M 3F03 Assignment 1.

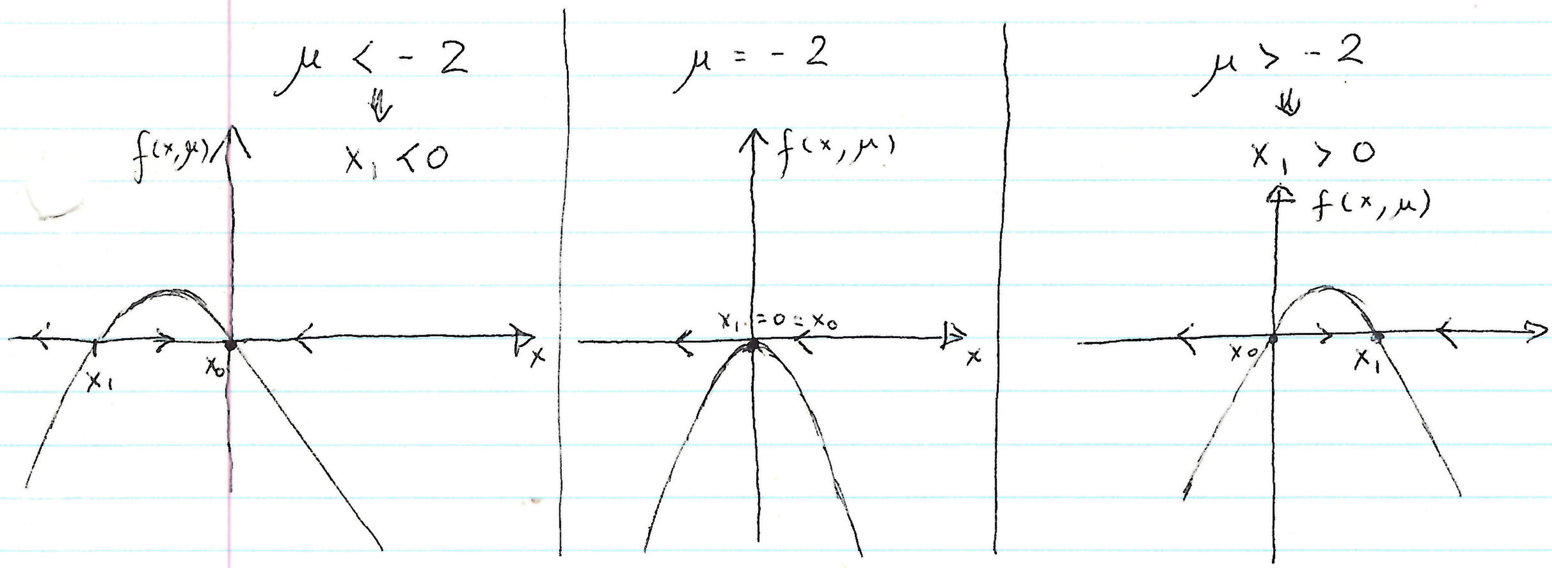
1= $x' = (2+\mu)x - 5x^2$, $\mu \in \mathbb{R}$.

Equilibrium Points: $0 = x' = (2+\mu)x - 5x^2 \Rightarrow x(2+\mu-5x) = 0$

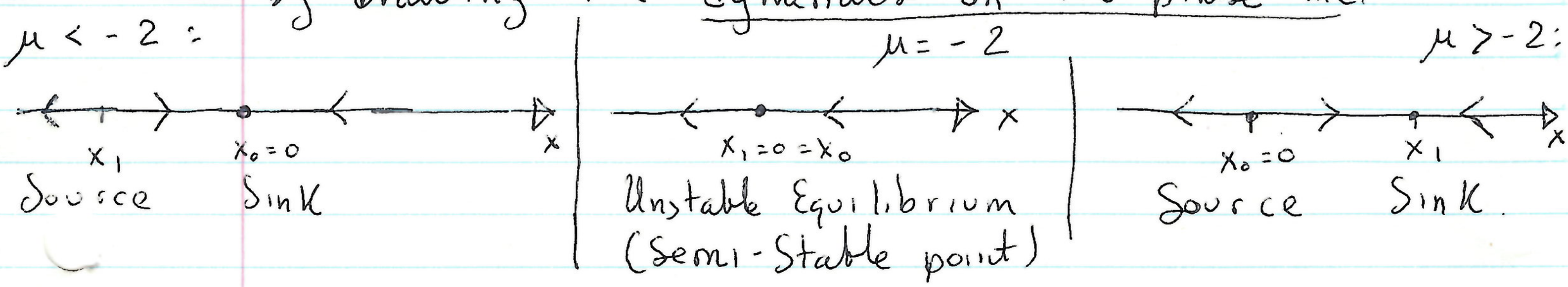
$\hookrightarrow$ $x_0 = 0$, $x_1 = \frac{2+\mu}{5}$.

