Optimal Transport for Applied Mathematicians

—

Calculus of Variations, PDEs and Modelling

Filippo Santambrogio[1]

[1] Laboratoire de Mathématiques d'Orsay, Université Paris Sud, 91405 Orsay cedex, France
filippo.santambrogio@math.u-psud.fr, http://www.math.u-psud.fr/∼santambr.

Contents

Introduction

Why this book ?

blablabla

Introduction to optimal transport

The motivation for the whole subject is the following problem proposed by Monge in 1781 ([89]): given two densities of mass $f, g \geq 0$ on $\mathbb{R}^d$, with $\int f = \int g = 1$, find a map $T : \mathbb{R}^d \to \mathbb{R}^d$ pushing the first one onto the other, i.e. such that

$$\int_A g(x)dx = \int_{T^{-1}(A)} f(y)dy \quad \text{for any Borel subset } A \subset \mathbb{R}^d \tag{0.1}$$

and minimizing the quantity

$$M(T) := \int_{\mathbb{R}^d} |T(x) - x| f(x)dx$$

among all the maps satisfying this condition. This means that we have a collection of particles, distributed with density f on $\mathbb{R}^d$, that have to be moved, so that they arrange according to a new distribution, whose density is prescribed and is g. The movement has to be chosen so as to minimize the average displacement. The map T describes the movement (that we must choose in an optimal way), and $T(x)$ represents the destination of the particle originally located at x. The constraint on T precisely accounts for the fact that we need to reconstruct the density g. In the following, we will always define, similarly to (0.1), the image measure of a measure μ on X (measures will indeed replace the densities f and g in the most general formulation of the problem) through a measurable map $T : X \to Y$: it is the measure denoted by $T_\# \mu$ on Y and characterized by

$$T_\#\mu(A) = \mu(T^{-1}(A)) \quad \text{for every measurable set } A,$$
$$\text{or } \int_Y \phi \, d\,(T_\#\mu) = \int_X \phi \circ T \; d\mu \quad \text{for every measurable function } \phi.$$

It is easy – just by a change-of-variables formula – to transform the equality $\nu = T_\#\mu$ into the PDE $g(T(x)) = f(x)/|JT|(x)$, if we suppose f, g and T to be regular enough and T to be injective. J denotes here the determinant of the Jacobian matrix.
Yet, this equation is highly nonlinear in T and this is one of the difficulties preventing from an easy analysis of Monge's problem. For instance: how to prove existence of a minimizer? usually what one does is the following: take a minimizing sequence T_n, find a bound on it giving compactness in some topology (here, if the support of ν is compact the maps T_n take value in a common bounded set, $\operatorname{spt}\nu$, and so one can get compactness of T_n in the weak-* L^∞ convergence), take a limit $T_n \to T$ and prove that T is a minimizer. This requires semicontinuity of the functional M with respect to this convergence (which is ok for the weak-* L^∞ convergence: we need $T_n \rightharpoonup T \Rightarrow \liminf_n M(T_n) \geq M(T)$), but also proving that the limit T still satisfies the constraint. Yet, the nonlinearity of the PDE we mentioned above prevents from proving this stability when we only have weak convergence.

Hence, the problem of Monge has stayed with no solution (does a minimizer exist? how to characterize it?...) till the progress made in the 1940s. Indeed, only with the work by Kantorovich (1942) it has been inserted into a suitable framework which gave the possibility to approach it and, later, to find that solutions actually exist and to study them. The problem has been widely generalized, with very general cost functions $c(x, y)$ instead of the Euclidean distance $|x - y|$ and more general measures and spaces. For simplicity, here we will not try to present a very wide theory on generic metric spaces, manifolds and so on, but we will deal only with the Euclidean case.

Chapter 1

Primal and dual problems

In this chapter we will start with the generalities on the transport problem from a measure μ on a space X to another measure ν on a space Y. X and Y can be complete and separable metric spaces, but very soon we will focus on the case where they are the same subset $\Omega \subset \mathbb{R}^d$ (very often compact). The cost function $c : X \times Y \to [0, +\infty]$ will be possibly supposed to be continuous or semicontinuous, and then we will analyze particular cases (such as $c(x, y) = h(x - y)$ for convex or strictly convex h).

1.1 Kantorovich and Monge problems

The generalization that appears as natural from the work of Kantorovich ([80]) of the problem raised by Monge is the following:

Problem 1. Given two probability measures μ and ν on Ω and a cost function $c : X \times Y \to [0, +\infty]$ we consider the problem

$$(PK) \quad \min \left\{ K(\gamma) := \int_{X\times Y} c \, d\gamma \, | \gamma \in \Pi(\mu, \nu) \right\}, \tag{1.1}$$

where $\Pi(\mu, \nu)$ is the set of the so-called *transport plans*, i.e. $\Pi(\mu, \nu) = \{\gamma \in \mathcal{P}(X \times Y) : (\pi^x)_{\#}\gamma = \mu, (\pi^y)_{\#}\gamma = \nu, \}$ where π^x and π^y are the two projections of $X \times Y$ onto Ω. These probability measures over $X \times Y$ are an alternative way to describe the displacement of the particles of μ: instead of saying, for each x, which is the destination $T(x)$ of the particle originally located at x, we say for each pair (x, y) how many particles go from x to y. It is clear that this description allows for more general movements, since from a single point x particles can a priori move to different destinations

y. If multiple destinations really occur, then this movement cannot be described through a map T. Notice that the constraints on $(\pi^x)_\#\gamma$ and $(\pi^y)_\#\gamma$ exactly mean that we restrict our attention to the movements that really take particles distributed according to the distribution μ and move them onto the distribution ν.

The minimizers for this problem are called *optimal transport plans* between μ and ν. Should γ be of the form $(id \times T)_\#\mu$ for a measurable map $T : X \to Y$ (i.e. when no splitting of the mass occurs), the map T would be called *optimal transport map* from μ to ν.

Remark 1. It can be easily checked that if $(id \times T)_\#\mu$ belongs to $\Pi(\mu, \nu)$ then T pushes μ onto ν (i.e. $\nu(A) = \mu(T^{-1}(A))$ for any Borel set A) and the functional takes the form $\int c(x, T(x))\mu(dx)$, thus generalizing Monge's problem.

This generalized problem by Kantorovich is much easier to handle than the original one proposed by Monge: for instance in the Monge case we would need existence of at least a map T satisfying the constraints. This is not verified when $\mu = \delta_0$, if ν is not a single Dirac mass. On the contrary, there always exist transport plan in $\Pi(\mu, \nu)$ (for instance $\mu \otimes \nu \in \Pi(\mu, \nu)$). Moreover, one can state that (K) is the relaxation of the original problem by Monge: if one considers the problem in the same setting, where the competitors are transport plans, but sets the functional at $+\infty$ on all the plans that are not of the form $(id \times T)_\#\mu$, then one has a functional on $\Pi(\mu, \nu)$ whose relaxation is the functional in (K). (see below).

Anyway, it is important to notice that an easy use of the Direct Method of Calculus of Variations (i.e. taking a minimizing sequence, saying that it is compact in some topology - here it is the weak convergence of probability measures - finding a limit, and proving semicontinuity (or continuity) of the functional we minimize, so that the limit is a minimizer) proves that a minimum does exist.

Memo – *Weierstrass criterion for the existence of minimizers, semicontinuity*

Here is the most common way to prove that a function admits a minimizer. It is called "direct method in calculus of variations" but simply recalls in a more abstract way what is taught in first-year calculus courses about Weierstrass' Theorem, possibly replacing continuity with semicontinuity.

Definition: A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be lower-semicontinuous if for every sequence $x_n \to x$ we have $f(x) \leq \liminf_n f(x_n)$.

Definition: A metric space X is said to be compact if from any sequence x_n we can extract a converging subsequence $x_{n_k} \to x \in X$.

Theorem (Weierstrass): If $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous and X is compact, then there exists $\bar{x} \in X$ such that $f(\bar{x}) = \min\{f(x) : x \in X\}$.

Sketch of proof: Define $\ell \in \mathbb{R} \cup \{-\infty\}$ the infimum of $\{f(x) : x \in X\}$ (this infimum is not $+\infty$ unless f is identically $+\infty$, but in this case any point in X minimizes). By definition there exists a minimizing sequence x_n, i.e. a sequence of points in X such that $f(x_n) \to \ell$. By compactness, up to extracting a subsequence we can assume $x_n \to \bar{x}$ (since the extracted subsequence will still be minimizing, so we can replace the previous one with it). By lower semi-continuity, we have $f(\bar{x}) \leq \liminf_n f(x_n) = \ell = \inf\{f(x) : x \in X\}$. On the other hand, we have $f(\bar{x}) \geq \ell$ since ℓ is the infimum, so we have $f(\bar{x}) = \ell$. This proves that $\ell \in \mathbb{R}$ and and this value is the minimum of f, realized at $\bar{x}$.

Memo – *l.s.c. functions as suprema of Lipschitz functions*

Theorem : If f_α is an arbitrary family of lower semi-continuous functions on X, then the function f defined through $f(x) := \sup_\alpha f_\alpha(x)$ is also lower semi-continuous.

Proof : Take $x_n \to x$ and write $f_\alpha(x) \leq \liminf_n f_\alpha(x_n) \leq \liminf_n f(x_n)$. Then pass to the sup in α and get $f(x) \leq \liminf_n f(x_n)$.

Theorem : Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function bounded from below. Then f is lower semi-continuous if and only there exists a sequence f_k of $k-$Lipschitz functions such that for every $x \in X$, $f_k(x)$ converges increasingly to $f(x)$

Proof : One implication if easy, since the functions f_k are continuous, hence lower semi-continuous, and f is the sup of f_k. The other is more delicate. Given f lowr semi-continuous and bounded from below, let us define

$$f_k(x) = \inf_y f(y) + kd(x, y).$$

These functions are $k-$Lipschitz continuous since $x \mapsto f(y)+kd(x, y)$ is $k-$Lipschitz. It is obvious that, for fixed x, the sequence $f_k(x)$ is increasing and we also have $\inf f \leq f_k(x) \leq f(x)$ (this last inequality being proven if one takes $y = x$ in the definition of f_k). We just need to prove that $\ell := \lim_k f_k(x) = \sup_k f_k(x) = f(x)$. Suppose by contradiction $\ell < f(x)$, which implies in particular $\ell < +\infty$. For every k, let us chose a point y_k such that $f_k(x) \leq f(y_k) + kd(y_k, x) < f_k(x) + 1/k$. We get in particular

$$d(y_k, x) \leq \frac{\ell + 1/k - f(y_k))}{k} \leq \frac{C}{k},$$

thanks to the lower bound on f and to $\ell < \infty$. Hence we know $y_k \to x$. Yet, we have $f_k(x) + 1/k \geq f(y_k)$ and hence $\lim_k f_k(x) \geq \liminf_k f(y_k) \geq f(x)$ (by semi-continuity). This proves $f_k(x) \to f(x)$.

Memo – *Weak compactness in dual spaces; duality between C_0 and $\mathcal{M}$; weak convergence of measures; Prokhorov's theorem*

Definition: A sequence x_n in a Banach space X is said to be weakly converging to x - and we write $x_n \rightharpoonup x$ - if for every $\xi \in X'$ we have $< \xi, x_n > \to < \xi, x >$. A sequence $\xi_n \in X'$ is said to be weakly-* converging to ξ - and we write $\xi_n \overset{*}{\rightharpoonup} \xi$ - it for every $x \in X$ we have $< \xi_n, x > \to < \xi, x >$.

Theorem (Banach-Alaoglu): If X is separable and ξ_n is a bounded sequence in X' then there exists a subsequence ξ_{n_k} weakly converging to some $\xi \in X'$.

Definition: A finite measure λ on a set Ω is a map associating to every Borel subset $A \subset \Omega$ a value $\lambda(A) \in \mathbb{R}$ (we will see in Chapter 4 the case of vector measure, where this application is valued in $\mathbb{R}^d$) such that, for every disjoint union $A = \bigcup_i A_i$ with $A_i \cap A_j = \emptyset$ for $i \neq j$, we have

$$\sum_i |\lambda(A_i)| < +\infty \quad \text{and} \quad \lambda(A) = \sum_i \lambda(A_i).$$

We denote by $\mathcal{M}(\Omega)$ the set of finite signed measures on Ω. To such measures we can associate a positive scalar measure $\lambda \in \mathcal{M}_+(\Omega)$ through

$$|\lambda|(A) := \sup \left\{ \sum_i |\lambda(A_i)| \ : \ A = \bigcup_i A_i \ \text{ with } A_i \cap A_j = \emptyset \text{ for } i \neq j \right\}.$$

Theorem (Riesz representation theorem): Let $X = C_0(\Omega)$ be the space of continuous function on Ω vanishing at infinity, i.e. $f \in C_0(\Omega) \iff f \in C(\Omega)$ and for every $\varepsilon > 0$ there exists a compact subset $K \subset \Omega$ such that $|f(x)| < \varepsilon$ for every $x \in \Omega \setminus K$. Let us endow this space with the sup norm since $C_0(\Omega) \subset C_b(\Omega)$ (this last space being the space of bounded continuous functions on Ω). Then every element of X' is represented in a unique way as an element of $\mathcal{M}(\Omega)$: for all $\xi \in X'$ there exists unique a measure $\lambda \in \mathcal{M}(\Omega)$ such that $\xi, \phi >= \int \phi d\lambda$ for every $\phi \in X$; moreover, X' is isomorphic to $\mathcal{M}(\Omega)$ endowed with the norm $||\lambda|| := |\lambda|(\Omega)$.

For signed measures of $\mathcal{M}(\Omega)$ we should call weak-* convergence the convergence in the duality with $C_0(\Omega)$. Yet, another interesting notion of convergence exists, that in the duality with $C_b(\Omega)$. We will call it (by abuse of notation) weak convergence and denote it through the symbol $\rightharpoonup$: $\mu_n \rightharpoonup \mu$ if and only if for every $\phi \in C_b(\Omega)$ we have $\int \phi d\mu_n \to \int \phi d\mu$ (notice that, taking $\phi = 1$, the total mss of μ_n also converges to that of μ, which is not the case for the $\overset{*}{\rightharpoonup}$ convergence in X'). Notice that $\mathcal{M}(\Omega)$ is not the dual of $C_b(\Omega)$, which is much more complicated, since, by Hahn-Banach's theorem, it is possible to produce elements of $C_b(\Omega)'$ that only look at the behavior of functions of $C_b(\Omega)$ at infinity. Notice also that $C_0(\Omega) = C_b(\Omega) = C(\Omega)$ if Ω is compact, and that the two notions of convergence are compact.

Probability measures are particular measures in $\mathcal{M}(\Omega)$: those that are positive and have unit mass. We can say $\mu \in \mathcal{P}(\omega) \iff \mu \in \mathcal{M}_+(\Omega)$ and $\mu(\Omega) = 1$ (for positive measures μ and $|\mu|$ coincide).

Definition: A sequence μ_n of probability measures over Ω is said to be tight if for every $\varepsilon > 0$ there exists a compact subset $K \subset \Omega$ such that $\mu_n(\Omega \setminus K) < \varepsilon$ for every n.

Theorem (Prokhorov): Suppose that μ_n is a tight sequence of probability measures over a space Ω. Then there exists a probability measure $\mu \in \mathcal{P}(\Omega)$ and a subsequence μ_{n_k} such that $\mu_{n_k} \rightharpoonup \mu$ (in the duality with $C_b(\Omega)$).

Sketch of proof: Since μ_n are probabilities, they are bounded in $\mathcal{M}(\Omega)$ and hence admit a converging subsequence μ_{n_k}. We only need to use the tightness assumption to prove that the convergence holds in duality with $C_b(\Omega)$ and that μ is a probability. If we fix $\phi \in C_b(\Omega)$ the tightness assumption allows, up to a small error, to neglect what happens outside a compact set K. By modifying ϕ on a small neighborhood of K, we can transform into $\tilde{\phi} \in C_0(\Omega)$ and so $\int \tilde{\phi} d\mu_n \to \int \tilde{\phi} d\mu$, which implies in the end (by estimating the difference $\int |\phi - \tilde{\phi}| d\mu_n$) to get $\int \phi d\mu_n \to \int \phi d\mu$ and $\mu_n \rightharpoonup \mu$. since μ is a limit of positive measures it is also positive, and since the convergence holds tested against $\phi = 1$ the total mass of μ is also 1, hence $\mu \in \mathcal{P}(\omega)$.

We are now ready to state some existence results.

Theorem 1.1.1. *Let X and Y be compact, $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$ and $c : X \times Y \to \mathbb{R}$ a continuous function. Then Problem (PK) admits a solution.*

Proof. We just need to show that the set $\Pi(\mu, \nu)$ is compact and that $\gamma \mapsto K(\gamma) = \int c\, d\gamma$ is continuous, and apply Weierstrass's theorem. We have to choose a notion of convergence for that and we choose to use the weak convergence of probability measures (i.e. convergence in the duality with $C_b(X \times Y)$, which is the same here as $C(X \times Y)$ or $C_0(X \times Y)$). This gives continuity of K by definition, since c is a continuous function on $X \times Y$, which is compact, and it belongs to C_b.

