^2$

$$= a\left(v^2\left(\frac{\partial f}{\partial x}\right)^2 + u^2\left(\frac{\partial f}{\partial y}\right)^2 + 2uv\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right)$$
$$- b\left(u^2\left(\frac{\partial f}{\partial x}\right)^2 + v^2\left(\frac{\partial f}{\partial y}\right)^2 - 2uv\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right)$$
$$= \left(\frac{\partial f}{\partial x}\right)^2(av^2 - bu^2) + \left(\frac{\partial f}{\partial y}\right)^2(au^2 - bv^2) + 2uv\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}(a+b)$$

Hence required value for a and b are $a = \frac{1}{2}$, $b = \frac{-1}{2}$.

Ex. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. If $\vec{A}$ and $\vec{B}$ are constant vector show that

(a) $\vec{A} \cdot \nabla\left(\frac{1}{r}\right) = -\frac{\vec{A} \cdot \vec{r}}{r^3}$

(b) $\vec{B} \cdot \nabla\left(\vec{A} \cdot \nabla\left(\frac{1}{r}\right)\right) = \frac{3\vec{A} \cdot \vec{r}\ \vec{B} \cdot \vec{r}}{r^5} - \frac{\vec{A} \cdot \vec{B}}{r^3}$.

(a) We know if $f(x) = \|x\|^2$ then

$$f'(x)(y) = 2\langle x, y \rangle.$$

Let $\alpha : \mathbb{R}_{>0} \to \mathbb{R}$

$$t \mapsto t^{-\frac{1}{2}}$$

Then if $\varphi : \vec{r} \mapsto \frac{1}{\|r\|}$ then $\varphi(\vec{r}) = \alpha(f(\vec{r}))$

Thus $\varphi'(\vec{r})(\vec{y}) = \alpha'(f(\vec{r})) \cdot f'(\vec{r})(y)$

$$= -\frac{1}{2} \|r\|^{-3} \cdot 2\langle \vec{r}, \vec{y} \rangle$$

$$= -\frac{1}{\|r\|^3} \langle \vec{r}, \vec{y} \rangle$$

Now $\nabla \varphi(\vec{r}) = (\varphi'(\vec{r})(e_1), \varphi'(r)(e_2), \varphi'(r)(e_3))$

$$= -\frac{1}{\|\vec{r}\|^3}(x, y, z) \qquad \vec{r} = (x, y, z)$$

Thus $\vec{A} \cdot \nabla \varphi(\vec{r}) = -\frac{\vec{A} \cdot \vec{r}}{\|\vec{r}\|^3}$

(b) Consider the linear map

$$B : \mathbb{R}^3 \to \mathbb{R}$$
$$\vec{r} \mapsto \vec{A}\cdot\vec{r}$$

Let $\alpha : \mathbb{R}_{>0} \to \mathbb{R}$
$$t \mapsto -t^{-3}$$

Thus $T : \vec{r} \mapsto A\cdot\nabla\left(\frac{1}{\|\vec{r}\|}\right)$ then

$$T(\vec{r}) = \alpha(\|r\|)\,B(\vec{r})$$

Then $T'(\vec{r}) = \alpha'(\|r\|)B(\vec{r}) + \alpha(\|r\|)B'(\vec{r})$

$$= 3\|r\|^{-4} \underbrace{\frac{r}{\|r\|}}_{\text{derivative of } r \mapsto \|r\|} \vec{A}\cdot\vec{r} +$$

$$\left(-\frac{1}{\|r\|^3}\ \cancel{\frac{r}{\|r\|}} \underbrace{\langle \vec{A}, \cdot \rangle}_{\text{derivative of } B}\right)$$

Hence $T'(\vec{r})\cdot\vec{B} = \frac{3}{\|r\|^5}\,\vec{A}\cdot\vec{r}\ \vec{r}\cdot\vec{B}$

$$-\frac{1}{\|r\|^3}\,\vec{A}\cdot\vec{B}$$

(e3)

(,y,z)

Inverse Function Thm / Implicit Function Thm

IFT: Let $f : E\,(\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ be a cont. diff. function and $x_0 \in E^\circ$. Let $f'(x_0)$ is nonsingular. Then $\exists$ a nbd. U of x_0, V of $f(x_0)$ s.t. $f : U \to V$ is one one onto and $f^{-1} : V \to U$ is diff at $f(x_0)$ with $(f^{-1})'(f(x_0)) = f'(x_0)^{-1}$.