μ value where bifurcation happens: at $\mu = -2$, $x_0 = 0 = x_1$; for $\mu \neq -2$, $x_1 \neq 0 = x_0$.

Moreover, $x' = f(x;\mu) = (2+\mu)x - 5x^2$, and plotting f:

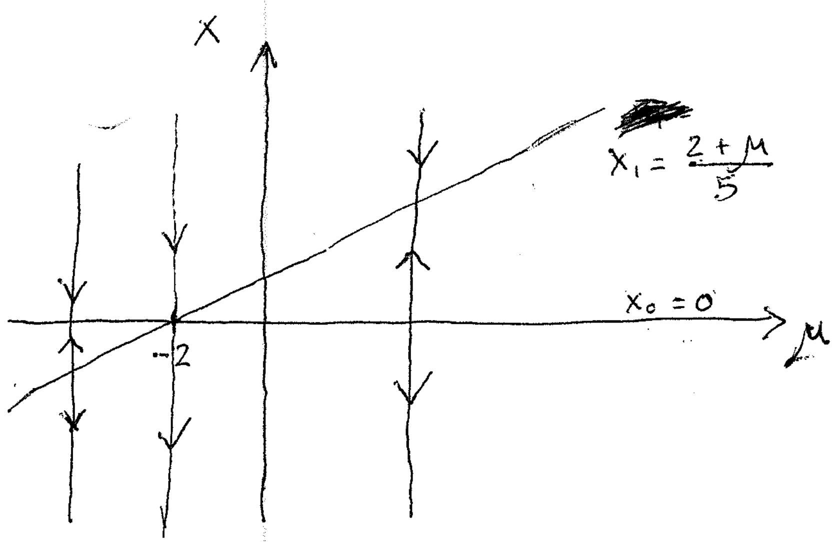


a bifurcation occurs at $\mu = -2$ [redacted] also because the behavior of $x_0 = 0$ changes there, as can be seen by drawing the <u>dynamics on the phase line</u>:



and the behavior of $x_1 = \frac{1}{5}(2+\mu)$ changes as μ goes from $\mu < -2$ to $\mu > -2$ from a source to a sink.

Bifurcation Diagram:



2. $x' = ax + 3$, $a \in \mathbb{R}$.

General Solution: i) If $a = 0$: $x' = 3 \Rightarrow$ given $x(0) = x_0$, $x(t) = x_0 + 3t$.

ii) If $a \neq 0$: $\int \frac{dx}{ax+3} = \int dt \Rightarrow \int \frac{a\,dx}{ax+3} = a\int dt$

Given the initial condition $x(0) = x_0$, then $(x_0 \neq -3/a)$:

$$\ln|ax+3| - \ln|ax_0+3| = at = \ln\left|\frac{ax+3}{ax_0+3}\right|$$

$$e^{at} = \left|\frac{ax+3}{ax_0+3}\right| = \frac{ax(t)+3}{ax_0+3} \quad \text{Same sign, so}$$

$$(*) \quad x(t) = -\frac{3}{a} + \left(x_0 + \frac{3}{a}\right)e^{at}.$$

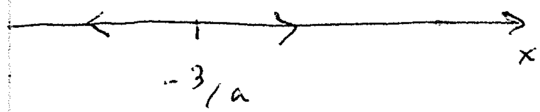
The initial condition $x(0) = -3/a$ is related to an equilibrium point, as:

Equilibrium Points: $0 = x' = ax + 3 \Rightarrow x = -3/a$, so $x(t) = -3/a \;\; \forall\, t \geq 0$, so the formula $(*)$ also holds for $x_0 = -3/a$.