As for the compactness, given a sequence $\gamma_n \in \Pi(\mu, \nu)$. They are probability measures, so that their mass is 1, and hence they are bounded in the dual of $C(X \times Y)$. Hence usual weak-* compactness in dual spaces guarantees the existence of a subsequence $\gamma_{n_k} \rightharpoonup \gamma$ converging to a probability γ. We just need to check $\gamma \in \Pi(\mu, \nu)$. This may be done by fixing $\phi \in C(X)$ and using $\int \phi(x) d\gamma_{n_k} = \int \phi\, d\mu$ and passing to the limit, which gives $\int \phi(x) d\gamma = \int \phi\, d\mu$. This shows $(\pi^x)_\# \gamma = \mu$. The same may be done for π^y. More generally, the image measure through continuous maps preserves weak convergence (and here we use the map $(x, y) \mapsto x$ or $(x, y) \mapsto y$). □

Theorem 1.1.2. *Let X and Y be compact, $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$ and $c : X \times Y \to [0, +\infty]$ lower semi-continuous (and bounded from below). Then Problem (PK) admits a solution.*

Proof. Only difference : K is no more continuous, it is l.s.c. for the weak convergence of probabilities. This is a consequence of the following lemma, applied to $f = c$ on the space $X \times Y$. □

Lemma 1.1.3. *If $f : X \to \mathbb{R} \cup \{+\infty\}$ is a lower-semi continuous function bounded from below on a metric space X and then the functional $K : \mathcal{M}_+(X) \to \mathbb{R} \cup \{+\infty\}$ defined on positive measures on X through $K(\mu) := \int f\, d\mu$ is lower semicontinuous for the weak convergence of measures.*

Proof. Consider a sequence f_j of continuous and bounded functions converging increasingly to f. Then write $K(\mu) = \sup_j K_j(\mu) := \int f_j\, d\mu$ (actually $K_j \leq K$ and $K_j(\mu) \to K(\mu)$ for every μ by monotone convergence $f_j \to f$). Every K_j is continuous for the weak convergence, and hence K is l.s.c. as a supremum of continuous functionals. □

Theorem 1.1.4. *Let X and Y be complete separable metric spaces, $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$ and $c : X \times Y \to [0, +\infty]$ lower semi-continuous (and bounded from below). Then Problem (PK) admits a solution.*

Proof. It is now the compactness which is no more evident. We need to use Prokhorov theorem. This means showing that any sequence in $\Pi(\mu, \nu)$ is tight. To do that, fix $\varepsilon > 0$ and find two compact sets $K_X \subset X$ and $K_Y \subset Y$ such that $\mu(X \setminus K_X), \nu(Y \setminus K_Y) < \frac{1}{2}\varepsilon$. Then the set $K_X \times K_Y$ is compact in $X \times Y$ and, for any $\gamma_n \in \Pi(\mu, \nu)$, we have

$$\begin{aligned}\gamma_n((X \times Y) \setminus (K_X \times K_Y)) &\leq \gamma_n((X \setminus K_X) \times Y) + \gamma_n(X \times (Y \setminus K_Y)) \\ &= \mu(X \setminus K_X) + \nu(Y \setminus K_Y) < \varepsilon.\end{aligned}$$

This shows that any sequence in $\Pi(\mu, \nu)$ is tight and gives the desired compactness. □

We add to this section an improvement of the continuity and semi-continuity results above, which could be useful when the cost functions is not continuous.

Lemma 1.1.5. *Let γ_n and γ be measures on $X \times Y$, all with marginals μ and ν, respectively and $a : X \to \tilde{X}$ and $b : Y \to \tilde{Y}$ be two measurable maps valued in two separable metric spaces $\tilde{X}$ and $\tilde{Y}$. Let $c : \tilde{X} \times \tilde{Y} \to [0, +\infty]$ be a continuous function with $c(a, b) \leq f(a) + g(b)$ with f, g continuous and $\int f(a(x))d\mu$, $\int g(b(y))d\nu < +\infty$. Then we have*

$$\gamma_n \rightharpoonup \gamma \Rightarrow \int_{X \times Y} c(a(x), b(y))d\gamma_n \to \int_{X \times Y} c(a(x), b(y))d\gamma.$$

Proof. We start from the case where c is bounded, say $0 \leq c \leq M$. Since we can apply the weak version of Lusin's Theorem (see the observations in the next *memo*) to maps valued in $\tilde{X}$ and $\tilde{Y}$, then we can fix a $\delta > 0$ and find two compact sets $K_X \subset X$, $K_Y \subset Y$, with $\mu(X \setminus K_X) < \delta$ and $\nu(Y \setminus K_Y) < \delta$, such that a and b are continuous when restricted to K_X and K_Y, respectively. Let us set $K := K_X \times K_Y \subset X \times Y$, which is a compact set in the product.

We can write

$$\int c(a,b)d\gamma_n \leq \int I_K c(a,b)d\gamma_n + 2M\delta,$$

and the function $I_K c(a,b)$ is upper semi-continuous on $X \times Y$ (since it is continuous and positive on a closed set, and vanishes outside it). This implies

$$\limsup_n \int c(a,b)d\gamma_n \leq \int I_K c(a,b)d\gamma + 2M\delta \leq \int c(a,b)d\gamma + 2M\delta$$

and, since δ is arbitrary, $\limsup_n \int c(a,b)d\gamma_n \leq \int c(a,b)d\gamma$. This proves upper semicontinuity of the integral functional when c is bounded by M. An analogous computation with $M - c$ instead of c proves lower semi-continuity.

If c is positive but unbounded, just approximate it from below with its truncations $c_M = c \wedge M$, and lower semicontinuity is proven for the integral functional, which will be a sup of lower semicontinuous functionals. By replacing c with the function $\tilde{X} \times \tilde{Y} \ni (\tilde{x}, \tilde{y}) \mapsto f(\tilde{x}) + g(\tilde{y}) - c(\tilde{x}, \tilde{y})$ the upper semicontinuity is proven as well, which finally gives the desired result. $\square$

Memo – *Lusin's theorem.*

A quite well known theorem in measure theory states that every measurable function f on a reasonable space X, endowed with a finite and positive measure μ, is actually continuous on a set K which fills almost all the measure of X. This set K can be taken compact. Actually, there can be at least two statements: either we want f to be merely continuous on K (which is easier), or we want f to coincide on K with a continuous function g, defined on the whole X. This theorem is usually stated for real-valued functions, but we happen to need it for functions valued in more general spaces. Let us give precise statements and prove what we are able to prove. We will take a topological measure space X endowed with a regular measure μ (i.e. any Borel set $A \subset X$ satisfies $\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ compact}\}$. The arrival space Y will be supposed to be second-countable (i.e. to admit a countable family $(B_i)_i$ of open sets such that any other open set $B \subset Y$ may be expressed as a union of B_i).

Theorem (weak Lusin): If X is a topological measure space endowed with a regular measure μ, if Y is second-countable and $f : X \to Y$ is measurable, then for every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $\mu(X \setminus K) < \varepsilon$ and the restriction of f to K is continuous.

Proof: For every $i \in \mathbb{N}$, set $A_i^+ = f^{-1}(B_i)$ and $A_i^- = f^{-1}(B_i^c)$. Consider compact sets $K_i^+ \subset A_i^+$ such that $\mu(A_i^+ \setminus K_i^+) < \varepsilon 2^{-i}$ and $K_i^- \subset A_i^-$ with $\mu(A_i^- \setminus K_i^-) < \varepsilon 2^{-i}$. Set $K_i = K_i^+ \cup K_i^-$ and $K = \bigcap_i K_i$. For each i we have $\mu(X \setminus K_i) < 2\varepsilon 2^{-i}$. By construction, K is compact and $\mu(X \setminus K) < 4\varepsilon$. To prove that f is continuous on K it is sufficient to check that $f^{-1}(B) \cap K$ is relatively open in K for each open set B, and it is enough to check this for $B = B_i$ (the other pre-images are just unions of these sets, and unions of open sets are open). To prove that $f^{-1}(B_i) \cap K$ is relatively open, it is enough to prove that $f^{-1}(B_i^c) \cap K$ is closed, and this is true since it coincides with $K_i^+ \cap K$.

Theorem (strong Lusin): If X is a metric space endowed with a regular measure μ and $f : X \to \mathbb{R}$ is measurable, then for every $\varepsilon > 0$ there exists a compact set $K \subset X$ and a continuous function $g : X \to \mathbb{R}$ such that $\mu(X \setminus K) < \varepsilon$ and $f = g$ on K.

Proof: First apply weak Lusin's theorem, since $\mathbb{R}$ is second countable. Then we just need to prove that the restriction of $f_{|K}$ of f to K may be extended as a continuous function g on the whole X. This is possible since $f_{|K}$ is uniformly continuous (as a continuous function on a compact set) and hence has a modulus of continuity ω, such that $|f(x) - f(x')| \le \omega(d(x, x'))$. Then define $g(x) = \inf\{f(x') + \omega(d(x, x')) : x' \in K\}$. It can be easily checked that g is continuous and coincides with f on K.

Notice that this last proof strongly uses the fact that the arrival space is $\mathbb{R}$. It could be adapted to the case of $\mathbb{R}^d$ just by extending componentwise. It is also interesting to remark that if Y is a separable metric space, than it can be embedded into ℓ^∞, and extend componentwise as well (but the extension will not be valued in Y, but in ℓ^∞). On the other hand, it is clear that the strong version of Lusin's Theorem cannot hold (without extension to a bigger space) for any space Y, and the counter-example is easy to build : just take X a connected space and Y disconnected (for instance, a space composed by two points). A measurable function $f : X \to Y$ taking two different values in two different connected components on two sets of positive measure cannot be approximated by continuous functions in the sense of the strong Lusin's Theorem.

The consequence of all these continuity, semi-continuity, and compactness results is the existence, under very mild assumptions on the cost and the space, of an optimal transport plan γ. Then, if one is interested in the problem of Monge, the question may become "does this minimal γ come from a transport map T?". Actually, if the answer to this question is yes, then it is evident that the problem of Monge has a solution, which also solves a wider problem, that of minimizing among transport plans. This is the object of the next two sections 1.2 and 1.3. On the other hand, in some cases proving that the optimal transport plan comes from a transport map (or proving that there exists at least one optimal plan coming from a map) is equivalent to proving that the problem of Monge has a solution, since very often the infimum among transport plans and among transport maps is the same. This depends on the presence of atoms (see Sections 1.4 and 1.5).

1.2 Duality

Since the problem (PK) is a linear optimization under linear constraints, an important tool will be duality theory, which is typically used for convex problems. We will find a dual problem (D) for (K) and exploit the relations between dual and primal.

The first thing we will do is finding a formal dual problem, by means of an inf-sup exchange.

First express the constraint $\gamma \in \Pi(\mu, \nu)$ in the following way : notice that, if γ is a non-negative measure on $X \times Y$, then we have

$$\sup_{\phi, \psi} \int \phi \, d\mu + \int \psi \, d\nu - \int (\phi(x) + \psi(y)) \, d\gamma = \begin{cases} 0 & \text{if } \gamma \in \Pi(\mu, \nu) \\ +\infty & \text{otherwise} \end{cases}.$$

Hence, one can remove the constraints on γ if he adds the previous sup, since if they are satisfied nothing has been added and if they are not one gets $+\infty$ and this will be avoided by the minimization. Hence we may look at the problem we get and interchange the inf in γ and the sup in ϕ, ψ:

$$\min_{\gamma} \int c \, d\gamma + \sup_{\phi,\psi} \left(\int \phi \, d\mu + \int \psi \, d\nu - \int (\phi(x) + \psi(y)) \, d\gamma \right) =$$
$$\sup_{\phi,\psi} \int \phi \, d\mu + \int \psi \, d\nu + \inf_{\gamma} \int \left(c(x, y) - (\phi(x) + \psi(y)) \right) \, d\gamma.$$

Obviously it is not always possible to exchange inf and sup, and the main tool to do it is a theorem by Rockafellar (see [94], Section 37) requiring concavity in one variable, convexity in the other one, and some compactness assumption. Yet, Rockafellar's statement concerns finite-dimensional spaces, which is not the case here. To handle infinite-dimensional situations one needs to use a mini-max theorem stated in [40], Chapter 1. A complete proof ok Kantorovich duality, by applying the theorem in [40] to well-chosen functions and space, can be found in [104], Section 1.1.

Yet, e will not investigate anymore the question of obtaining the duality equality between inf-sup and sup-inf through general convex analysis theorems. The result is true and we will see later on how to prove it in this particular case (Section 1.6). For the moment, let us take it as true.

Afterwards, one can re-write the inf in γ as a constraint on ϕ and ψ, since one has

$$\inf_{\gamma\geq 0}\int (c-\phi\oplus\psi))\,d\gamma = \begin{cases} 0 & \text{if } \phi\oplus\psi\leq \text{ on } X\times Y \\ -\infty & \text{otherwise} \end{cases},$$

where $\phi\oplus\psi$ denotes the function defined through $(\phi\oplus\psi)(x,y) := \phi(x)+\psi(y)$. This leads to the following dual optimization problem.

Problem 2. Given the two probabilities μ and ν on Ω and the cost function $c : X\times Y \to [0,+\infty[$ we consider the problem

$$(PD)\quad \max\left\{\int_X \phi\,d\mu + \int_Y \psi\,d\nu \;\middle|\; \phi\in C(X), \psi\in C(Y) : \phi\oplus\psi\leq c \text{ on } X\times Y\right\}, \tag{2.2}$$

This problem does not admit a straightforward existence result, since the class of admissible functions lacks compactness. Let us recall the main result concerning compactness in the space of continuous functions.

Memo – *Compactness for the uniform convergence.*

Theorem (Ascoli-Arzelà): If X is a compact metric space and $f_n : X \to \mathbb{R}$ are equi-continuous (i.e. for every $\varepsilon > 0$ there exists a common $\delta > 0$ such that $|f_n(x)-f_n(y)| < \varepsilon$ for all pairs x, y with $d(x,y) < \delta$ and for all n) and equi-bounded (i.e. there is a common constant C with $|f_n(x)| \leq C$ for all $x\in X$ and all n), then the sequence (f_n) admits a subsequence f_{n_k} uniformly converging to a continuous function $f : X \to \mathbb{R}$.

Conversely, a subset of $C(X)$ is relatively compact for the uniform convergence (if and) only if its elements are equi-continuous and qui-bounded.

Morevoer, the same result is true if the arrival space $\mathbb{R}$ and the equiboundedness assumption are replaced with an arrival space which is a compact metric space.

Definition 1. Given a function $\chi : X \to \overline{\mathbb{R}}$ we define its $c-$*transform* (or $c-$conjugate function) $\chi^c : Y \to \overline{\mathbb{R}}$ by

$$\chi^c(y) = \inf_{x\in X} c(x,y) - \chi(x).$$

We also define the $\bar{c}-$transform of $\xi : Y \to \overline{\mathbb{R}}$ by

$$\xi^{\bar{c}}(x) = \inf_{y\in\Omega} c(x,y) - \xi(y).$$

Moreover, we say that a function ψ defined on Y is $\bar{c}-$*concave* if there exists χ such that $\psi = \chi^c$ (and, analogously, a function ϕ over X is said to be $c-$concave if there is $\xi : Y \to \overline{\mathbb{R}}$ such that $\phi = \xi^{\bar{c}}$) and we denote by $c-conc(X)$ and $\bar{c}-conc(Y)$ the sets of $c-$ and $\bar{c}-$concave functions, respectively (when $X = Y$ and c is symmetric this distinction between c and $\bar{c}$ will play no more any role and will be dropped).

It is important to notice that the notion of $c-$concavity implies a bound on the continuity modulus.

Memo – *Continuity of functions defined as an* inf *or* sup

Proposition: Let $(f_\alpha)_\alpha$ be a family (finite, infinite, countable, uncountable…) of functions all satisfying the same condition

$$|f_\alpha(x) - f_\alpha(x')| \leq \omega(d(x, x')).$$

Consider f defined through $f(x) := \inf_\alpha f_\alpha(x)$. Then f also satisfies the same estimate.

This can be easily seen if one writes $f_\alpha(x) \leq f_\alpha(x') + \omega(d(x, x'))$, which implies $f(x) \leq f_\alpha(x') + \omega(d(x, x'))$ since $f \leq f_\alpha$. Then, taking the infimum over α at the r.h.s. one gets $f(x) \leq f(x') + \omega(d(x, x'))$. Interchanging x and x' one obtains

$$|f(x) - f(x')| \leq \omega(d(x, x')).$$

In particular, if the function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\lim_{t\to 0} \omega(t) = 0$ (which means that the family $(f_\alpha)_\alpha$ is equicontinuous), then f has the same modulus of continuity (i.e., the same function ω) as the functions f_α. The same idea obviously work for the supremum instead of the infimum.

In our case, if c is continuous on a compact set, and hence equicontinuous, this means that there exists an increasing continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(0) = 0$ such that

$$|c(x, y) - c(x', y')| \leq \omega(d(x, x') + d(y, y')).$$

Hence, when we take the definition of χ^c, we have $\chi^c(y) = \inf_x g_x(y) := c(x, y) - \chi(x)$ and the functions g_x satisfy $|g_x(y) - g_x(y')| \leq \omega(d(y, y'))$, which finally proves that χ^c shares the same continuity modulus also.