proof: W.log assume $x_0 = 0 = f(x_0)$ and $f'(x_0) = I_{n\times n}$

Infact otherwise define

$$T_1(x) = x + x_0,\ T_2(x) = x - f(x_0)\ \&\ T_3(x) = f'(x_0)^{-1}$$

and define $f_1(x) = T_3(T_2(f(T_1(x))))$

Then $f_1(0) = 0$ and

$$f_1'(0) = T_3'(T_2 f T_1(0))(T_2 f T_1)'(0)$$
$$= T_3(T_2 f T_1(0))\, T_2'(f T_1(0))(f T_1)'(0)$$
$$= f'(x_0)^{-1} \circ I \circ f'(x_0) \circ I = I.$$

The last identity follows from

$$T_3'(x) = f'(x_0)^{-1} \quad \forall x$$
$$T_2'(x) = I = T_1'(x) \quad \forall x.$$

Again $f_1^{-1} = T_1^{-1} f^{-1} T_2^{-1} T_3^{-1}$

where $T_1^{-1}(x) = x - x_0$

$T_2^{-1}(x) = x + f(x_0)$

$T_3^{-1}(x) = f'(x_0)\, x$

Let us assume $(f_1^{-1})'(0) = I$

Now $(f_1^{-1})'(0) = (T_1^{-1})'(f^{-1}T_2^{-1}T_3^{-1}(0))(f^{-1}T_2^{-1}T_3^{-1})'(0)$

$= I \circ (f^{-1}T_2^{-1}T_3^{-1})'(0)$

$= (f^{-1})'(T_2^{-1}T_3^{-1}(0))(T_2^{-1}T_3^{-1})'(0)$

$= (f^{-1})'(f(x_0))(T_2^{-1})'(T_3^{-1}(0))(T_3^{-1})'(0)$

$= (f^{-1})'(f(x_0)) \circ I \circ f'(x_0)$

Thus $(f^{-1})'(f(x_0)) = f'(x_0)^{-1}$.

Hence we can assume $x_0 = 0 = f(x_0)$ and $f'(0) = I$.

It remain to prove $(f^{-1})'(0) = I$. and to find such open sets U and V nbd. of 0

Let us define $g(x) = f(x) - x$.

Clearly $g(0) = 0$ and $\frac{\partial g}{\partial x_i}(0) = f'(0)(e_i) - I(e_i)$
$= 0$
for $1 \le i \le n$.

Since f is cont diff at 0. $\exists r > 0$ s.t.
$\| \frac{\partial g}{\partial x_i}(x) - 0 \| < \frac{1}{2n^2}$ whenever $x \in B(0,r)$.

Now for any $\vec{v} \in \mathbb{R}^n$ and $\vec{x} \in B(0,r)$

$$\begin{aligned}\| D_v g(x) \| &= \| \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(\vec{x}) v_i \| \\ &\le \sum_{i=1}^{n} |v_i| \, \| \frac{\partial g}{\partial x_i}(\vec{x}) \| \\ &\le \|\vec{v}\| \sum_{i=1}^{n} \| \frac{\partial g}{\partial x_i}(\vec{x}) \| \\ &< \frac{1}{2n} \|\vec{v}\| \qquad \because \|\frac{\partial g}{\partial x_i}(x)\| < \frac{1}{2n^2} \ \forall i\end{aligned}$$

Consider the function $G(t) = g(x+t(y-x)) : [0,1] \to \mathbb{R}^n$

By FTC we have

$$g(y) - g(x) = \int_0^1 \frac{d}{dt} g(x+t(y-x))\, dt$$

$G(1) - G($

By Chain Rule we have

$$\frac{d}{dt} g(x+t(y-x)) = D_{y-x}\, g(x+t(y-x))$$

Now if $D_{y-x}\, g(x+t(y-x)) = (z_1(t), \ldots, z_n(t))$

then $|z_i(t)| \leq \|(z_1(t), \ldots, z_n(t))\| < \frac{1}{2n}\|y-x\|$

Thus we have $\|g(y) - g(x)\| < \frac{1}{2}\|y-x\|, \forall x, y \in B(0,r)$

By the Lemma (stated at the end of the proof) $f(x) = g(x) + x$ is one one on $B(0,r)$ and $f(B(0,r)) \supseteq B(0, \frac{r}{2})$.

Let $U = f^{-1}(B(0,\frac{r}{2}))$ and $V = B(0, \frac{r}{2})$.

clearly both U and V are open and

$f : U \to V$ is a bijection.