Clearly for the other case $\underline{a = 0} \Rightarrow x' = 3$ there are no equilibrium points as $0 \neq x'$.

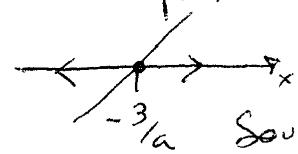
$x' = ax + 3 = f(x)$, $a \neq 0$: $\quad f(-3/a) = 0$.

u) $a > 0$: $\quad f(x) > 0$ for $x > -3/a$, $\quad f(x) < 0$ for $x < -3/a$



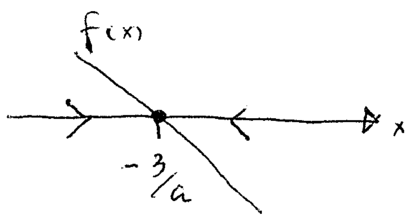
Source

$f'(x) = a \quad (a \neq 0)$, so $\quad a > 0 \Rightarrow$



Source

u) $a < 0$: $\quad f(x) < 0$ for $x > -3/a$, $\quad f(x) > 0$ for $x < -3/a$



Sink

$f'(x) = a < 0$.

3= a) $y' = y^4$, $\quad y(0) = 1$.

$$\Rightarrow \int y^{-4}\, dy = \int dt \Rightarrow \left.\frac{y^{-3}}{-3}\right|_{y(0)}^{y(t)} = t = \frac{y^{-3}(t) - 1}{-3}$$

$$y^{-3}(t) = 1 - 3t \Rightarrow y(t) = \left(\frac{1}{1-3t}\right)^{1/3}$$

as $t \to \underline{\underline{1/3 = T}}$, then $y(t) \xrightarrow[t \to 1/3]{} \infty$: it blows up.

b) $y' = y^2 + 3y + 2$, $\quad y(0) = 0$.

$$y' = (y+2)(y+1) \Rightarrow \int \frac{dy}{(y+2)(y+1)} = \int dt = \int \frac{dy}{y+1} - \int \frac{dy}{y+2}$$

$$\frac{1}{(y+2)(y+1)} = \frac{A}{y+2} + \frac{B}{y+1} = \frac{(A+B)y + (A+2B)}{(y+2)(y+1)} \Rightarrow \begin{array}{l} B = -A \\ B = 1, \ A = -1 \end{array}$$

$$\Rightarrow \quad t + c = \ln|y+1| - \ln|y+2| = \ln\left|\frac{y+1}{y+2}\right|$$

$$e^{c} e^{t} = \left|\frac{y+1}{y+2}\right| = \left|1 - \frac{1}{y+2}\right|, \quad y(0)=0 \Rightarrow e^{c} = \left|1-\tfrac{1}{2}\right| = \tfrac{1}{2}.$$

$$\frac{e^{t}}{2} = \left|1 - \frac{1}{y(t)+2}\right| = 1 - \frac{1}{y(t)+2} \Rightarrow y(t)+2 = \left(1 - \frac{e^{t}}{2}\right)^{-1}$$

When $t \to \underline{\ln 2 = T}$, then $y(t) \xrightarrow[t \to \ln 2]{} \infty$ blows up.

4 : a) $f(x) = \begin{cases} 1, & \text{if } x \le 1 \\ 2, & \text{if } x > 1. \end{cases}$ $\quad x' = f(x)$, $x(0) = 1$ $\quad$ (f discontinuous at 1)

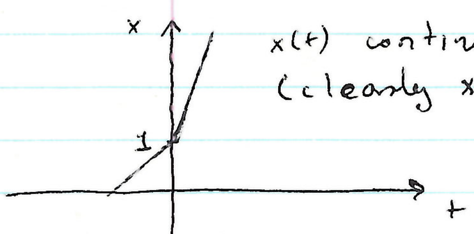
f is not continuous at the initial value 1, so we can't apply our Thm. of Existence & Uniqueness.