It is quite easy to realize that, given a pair (ϕ, ψ) in the maximization problem (D), one can always replace it with (ϕ, ϕ^c), and then with $(\phi^{c\bar{c}}, \phi^c)$, and the constraints are preserved and the integrals increased. Actually one could go on but it is possible to prove that $\phi^{c\bar{c}c} = \phi^c$ for any function ϕ.

This is the same as saying that $\psi^{c\bar{c}} = \psi$ for any $\bar{c}-$concave function ψ, and this perfectly recalls what happens for the Legendre transform of convex functions (which corresponds to the particular case $c(x, y) = x \cdot y$). The goal of these transformations is to "improve" the maximizing sequence so that to get a uniform bound on its continuity.

A consequence of these considerations is the following existence result.

Proposition 1.2.1. *Suppose that X and Y are compact and c is continuous. Then there exists a solution (ϕ, ψ) to problem (PD) and it has the form $\phi \in c-conc(X), \psi \in c-conc(Y)$, and $\psi = \phi^c$. In particular*

$$\max(PD) = \max_{\phi \in c-conc(X)} \int_\Omega \phi \, d\mu + \int_\Omega \phi^c \, d\nu.$$

Proof. From the considerations above we can take a maximizing sequence (ϕ_n, ψ_n) and improve it, by means of $c-$ and $\bar{c}-$transforms, so that we can assume a uniform bound on the continuity of these functions (the same modulus of continuity as c). Instead of renaming the sequence, we will still call (ϕ_n, ψ_n) the new sequence obtained after these transforms. We only need to check equi-boundedness so as to apply Ascoli-Arzelà's theorem. This may be done if we notice that adding a constant to ϕ and subtracting it to ψ is always possible: the value of the functional does not change, nor the constraints are affected. Hence, since ϕ_n is continuous on a compact set and hence bounded, we can always subtract its minimum and suppose without loss of generality that $\min \phi_n = 0$. Hence we get $\max \phi_n \leq \omega(\operatorname{diam} X)$ (since the oscillation of a function is always less than its modulus of continuity computed at the highest possible distance in the set). So, if we have chose $\psi_n = \phi_n^c$, we also have $\psi_n(y) = \inf c(x, y) - \phi_n(x) \in [\min c - \omega(\operatorname{diam} X), \max c]$. This gives uniform bounds on ϕ_n and ψ_n and allows to apply Ascoli-Arzelà's theorem.

Passing to a subsequence and without relabeling the sequence we can assume $\phi_n \to \phi$ and $\psi_n \to \psi$, both convergence being uniform. It is easy to see that

$$\lim_n \int \phi_n \, d\mu + \int \psi_n \, d\nu = \int \phi \, d\mu + \in \psi \, d\nu,$$

as a consequence of uniform convergence, and that

$$\phi_n(x) + \psi_n(y) \leq c(x, y) \;\Rightarrow\; \phi(x) + \psi(y) \leq c(x, y)$$

(where pointwise convergence would have been enough). This shows that (ϕ, ψ) is an admissible pair for (PD) and that it is optimal. $\square$

If we admit the duality result $\min(PK) = \max(PD)$ (the proof is postponed to Section 1.6), then we also have

$$\min(PK) = \max_{\phi \in c\text{–}conc(X)} \int_X \phi \, d\mu + \int_\Omega \phi^c \, d\nu,$$

which also shows that the minimum value of (K) is a convex function of (μ, ν), as it is a supremum of linear functionals.

Definition 2. The functions ϕ realizing the maximum in (5.2) are called *Kantorovich potentials* for the transport from μ to ν. This is in fact a small abuse, because usually this term is used only in the case $c(x, y) = |x - y|$, but it is usually understood in the general case as well.

1.3 The case $c(x, y) = h(x - y)$ for h strictly convex, and the existence of an optimal T

This section is devoted to the results taht we can obtain in the case where $X = Y = \Omega \subset \mathbb{R}^d$ and the cost c is of the form $c(x, y) = h(x - y)$, for a strictly convex function h. This case allows for very strong results, and in particular we will find existence, as well as a representation formula, for the optimal T. Anyway, the first few lines of this section will be concerned with a more general case that of costs functions c satisfying a *twist condition*, the most remarkable case being exactly thoseof the form $h(x - y)$ with h strictly convex.

The main tool is the duality result. If we have equality between the minimum of (K) and the maximum of (D) and both extremal values are realized, one can consider an optimal transport plan γ and a Kantorovich potential ϕ and write

$$\phi(x) + \phi^c(y) \leq c(x, y) \text{ on } \Omega \times \Omega \text{ and } \phi(x) + \phi^c(y) = c(x, y) \text{ on } \operatorname{spt} \gamma.$$

The equality on $\operatorname{spt} \gamma$ is a consequence of the inequality which is valid everywhere and of

$$\int c \, d\gamma = \int \phi \, d\mu + \int \phi^c \, d\nu = \int (\phi(x) + \phi^c(y)) \, d\gamma,$$

which implies equality $\gamma-$a.e. These functions being continuous, the equality passes to the support of the measure.

Once we have that, let us fix a point $(x_0, y_0) \in \operatorname{spt}\gamma$. One may deduce from the previous computations that

$$x \mapsto \phi(x) - c(x, y_0) \quad \text{is minimal at } x = x_0$$

and, if ϕ and $c(\cdot, y_0)$ are differentiable at x_0 and $x_0 \notin \partial\Omega$, one gets $\nabla\phi(x_0) = \nabla_x c(x_0, y_0)$. We resume this fact in a very short statement (where we do not put the sharpest assumptions on c) since we will use it much later on.

Proposition 1.3.1. *If c is C^1, ϕ is a Kantorovich potential for the cost c in the transport from μ to ν, and (x_0, y_0) belong to the support of an optimal transport plan γ, then $\nabla\phi(x_0) = \nabla_x c(x_0, y_0)$, provided ϕ is differentiable at x_0. In particular, the gradients of two different Kantorovich potentials coincide on every point $x_0 \in \operatorname{spt}(/mu)$ where both the potentials are differentiable.*

The equality $\nabla\phi = \nabla_x c$ is especially useful when c satisfies the following definition.

Definition 3. For $X \subset \mathbb{R}^d$ we say that $c : X \times Y \to \mathbb{R}$ satisfies the Twist condition whenever c is differentiable w.r.t. x for every y and the map $y \mapsto \nabla_x c(x_0, y)$ is injective for every x_0. This condition is also known in economics as Spence-Mirlees condition. On a "nice" domain, it corresponds for $C \in C^2$ to $\det\left(\frac{\partial^2 c}{\partial x_i \partial x_j}\right) \neq 0$.

The goal of this condition is to deduce from $(x_0, y_0) \in \operatorname{spt}\gamma$, that y_0 is indeed uniquely defined from x_0. This shows that γ is concentrated on a graph, that of the mapping associating y_0 to each x_0, and this map will be the optimal transport. By the way, since this map has been constructed using ϕ an c only, and not γ, it also provides uniqueness for the optimal γ.

We will see this strategy with more details in the particular case where $c(x, y) = h(x - y)$, with h strictly convex, but the reader can see how to translate it into the most general case.

For this choiceof c, if ϕ and h are differentiable at x_0 and $x_0 - y_0$, respectively, and $x_0 \notin \partial\Omega$, one gets $\nabla\phi(x_0) = \nabla h(x_0 - y_0)$. This works if the function h is differentiable, if it is not we shall write $\nabla\phi(x_0) \in \partial h(x_0 - y_0)$, but we will see what this notation means in Section 1.6. For a strictly convex function h one may inverse the relation passing to $(\nabla h)^{-1}$ thus getting

$$x_0 - y_0 = (\nabla h)^{-1}(\nabla\phi(x_0)).$$

This solves several questions concerning the transport problem with this cost, provided ϕ is differentiable a.e. with respect to μ. This is usually guaranteed by requiring μ to be absolutely continuous with respect to the Lebsgue measure, and using the fact that ϕ may be proven to be Lipschitz.

Memo – *Differentiability of Lipschitz functions*

Theorem (Rademacher): Let $\Omega \subset \mathbb{R}^d$ be an open set and $f : \Omega \to \mathbb{R}$ a Lipschitz continuous function. Then the set of points where f is not differentiable is negligible for the Lebesgue measure.

Then, one may use the previous computation to deduce that, for every x_0, the point y_0 such that $(x_0, y_0) \in \operatorname{spt}\gamma$ is unique (i.e. γ is of the form $(id \times T)_{\#}\mu$ where $T(x_0) = y_0$). Moreover, this also gives uniqueness of the optimal transport plan and of the gradient of the Kantorovich potential.

We may summarize everything in the following theorem:

Theorem 1.3.2. *Given μ and ν probability measures on a compact domain $\Omega \subset \mathbb{R}^d$ there exists an optimal transport plan π. It is unique and of the form $(id \times T)_{\#}\mu$, provided μ is absolutely continuous and $\partial\Omega$ is negligible. Moreover there exists also at least a Kantorovich potential ϕ, and the optimal transport map T and the potential ϕ are linked by*

$$T(x) = x - (\nabla h)^{-1}(\nabla\phi(x)).$$

Proof. The previous theorems give the existence of an optimal γ and an optimal ϕ. The previous considerations show that if we take a point $(x_0, y_0) \in \operatorname{spt}\gamma$ where $x_0 \notin \partial\Omega$ and $\nabla\phi(x_0)$ exists, then necessarily we have $y_0 = x_0 - (\nabla h)^{-1}(\nabla\phi(x_0))$. The points x_0 on the boundary (by assumption) and/or where the differentiablity fails (by Rademacher's theorem, since ϕ shares the same modulus of continuity of c, which is a Lipschitz function on $\Omega\times\Omega$ since h is locally Lipschitz continuous and Ω is bounded) are negligible for the Lebesgue measure, and hence for μ as well. This shows at the same time that every optimal transport plan is induced by a transport map and that this transport map is uniquely determined (since the potential ϕ does not depend on γ). As a consequence, we also have uniqueness of the optimal γ. $\square$

Remark 2. All the costs of the form $c(x, y) = |x-y|^p$ with $p > 1$ fall under Theorem 1.3.2.

Remark 3. Notice also that strict convexity of h did not play really any role up to the fact that we could invert its gradient. More generally, the same existence result is true if one takes a cost $c(x,y)$ which is Lipschitz continuous, differentiable, and such that for every x the map $y \mapsto \nabla_x c(x,y)$ is injective. This is usually known as *twist condition* and is guaranteed, for instance, if $c \in C^2$ and the mixed Hessian $D^2_{xy}c$ (the matrix made of the second derivatives $\partial^2/(\partial x_i \partial y_j)$) is non-singular, i.e. $\det(D^2_{xy}c) \neq 0$.

Remark 4. In the previous theorem we showed the uniqueness of the optimal plan by giving an explicit expression for the optimal map. Yet, it is possible to use a more general argument: every time that we know that any optimal γ actually derives from a map T, this implies uniqueness. Indeed, suppose that two different plans $\gamma_1 = \gamma_{T_1}$, $\gamma_2 = \gamma_{T_2}$ are optimal: consider $\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2$, which is optimal as well by convexity. This last transport plan cannot be induced by a map unless $T_1 = T_2$ $\mu-$a.e., which gives a contradiction.

Remark 5. In Theorem 1.3.2 only the part concerning the optimal map T is not symmetric in μ and ν: hence the uniqueness of the Kantorovich potential is true even if it ν (and not μ) has positive density a.e. (since one can retrieve ϕ from ϕ^c and viceversa).

1.3.1 The quadratic case

Theorem 1.3.2 may be particularized to the quadratic case $c(x,y) = \frac{1}{2}|x-y|^2$, thus getting the existence of an optimal transport map

$$T(x) = x - \nabla\phi(x) = \nabla\left(\frac{x^2}{2} - \phi(x)\right) = \nabla\psi(x)$$

for a convex function ψ. Since we will also see the converse implication (sufficient optimality conditions), this will also prove the existence and the uniqueness of a gradient of a convex function transporting μ onto ν. This well known fact has been investigated first by Brenier (see [36]) and is usually referred to as Brenier's Theorem. Section 1.7.2 will present the original approach by Brenier, called "polar factorization".

Let us moreover notice that a specific approach for the case $|x-y|^2$, based on the fact that we can withdraw the parts of the cost depending on x or y only and maximize $\int x \cdot y \, d\gamma$, gives the same result in a easier way: we actually get $\phi(x_0) + \phi^*(y_0) = x_0 \cdot y_0$ for a convex function ϕ and its Legendre transform ϕ^* and we deduce $y_0 \in \partial\phi(x_0)$.

Moreover, the existence of an optimal transport map is true under weaker assumptions: we can replace the condition of being absolutely continuous with the condition "$\mu(A) = 0$ for any $A \subset \mathbb{R}^d$ such that $\mathcal{H}^{d-1}(A) < +\infty$" or with any condition which ensures that the non-differentiability set of ϕ is negligible (and, since the differentiability of ϕ is the same of that of a convex function, this occurs everywhere but on a $(d-1)-$dimensional set). More generally, in the theorem we used the Lipschitz behavior of $\phi \in c-conc(\Omega)$ and applied Rademacher Theorem, but $c-$concave functions are often more regular than only Lipschitz.

Remark 6. As a consequence of all these considerations, the quadratic case gives a very interesting result in dimension one. Suppose that $\mu \in \mathcal{P}(\mathbb{R})$ is non-atomic. Then every convex function is differentiable $\mu-$a.e., since we know that the set of non-differentiability points of a convex function is at most countable (this is a consequence of the fact that, if ψ is convex, then the intervals $]\psi_l'(x), \psi_r'(x)[$, where ψ_l' and ψ_r' denote the left and right derivatives, are all non-empty and disjoint when x ranges among non-differentiability points). This implies the existence on an optimal transport map for the quadratic cost between μ and any measure $\nu \in \mathcal{P}(\mathbb{R})$. This transport map will be the derivative of a convex function, i.e. an increasing map.

1.4 Counter-examples to existence

We want to give at least two interesting examples that are not included in the statement of Theorem 1.3.2 and where an optimal transport does not exist.

No transport may exist The first one is very easy. Consider the case $\mu = \delta_a$, $a \in X$ and suppose that ν is not a Dirac mass. In this case there is no transport map at all. Indeed, it is easy to check that $T_\# \delta_a = \delta_{T(a)}$ and hence no transport map $T : X \to Y$ can exist if ν is not of the form δ_b for some $b \in Y$.

More generally, we can say that the image measure $T_\# \mu$ always includes an atom of mass at least $\mu(\{a\})$ for every atom a of μ. This implies in particular that measures with atoms cannot be sent through a transport map onto measures without atoms. For these reasons, the absence of atoms is a typical assumption on the starting measure μ when one wants to solve the problem of Monge (i.e. finding an optimal transport map).

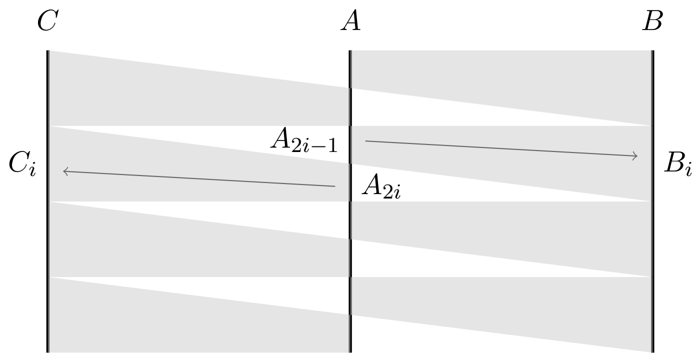
On the contrary, the problem (PK) by Kantorovitch still makes sense even for atomic measures, since we already said that there always exists a transport plan γ between any two given probability measures. In particular, the situation is particulary easy when $\mu = \delta_a$. In this case the set $\Pi(\mu, \nu)$ contains a unique element, which is $\gamma = \mu \otimes \nu = \delta_a \otimes \nu$. This can be checked in this way: take $\gamma \in \Pi(\mu, \nu)$ and a test function $\xi : X \times Y \to \mathbb{R}$. Let us integrate $\int \xi(x, y) d\gamma$: if we consider $\gamma((\{a\} \times Y)^c) = \gamma(\{a\}^c \times Y) = \mu(\{a\}^c) = 0$, we deduce that γ is concentrated on $\{a\} \times Y$, i.e. that $x = a$ $\gamma-$a.e. This allows to replace the variable x with a in the integration, thus getting $\int \xi(x, y) d\gamma(x, y) = \int \xi(a, y) d\gamma(x, y) = \int \xi(a, y) d\nu(y) = \int \xi d\delta_a \otimes \nu$, which proves $\gamma = \int \xi(x, y) d\gamma$.