It remains to prove $f^{-1} : V \to U$ is diff at $f(0) = 0$

Hence we have to prove

$$\lim_{\substack{y \to 0 \\ y \in V}} \frac{\|f^{-1}(0+y) - f^{-1}(0) - I(y-0)\|}{\|y\|} = 0$$

ie $$\lim_{\substack{y \to 0 \\ y \in V}} \frac{\|f^{-1}(y) - I(y)\|}{\|y\|} = 0$$

ie $$\lim_{\substack{y \to 0 \\ y \in V}} \frac{\|f^{-1}(y) - y\|}{\|y\|} = 0$$

Now we have $\|g(x)\| < \frac{1}{2}\|x\|$

ie $\|f(x) - x\| < \frac{1}{2}\|x\|$

ie $\frac{1}{2}\|x\| \le \|f(x)\| \le \frac{3}{2}\|x\|$

Hence by choosing a seqⁿ: $(y_n) \subseteq V$ with $y_n \to 0$ it is enough to prove

$$\lim_{n\to\infty} \frac{\|f^{-1}(y_n) - y_n\|}{\|y_n\|} = 0$$

If $x_n = f^{-1}(y_n)$ then $x_n \to 0$ and also

$$\frac{1}{2} \le \frac{\|y_n\|}{\|x_n\|} \le \frac{3}{2}.$$

Now
$$\frac{\|f^{-1}(y_n) - y_n\|}{\|y_n\|} = \frac{\|x_n - f(x_n)\|}{\|y_n\|}$$
$$= \frac{\|f(x_n) - f(0) - I(x_n)\|}{\|x_n\|} \frac{\|x_n\|}{\|y_n\|}$$
$$\le 2 \frac{\|\quad\|}{\|\ \|}$$

$\to 0$ as $n \to \infty$, follows from f is diff at 0.

Lemma: Let $B(0,r)$ be a ball (open) in $\mathbb{R}^n$ and let $g: B(0,r) \to \mathbb{R}^n$ be a map s.t. $g(0) = 0$ and $\|g(x) - g(y)\| \le \frac{1}{2}\|x - y\|$ for all $x, y \in B(0,r)$. Then the function $f: B(0,r) \to \mathbb{R}^n$ defined by $f(x) = x + g(x)$ is one to one and the image $f(B(0,r))$ of this map contains the ball $B(0, \frac{r}{2})$.

Ref: Analysis II (by T. Tao) Pg 499.

Some Applications of IFT

1. Show that the conclusion of IFT not necessarily true if the function is not cont. diff.

Consider the function $f(x) = x + x\sin\frac{1}{x}$ when $x \neq 0$ and $f(x) = 0$ when $x = 0$.

Then $f'(x) = 1 + \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}$ when $x \neq 0$

and $f'(0) = 1$.

Now f' is not cont at 0. Near $x = 0$ the function is unbounded.

We now show that the conclusion of IFT is not true for this f at $x = 0$

If we choose any nbd. of 0, the function f is increasing and decreasing both, hence f can not be invertible in any nbd. of $f(0) = 0$. In fact f^{-1} exists on some nbd. of $f(0)$ implies f can not take same value at two points in that nbd. and hence in particular it can not be increasing and decreasing both.

2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be cont. diff with $f'(x)$ is non singular for all $x \in \mathbb{R}^n$. Show that f is open mapping.

Let $U \subseteq \mathbb{R}^n$ be open and $V = f(U)$.

Let $v \in V$ and choose $u \in U$ s.t. $f(u) = v$.
Since $f'(u)$ is invertible get nbd $U_1 \ni u$ and $V_1 \ni v$ s.t. $f|_{U_1} : U_1 \to V_1$ is invertible and $V_1 = f(U_1)$. Since $U_1 \subseteq U$ we have $V_1 \subseteq f(U) = V$
Hence v is an interior point.

Implicit Function Thm: Let E be an open subset of $\mathbb{R}^2$, let $f : E \to \mathbb{R}$ be cont diff and let $y_0 = (y_1, y_2)$ be a point in E s.t. $f(y_0) = 0$ and $\frac{\partial f}{\partial y}(y_0) \neq 0$. Then $\exists$ an open subset U of $\mathbb{R}$ containing y_1 an open subset V of E containing $\cancel{y_2}$ y_0 and a function $g : U \to \mathbb{R}$ s.t. $g(y_1, \cancel{y_2}) = y_2$ and
$\{x \in V : f(x) = 0\} = \{(y, g(y)) : y \in U\}$
In otherwords the set $\{x \in V : f(x) = 0\}$ is a graph of a function over U. Moreover g is differentiable at y_1 and we have $\frac{dg}{dx}(y_1) =$
$$\frac{\partial f}{\partial x}(y_0) \Big/ \frac{\partial f}{\partial y}(y_0)$$

__proof__: Let us define $F : E \to \mathbb{R}^2$
$$F(x, y) = (x, f(x, y))$$
then $F(y_1, y_2) = (y_1, 0)$ and $F'(y_0) = \begin{pmatrix} 1 & 0 \\ \frac{\partial f}{\partial x}(y_0) & \frac{\partial f}{\partial y}(y_0) \end{pmatrix}$

From the assumption on $\frac{\partial f}{\partial y}(y_0)$ we have $F'(y_0)$ is invertible.