If there was a 'solution' to $x' = f(x)$, $x(t)$ can't be C^1 because $x' = f$ would be discontinuous at $x(0) = 1$.

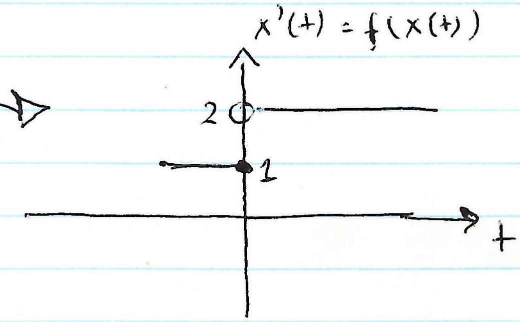
There exists a C^0 solution $x(t)$ for $t \geq 0$ (continuous only)

$$x(t) = \begin{cases} 1 + t, & t \le 0 \\ 1 + 2t, & t > 0 \end{cases} \Rightarrow x'(t) = \begin{cases} 1, & t \le 0 \\ 2, & t > 0 \end{cases}$$

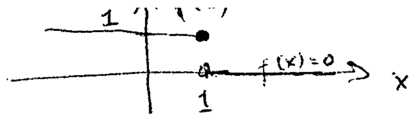
$(x(0) = 1)$



$x(t)$ continuous
(clearly x' discontinuous) $\longrightarrow$



so $x(t)$ is clearly not continuously differentiable at $t=0$.



b) $f(x) = \begin{cases} 1, & \text{if } x \le 1 \\ 0, & \text{if } x > 1 \end{cases}$, $\quad x' = f(x)$, $x(0) = 1$ $\quad$ (f discontinuous at 1)

Can we even find a solution for this IVP?
(continuous)

$f(x(0)) = f(1) = 1 > 0$ (increase)

but if $x(t) > 1$ for $t > 0$

then $f(x(t)) = 0$ for $t > 0$, so $x(t) = 1$ for $t \geq 0$

which is in contradiction with

$$\frac{dx}{dt} = f(x(t)) = f(1) = 1 > 0 \quad \text{for } t > 0.$$

So no continuous solution satisfying this IVP exists.

5 = $x'' + x = 0$, $\quad x(0) = 1$, $\quad x'(0) = 0$. $\quad x(t) = \cos t$.

This second order diff. eq. is equivalent to the first order system: $y = x'$, $y' = -x$, $x(0) = 1$, $y(0) = 0$,

with solution $x(t) = \cos t$, $y(t) = x'(t) = -\sin(t)$.

So:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \bar{X}' = \begin{pmatrix} y \\ -x \end{pmatrix} = F(x, y) = F(\bar{X}),$$

Solving this first order system by Picard iterations: $\bar{X} = (x, y)$.

$$\bar{X}_0 = \bar{X}(0) = (1, 0)$$

$$\bar{X}_{n+1}(t) = \bar{X}(0) + \int_0^t F(\bar{X}_n(s))\, ds, \quad n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}.$$

Expressing Picard's method component-wise:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} y_n(s) \\ -x_n(s) \end{pmatrix} ds = \begin{pmatrix} 1 + \int_0^t y_n(s)\,ds \\ -\int_0^t x_n(s)\,ds \end{pmatrix}; \quad n \geq 0.$$

$$x_1(t) = 1 + \int_0^t 0\,ds = 1 \qquad x_2(t) = 1 + \int_0^t -s\,ds = 1 - t^2/2$$

$$\Rightarrow \qquad\qquad \Rightarrow$$

$$y_1(t) = -\int_0^t 1 \cdot ds = -t \qquad y_2(t) = -\int_0^t 1 \cdot ds = -t$$

$$x_3(t) = 1 + \int_0^t -s\,ds = 1 - t^2/2$$

$$y_3(t) = -\int_0^t \left(1 - \frac{s^2}{2}\right) ds = -t + \frac{t^3}{3!}$$

We see the pattern $\longrightarrow$

$x_0 = 1$	$y_0 = 0$
$x_1 = 1$	$y_1(t) = -t$
$x_2(t) = 1 - t^2/2$	$y_2(t) = -t$
$x_3(t) = 1 - t^2/2$	$y_3(t) = -t + t^3/3!$
	$y_4(t) = -t + t^3/3!$

$$y_4(t) = -\int_0^t \left(1 - \frac{s^2}{2}\right) ds = -t + t^3/3!$$ (Remember Taylor Series for sin, cos):