Transport can exist, but no optimal one Set

$$\mu = \mathcal{H}^1 \llcorner A \text{ and } \nu = \frac{\mathcal{H}^1 \llcorner B + \mathcal{H}^1 \llcorner C}{2}$$

where A, B and C are three vertical parallel segments in $\mathbb{R}^2$ whose vertexes lie on the two line $y = 0$ and $y = 1$ and the abscissas are 0, 1 and -1, respectively, and $\mathcal{H}^1$ is the $1-$dimensional Haudorff measure. It is clear that no transport plan may realize a cost better than 1 since, horizontally, every point needs to be displaced of a distance 1. Moreover, one can get a sequence of maps $T_n : A \to B \cup C$ by dividing A into $2n$ equal segments $(A_i)_{i=1,\dots,2n}$ and B and C into n segments each, $(B_i)_{i=1,\dots,n}$ and $(C_i)_{i=1,\dots,n}$ (all ordered downwards). Then define T_n as a piecewise affine map which sends A_{2i-1} onto B_i and A_{2i} onto C_i. In this way the cost of the map T_n is less than $1 + 1/n$, which implies that the infimum of the Kantorovich problem is 1, as well as the infimum on transport maps only. Yet, no map T may obtain a cost 1, as this would imply that all points are sent horizontally,but this cannot respect the push-forward constraint. On the other hand, the transport plan associated to T_n weakly converge to the transport plan $\frac{1}{2} T^+_\# \mu + \frac{1}{2} T^-_\# \mu$, where $T^\pm(x) = x \pm e$ and $e = (1, 0)$. This transport plan turns out to be the only optimal transport plan and its cost is 1.

Notice that in this last example we also saw that the infimum among transport maps was equal to the infimum (i.e. the minimum) among transport plans. This is a general fact, which relies on the notion of relaxation, as we can see in the following section.



1.5 Kantorovich as a relaxation of Monge

Let us set $J(\gamma) := \int_{\Omega\times\Omega} c\,d\gamma$. Since we know that for any map T we have $\int_\Omega c(x,T(x))d\mu = \int_{\Omega\times\Omega} c\,d\gamma_T = J(\gamma_T)$, Monge's problem may be re-written as

$$\min\ \tilde{J}(\gamma)\ :\ \gamma \in \Pi(\mu,\nu),$$

where

$$\tilde{J}(\gamma) = \begin{cases} J(\gamma) & \text{if } \gamma = \gamma_T, \\ +\infty & \text{otherwise.} \end{cases}$$

This is simple to understand : the definition of $\tilde{J}$ forces to restrict the minimization to those plan induced by a transport map. This fact is useful in order to consider Monge's and Kantorovich's problems as two problems on the *same set of admissible objects*, where the only difference is the functional to be minimized, $\tilde{J}$ or J.

The question is now : why did Kantorovitch decided to replace $\tilde{J}$ with J? Can we easily prove that $\inf J = \inf \tilde{J}$? this is obviously true when, by chance, the minimizer of J is of the form $\gamma = \gamma_T$, since in this case we would have equality of the two minima. But is it possible to justify the procedure in general? The main mathematical justification comes from the following notion of *relaxation*.

Definition 4. Lef $F : X \to \mathbb{R} \cup \{+\infty\}$ be a given functional on a metric space X. We define the relaxation of F as the functional $\overline{F} : X \to \mathbb{R}\cup\{+\infty\}$ which is the maximal functional among those $G : X \to \mathbb{R}\cup\{+\infty\}$ which are lower semicontinuous and such that $G \le F$. This functional exists since the supremum of an arbitrary family of l.s.c. functions is also l.s.c.. Moreover, we also have a representation formula, which is easy to prove:

$$\overline{F}(x) = \inf\{\liminf_n F(x_n)\ :\ x_n \to x\}.$$

A consequence of the definition is also that $\inf F = \inf \overline{F}$ (this latter infimum, that of $\overline{F}$, being often a minimum, when X is compact). This is easy to check if one considers that $F \geq \overline{F}$, which implies $\inf F \geq \inf \overline{F}$, but we also have that F is larger than the constant $l := \inf F$, and a constant function is l.s.c.. Hence $\overline{F} \geq l$ and $\inf \overline{F} \geq \inf F$.

Here we claim that, under some assumptions, J is actually the relaxation of $\tilde{J}$. It will happen, in this case, by chance, that this relaxation is also continuous, instead of only semi-continuous, and that it coincides with $\tilde{J}$ on $\{\tilde{J} < +\infty\}$.

The assumptions are the following: we take $\Omega \subset \mathbb{R}^n$ to be compact, c continuous and μ atomless (i.e. for every $x \in \Omega$ we have $\mu(\{x\}) = 0$).

We need some preliminary results. The first concerns the one-dimensional case, that we will analyze in details in Chapter 2, but can be stated right now.

Lemma 1.5.1. *If μ, ν are two probability measures on the real line $\mathbb{R}$ and μ is atomless, then there exists at least a transport map T such that $T_{\#}\mu = \nu$.*

Proof. Just consider the monotone increasing map T provided by Remark 6. This map also optimizes the quadratic cost, but here we don't care about it. □

Lemma 1.5.2. *There exists a Borel map $\sigma_n : \mathbb{R}^n \to \mathbb{R}$ which is injective, its image is a Borel subset of $\mathbb{R}$, and its inverse map is Borel measurable as well.*

Proof. First notice that it is sufficient to prove this result for $n = 2$, since then one can proceed by induction: if a map σ_{n-1} is given, defined on $\mathbb{R}^{n-1}$, then one can produce a map σ_n by considering $\sigma_n(x_1, x_2, \ldots, x_n) = \sigma_2(x_1, \sigma_{n-1}(x_2, x_3, \ldots, x_n))$.

Then, notice also that it is enough to define a map from $]0,1[^2$, since one can go from $\mathbb{R}^2$ to $]0,1[^2$ by considering the map $(x, y) \mapsto (\frac{1}{2} + \frac{1}{\pi} \arctan x, \frac{1}{2} + \frac{1}{\pi} \arctan y)$.

Then, consider the map which associates to the pair (x, y), where $x = 0, x_1 x_2 x_3 \ldots$ and $y = 0, y_1 y_2 y_3 \ldots$ in decimal (or binary) notation, the point $0, x_1 y_1 x_2 y_2 x_3 y_3 \ldots$. In order to avoid ambiguities, we can decide that no decimal notation is allowed to end with a periodic 9 (i.e. $0, 347299999\ldots$ is to be written as $0, 3473$). This is why the image of this map will not be the whole interval, since the points like $0, 39393939\ldots$ are not obtained through this map. But this set of points is actually Borel measurable.

It is not difficult neither to check that the map is Borel measurable, as well as its inverse, since the pre-image of every interval defined by prescribing the first $2k$ digits of a number in $\mathbb{R}$ is just a rectangle in $\mathbb{R}^2$, the product of two intervals defined by prescribing the first k digits of every component. These particular intervals being a base for the Borel tribe, this proves the measurability we need. □

Corollary 1.5.3. *If μ, ν are two probability measures on $\mathbb{R}^n$ and μ is atomless, then there exists at least a transport map T such that $T_\# \mu = \nu$.*

Proof. This is just obtained by considering a transport map T from $(\sigma_n)_\# \mu$ to $(\sigma_n)_\# \mu$ and then composing with σ_n and σ_n^{-1}. □

Remark 7. We provided here, through Lemma 1.5.2, an explicit way of constructing a transport map between a non-atomic measure μ and an arbitrary measure ν, when the ambient space is $\mathbb{R}^n$. Actually, this fact is much more general, since it is well known that any measure space endowed with a non-atomic measure is isomorphic to $[0,1]$ with the Lebesgue measure. Yet, we do not want to introduce this kind of arguments when an explicit construction is sufficient to deal with the Euclidean case, which already meets our scopes.

A last lemma

Lemma 1.5.4. *Consider on a compact metric space X, endowed with a probability $\lambda \in \mathcal{P}(X)$, a sequence of partitions G_n, each G_n being a family of disjoint subsets $C_{i,n}$ such that $\bigcup_{i \in I_n} C_{i,n} = X$ for every n. Suppose that size$(G_n) := \max_i \operatorname{diam}(C_{i,n})$ tends to 0 and consider a sequence of probability measures λ_n on X such that, for every n and $i \in I_n$, the equality $\lambda_n(C_{i,n}) = \lambda(C_{i,n}) = m_{i,n}$ holds. Then $\lambda_n \rightharpoonup \lambda$.*

Proof. It is sufficient to take a continuous function $\phi \in C(X)$ and notice that

$$\begin{aligned} \left| \int_X \phi d\lambda_n - \int_X \phi d\lambda \right| &\leq \sum_{i \in I_n} \left| \int_{C_{i,n}} \phi d\lambda_n - \int_{C_{i,n}} \phi d\lambda \right| \\ &\leq \omega(\operatorname{diam}(C_{i,n})) \sum_{i \in I_n} m_{i,n} = \omega(\operatorname{diam}(C_{i,n})) \to 0, \end{aligned}$$

where ω is the modulus of continuity of ϕ. This is justified by the fact that, whenever two measures have the same mass on a set $C \subset X$, since the oscillation of ϕ on the same set does not exceed $\omega(\mathrm{diam}(C))$, the difference of the two integrals is no more than this number times the common mass.

This proves $\int \phi d\lambda_n \to \int \phi d\lambda$ and hence $\lambda_n \to \lambda$. □

We can now prove the following

Theorem 1.5.5. *On a compact subset Ω of $\mathbb{R}^n$, the set of plans γ_T induced by a transport is dense in the set of plans $\Pi(\mu,\nu)$ whenever μ is atomless.*

Proof. Fix n, and consider any partition of Ω into sets $K_{i,n}$ of diameter smaller than $1/(2n)$ (for instance, small cubes). The sets $C_{i,j,n} := K_{i,n} \times K_{j,n}$ make a partition of $\Omega \times \Omega$ with size smaller than $1/n$.

Let us now take any measure $\gamma \in \Pi(\mu,\nu) \subset \mathcal{P}(\Omega\times\Omega)$. Thanks to Lemma 1.5.4, we will get the desired density if we are able to build a transport T sending μ to ν such that γ_T gives the same mass as γ to each one of the sets $C_{i,j,n}$. To do this, define the columns $Col_{i,n} := K_{i,n} \times \Omega$ and denote by $\gamma_{i,n}$ the restriction of γ on $Col_{i,n}$. Its marginal will be denoted by $\mu_{i,n}$ and $\nu_{i,n}$. Consider now, for each i, a transport map $T_{i,n}$ sending $\mu_{i,n}$ to $\nu_{i,n}$. It exists thanks to Corollary 1.5.3, since each $\mu_{i,n}$ is a submeasure of μ and is atomless as well. Since the $\mu_{i,n}$ are concentrated on disjoint sets, by "gluing" the transports $T_{i,n}$ we get a transport T sending μ to ν (using $\sum_i \mu_{i,n} = \mu$ and $\sum_i \nu_{i,n} = \nu$).

It is enough to check that γ_T gives the same mass as γ to every $C_{i,j,n}$, but it is easy to prove. Actually, this mass equals that of $\gamma_{T_{i,n}}$ and $\gamma_{T_{i,n}}(C_{i,j,n}) = \mu_{i,n}(\{x : x \in K_{i,n}, T_{i,n}(x) \in K_{j,n}\}) = \mu_{i,n}(\{x : T_{i,n}(x) \in K_{j,n}\}) = \nu_{i,n}(K_{j,n}) = \gamma(K_{i,n} \times K_{j,n})$. □

The relaxation result is just a consequence.

Theorem 1.5.6. *Under the abovementioned assumptions, J is the relaxation of $\tilde{J}$.*

Proof. First notice that, since J is continuous, then it is l.s.c. and since, due to the definition, we have $J \leq \tilde{J}$, then J is necessarily smaller than the relaxation of $\tilde{J}$. We only need to prove that, for each γ, we can find a sequence of transports T_n such that $\gamma_{T_n} \rightharpoonup \gamma$ and $\tilde{J}(\gamma_{T_n}) \to J(\gamma)$, so that the infimum in the sequential characterization of the relaxed functional (see definition) will be smaller than J, thus proving the equality. Actually, since for $\gamma = \gamma_{T_n}$ the two functionals J and $\tilde{J}$ coincide, and since J is continuous, we only need to produce a sequence T_n such that $\gamma_{T_n} \rightharpoonup \gamma$. This is possible thanks to Theorem 1.5.5 □

1.6 Convexity, c−concavity, cyclical monotonicity, duality and optimality

1.6.1 Convex and c−concave functions

In this section we analyze properties of c−concave functions in comparison with convex functions, which are better known. We start from recalling some notions from convex analysis.

Memo – *Convex functions, Legendre transform, subdifferential*

The definition of convex function is always the same: $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is convex if and only if for all $x, y \in /R^d$ and $t \in [0,1]$ we have $f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$. We do not care about convex functions defined on subsets $\Omega \subset \mathbb{R}^d$ (which should be convex, for the definition to make sense) since one can always extend f outside Ω by setting it to $+\infty$, and the extension is convex on $\mathbb{R}^d$ if and only if f was convex on Ω.

An easy stability property is the following: if f_α is a family of convex functions, then f defined through $f(x) := \sup_\alpha f_\alpha(x)$ is also convex. This can be checked easily by writing

$$f(_\alpha(1-t)x + ty) \leq (1-t)f_\alpha(x) + tf_\alpha(y) \leq (1-t)f(x) + tf(y)$$

and passing to the sup in α on the left hand side.

Real-valued convex functions defined on $\mathbb{R}^d$ are automatically continuous and locally Lipschitz (and in general, they are continuous and locally Lipschitz in the interior of their domain $\{f < +\infty\}$). But on the boundary between $\{f < +\infty\}$ and $\{f = +\infty\}$ there could be discontinuities. This is why typically we require at least lower semi-continuity.

Theorem: A function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is convex and l.s.c. if and only if there exists a family of affine functions f_α such that $f(x) := \sup_\alpha f_\alpha(x)$. This family can also be chose to be the family of all affine functions smaller than f.

One implication is easy (if f is a sup of affine functions, since affine implies convex, then it is convex). The geometrical intuition behind the other is the fact that tangent planes to the graph of f will stay below this graph, so at every point x where $f(x) < +\infty$ we can find an affine function smaller than f with value at x very close to that of f (actually, if x in the interior of the domain we can also guarantee the equality of the two values). Yet, a precise proof requires the use of Hahn-Banach Theorem in its geometrical form.

Definition (Legendre-Flenchel transform): for any given function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ we can define its transform $f^*(y) := \sup_x x \cdot y - f(x)$.

A function f is convex and l.s.c. if and only if there exists g such that $f = g^*$ (since te class of functions that are Legendre transform of something exactly agrees with that of functions which are expressed as suprema of affine functions). To be convinced that every supremum of affine functions can be expressed as g^*, just take

$f(y) = \sup x_\alpha \cdot y + b_\alpha$; For each vector x, set $g(x) = -\sup\{b_\alpha : x_\alpha = x\}$ (by setting $g(x) = +\infty$ if no α is such that $x_\alpha = x$) and check that we have $f = g^*$.

Proposition: A function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is convex and l.s.c. if and only if $f^{**} = f$.

Definition (subdifferential): For every convex function f we define its subdifferential at x as the set $\partial f(x) = \{p \in \mathbb{R}^d : f(y) \geq f(x) + p \cdot (y - x)\ \forall y \in \mathbb{R}^d\}$.

It is possible to prove that $\partial f(x)$ is never empty if x lies in the interior of the set $\{f < +\infty\}$. At every pint where the function f is differentiable, then ∂f reduces to the singleton $\{\nabla f\}$.

For the subdifferential of the convex functions f and f^* we have $p \in \partial f(x) \Leftrightarrow x \in \partial f^*(p)$.

The subdifferential of a convex function satisfies this monotonicity property: if $p_1 \in \partial f(x_1)$ and $p_2 \in \partial f(x_2)$, then $(p_1 - p_2) \cdot (x_1 - x_2) \geq 0$.

Analogously, $c-$concave functions (or $\bar{c}-$concave, this distinction being meaningless for the convex case, because $x \cdot y$ is symmetric), are defined as being the $\bar{c}-$transform (or $c-$transform) of something. Notice that we admit the value $+\infty$ for convex functions and $-\infty$ for concave and $c-$concave ones. Yet, we do not like the convex function which is identically $+\infty$; nor any $c-$concave function which is identically $-\infty$.

Instead of proving the result on $f^{**} = f$ we may prove this more general one concerning $c-$concave functions.

Proposition 1.6.1. *Suppose that c is real valued. For any $\phi : X \to \mathbb{R} \cup \{-\infty\}$ we have $\phi^{c\bar{c}} \geq \phi$. We have the equality $\phi^{c\bar{c}} = \phi$ if and only if ϕ is $c-$concave and for any other function $\phi^{c\bar{c}}$ is the smallest $c-$concave function larger than ϕ.*

Proof. First we prove $\phi^{\bar{c}c} \geq \phi$. Write

$$\phi^{c\bar{c}}(x) = \inf_y c(x, y) - \phi^c(y) = \inf_y c(x, y) - \inf_{x'} c(x', y) - \phi(x').$$

Consider that $\inf_{x'} c(x', y) - \phi(x') \leq c(x, y) - \phi(x)$, and hence

$$\phi^{c\bar{c}}(x) \geq \inf_y c(x, y) - c(x, y) + \phi(x) = \phi(x).$$

Analogously, we have $\xi^{\bar{c}c} \geq \xi$.

Then, let us prove $\phi^{c\bar{c}} = \phi$ if ϕ is $c-$concave. In such a case, we may write

$$\phi = \xi^{\bar{c}} \Rightarrow \phi^c = \xi^{\bar{c}c} \geq \xi \Rightarrow \phi c\bar{c} \leq \xi^{\bar{c}} = \phi,$$

where the last inequality is obtained by noticing that $c-$ and $\bar{c}-$transforms revert the inequalities between functions (due to the minus sign in the definition). This proves that in such a case we have $\phi^{c\bar{c}} \leq \phi$, and hence $\phi c\bar{c} = \phi$.