By IFT $\exists$ open set V of E containing $\vec{y}_0$ and open set G of $\mathbb{R}^2$ containing $(y_1, 0)$ s.t. $F : V \to G$ is 1-1 and onto.

clearly $F^{-1} : G \to V$ is also a bijection.

Let $F^{-1}(u,v) = (h_1(u,v), h_2(u,v))$

where $h_i : G \to \mathbb{R}$.

Now $F(F^{-1}(u,v)) = (u,v)$

ie $F(h_1(u,v), h_2(u,v)) = (u,v)$

ie $(h_1(u,v), f(h_1(u,v), h_2(u,v))) = (u,v)$

ie $h_1(u,v) = u$ & $f(h_1(u,v), h_2(u,v)) = v$

ie $f(u, h_2(u,v)) = v$

Define $U = \{ s \in \mathbb{R} : (s,0) \in G \}$ then ~~G~~ U is open and $y_1 \in U$. and define $g(s) = h_2(s,0)$

We now show that,

$$\{(x,y) \in V : f(x,y) = 0\} = \{(x, g(x)) : x \in U\} \quad *$$

Let $x \in U$ then $(x,0) \in G$

Hence $F^{-1}((x,0)) \in V$

ie $(h_1(x,0), h_2(x,0)) \in V$

ie $(x, h_2(x,0)) \in V$

ie $(x, g(x)) \in V$

Now it remains to show $f(x, g(x)) = 0$

But $f(x, g(x)) = f(x, h_2(x,0)) = 0$

Completes the proof of RHS $\subseteq$ LHS.

Let $(x,y) \in$ LHS then $F(x,y) \in G_2$

ie $(x, f(x,y)) \in G_2$

ie $(x,0) \in G_2$

ie $x \in U$

It remains to show $(x,y) = (x, g(x))$

Now $g(x) = h_2(x,0)$ and finally

$$(x,y) = F^{-1}((x,0)) = (h_1(x,0), h_2(x,0))$$
$$= (x, g(x))$$

Which completes the proof of $(*)$. It is not difficult to observe $g(x_1) = y_2$

Now we have $\frac{\partial f}{\partial x}(y_0) = 0$

ie $\frac{\partial f}{\partial x}(y_0) + \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}(y_0) = 0$

ie $\frac{\partial g}{\partial x}(y_0) = -\frac{\partial f}{\partial x}(y_0) \Big/ \frac{\partial f}{\partial y}(y_0)$.

Remark: IFT says that in V, f has a graphical representation viz. it is graph of g. To get a graphical representation of an arb. function $\{x \in U : f(x) = 0\}$ we can approach the 'vertical line test'. Consider the function $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ then for every x $\nexists$ unique y s.t. (x,y) is on the graph (or (x,y) satisfies the above condition) But if we restrict ourselves on $\{(x,y) : x^2 + y^2 = 1, ~~x~~ y \geq 0\}$ then it satisfies the above cond^n.

A slight modification of the above IFT lead to the following result:

<u>Implicit function Thm</u>: Let E be an open subset of $\mathbb{R}^n$, let $f: E \to \mathbb{R}$ be continuously diff and let $\vec{y_0} = (y_1, \dots, y_n)$ be a point in E s.t. $f(\vec{y_0}) = 0$ and $\frac{\partial f}{\partial x_n}(\vec{y_0}) \neq 0$. Then $\exists$ open subset U of $\mathbb{R}^{n-1} \ni (y_1, \dots, y_{n-1})$ and open subset V of $E \ni \vec{y_0}$ and $g: U \longrightarrow \mathbb{R}$ s.t. $g(y_1, \dots, y_{n-1}) = y_n$ and

$$\{(x_1, \dots, x_n) \in V : f(x_1, \dots, x_n) = 0\} = \{(x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1})) : (x_1, \dots, x_{n-1}) \in U\}$$

Finally $\frac{\partial g}{\partial x_j}(y_1, \dots, y_{n-1}) = -\frac{\partial f}{\partial x_j}(\vec{y_0}) \Big/ \frac{\partial f}{\partial x_n}(\vec{y_0})$

Maxima / Minima

Defn: Let $f(x,y)$ be a function of two variables. We say that (x_0,y_0) is a local minimum point of f if $\exists$ a disc D about (x_0,y_0) s.t. $f(x,y) \geq f(x_0,y_0)$ for all $(x,y) \in D$.