Let's prove by induction that if

$$x_{2n} = \sum_{i=0}^{n} \frac{(-1)^i t^{2i}}{(2i)!}, \qquad y_{2n} = \sum_{i=0}^{n-1} \frac{(-1)^{i+1} t^{2i+1}}{(2i+1)!} = \sum_{i=1}^{n} \frac{(-1)^i t^{2i-1}}{(2i-1)!}, \ (n \geq 1)$$

$$x_{2n+1} = 1 + \int_0^t y_{2n}(s)\,ds = 1 + \int_0^t \sum_{i=1}^{n} \frac{(-1)^i s^{2i-1}}{(2i-1)!}\,ds = 1 + \sum_{i=1}^{n} \frac{(-1)^i}{(2i-1)!}\frac{t^{2i}}{(2i)}$$

$$= 1 + \sum_{i=1}^{n} \frac{(-1)^i t^{2i}}{(2i)!}.$$

$$y_{2n+1} = -\int_0^t x_{2n}(s)\,ds = -\int_0^t \sum_{i=0}^{n} \frac{(-1)^i s^{2i}}{(2i)!} = \sum_{i=0}^{n} \frac{(-1)^{i+1} t^{2i+1}}{(2i+1)!}$$

$$x_{2n+2} = 1 + \int_0^t y_{2n+1}(s)\,ds = 1 + \int_0^t \sum_{i=0}^{n} \frac{(-1)^{i+1} s^{2i+1}}{(2i+1)!} = 1 + \sum_{i=0}^{n} \frac{(-1)^{i+1} t^{2(i+1)}}{(2i+2)!} =$$

$$= \sum_{i=0}^{n+1} \frac{(-1)^i t^{2i}}{(2i)!} = x_{2(n+1)}$$ with the proposed expression above.

Ex 5 (Continued):

$$y_{2n+2} = -\int_0^t x_{2n+1}(s)\,ds = -\int_0^t \left[1 + \sum_{i=1}^{n} \frac{(-1)^i s^{2i}}{(2i)!} \right] ds =$$

$$= -t + \sum_{i=1}^{n} \frac{(-1)^{i+1} t^{2i+1}}{(2i+1)!} = \sum_{i=0}^{n} \frac{(-1)^{i+1} t^{2i+1}}{(2i+1)!} = y_{2(n+1)},$$

satisfying the formula proposed for y_{2n} in the induction claim above. So the recurrence is proved,

and $$x_{2n} = \sum_{i=0}^{n} \frac{(-1)^i t^{2i}}{(2i)!} \xrightarrow[n\to\infty]{} \sum_{i=0}^{\infty} \frac{(-1)^i t^{2i}}{(2i)!} = \cos t$$

$$y_{2n} = \sum_{i=0}^{n-1} \frac{(-1)^{i+1} t^{2i+1}}{(2i+1)!} \xrightarrow[n\to\infty]{} \sum_{i=0}^{\infty} \frac{(-1)^{i+1} t^{2i+1}}{(2i+1)!} = -\sum_{i=0}^{\infty} \frac{(-1)^i t^{2i+1}}{(2i+1)!} = y_\infty$$

so $y_\infty(t) = -\sin t$ and $x_\infty(t) = \cos t$.

by recognizing the Taylor series for $\cos t$ and $-\sin t$.

$\therefore$ $x_\infty(t) = \cos(t) = x(t)$

$y_\infty(t) = -\sin(t) = y(t) = x'(t)$

and the Picard iterative method has let us recover the exact solution of our problem.