Finally, we can prove that for any ϕ, the function $\phi^{c\bar{c}}$ is the smallest $c-$concave function larger than ϕ. To prove that, take $\psi = \chi^{\bar{c}}$ any $c-$concave function and suppose $\psi \geq \phi$. Then consider

$$\chi^{\bar{c}} \geq \phi \Rightarrow \chi^{\bar{c}c} \leq \phi^c \Rightarrow \chi \leq \phi^c \Rightarrow \psi = \chi^{\bar{c}} \geq \phi c\bar{c},$$

which proves the thesis. □

We finish this section with a last concept about convex functions, that we will then translate into the framework of $c-$concave functions later on.

Let us define the graph of the subdifferential of a convex function as

$$G\partial f := \{(x,p) \,:\, p \in \partial f(x)\} = \{(x,p) \,:\, f(x) + f^*(p) = x \cdot p\}.$$

We already know that this graph is monotone in the sense that $(x_i, p_i) \in G\partial f$ for $i = 1, 2$ implies

$$(p_2 - p_1) \cdot (x_2 - x_1) \geq 0.$$

Yet, not all monotone graphs are the graphs of the subdifferential of a convex functions, nor they are contained in one of such graphs. Take for instance a 90° rotation R in $\mathbb{R}^2$ and consider the set $A = \{(x, Rx),\, x \in \mathbb{R}^n\}$. This set satisfies the monotonicity inequality above (actually, we always have $(Rx_1 - Rx_2) \cdot (x_1 - x_2) = 0$ for any x_1 and x_2). Yet, the map $x \mapsto Rx$ is not the gradient of a convex functions (since it is not a gradient at all), nor it can be contained in the graph of the subdifferential of a convex function. To check this last claim it is sufficient to notice that nothing can be added to A if we want to keep the monotonicity property. Actually, suppose $A \subset B$ for a set B still satisfying the monotonicity property. Take $(x_0, p_0) \in B$: then we have

$$(p_0 - Rx) \cdot (x_0 - x) \geq 0 \quad \forall x;$$

yet, if we subtract $(Rx_0 - Rx) \cdot (x_0 - x) = 0$, we get

$$(p_0 - Rx_0) \cdot (x_0 - x) \geq 0 \quad \forall x$$

ant this implies $p_0 - Rx_0 = 0$ since $x - x_0$ is an arbitrary vector. This means that any point $(x_0, p_0) \in B$ is of the form $p_0 = Rx_0$ and hence $B = A$.

Hence, monotonicity is not enough to characterize gradients and subdifferential of convex functions (to be more precise, gradient vector functions that are monotone are gradient of convex functions, but monotonicity alone does not imply gradientness).

A stronger notion is that of cyclical monotonicity.

Definition 5. A set $A \subset \mathbb{R}^d \times \mathbb{R}^d$ is said to be cyclically monotone if for every $k \in \mathbb{N}$, every permutation σ and every finite family of points $(x_1, p_1), \dots, (x_k, p_k) \in A$ we have

$$\sum_{i=1}^{k} x_i \cdot p_i \geq \sum_{i=1}^{k} x_i \cdot p_{\sigma(i)}.$$

The word "cyclical" refers to the fact that, since every σ is the disjoint composition of cycles, it is enough to check this property for cyclical permutations, i.e. replacing $\sum_{i=1}^{k} x_i \dots p_{\sigma(i)}$ with $\sum_{i=1}^{k} x_i \dots p_{i+1}$ in the definition (with the obvious convention $p_{k+1} = p_1$).

Notice that if we take $k = 2$ we get the usual definition of monotonicity, since

$$x_1 \cdot p_1 + x_2 \cdot p_2 \geq x_1 \cdot p_2 + x_2 \cdot p_1 \iff (p_1 - p_2) \cdot (x_1 - x_2) \geq 0.$$

A famous theorem by Rockafellar **CITE !!!** states that every cyclically monotone set is contained in the graph of the subdifferential of a convex function. We will not prove this theorem here, both since we do not really need it, and since we will see it as a particular case of a theorem on $c-$concave functions.

1.6.2 $c-$cyclical monotonicity and duality

We start from the translation to the $c-$concave case of the definition of cyclical monotonicity.

Definition 6. Once a function $c : \Omega \times \Omega \to \mathbb{R} \cup +\infty$ is given, we say that a set $\Gamma \subset \Omega \times \Omega$ is $c-$cyclically monotone (briefly $c-$CM) if, for every $k \in \mathbb{N}$, every permutation σ and every finite family of points $(x_1, y_1), \dots, (x_k, y_k) \in \Gamma$ we have

$$\sum_{i=1}^{k} c(x_i, y_i) \leq \sum_{i=1}^{k} c(x_i, y_{\sigma(i)}).$$

As for the convex case, the word "cyclical" refers to the fact that we can restrict our attention to cyclical permutations. The word "monotone" is a left-over from the case $c(x, y) = -x \cdot y$.

It is useful to recall the following theorem, which is a generalization of a theorem by Rockafellar in convex analysis, and whose proof may be found in [2] (Theorem 2.3), even if it is originally taken from [92]

Theorem 1.6.2. *If $\Gamma \neq \emptyset$ is a $c-CM$ set in $X \times Y$ and c is real valued, then there exist a $c-$concave function $\phi : X \to \mathbb{R} \cup \{-\infty\}$ (different from the constant $-\infty$ function) such that*

$$\Gamma \subset \{(x, y) \in X \times Y \,:\, \phi(x) + \phi^c(y) = c(x, y)\}$$

Proof. We will give an explicit formula for the function ϕ, prove that it is well defined and that it satisfies the properties that we want to impose.

Let us fix a point $(x_0, y_0) \in \Gamma$: for $x \in X$ set

$$\begin{aligned}\phi(x) = \inf \{&c(x, y_n) - c(x_n, y_n) + c(x_n, y_{n-1}) - c(x_{n-1}, y_{n-1}) + \cdots + \\ &+c(x_1, y_0) - c(x_0, y_0) \,:\, n \in \mathbb{N},\, (x_i, y_i) \in \Gamma \text{ for all } i = 1, \ldots, n\}\,.\end{aligned}$$

Since c is real-valued and Γ is non-empty, ϕ never takes the value $+\infty$. Moreover, c is real-valued avoids ambiguity such as $+\infty - \infty$ in the definition.

If we set, for $y \in Y$,

$$\begin{aligned}-\psi(y) = \inf \{&-c(x_n, y) + c(x_n, y_{n-1}) - c(x_{n-1}, y_{n-1}) + \cdots + c(x_1, y_0)+ \\ &-c(x_0, y_0) \,:\, n \in \mathbb{N},\, (x_i, y_i) \in \Gamma \text{ for all } i = 1, \ldots, n,\ y_n = y\}\end{aligned}$$

(which implies in particular $\psi = -\infty$ for $y \notin (\pi_y)(\Gamma))$ we have by construction $\phi = \psi^c$. This proves that ψ is $c-$concave. The fact that ϕ is not constantly $-\infty$ can be seen from $\phi(x_0) \geq 0$: indeed, if we take $x = x_0$, then for any chain of points $(x_i, y_i) \in \Gamma$ we have

$$\sum_{i=0}^{n} c(x_{i+1}, y_i) \geq \sum_{i=0}^{n} c(x_i, y_i),$$

where we consider $x_{n+1} = x_0$. This shows that the infimum in the definition of $\phi(x_0)$ is non-negative.

To prove $\phi(x) + \phi^c(y) = c(x, y)$ on Γ it is enough to prove the inequality $\phi(x) + \phi^c(y) \geq c(x, y)$ on the same set, since by definition of the $c-$transform the opposite inequality is always true. Moreover, since $\phi^c = \psi^{cc}$ **CHECK TRANSFORMS**, and $\psi^{cc} \geq \psi$, it is enough to check $\phi(x) + \psi(y) \geq c(x, y)$.

Suppose $(x, y) \in \Gamma$ and fix $\varepsilon > 0$. From $\phi = \psi^c$ one can find a point $\bar{y} \in (\pi_y)(\Gamma)$ such that $c(x, \bar{y}) - \psi(\bar{y}) < \phi(x) + \varepsilon$. From the definition of ψ one has the inequality $-\psi(y) \leq -c(x, y) + c(x, \bar{y}) - \psi(\bar{y})$ (since every chain starting from $\bar{y}$ may be completed adding the point $(x, y) \in \gamma$).

Putting together these two informations one gets

$$-\psi(y) \leq -c(x, y) + c(x, \bar{y}) - \psi(\bar{y}) < -c(x, y) + \phi(x) + \varepsilon,$$

which implies the inequality $c(x, y) \leq \phi(x) + \psi(y)$ since ε is arbitrary. $\square$

To introduce the following theorem we recall the definition

Definition 7. On a separable metric space X, the support of a measure γ is defined as the smallest closed set on which γ is concentrated, i.e.

$$\mathrm{spt}(\gamma) := \bigcap \{A \,:\, A \text{ is closed and } \gamma(X \setminus A) = 0\}.$$

This is well defined since the intersection may be taken countable, due to the assumption on the space X. Moreover, there exists also this characterization

$$\mathrm{spt}(\gamma) = \{x \in X \,:\, \gamma(B(x,r)) > 0 \text{ for all } r > 0\}.$$

We can now prove

Theorem 1.6.3. *If γ is an optimal transport plan for the cost c (i.e. it minimizes $\int c\,d\gamma$ over $\Pi(\mu,\nu)$) and c is continuous, then* $\mathrm{spt}(\gamma)$ *is a $c-$CM set.*

Proof. Suppose by contradiction that there exist k, σ and $(x_1,y_1),\dots,(x_k,y_k) \in \mathrm{spt}(\gamma)$ such that

$$\sum_{i=1}^k c(x_i,y_i) > \sum_{i=1}^k c(x_i,y_{\sigma(i)}).$$

Take now $\varepsilon < \frac{1}{2k}\left(\sum_{i=1}^k c(x_i,y_i) - \sum_{i=1}^k c(x_i,y_{\sigma(i)})\right)$. By continuity of c, there exists r such that for all $i = 1,\dots,k$ and for all $(x,y) \in B(x_i,r) \times B(y_i,r)$ we have $c(x,y) > c(x_i,y_i) - \varepsilon$ and for all $(x,y) \in B(x_i,r) \times B(y_{\sigma(i)},r)$ we have $c(x,y) < c(x_i,y_{\sigma(i)}) + \varepsilon$.

Now consider $V_i := B(x_i,r) \times B(y_i,r)$ and notice that $\gamma(V_i) > 0$ for every i, due to the condition $(x_i,y_i) \in \mathrm{spt}(\gamma)$. Define the measures $\gamma_i := \gamma \llcorner V_i/\gamma(V_i)$ and $\mu_i := (\pi_x)_\#\gamma_i$, $\nu_i := (\pi_y)_\#\gamma_i$. Take $\alpha < \frac{1}{k}\min_i \gamma(V_i)$.

For every i, build a measure $\tilde\gamma_i \in \Pi(\mu_i,\nu_{\sigma(i)})$ at will (for instance take $\tilde\gamma_i = \mu_i \otimes \nu_i$).

Now define

$$\tilde\gamma := \gamma - \alpha\sum_{i=1}^k \gamma_i + \alpha\sum_{i=1}^k \tilde\gamma_i.$$

We want to find a contradiction by proving that $\tilde\gamma$ is a better competitor than γ in the transport problem, i.e. $\tilde\gamma \in \Pi(\mu,\nu)$ and $\int c\,d\tilde\gamma < \int c\,d\gamma$.

First we check that $\tilde\gamma$ is a positive measure. It is sufficient to check that $\gamma - \alpha\sum_{i=1}^k \gamma_i$ is positive, and, for that, the condition $\alpha\gamma_i < \frac{1}{k}\gamma$ will be enough. This condition is satisfied since $\alpha\gamma_i = (\alpha/\gamma(V_i))\gamma\llcorner V_i$ and $\alpha/\gamma(V_i) \le \frac{1}{k}$.

Now, let us check the condition on the marginals of $\tilde{\gamma}$. We have

$$(\pi_x)_\# \tilde{\gamma} = \mu - \alpha \sum_{i=1}^{k} (\pi_x)_\# \gamma_i + \alpha \sum_{i=1}^{k} (\pi_x)_\# \tilde{\gamma}_i = \mu - \alpha \sum_{i=1}^{k} \mu_i + \alpha \sum_{i=1}^{k} \mu_i = \mu,$$

$$(\pi_y)_\# \tilde{\gamma} = \nu - \alpha \sum_{i=1}^{k} (\pi_y)_\# \gamma_i + \alpha \sum_{i=1}^{k} (\pi_y)_\# \tilde{\gamma}_i = \nu - \alpha \sum_{i=1}^{k} \nu_i + \alpha \sum_{i=1}^{k} \nu_{\sigma(i)} = \nu.$$

Finally, let us estimate $\int c\, d\gamma - \int c\, d\tilde{\gamma}$ and prove that it is positive, thus concluding the proof. We have

$$\begin{aligned}
\int c\, d\gamma - \int c\, d\tilde{\gamma} &= \alpha \sum_{i=1}^{k} \int c\, d\gamma_i - \alpha \sum_{i=1}^{k} \int c\, d\tilde{\gamma}_i \\
&\geq \alpha \sum_{i=1}^{k} (c(x_i, y_i) - \varepsilon) - \alpha \sum_{i=1}^{k} (c(x_i, y_{\sigma(i)}) + \varepsilon) \\
&= \alpha \left(\sum_{i=1}^{k} c(x_i, y_i) - \sum_{i=1}^{k} c(x_i, y_{\sigma(i)}) - 2k\varepsilon \right) > 0,
\end{aligned}$$

where we used the fact that γ_i is concentrated on $B(x_i, r) \times B(y_i, r)$, $\tilde{\gamma}_i$ on $B(x_i, r) \times B(y_{\sigma(i)}, r)$, and that they have unit mass (due to the rescaling by $\gamma(V_i)$). □

From the previous theorem, together with Theorem 1.6.2, we get the duality result that we were waiting for.

Theorem 1.6.4. *Suppose that $c : X \times Y \to \mathbb{R}$ is continuous and X, Y are compact. Then the problem (PD) admits a solution (ϕ, ϕ^c) and we have $(PD) = (PK)$.*

Proof. First consider the minimization problem PK. Since c is continuous, it admits a solution γ. Moreover, the support Γ of γ is $c-$cyclically monotone, thanks to Theorem 1.6.3. Then, since c is real valued, we can apply Theorem 1.6.2 and obtain the existence of a $c-$concave function ϕ such that

$$\Gamma \subset \{(x, y) \in \Omega \times \Omega \,:\, \phi(x) + \phi^c(y) = c(x, y)\}.$$

The pair (ϕ, ϕ^c) is obviously admissible in PD. Consider now

$$\int \phi\, d\mu + \int \phi^c\, d\nu = \int (\phi(x) + \phi^c(y)) d\gamma = \int c(x, y) d\gamma,$$

where the last equality is due to the fact that γ is concentrated over Γ and there we have $\phi(x) + \phi^c(y) = c(x, y)$. The equality just before comes from the fact that the marginals of γ are μ and ν, respectively. This finally shows, using the optimality of γ,

$$PD \geq \int \phi \, d\mu + \int \phi^c \, d\nu = \int c(x, y) d\gamma = (PK),$$

and implies $(PD) = (PK)$, since we already know $(PD) \leq (PK)$. As a byproduct of this proof, the pair (ϕ, ϕ^c) turns out to be optimal for PD. $\square$

We also want to give some details about the case where c is not continuous. Let us recall that this assumption is sufficient for the existence of an optimal transport plan. We will give two results about l.s.c. costs.

The first concerns the validity of the duality formula. This means the equality

$$\min \left\{ \int c \, d\gamma, \ \gamma \in \Pi(\mu, \nu) \right\} = \sup \left\{ \int \phi d\mu + \int \psi d\nu \ : \ \phi, \psi \in C(\Omega), \ \phi \oplus \psi \leq c \right\}. \tag{6.3}$$

By now, we have established this equality when c is continuous, also proving that the dual problem admits a maximizing pair. We also know that an inequality is always true : the minimum on the left is always larger than the maximum on the right (just integrate w.r.t. γ the condition on (ϕ, ψ)). More precisely, we are able to deal with the uniformly continuous case (since we want to guarantee continuity of $c-$concave functions of the form $\phi(x) = \inf_y c(x, y) - \psi(y)$). This means that we can handle the case where Ω is compact, or we need to add the uniform continuity assumption on c.

To deal with a l.s.c. cost c bounded from below, we will use the fact there exists a sequence c_k of continuous functions (each one being $k-$Lipschitz) increasingly converging to c. We need the following lemma.