A similar definition can be defined for local maximum.

A point is either local maximum or minimum point is called a local extremum.

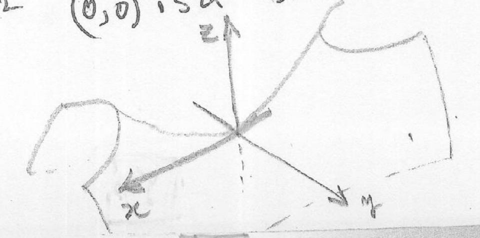
Critical point : A point (x_0,y_0) is a critical point of $f(x,y)$ if $\frac{\partial f}{\partial x}(x_0,y_0) = 0 = \frac{\partial f}{\partial y}(x_0,$

Thus the tangent plane of $z = f(x,y)$ at (x_0,y_0) is horizontal.

Remark: It is an easy consequence that if (x_0,y_0) is a local extremum of f and the partial derivatives of f exist at (x_0,y_0) then (x_0,y_0) is a critical point.

Similar to single variable case there exist another type of critical point viz. Saddle point.

Ex: $z = x^2 - y^2$ $(0,0)$ is a saddle point.



Absolute maxima/minima : If $x_1, \ldots, x_n$ are the critical point for f on an interval $[a, b]$ then the maximum and minimum value of $f(a), f(x_1), \ldots, f(x_n), f(b)$ represents the absolute minima/maxima.

Ex 1) Find the minimum distance from the point $(1,2,0)$ to the cone $z^2 = x^2 + y^2$

2) Write the number 120 as a sum of three numbers so that the sum of the product of two taken at a time is maximum.

3) Find the point nearest to the origin in the plane $2x - y + 2z = 20$.

Taylor's thm for a function of two variable

Statement: If $f(x,y)$ possess cont. partial derivatives of the nth order in any nbd of any point (a,b) and if $(a+h, b+k)$ be any point of this nbd. then there exists a positive number θ $0 < \theta < 1$, s.t.

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + \cdots + \frac{1}{(n-1)!}\left(\quad \right)^{n-1} f(a,b) + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n \cancel{f(a,b)}\ f(a+\theta h, b+\theta k)$$

Pf: Write $z = f(x,y)$ and we put

$x = a+ht$; $y = b+kt$ so that z becomes a function of t alone

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) z$$

Thus $\frac{d}{dt} = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$

Again $\frac{d^2 z}{dt^2} = \frac{d}{dt}\left(\frac{dz}{dt}\right)$

$$= \frac{d}{dt}\left\{h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right\}(z)$$

$$= h\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right) + k\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right)$$

$$= h\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right) + k(\quad)\left(\frac{\partial z}{\partial y}\right)$$

$$= h^2\frac{\partial^2 z}{\partial x^2} + 2hk\frac{\partial^2 z}{\partial x \partial y} + k^2\frac{\partial^2 z}{\partial y^2}$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 z$$

ie $\frac{d^2}{dt^2} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2$

Similarly, $\frac{d^n}{dt^n} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n$

Now write $f(x,y) = f(a+ht, b+kt) = F(t)$

Being a function of single variable there exist $\theta \in (0,1)$ s.t.

$$F(t) = F(0) + tF'(0) + \cdots + \frac{t^{n-1}}{(n-1)!}F^{(n-1)}(0) + \frac{t^n}{n!}F^{(n)}(\theta t)$$

Considering $t=1$ and putting the values of

$$F^{(n)}(t) = \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x,y)$$

$$F'(0) = \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right) f(a,b)$$

$$F''(0) = \frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)$$

$$F^{(n-1)}(0) = \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a,b)$$

$$F^{(n)}\theta = \frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k)$$

we get the result.