Lemma 1.6.5. *Suppose that c_k and c are l.s.c. and bounded from below and that c_k converges increasingly to c. Then*

$$\lim_{k \to \infty} \min \left\{ \int c_k \, d\gamma, \ \gamma \in \Pi(\mu, \nu) \right\} = \min \left\{ \int c \, d\gamma, \ \gamma \in \Pi(\mu, \nu) \right\}.$$

Proof. Due to the increasing limit condition, we have $c_k \leq c$ and hence the limit on the left (which exists by monotonicity) is obviously smaller than the quantity on the right. Now consider a sequence $\gamma_k \in \Pi(\mu, \nu)$, built by picking an optimizer for each cost c_k. Up to subsequences, due to the tightness of $\Pi(\mu,\nu)$, we can suppose $\gamma_k \rightharpoonup \bar\gamma$. Fix now an index j. Since for $k \geq j$ we have $c_k \geq c_j$, we have

$$\lim_k \min\left\{\int c_k\,d\gamma,\ \gamma\in\Pi(\mu,\nu)\right\} = \lim_k \int c_k\,d\gamma_k \geq \liminf_k \int c_j\,d\gamma_k.$$

By semicontinuity of the integral cost c_j we have

$$\liminf_k \int c_j\,d\gamma_k \geq \int c_j\,d\bar\gamma.$$

Hence we have obtained

$$\lim_k \min\left\{\int c_k\,d\gamma,\ \gamma\in\Pi(\mu,\nu)\right\} \geq \int c_j\,d\bar\gamma.$$

Since j was arbitrary and $\lim_j \int c_j\,d\bar\gamma = \int c\,d\bar\gamma$ by monotone convergence, we also have

$$\lim_k \min\left\{\int c_k\,d\gamma,\ \gamma\in\Pi(\mu,\nu)\right\} \geq \int c\,d\bar\gamma \geq \min\left\{\int c\,d\gamma,\ \gamma\in\Pi(\mu,\nu)\right\}.$$

This concludes the proof. Notice that it also gives, as a byproduct, the optimality of $\bar\gamma$ for the limit cost c. □

We can now establish the validity of the duality formula for semi-continuous costs.

Theorem 1.6.6. *If c is l.s.c. and bounded from below, then* (5.2) *holds.*

Proof. Consider a sequence c_k of $k-$Lipschitz functions approaching c increasingly. Then the same duality formula holds for c_k, and hence we have

$$\begin{aligned}\min\left\{\int c_k\,d\gamma,\ \gamma\in\Pi(\mu,\nu)\right\} &= \max\left\{\int \phi d\mu + \int \psi d\nu\ :\ \phi,\psi\in C(\Omega),\ \phi(x)+\psi(y)\leq c_k(x,y)\right\}\\ &\leq \sup\left\{\int \phi d\mu + \int \psi d\nu\ :\ \phi,\psi\in C(\Omega),\ \phi(x)+\psi(y)\leq c(x,y)\right\},\end{aligned}$$

where the inequality is justified by the fact that $c_k \leq c$ and hence every pair (ϕ,ψ) satisfying $\phi(x)+\psi(y)\leq c_k(x,y)$ also satisfies $\phi(x)+\psi(y)\leq c(x,y)$. The conclusion follows by letting $k\to+\infty$, using Lemma 1.6.5. Notice that for the cost c we cannot guarantee the existence of a maximizing pair (ϕ,ψ). □

The duality formula also allows to prove the following $c-$cyclical monotonicity theorem.

Theorem 1.6.7. *If c is l.s.c. and γ is an optimal transport plan, then γ is concentrated on a $c-$CM set Γ (which will not be closed in general).*

Proof. Thanks to the previous theorem the duality formula holds, which means that, if we take a maximizing pair (ϕ_h, ψ_h) in the dual problem, we have

$$\int (\phi_h(x) + \psi_h(y))d\gamma = \int \phi_h d\mu + \int \psi_h d\nu \to \int c\, d\gamma,$$

since the value of $\int c\, d\gamma$ is the minimum of the primal problem, which is also the maximum of the dual. Yet, we also have $c(x, y) - \phi_h(x) + \psi_h(y) \geq 0$, which implies that the functions $f_h := c(x, y) - \phi_h(x) - \psi_h(y)$, defined on $\Omega\times\Omega$, converge to 0 in $L^1(\Omega\times\Omega, \gamma)$ (since they are positive and their integral tends to 0). As a consequence, up to a subsequence (not relabeled) they also converge pointwisely $\gamma-$a.e. to 0. Let $\Gamma \subset \Omega \times \Omega$ be a set with $\gamma(\Gamma) = 1$ where the convergence happens. Let us prove that this set is $c-$CM. This is true since, for any k, σ and $(x_1, y_1), \dots, (x_k, y_k) \in \Gamma$ we have

$$\begin{aligned}\sum_{i=1}^{k} c(x_i, y_i) &= \lim_h \sum_{i=1}^{k} \phi_h(x_i) + \psi_h(y_i) \\ &= \lim_h \sum_{i=1}^{k} \phi_h(x_i) + \psi_h(y_{\sigma(i)}) \leq \sum_{i=1}^{k} c(x_i, y_{\sigma(i)}). \quad \square\end{aligned}$$

Remark 8. The duality formula that we proved for l.s.c. costs c differs from that for continuous costs in that there is no existence for the dual problem (PD). Indeed, if one restates the dual problem as a maximisation among measurable functions (ϕ, ψ) instead of requiring their continuity (the supremum among measurable functions and among continuous ones is easily seen to be the same, thanks to the approximation of each measurable function via Lusin's Theorem), then one can produce an optimizer by first applying Theorem 1.6.7 to say that the optimal γ is concentrated on a $c-$CM set Γ, and then build a potential ϕ through Theorem 1.6.2. This works under the assumption that c is real-valued, and does not depend on its continuity.

We conclude this Section by stressing that the problems about the $c-$cyclical monotonicity, the duality formula and the existence of optimal potentials (ϕ, ϕ^c) in general situations are a matter of current investigation. Typical questions are the equivalence between the optimality of γ and the fact that

γ is concentrated on a $c-$CM set, as well as the existence of a solution to the dual problem in a suitable functional class (probably not C^0 nor Lipschitz functions but rather BV) for costs taking the value $+\infty$. Possible answers to these questions could find applications also in easier problems for "normal" costs (i.e. uniformly continuous and real-valued, say), since some strategies about these costs pass through decompositions that add additional constraints or through secondary variational problems.

1.6.3 Sufficient conditions for optimality and stability

The consideration of the Subsection 1.6.1 about $c-$transforms allow us to prove the following theorem giving sufficient conditions for optimality in optimal transportation whenever the cost has the standard form $c(x,y) = h(x-y)$.

Theorem 1.6.8. *Suppose that $\mu \in \mathcal{P}(\Omega)$ and $\phi \in c-conc(\Omega)$ are given, that ϕ is differentiable $\mu-$a.e. and that $\mu(\partial\Omega) = 0$. Given a strictly convex function $h : \mathbb{R}^d \to \mathbb{R}$, the map $T(x) = x - (\nabla h)^{-1}(\nabla\phi(x))$ is optimal for the transport cost $c(x,y) := h(x-y)$ between the measures μ and $\nu := T_{\#}\mu$.*

Proof. Consider the function ϕ, which is $c-$concave and hence may be written as $\phi(x) = \inf_y c(x,y) - \psi(y)$ for a certain function $\psi : \Omega \to \mathbb{R}$. The function ψ may be supposed to be $\bar{c}-$concave and hence continuous (actually, we can take $\psi = \phi^c$, for the considerations of the previous section). Fix now $x_0 \in \Omega$ such that $\nabla\phi(x_0)$ exists and $x_0 \notin \partial\Omega$. By compactness and continuity, one can say that $\inf_y c(x,y) - \psi(y)$ is realized by a certain point y_0. This gives

$$\phi(x) \leq h(x-y_0) - \psi(y_0) \text{ for every } x \quad \phi(x_0) \leq h(x_0-y_0) - \psi(y_0),$$

and hence $x \mapsto h(x-y_0) - \phi(x)$ is minimal at $x = x_0$. Notice that we defined y_0 by the optimality in y but now we use the optimality in x. As a consequence of the optimality and of the differentiability, we get

$$\nabla\phi(x_0) = \nabla h(x_0 - y_0),$$

and, inverting ∇h, we get $y_0 = x_0 - (\nabla h)^{-1}(\nabla\phi(x_0)) = T(x_0)$. This proves that, up to a $\mu-$negligible set, we have $\phi(x) + \psi(T(x)) = h(x - T(x))$. If we integrate with respect to μ we get

$$\int \phi \, d\mu + \int \psi \, d\nu = \int \phi \, d\mu + \int \psi \circ T \, d\mu = \int h(x - T(x))\mu(dx),$$

which proves the optimality of T since the last integral equals the cost of T in the problem of Monge, and we have $\int \phi \, d\mu + \int \psi \, d\nu \le (PD) \le (PK)$, since (ϕ, ψ) is admissible in (PD). $\square$

On the other hand, a more general criterion can be used for almost arbitrary costs. The main idea is that we proved that optimal plans are necessarily concentrated on $c-$CM sets, but the converse is also true: a plan which is concentrated on a $c-$CM set is optimal, at least in the most reasonable cases. We will prove this fact in the easiest case, i.e. if $c : X \times Y \to \mathbb{R}$ is continuous and X and Y are compact metric spaces. Actually, we will see in the next proof that the main ingredient is Theorem (1.6.2), which only requires finiteness of the cost, but continuity and compactness are needed to avoid integrability issues (in this case all the functions are bounded). Moreover, we will apply this sufficient criterion to another interesting issue, i.e. stability of the optimal plans, which will require these assumptions for other reasons. The fact that this converse implication stays true when these assumptions are withdrawn (and in particular for infinite costs) is a delicate matter, and we refer to **GIVE REFERENCES** for it.

Theorem 1.6.9. *Suppose that $\gamma \in \mathcal{P}(X \times Y)$ is given, that X and Y are compact metric spaces, that $c : X \times Y \to \mathbb{R}$ is a continuous cost, and that* $\mathrm{spt}(\gamma)$ *is $c-$CM. Then γ is an optimal transport plan between its marginals $\mu = (\pi_x)_\# \gamma$ and $\nu = (\pi_y)_\# \gamma$ for the cost c.*

Proof. Theorem (1.6.2) gives the existence of a $c-$concave function ϕ such that $\mathrm{spt}(\gamma)$ is contained in the set $\{(x, y) : \phi(x) + \psi^c(y) = c(x, y)\}$. Both ϕ and ϕ^c are continuous thanks to the continuity of c, and hence bounded on X and Y, respectively.

Thanks to what we know about duality we have

$$(PK) \le \int c(x, y) \, d\gamma = \int (\phi(x) + \phi^c(y)) \, d\gamma = \int \phi \, d\mu + \int \phi^c \, d\nu \le (PD) = (PK)$$

which shows that γ is optimal (and that (ϕ, ϕ^c) solves the dual problem). $\square$

We are now able to prove the following stability result.

Theorem 1.6.10. *Suppose that X and Y are compact metric spaces and that $c : X \times Y \to \mathbb{R}$ is continuous. Suppose that $\gamma_n \in \mathcal{P}(X \times Y)$ is a sequence of transport plan which are optimal for the cost c between their own marginals $\mu_n := (\pi_x)_\# \gamma_n$ and $\nu_n := (\pi_y)_\# \gamma_n$, and suppose $\gamma_n \rightharpoonup \gamma$. Then $\mu_n \rightharpoonup \mu := (\pi_x)_\# \gamma$, $\nu_n \rightharpoonup \nu := (\pi_y)_\# \gamma$ and γ is optimal in the transport between μ and ν.*

Proof. Set $\Gamma_n := \mathrm{spt}(\gamma_n)$. Up to subsequences, we can assume $\Gamma_n \to \Gamma$ in the Hausdorff topology (see below). Each support Γ_n is a $c-$CM set (Theorem (1.6.3)) and the Hausdorff limit of $c-$CM sets is also $c-$CM. Indeed, if one fixes $(x_1, y_1), \dots, (x_k, y_k) \in \Gamma$ there are points $(x_1^n, y_1^n), \dots, (x_k^n, y_k^n) \in \Gamma_n$ such that, for each $i = 1, \dots, k$, we have $(x_i^n, y_i^n) \to (x_i, y_i)$. The cyclical monotonicity of Γ_n gives $\sum_{i=1}^k c(x_i^n, y_i^n) \le \sum_{i=1}^k c(x_i^n, y_{\sigma(i)}^n)$, which implies, taking the limit $n \to \infty$, $\sum_{i=1}^k c(x_i, y_i) \le \sum_{i=1}^k c(x_i, y_{\sigma(i)})$. This proves that Γ is $c-$CM.

Yet, we also know from the convergence $\gamma_n \rightharpoonup \gamma$ together with $\Gamma_n \to \Gamma$, that $\mathrm{spt}(\gamma) \subset \Gamma$. This shows that $\mathrm{spt}(\gamma)$ is $c-$CM and implies the optimality of γ □

We finish with an easy but useful consequence of Theorem 1.6.10. To fix the notations, for a given cost $c : X \times Y \to \mathbb{R}$ and $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$, let us define

$$\mathcal{T}_c(\mu, \nu) := \min \left\{ \int_{X \times Y} c \, d\gamma \, : \, \gamma \in \Pi(\mu, \nu) \right\}.$$

Theorem 1.6.11. *Suppose that X and Y are compact metric spaces and that $c : X \times Y \to \mathbb{R}$ is continuous. Suppose that $\mu_n \in \mathcal{P}(X)$ and $\nu_n \in \mathcal{P}(Y)$ are two sequences of probability measures, with $\mu_n \rightharpoonup \mu$ and $\nu_n \rightharpoonup \nu$. Then we have $\mathcal{T}_c(\mu_n, \nu_n) \to \mathcal{T}_c(\mu, \nu)$.*

Proof. Let γ_n be an optimal transport plan for from μ_n to ν_n for the cost c. Up to subsequences, we can assume $\gamma_n \rightharpoonup \gamma$. Theorem 1.6.10 provides the optimality of γ. This means that we have (along this subsequence, but since it is arbitrary…)

$$\mathcal{T}_c(\mu_n, \nu_n) = \int_{X \times Y} c \, d\gamma_n \to \int_{X \times Y} c \, d\gamma = \mathcal{T}_c(\mu, \nu). \quad \square$$

Memo – *Hausdorff convergence*

Definition In a compact metric space X we the Hausdorff distance is defined on pair of compact subsets of X by setting

$$d_H(A, B) := \max \{ \max\{ d(x, A) \, : \, x \in B\}, \max\{ d(x, B) \, : \, x \in A\} \}.$$

Properties We have the following equivalent definition:
1) $d_H(A, B) = \max\{ |d(x, A) - d(x, B)| \, : \, x \in X\} = ||d(\cdot, A) - d(\cdot, B)||_\infty$
2) $d_H(A, B) = \inf\{\varepsilon > 0 \, : \, A \subset B_\varepsilon, \, B \subset A_\varepsilon\}$, where A_ε and B_ε stand for the $\varepsilon-$neighborhood of A and B, respectively.

Theorem (Blashke) d_H is a distance: it is positive and symmetric, it only vanishes if the two sets coincide, and satisfies triangle inequality. With this distance, the set of compact subsets of X becomes a compact metric space itself.

We refer to [10] for a detailed discussion (with proofs) of this topic. Here we only prove a simple fact that we need.

Proposition If $d_H(A_n, A) \to 0$ and μ_n is a sequence of positive measures such that $\mathrm{spt}(\mu_n) \subset A_n$ with $\mu_n \rightharpoonup \mu$, then $\mathrm{spt}(\mu) \subset A$.

Sketch of proof For each n we have $\int d(x, A_n) d\mu_n = 0$, since μ_n is supported on A_n. Since $d(x, A_n) \to d(x, A)$ uniformly and $\mu_n \rightharpoonup \mu$, thanks to the duality between uniform convergence and weak converge of measures we get $\int d(x, A_n) d\mu_n \to \int d(x, A) d\mu$. This implies $\int d(x, A) d\mu = 0$ and hence μ is concentrated on A.

1.7 Discussions

1.7.1 Probabilistic interpretation

We completely skipped up to now any interpretations involving probabilities, which are actually natural in this setting. By changing for a while our language, let us describe the situation in the following way.

Two probability measures μ and ν on some spaces (often, $\mathbb{R}^d$) are given, and can be interpreted as the laws of two random variables. Yet, we do not prescribe the joint law (which corresponds to γ) of these two random variables and we consider the optimization problem

$$\min\{\mathbb{E}[c(X, Y)] \; : \; X \sim \mu, \, Y \sim \nu\}$$

where $\mathbb{E}$ denotes the expected value (according to a probability $\mathbb{P}$ on a probability space Ω, which is not relevant for the minimization, but could be considered either "big enough" or to be part of the optimization). This expected value obviously depends on the joint law of (X, Y), which is the main unknown.

The particular case of real (or vector) valued r.v. with $c(X, Y) = |X-Y|^p$ reads

$$\min\{||X - Y||_{L^p} \; : \; X \sim \mu, \, Y \sim \nu\}.$$

More interesting is the case $p = 2$, where the problem can be expressed in terms of covariance. Indeed, let us set $x_0 = \mathbb{E}[X] = \int x \, d\mu$ and $y_0 = \mathbb{E}[Y] = \int y \, d\nu$, these two values being the mean values of X and Y. We have

$$\begin{aligned}\mathbb{E}[|X-Y|^2] = \mathbb{E}[|(X-x_0)-(Y-y_0)+(x_0-y_0)|^2] = \\ \mathbb{E}[|X-x_0|^2] + \mathbb{E}[|Y-y_0|^2] + |x_0-y_0|^2 \\ + 2\mathbb{E}[X-x_0]\cdot(x_0-y_0) - 2\mathbb{E}[Y-y_0]\cdot(x_0-y_0) \\ - 2\mathbb{E}[(X-x_0)\cdot(Y-y_0)].\end{aligned}$$

In this expression, the three first terms only depend on the laws of X and Y separately (the first being the variance of X, the second the variance of Y, and the third the squared distance between the two mean values), and the next two terms vanish (since the mean value of $X-x_0$ is 0, and so for the mean value of $Y-y_0$). The problem is hence reduced to the maximization of $\mathbb{E}[(X-x_0)\cdot(Y-y_0)]$. This means that we need to find the joint law which guarantees maximal covariance (i.e. somehow maximal dependence) of two r.v. with given laws. In the case of real valued r.v. (see next chapter) the answer will be that the optimal coupling is obtained in the case where X and Y are completely dependent, and one is an increasing function of the other, which seems logical. The multidimensional case obviously replaces this increasing behavior with other monotone behaviors.

1.7.2 Polar Factorization

A classical result in linear algebra states that every matrix $A \in \mathbb{R}^{n\times n}$ can be decomposed as a product $A = R\cdot U$, where R is symmetric and positive-semidefinite, and U is a unitary matrix, i.e. $U\cdot U^t = I$. The decomposition is unique if A is non-singular (otherwise U is not uniquely defined), and in such a case R is positive definite. Also, one can see that the matrix U of this decomposition is also a solution (the unique one if A is non singular) of

$$\max\{A : V \;:\; V\cdot V^t = I\},$$

where $A : V$ stands for the scalar product between matrices, defined as $A : V := Tr(A\cdot V^t)$.

Indeed, one can write

$$A : V = (R\cdot U) : V = Tr(R\cdot U\cdot V^t) = \sum_{i,j} R^{ii}U^{ij}V^{ij}$$

where the coordinates are chosen so that R is diagonal (and hence $R^{ii}\geq 0$). Let us use the fact that the condition $U\cdot U^t = I$ for unitary matrices imposes that the vectors $u(i)$ with components $u(i)^j := U^{ij}$ are unit vectors (let us also define the unit vectors $v(i)$ in the same way from the matrix V), so that

$$A : V = \sum_{i,j} R^{ii} U^{ij} V^{ij} = \sum_i R^{ii} u(i) \cdot v(i)$$
$$\leq \sum_i R^{ii} = Tr(R) = Tr(R \cdot U \cdot U^t) = Tr(A \cdot U^t) = A : U.$$

Also notice that maximizing $A : V$ is the same as minimizing $||A - V||^2$ (where the norm of a matrix is defined as the square root of its scalar product with itslef). Indeed, we have $||A - V||^2 = ||A||^2 - 2A : V + V : V$ and both $||A||^2$ and $V : V = Tr(V \cdot V^t) = Tr(I) = n$ are constants).

Analogously, in his first works about the quadratic optimal transport, Y. Brenier noticed that Monge-Kantorovich theory allowed to give a similar interpretation for vector fields instead of linear maps.

The statement is the following:

Theorem 1.7.1. *Given a vector map $u : \Omega \to \mathbb{R}^n$ with $\Omega \subset \mathbb{R}^n$, consider the rescaled Lebesgue measure $\mathcal{L}$ on Ω and suppose that $u_\# \mathcal{L}$ is absolutely continuous; then, one can find a convex function $\psi : \Omega \to \mathbb{R}$ and a measure-preserving map $s : \Omega \to \Omega$ (i.e. such that $s_\# \mathcal{L} = \mathcal{L}$) such that $u = (\nabla \psi) \circ s$. Moreover, both s and $\nabla \psi$ are uniquely defined a.e. and s solves*

$$\max \left\{ \int u(x) \cdot r(x)\, dx \ : \ r_\# \mathcal{L} = \mathcal{L} \right\}.$$

Notice that the statement concerning non-singular matrices exactly corresponds to this one when one takes $\Omega = B(0,1)$, since the assumption on the image measure corresponds to the non-degeneracy of the linear map $x \mapsto Ax$, the matrix R is the gradient of the convex function $x \mapsto \frac{1}{2}(Rx) \cdot x$ and the unitary matrix U is measure preserving.

We give a proof of this statement, see also [37].

Proof. Consider $\mu = u_\# \mathcal{L}$ and take the optimal transport for the quadratic cost between μ and $\mathcal{L}$. This transport is the gradient of a convex function, that we will call ψ^*. Set $s := (\nabla \psi^*) \circ u$, and notice that $s_\# \mathcal{L} = (\nabla \psi^*)_\# (u_\# \mathcal{L}) = (\nabla \psi^*)_\# \mu = \mathcal{L}$, hence s is measure preserving. Then, take the Legendre transform ψ of ψ^* and notice $(\nabla \psi) \circ s = (\nabla \psi) \circ (\nabla \psi^*) \circ u = u$ (indeed, the gradients of both ψ and ψ^* are defined a.e. and we always have $(\nabla \psi) \circ (\nabla \psi^*) = id$). In this way we have the desired decomposition.

The uniqueness of $\nabla \psi$ comes from the fact that $u = (\nabla \psi) \circ s$ with s measure-preserving implies $(\nabla \psi)_\# \mathcal{L} = \mu$ and there is only one gradient of a convex function transporting a given measure (here, $\mathcal{L}$) to another given measure (here, μ), and it is the optimal transport between them.

Once $\nabla\psi$ is unique, the uniqueness of s is obtained by composing with $\nabla\psi^*$.

Concerning the optimality of s, use the optimality of $\nabla\psi$. If we use, instead of the minimization of the quadratic cost the maximization of the scalar product, we get

$$\int \nabla\psi(x) \cdot x \, dx \geq \int_{\Omega\times\mathbb{R}^n} y \cdot x \, d\gamma(x,y)$$

for every γ with first marginal equal to $\mathcal{L}$ ad second to μ. If we consider a measure-preserving map $r : \Omega \to \Omega$ and build $\gamma = (r,u)_\#\mathcal{L}$, we get

$$\begin{aligned}\int u(x) \cdot r(x) \, dx &= \int y \cdot x \, d\gamma(x,y) \\ &\leq \int \nabla\psi(x) \cdot x \, dx = \int \nabla\psi(s(x)) \cdot s(x) \, dx = \int u(x) \cdot s(x) \, dx. \quad \square\end{aligned}$$

1.7.3 Matching problems and economic interpretations

Optimal transport problems have had a lot of "abstract" economical interpretations, where the role transport plans γ is that of matchings between different actors of an economy, and the function $c(x,y)$ do not represent anymore a cost for moving from x to y but rather a compatibility of the two objects x and y.

A first easy and well known example is that of the maximization of the productivity: if a company has a certain number of employees of different types (let us use the variable x for the types and the measure μ for the distribution of these types, i.e. the quantity of employees for each type) and some tasks to attribute (we use y for the different kind of tasks, and ν for the distribution of different tasks), and if the productivity $p(x,y)$ of the employees of type x when they work out the task y is known, then the goal of the company is to solve

$$\max\left\{\int p(x,y) d\gamma \; : \; \gamma \in \Pi(\mu,\nu)\right\}.$$

Yet, we want to analyze some more interesting problems where also the Kantorovich potential ϕ plays a role. Suppose indeed that the variable x represents the goods that are sold on the market and that μ is the distribution (in quantity) of these goods, that we consider as fixed. The variable

y plays the role of the type of consumers, and ν is their distribution. Let $u(x, y)$ be the utility of the consumer y when he buys the good x. The goal is to determine the prices of the goods and who buys what. Suppose that the price $\phi(x)$ of each good is known; then, each consumer will choose what to buy by solving $\max_x u(x, y) - \phi(x)$. Let us denote (by abuse of notation since usually we used minimization instead of maximization) this quantity as $\phi^u(y)$. We describe the choices of the consumers through a measure $\gamma \in \Pi(\mu, \nu)$ where $\gamma(A \times B)$ stands for the number of consumers of type $y \in B$ buying a good $x \in A$. The constraint $\gamma \in \Pi(\mu, \nu)$ stands for the constraints given by the offer and the demand on the market (we say that the market is cleared). Another natural condition to impose is the fact that each consumer only buys goods which are optimal for him, i.e. that γ is concentrated over the set of pairs (x, y) with $\phi^u(y) = u(x, y) - \phi(x)$, i.e. such that x is an optimal choice, given ϕ, for y.

This means that we are lead to the following problem

$$\text{find } (\gamma, \phi) \text{ such that } \gamma \in \Pi(\mu, \nu) \text{ and } \phi(x) + \phi^u(y) = u(x, y)\ \gamma - a.e.,$$

and we can see here an optimal transport problem for the maximisation of the cost u. Indeed, the pairs (γ, ϕ) can be characterized as the solutions of the Kantorovich problem

$$\max \int u\, d\gamma\ :\ \gamma \in \Pi(\mu, \nu)$$

and of the dual problem

$$\min \int \phi\, d\mu + \int \phi^u\, d\nu.$$

By the way, this optimal transport interpretation shows that a simple equilibrium condition (clearing the market and only using rational choices of the consumers) implies good news, i.e. the fact that the general satisfaction $\int u\, d\gamma$ is maximized.

One could remark that, even in the most likely cases, the values of the functions ϕ and ϕ^u are only defined up to additive constants, which means that this procedures only selects the relative differences of prices between goods, and not the complete price chart. Yet, under natural assumptions to guarantee uniqueness up to additive constants of the solution of the dual problem (which is the case, for instance, x and y belong to Euclidean spaces, $u(x, y)$ is differentiable and one of the two measures has strictly positive density a.e.), as soon as the price of one special good $\bar{x}$ is known, then everything is uniquely determined. A typical example can be obtained if the "empty" good is included in the market. Let us denote by x_0 a special good which corresponds to "not buying anything at all". We can normalize utilities by setting $u(y, x_0) = 0$ for every y, but this is not really important. However, we can assume that no seller will charge a price different than 0 when selling this empty good. This fixes $\phi(x_0) = 0$ and allows for computing the other prices. If the measure μ is such that $\mu(\{x_0\}) > 0$, then this means that there are not enough goods on the market so as to satisfy all the demand, and that some of the consumers will stay "out of the market", i.e. they will buy the empty good.

Another interesting problem is obtained when the measure μ on the sets of goods is not fixed, and we consider a class of goods which are sold by a unique compony which has the monopole on this market. We suppose that the set X of feasible goods (those that the company knows how to produce) is known, and that it includes the empty good x_0. The goal of the company is, hence, to select at the same time a measure μ (how much production for each type of goods) and a price list $\phi : X \to \mathbb{R}$, satisfying $\phi(x_0) = 0$, so as to maximize its profit, assuming that each consumer will buy according to the maximization of $x \mapsto u(x - y) - \phi(x)$.

This problem may be expressed in two ways: indeed, the equilibrium condition of the previous problem (when μ was fixed) induces a relationship between μ and ϕ, and this allows to consider either one or the other as the variable of the problem.

The easiest idea is to think that the company chooses the price list ϕ, that every consumer y selects its optimal good $X(y) \in \operatorname{argmin}_x u(x, y) - \phi(x)$, and that the total income (supposing zero production costs) of the company is $\int \phi(X(y))\, d\nu(y) = \int \phi\, d\mu$ for $\mu = X_\# \nu$. The measure μ is also the measure of the real production of goods that the company will implement. In other words, it will adapt its production to the choice of the consumers. Anyway ϕ has then to be chosen so as to maximize $\int \phi(X(y))\, d\nu(y)$, taking into account that the map X depends on ϕ. The problem can also be adapted so as to take into account production costs, in the form of a function $c : X \to \mathbb{R}$, and the maximization becomes that of $\int (\phi - c)(X(y))\, d\nu(y) = \int (\phi - c)\, d\mu$.This formulation should be revisited in case the optimal point $X(y)$ is not uniquely defined (since different optimizers for the consumer could lead to very different incomes for the company, see [74] where different relaxed formulations, from an optimistic or pessimistic point of view, are proposed). Anyway, we will see in a while that in a very reasonable case there is no ambiguity.

The other approach is complementary to this one: we can give the company the right to select the production measure μ, then the market is cleared thanks to the previous considerations, thus determining a measure $\gamma \in \Pi(\mu, \nu)$ and a potential ϕ. The goal of the company is to maximize $\int(\phi - c)d\mu$, where μ is the unknown and ϕ depends on μ. Notice that this is an optimization problem in the space of measures which involves the Kantorovitch potential ϕ in the transport from a given measure ν to μ.

The case which is studied the most is the case where both x and y belong to convex sets $X, Y \subset \mathbb{R}^d$ and the function $u(x, y) = x \cdot y$ is a scalar product. This is natural if one thinks that goods can be described through a set of parameters (think at cars: we can use number of places, size of the luggage van, fuel consumption, maximal speed...) and that consumers are described through the importance that they give to each one of these parameters. Customers of type $y = (y_1, \dots, y_n)$ are ready to spend y_i extra unit of money for every unit increase in the $i-$th parameter of the good. This means that the values of the coefficients y_i give the relative value of each feature of the good compared to money (in the case of cars, we can expect that they will be higher for richer people, who care less about money, or for people who like a lot, or need a lot, cars). The empty good can be chosen as the point $x_0 = 0 \in \mathbb{R}^d$.

In this case it is easy to check that ϕ^u is simply the Legendre transform ϕ^* of convex analysis. Standard considerations in convex analysis imply that, if ϕ^* is differentiable at y, then the optimal choice x in $\max_x x \cdot y - \phi(x)$ is exactly $x = \nabla\phi^*(y)$ and gives uniqueness a.e. in case ν is absolutely continuous (due to differentiability properties of convex functions). Then, the map $\nabla\phi^*(y)$ sends ν onto μ and, conversely, $\nabla\phi$ sends μ onto ν. This means that the optimization problem reads

$$\max\left\{\int \phi_\mu \, d\mu \ : \ \mu \in \mathcal{P}(X)\right\},$$

where, for every μ, ϕ_μ is the unique convex function with $(\nabla\phi)_\#\mu = \nu$ and $\phi(x_0) = 0$. It can also be rephrased as an optimization problem in the class of convex functions, if one takes $\psi = \phi^*$ as an unknown, since in this case one should maximize $\int \psi^* \, d(\nabla\psi)_\#\nu = \int (\psi^*(\nabla\psi(y)) - c(\nabla\psi(y))) \, d\nu$, but we know $\psi^*(\nabla\psi(y)) + \psi(y) = y \cdot \nabla\psi(y)$, which turns the problem into

$$\max\left\{\int (y \cdot \nabla\psi(y) - \psi(y) - c(\nabla\psi(y))) \, d\nu(y) \ : \ \min\psi = 0, \psi \text{ convex}\right\},$$

the constraint $\min\psi = 0$ coming from $0 = \psi^*(0) = \sup_y 0 \cdot y - \psi(y)$. This is a standard calculus of variations problem, but considered in the restricted class of convex functions. It has been deeply studied in [96].

1.7.4 Multi-marginal and martingale transport problems

This last part of the theory of optimal transport is becoming more and more popular and find applications in many different fields. We will only give a brief sketch of what is done or can be done on this topic.

It is not difficult to imagine that, instead fo using measures $\gamma \in \mathcal{P}(X \times Y)$ with prescribed marginals on the two factors X and Y, one could consider more than two marginals. Take some spaces $X_1, X_2, \dots, X_N$, define $\mathcal{X} := X_1 \times X_2 \times \dots \times X_N$, take a cost function $c : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$, some measures $\mu_i \in \mathcal{P}(X_i)$, and solve

$$\min \left\{ \int_{\mathcal{X}} c(x_1, \dots, x_N) d\gamma \ : \ \gamma \in \mathcal{P}(\mathcal{X}),\ (\pi_i)_\# \gamma = \mu_i \right\},$$

where $\pi_i : \mathcal{X} \to X_i$ is the canonical projection map. Duality could be performed in the same way as for the two-marginal case, getting to problems such as

$$\max \left\{ \sum_i \int_{X_i} \phi_i \, d\mu_i \ : \ \phi_i : X_i \to \mathbb{R},\ \sum_i \phi_i(x_i) \le c(x_1, \dots, x_N) \right\}.$$

The existence of a transport map (in the sense that there exist maps $T_i : X_1 \to X_i$, i≥ 2, such that the optimal γ is of the form $\gamma = (id, T_2, \dots, T_N)_\# \mu_1$) is a much more delicate question than the usual case.

This kind of problems may have several different interpretations. In most of the cases the fact that the optimal γ is induced or not from a transport map is not crucial.

For instance, in economics, one can think at contracts with more than two agents (when a seller given a good to a buyer in exchange of money, this is a contract between two agents; when somebody wants to build a house, he buys dome land and hires a carpenter and an architect, there are at least four agents). A point in the space $\mathcal{X}$ means in this case a possible contract between the agents $x_1, x_2, \dots, x_N$ and $c(x_1, \dots, x_N)$ the global utility of this contract.