Taylor's Formula near a Critical point

If (x_0, y_0) is a critical point for $z=f(x,y)$ then

$$f(x,y) = f(x_0,y_0) + A(x-x_0)^2 + 2B(x-x_0)(y-y_0) + C(y-y_0)^2 + R_2$$

with $R_2/\|h\|^2 \to 0$ as $\|h\| \to 0$, where $h=(x-x_0, y-y_0)$ and A, B, C are following constants.

$$A = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0,y_0),\quad B = \frac{1}{2}\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0),\quad C = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x_0,y_0)$$

If $(x_0,y_0)=(0,0)$, Taylor's formula becomes

$$f(x,y) = f(0,0) + Ax^2 + 2Bxy + Cy^2 + R_2$$

Since $f(0,0)$ is a constant and R_2 is small the quadratic formula $f(0,0)+Ax^2+2Bxy+Cy^2$ closely approximates $z=f(x,y)$ if (x,y) is near to $(0,0)$.

Thus the shape of the graph $z = f(x,y)$ should be close to the shape of the graph of $z = Ax^2 + 2Bxy + Cy^2$ for (x,y) close to $(0,0)$.

We now determine whether $(0,0)$ is a local maxima/minima or saddle point for $g(x,y)$ where $z = g(x,y) = Ax^2 + 2Bxy + Cy^2$.

$(0,0)$ is a critical pt. for $g(x,y)$ since

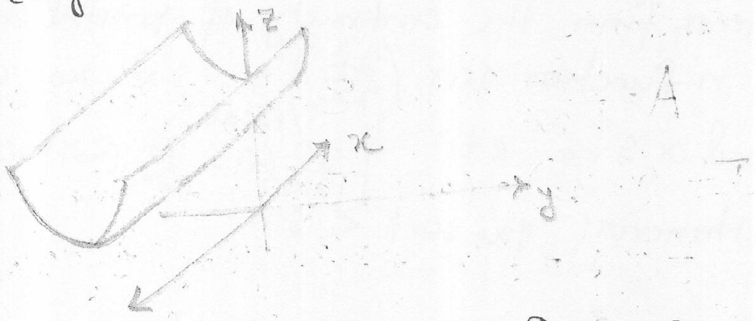
$$\frac{\partial g}{\partial x}(0,0) = 0 = \frac{\partial g}{\partial y}(0,0).$$

Since $\frac{\partial g}{\partial x} = 2Ax + 2By$

$$\frac{\partial g}{\partial y} = 2Bx + 2Cy$$

Hence if $AC - B^2 \neq 0$ then $(0,0)$ is the only critical point. Case for $AC - B^2 = 0$ is inconclusive

<u>Ex</u>: $z = y^2 + 1$ then $A = 0 = B$, $C = 1$



$\forall r \in \mathbb{R}$, $(r,0,1)$ is a minima for $z = f(x,y)$.

We'll assume that $AC - B^2 \neq 0$

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Case 1 : When $AC - B^2 > 0$

Now $A \neq 0 \neq C$ hence

$$g(x,y) = A\left(x^2 + \frac{2B}{A}xy + \frac{C}{A}y^2\right)$$

$$= A\left(x^2 + \frac{2B}{A}xy + \frac{B^2}{A^2}y^2 - \frac{B^2}{A^2}y^2 + \frac{C}{A}y^2\right)$$

$$= A\left(x + \frac{B}{A}y\right)^2 + \frac{AC - B^2}{A}y^2$$

Case 1.1 : When $A > 0$

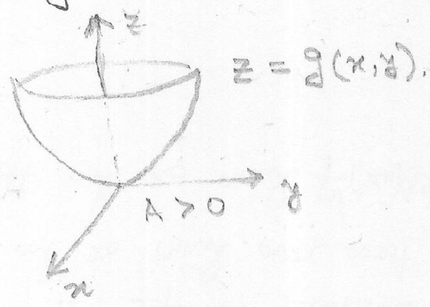
Then $C > 0$ (as $AC - B^2 > 0$) and $g(x,y) \geq 0$. and is equal to zero when both the summands are zero.

ie $(AC - B^2)y^2/A = 0$ ie $y = 0$ and

$$0 = A\left(x + \frac{B}{A}y\right)^2 = Ax^2$$

ie $x = 0$

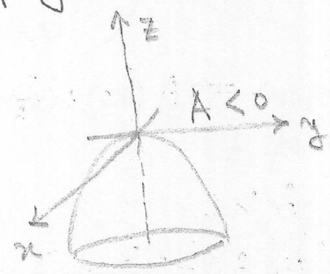
Thus $g(x,y) \geq 0$ and equals zero only when $(x,y) = (0,0)$. Thus $(0,0)$ is a minimum for g. ~~Here~~ In this case the graph of g looks like the following:



Case 1.2 : When $A < 0$

Then $-C < 0$ and we have $g(x,y) \le 0$ and it is 0 when $(x,y) = (0,0)$.

Thus $(0,0)$ is a maximum and the graph of g will look like the following



Case 2 : When $AC - B^2 < 0$

Case 2.1 : When $A \neq 0$.

Now $g(x,y) = A\left(x + \frac{B}{A}y\right)^2 + \frac{AC-B^2}{A}y^2$ is still valid.

If $A < 0$ then the first term in $g(x,y) \le 0$ but the second term satisfies $\frac{AC-B^2}{A}y^2 \ge 0$

If $x \neq 0$ then $g(x,0) = Ax^2 < 0$ and on the line $-By/A = x$,

$$g\left(-\frac{B}{A}y, y\right) = \frac{AC-B^2}{A}y^2 > 0$$

Since $g(x,y)$ can be both positive and negative in any nbd. of $(0,0)$, $(0,0)$ is a saddle pt. for f.

If $A > 0$ then on the line $-\frac{B}{A} y = x$

$g(x, y) < 0$ and $g(x, 0) > 0$

Again we have that $(0,0)$ is a saddle point for g.

<u>Case 2.2</u> : When $A = 0$

Then $g(x, y) = 2Bxy + Cy^2$

$$= y(2Bx + Cy)$$

Which can be both positive and negative if the signs of y and $2Bx+Cy$ same or opposite.

The Shape of graphs of Quadratic funs.

$g(x, y) = Ax^2 + 2Bxy + Cy^2$

1. If $AC - B^2 > 0$ and if

   (a) $A > 0$ then $(0,0)$ is a local minima

   (b) $A < 0$ then $(0,0)$ is a local maxima

2. If $AC - B^2 < 0$ then the graph of g looks like a saddle point.

<u>Ex</u> 1. Analyze the critical point at the origin for $g(x,y) = x^2 + 3xy + y^2$

2. Determine whether $(0,0)$ is a maximam point, a minimum point or neigther of $g(x,y) = 3x^2 - 5xy + 3y^2$.

If we now consider more general expression $g(x,y) = K + A(x-x_0)^2 + 2B(x-x_0)(y-y_0)^2 + C(y-y_0)^2$

where K, A, B, C are constants, then we can draw similar conclusions.

The result can now be used to give a second derivative test.

Let $z = f(x,y)$ have cont. partial derivatives upto second order and (x_0, y_0) is a critical point. By Taylor's thm we can write

$$f(x,y) = f(x_0,y_0) + A(x-x_0)^2 + 2B(x-x_0)(y-y_0) + C(y-y_0)^2 + R_2$$

where R_2 is very small near (x_0, y_0).

and $A = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x_0,y_0)$

$$B = \frac{1}{2}\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0), \quad C = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x_0,y_0)$$

For small values of R_2 the shape of $f(x,y)$ is close to

$$f(x_0,y_0) + A(x-x_0)^2 + 2B(x-x_0)(y-y_0) + C(y-y_0)^2$$

Using the values of A, B, C given by Taylor's thm we arrive at the following second derivative test.

Let $z = f(x,y)$ have continuous partial derivatives upto second order and suppose (x_0, y_0) is a crititical pt. of f. Consider the expression $D = \frac{\partial^2 f}{\partial x^2}(x_0,y_0)\,\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)\right)^2$

D is said to the discriminant of f at (x_0, y_0).

1. If $D > 0$ and $\frac{\partial^2 f}{\partial x^2}(x_0,y_0) > 0$ then (x_0,y_0) is a local minima.
2. If $D > \cancel{<} 0$ and $\frac{\partial^2 f}{\partial x^2}(x_0,y_0) < 0$ then (x_0,y_0) is a local maxima.
3. If $D < 0$ then (x_0,y_0) is a saddle pt.

If $D = 0$ then this test is inconclusive.

<u>Ex</u>. Investigate the extreme values of $f(x,y) = 2(x-y)^2 - x^4 - y^4$.

$\frac{\partial f}{\partial x} = 4x - 4y - 4x^3$ & $\frac{\partial f}{\partial y} = -4x + 4y - 4y^3$

Since in a critical point $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$

we have $x^3 + y^3 = 0$

ie $(x+y)(x^2 - xy + y^2) = 0$

Let $x - y - x^3 = 0$

$x + y = 0$

ie $2x - x^3 = 0$

ie $x(2 - x^2) = 0$

ie $x = 0, \pm\sqrt{2}$

Hence the critical points are
$(0,0), (\sqrt{2},-\sqrt{2}), (-\sqrt{2},\sqrt{2})$.

Let $x-y-x^3=0$ & $x^2-xy+y^2=0$.
then the real roots are $x=0=y$.

Now $D = f_{xx} f_{yy} - (f_{xy})^2 = 0$ at $(x,y)=(0,0)$.

$$\frac{\partial^2 f}{\partial x^2} = 4-12x^2 \; ; \; \frac{\partial^2 f}{\partial y^2} = 4-12y^2.$$

$$\frac{\partial^2 f}{\partial x \partial y} = -4$$

at $(\sqrt{2},-\sqrt{2})$ $\quad D = (4-12\cdot 2)\cdot(4-12\cdot 2)-16$
$\quad = 20^2 - 16 > 0$

at $(-\sqrt{2},\sqrt{2})$ $\quad D = (4-12\cdot 2)\cdot(4-12\cdot 2) - 16^2 > 0$

In both cases $(x=\pm\sqrt{2})$ $\frac{\partial^2 f}{\partial x^2}(\pm\sqrt{2},\mp\sqrt{2})<0$

Hence f attains its local maximum at $(\pm\sqrt{2},\mp\sqrt{2})$.

Now consider the point $(0,0)$.

on $y=x$, $f(x,x) = -2x^4 < 0$

on $y=-x$ $f(x,-x) = (2x)^2 - 2x^4$
$\quad = 4x^2 - 2x^4 > 0$ if $|x|<1$

Hence $(0,0)$ is a saddle point.

Remark : $D = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} (x_0, y_0)$

The above matrix is called Hassian matrix of f at (x_0, y_0).

We now consider the functions of three variables. Let $f : \mathbb{R}^3 \to \mathbb{R}$ is a function of three variables s.t. all second order partial derivative exist at (a,b,c) then the Hassian matrix of f at (a,b,c) is

$$H = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$$

All the partial derivatives are evaluated at (a,b,c).

Defn : Let A be an $n \times n$ matrix and for each $1 \le r \le n$, let A_r be the $r \times r$ matrix formed from the first r rows and r columns of A. The determinants $\det(A_r)$, $1 \le r \le n$ are called the leading minors of A.

Thm : Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a suff. smooth function of three var. with a critical point (a,b,c) and Hassian H at (a,b,c). If $\det(H) \neq 0$ then (a,b,c) is

(1) Local maximum if $0 > \det(H_1)$, $0 < \det(H_2)$ & $0 > \det(H_3)$

(2) A local minimum if $0 < \det(H_1)$
$0 < \det(H_2)$ & $0 < \det(H_3)$

(3) A saddle point if neither of the above hold.

Where the partial derivatives are evaluated at (a,b,c).
If $\det(H) = 0$ then (a,b,c) can be either a local extremum or a saddle point.

Ex. Find and classify the stationary points of the following funs.

(1) $f(x,y,z) = x^4 + x^2y + y^2 + z^2 + xz + 1$

(2) $f(x,y,z) = 2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z$

(3) Show that $f(x,y,z) = (x+y+z)^3 - 3(x+y+z) - 24xyz + a^3$ has a minimum at $(1,1,1)$ and a maximum at $(-1,-1,-1)$.

Sol of problem 2 : We have

$$\frac{\partial f}{\partial x} = 2yz - 4z + 2x - 2$$

$$\frac{\partial f}{\partial y} = 2xz - 2z + 2y - 4$$

$$\frac{\partial f}{\partial z} = 2xy - 4x - 2y + 2z + 4$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \;;\; \frac{\partial^2 f}{\partial y^2} = 2 = \frac{\partial^2 f}{\partial z^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2z$$

$$\frac{\partial^2 f}{\partial y \partial z} = 2x-2$$

$$\frac{\partial^2 f}{\partial z \partial x} = 2y-4.$$

For critical point we have,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0.$$

thus

$$yz - 2z + x - 1 = 0$$
$$zx - z + y - 2 = 0$$
$$xy - 2x - y + z + 2 = 0$$

Adding last two we get

$$zx + xy - 2x = 0$$

or $x(y+z-2) = 0$

ie $x = 0$ or $y+z-2 = 0$

Hence the critical points are given by

$$yz - 2z + x - 1 = 0$$
$$zx - z + y - 2 = 0$$
$$x = 0$$

And

$$yz - 2z + x - 1 = 0$$
$$zx - z + y - 2 = 0$$
$$y + z - 2 = 0$$

Solving these we get $(0,3,1)$, $(0,1,-1)$, $(1,2,0)$, $(2,1,1)$, $(2,3,-1)$

at $(0,3,1)$, $H = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix}$

Its principal minors are

$$2, \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 0, \begin{vmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{vmatrix} = -32$$

Hence $(0,3,1)$ is a saddle point.