A completely different setting is found in recent works from Physics, in the framework of Electronic Density Function Theory (see [41]). To describe in a very simplified way these issues, consider a family of N electrons moving around a nucleus. The law of the position of a single electron is described through a probability measure $\mu \in \mathcal{P}(\mathbb{R}^3)$ and this is supposed to be independent of time and of the electron itself. The unknown is the joint law of the positions of the N electrons, i.e. a probability $\gamma \in \mathcal{P}((\mathbb{R}^3)^N)$

with marginals μ. What is known is that the electrons minimize a joint interaction energy given by

$$c(x_1, \dots, x_N) := \sum_{i<j} \frac{1}{|x_i - x_j|},$$

so that γ should minimize $\int c\, d\gamma$ under marginal constraints. The fact that c is neither bounded nor finite (it goes to $+\infty$ as soon as two electrons approach to each other) is a strong difficulty of this problem even in the "easy" case $N = 2$. It is by the way an interesting transport problem where configurations with $x_i = x_j$ cost the most, so that the problem is meaningful and intriguing even when the marginals are all equal (which would not be the case for costs increasingly depending on the distances).

In evolution problems and particularly in fluid mechanics, a natural Lagrangian point of view is the description of the global movement by using a measure on the set of possible trajectories. This typically requires the use of probabilities on the set of continuous or Lipschitz paths on $[0, 1]$ but any kind of time-discretization of such a setting can be interpreted in terms of measures on Ω^N, where Ω is the state space and N represents the number of time steps, each one of length $\tau = 1/N$. In this case the marginals are somehow "ordered" and the cost could take into account this feature: for instance, natural costs could be

$$c(x_1, \dots, x_N) = \sum_{i=1}^{N-1} |x_i - x_{i+1}| \quad \text{or} \quad c(x_1, \dots, x_N) = \sum_{i=1}^{N-1} \frac{|x_i - x_{i+1}|^2}{\tau},$$

which are, respectively, the total length of a discrete path or the total kinetic energy (the term $|x - y|^2/\tau$ standing for the integration over a time interval of length τ of a speed $(x-y)/\tau$). Let us mention that the continuous counterpart of this last cost, i.e. $c(\omega) := \int_0^1 |\omega'(t)|^2 dt$ for $\omega \in H := H^1([0, 1]; \Omega)$ appears in Brenier's variational formulation of the incompressible Euler equation, which gives rise to a minimization problem such as

$$\min \left\{ \int_H c(\omega)\, dQ(\omega) \ : \ Q \in \mathcal{P}(H),\ (\pi_t)_\# Q = \mathcal{L}_\Omega,\ (\pi_0, \pi_1)_\# Q = \gamma_{0,1} \right\},$$

where $\pi_t : H \to \Omega$ is the evaluation map $\pi_t(\omega) := \omega(t)$, $\mathcal{L}_\Omega$ is the rescaled Lebesgue measure over Ω and $\gamma_{0,1}$ is a fixed measure in $\mathcal{P}(\Omega \times \Omega)$ with both marginals equal to $\mathcal{L}_\Omega$

Last but not least, we present two problems from mathematical finance involving multi-marginal optimal transport. To analyze them, one should know what is an option. An option is a contract giving the owner the right to buy or sell some financial assets at some given conditions, which could bring a monetary advantage. The most typical case is the European Call option, saying "at time T you will have the right to buy this asset at price K: should the price be S_T, higher than K, then you gain $S_T - K$; should it be lower, than simply do not use the option, so that finally the gain is $(S_T - K)_+$. More generally, we call option any financial contract giving the owner a gain which is a function of the value S_T at time T of a given asset (a share, for instance), this value being a random variable (based on a probability $\mathbb{P}$) evolving in time. More exotic options such that their payoff depends on the whole history $(S_t)_t$ of the asset value also exist. One of the main issue of financial mathematics is to give formulas to compute the correct price for these contracts. This is based on the no-arbitrage assumption: the only reasonable price for a contract is the one which avoids the existence of arbitrage opportunities on the market, i.e. the possibility of buy-and-sell this contract together with the related underlying asset and to produce a positive amount of money out of nothing (more precisely, since all the values are random variables: to produce with probability 1 a non-negative amount of money, which is strictly positive with strictly positive probability). A general theorem in financial mathematics states that all these no-arbitrage prices are actually the expected value of the contract according to a probability $\mathbb{Q}$ which is in general not $\mathbb{P}$ and which has the property that all the asset values on the market are martingales under $\mathbb{Q}$. This means, for instance, that the price of a European Call option should be $\mathbb{E}^{\mathbb{Q}}[(S_T - K)_+]$. Notice that the knowledge of this price for every K implies the knowledge of the law of S_T under $\mathbb{Q}$.

A first problem that one could find in finance is the following: consider a complicated option with payoff depending on the values at time T of several assets S^i_T at the same time T. One should compute $\mathbb{E}^{\mathbb{Q}}[f(S^1_T, S^2_T, \ldots, S^N_T)]$. Yet, this depends on the joint law of the different assets under the unknwon probability $\mathbb{Q}$. If we suppose that this option is new on the market, but that all the European Calls on each of the assets S^i are regularly sold on the market for each strike price K, then this means that we know the marginal laws μ^i under $\mathbb{Q}$ of each asset, and we only lack the joint law. We can have an estimate of the correct price of the option by computing

$$\min / \max \left\{ \int f(x_1, x_2, \ldots, x_N)\, d\gamma \ : \ (\pi_i)_\# \gamma = \mu_i \right\}$$

which is nothing but a transport problem. We ignored here the constraints on the fact that $\mathbb{Q}$ should make the asset value process a martingale since they are already included in the law of each asset and do not involve the joint law.

Another problem, much more intriguing, can be obtained when one considers one only asset, but uses several marginals for several time steps. This allows to consider exotic options depending on the whole history of the asset value. Consider a process $(S_t)_t$ defined for various instants of time (we will consider discrete time models for the sake of simplicity: $t = 0, 1, \ldots, N$). Suppose that an option pays $f(S_0, S_1, \ldots, S_N)$, a price depending on the trajectory $(S_t)_t$. Suppose that European Calls are traded in the market for each maturity time t and each strike price K, which provides the law of S_t under the unknown martingale measure $\mathbb{Q}$. Then, the price of the option may be estimated by solving

$$\min / \max \left\{ \mathbb{E}^{\mathbb{Q}}[f(S_0, S_1, \ldots, S_N)] \ : \ (S_i)_{\#}\mathbb{Q} = \mu^i, \ \ (S_t)_t \text{ is a martingale under } \mathbb{Q} \right\}.$$

This can also be expressed in terms of the measure $\gamma = (S_0, S_1, \ldots, S_N)_{\#}\mathbb{Q}$ as

$$\min / \max \left\{ \int f(x_1, x_2, \ldots, x_N) \, d\gamma \ : \ (\pi_i)_{\#}\gamma = \mu_i, \ \gamma \in \mathrm{Mart}_T \right\},$$

where the set Mart_T is the set of discrete time martingale measures γ, satisfying for each $t \leq T - 1$

$$\int x_{t+1}\phi(x_t) d\gamma = \int x_t \phi(x_t) d\gamma = \int x\phi(x) \, d\mu^t(x). \tag{7.4}$$

These problems, for which a duality theory is also possible, are now called martingale optimal transport problems and are out of the framework of usual transport problems that we presented before. By the way, in general there is no reason to hope for the existence of an optimal transport map. Indeed, even in the easiest case, i.e. two marginals, this will not be the case. The reason is the fact that the only martingale measure which is of the form $\gamma_T = (id, T)_{\#}\mu$ is that with $T = id$. This can be seen from Equation (7.4), since it gives

$$\int T(x)\phi(x) d\mu = \int y\phi(x) d\gamma_T = \int x\phi(x) d\gamma_T = \int x\phi(x) \, d\mu,$$

which implies, ϕ being an arbitrary test function, $T(x) = x$ $\mu-$a.e. This means that, unless $\nu = \mu$, the martingale transport problem with two marginals admits no admissible transport plan issued by a transport map. On the contrary, it is possible to realize γ as a combination of two transport maps, and optimality results of this kind are proven in [14] (**OTHER REF BY TOUZI**)

Chapter 2

One-dimensional issues

2.1 The monotone transport map in the 1D case

We have seen in Section 1.3 that the optimal transport map for the quadratic case in dimension one is well-defined as soon as the starting measure μ has no atoms and is monotone non-decreasing. If one also uses the sufficient optimality condition given in Theorem 1.6.8, then one can also obtain a uniqueness result. Indeed, any monotone map T sending a non-atomic μ onto ν should be optimal for the quadratic cost by Theorem 1.6.8; then, by applying Remark quad 1D and the proof of Theorem 1.3.2, we get the uniqueness of the optimizer, and hence the uniqueness of the monotone map transporting μ onto ν.

Yet, we prefer to give a full characterization of the monotone transport map between two given measures in terms of their cumulative distribution functions and independent of optimal transport considerations.

Definition 8. Given a probability measure $\mu\mathcal{P}(\mathbb{R})$ we define its cumulative distribution function F_μ through

$$F_\mu(x) = \mu((-\infty, x]).$$

The cumulative distribution function F_μ is easily seen to be non-decreasing and right-continuous since if $x_n \to x^+$ then $(-\infty, x] = \cap_n(-\infty, x_n]$ and hence $\mu((-\infty, x]) = \lim_n \mu((-\infty, x_n]) = \inf_n \mu((-\infty, x_n])$. It is continuous at any point where μ has no atom since if $x_n \to x^-$, then $\lim_n \mu((-\infty, x_n]) = \mu((-\infty, x[)$.

Now, consider two measures $\mu, \nu \in \mathcal{P}(\mathbb{R})$.

Theorem 2.1.1. *Given $\mu, \nu \in \mathcal{P}(\mathbb{R})$, suppose that μ is atomless. Then, there exists unique a non-decreasing map $T_{mon} : \mathbb{R} \to \mathbb{R}$ such that $(T_{mon})_{\#}\mu = \nu$.*

Proof. First, let us build one such a map. Let us consider the cumulative distribution function F_μ and F_ν, and define a map T through

$$T(x) := \min\{t \in \mathbb{R} \,:\, F_\nu(t) \geq F_\mu(t)\}.$$

To prove that the above minimum is well defined, just use the fact that F_ν is right-continuous, so that if a sequence $t_n \to t^+$ satisfies $F_\nu(t_n) \geq c$ for a constant c, then we also have $F_\nu(t) \geq c$. Hence, the minimum is attained, provided the set of t satisfying $F_\nu(t) \geq c$ is bounded from below, which is true for $c > 0$ (since $\lim_{t\to-\infty} F_\nu(t) = 0$). Since for $c = 0$ this does not work, the function T is well defined for every x such that $F_\mu(x) > 0$. The set $A = \{x \,:\, F_\mu(x) = 0\}$ is an interval of the form $]-\infty, a]$ (this interval is closed since F_μ is continuous, as a consequence of μ being atomless, and is bounded from above since $\lim_{t\to+\infty} F_\mu(t) = 1$). Moreover, we have $\mu(A) = 0$, by definition of F_μ, which means that T is well-defined $\mu-$a.e.

The fact that T is monotone non-decreasing is obvious; we just have to prove $T_{\#}\mu = \nu$. To check this, it is enough to prove $\nu(]-\infty, b]) = \mu(T^{-1}(]-\infty, b]))$ for every b. The condition $T(x) \leq b$ means $F_\nu(b) \geq F_\mu(x)$. Suppose $F_\nu(b) < 1$: in this case the set $B = T^{-1}(]-\infty, b])$ is an interval of the form $]-\infty, d]$ (since F_μ is continuous and B must be bounded), and $F_\mu(d) = F_\nu(b)$. This implies $\mu(B) = F_\nu(b)$ which is the desired result. On the other hand, if $F_\nu(b) = 1$, then the set $B = T^{-1}(]-\infty, b])$ coincides with $\mathbb{R}$ and one has $\mu(B) = 1$, which concludes the proof.

We now pass to the proof of uniqueness. Let us define the function $G : \mathbb{R} \to \mathbb{R}$ given by $G_-(y) := \nu(]-\infty, y[)$, which coincides with the left limit of F_ν at every point y.

Consider any non-decreasing function T such that $T_{\#}\mu = \nu$.

From the fact that T is non-decreasing we have $T^{-1}(]-\infty, T(x)]) \supset]-\infty, x]$. We deduce

$$F_\mu(x) = \mu(]-\infty, x]) \leq \mu(T^{-1}(]-\infty, T(x)])) = \nu(]-\infty, T(x)]) = F_\nu(T(x))$$

Moreover, $T^{-1}(]-\infty, T(x)[) \subset]-\infty, x[$, which gives $G(T(x)) \leq F(x)$.

In particular, this means that for every x, the point $T(x)$ must be such that

$$F(x) \in [G(T(x)), F_\nu(T(x))].$$

The question is: is it possible to have different values $t_1 <_2$ such that $F(x) \in [G(t_1), F_\nu(t_1)] \cap [G(t_2), F_\nu(t_2)]$? should it be the case, we should have si

$G(t_2) \leq F_\nu(t_1)$, thus G, which is non-decreasing, would be constant on $[t_1, t_2[$ and equal to $F_\mu(x)$.

Hence, the points where different values of T are allowed are the pre-images through F_μ of flat parts of F_ν. The intervals of positive length where F_ν is flat are countable, and hence the set L of corresponding values are also countable. Moreover, for each value $\ell \in \mathbb{R}$, we have $\mu(F_\mu^{-1}(\{\ell\})) = 0$, since the pre-image of a point is an interval, and the fact that F_μ is constant on an interval means that there is no mass on it (up to, possibly, an atom at the first point, which is not possible for μ). This means that $\mu(F_\mu^{-1}(L)) = 0$,which shows that, outside a negligible set, the value of the map T is prescribed, and gives uniqueness.

We just proved the existence, and then the uniqueness of a monotone map with given image measure. We will call this map T_{mon}. □

Remark 9. Notice that the previous proof was complicated by the possibility that the cumulative distribution functions could be either discontinous (that of ν) or not strictly increasing. Should F_ν be continuous and strictly monotone (which means that ν is non-atomic and supported on the whole $\mathbb{R}$), then one would simply have

$$T_{mon} = (F_\nu)^{-1} \circ F_\mu.$$

In the next section we will see that the map T_{mon} that we have just built optimizes a whole class of transport costs. To prove it,we will need the following characterization of T_{mon}.

Proposition 2.1.2. *Let $\gamma \in \Pi(\mu, \nu)$ be a transport plan between two measures $\mu, \nu \in \mathcal{P}(\mathbb{R})$. Suppose that it satisfies the property*

$$(x, y), (x', y') \in \operatorname{spt}(\gamma),\ x < x' \ \Rightarrow\ y \leq y'$$

and that μ has no atoms. Then $\gamma = \gamma_{T_{mon}}$, i.e. γ is induced by a transport map, and, more precisely, by T_{mon}. In particular one only transport plan can satisfy this condition, provided μ and ν are fixed and μ non-atomic.

Proof. For any point $x \in \mathbb{R}$ one can define the interval I_x as the minimal interval I such that $\operatorname{spt}(\gamma) \cap \{x\} \times \mathbb{R} \subset \{x\} \times I$. The assumption on γ implies that the interior of all these intervals are disjoint (and ordered). In particular, there can be at most a countable quantity of points such that I_x is not a singleton. Since these points are $\mu-$negligible (as a consequence of μ being atomless), we can define $\mu-$a.e. a map T such that γ is concentrated on the graph of T. This map will be monotone non-decreasing as a consequence of the implication above, and this gives $T = T_{mon}$ since we already know the uniqueness of a non-decreasing map with fixed marginals. $\square$

It is interesting to investigate the regularity of T_{mon} according to the regularity of the densities of μ and ν. If we suppose that μ has no atoms and that ν is supported on a whole interval $[a, b] \subset \mathbb{R}$ we get the equality $F_\nu \circ T_{mon} = F_\mu$. If $\mu = f(x)dx$ and $\nu = g(y)dy$ are absolutely continuous, then F_μ has one degree of regularity more than f and F_ν one more than g. Moreover, if $g \neq 0$ then $F'_\nu \neq 0$. This means that T_{mon} has one regularity more than the worse between f and g, provided $g \neq 0$.

2.2 The optimality of T_{mon}

Now that we know quite well the properties-, definitions and characterizations of the map T_{mon}, we can see that it is – in the one-dimensional case – the optimal map for several different costs, and not only for the quadratic one. This is really specific to the one-dimensional case, it will not be true in higher dimension.

Theorem 2.2.1. *Let $h : \mathbb{R} \to \mathbb{R}$ be a strictly convex function, and $\mu, \nu \in \mathcal{P}(\mathbb{R})$ be compactly supported measures with μ non-atomic. Then the optimal transport problem with cost $c(x, y) = h(y - x)$ over $\Pi(\mu, \nu)$ has a unique solution, which is given by $\gamma_{T_{mon}}$.*

Moreover, if the strict convexity assumption is withdrawn and h is only convex, then the same T_{mon} is actually an optimal transport map, but no uniqueness is guaranteed anymore.

Proof. We will use the fact that the support of any optimal γ is a $c-$CM set Γ. This means in particular that $(x, y), (x', y') \in \Gamma$ implies

$$h(y - x) + h(y' - x') \leq h(y' - x) + h(y - x'). \tag{2.1}$$

We only need to show that this implies (in the strictly convex case) a monotone behavior : we will actually deduce from (2.1) that $x < x'$ implies $y \leq y'$, and this will allow to conclude as we noticed above.

To prove $y \leq y'$ suppose by contradiction $y > y'$ and denote $a = y-x$, $b = y'-x'$ and $c = x'-x$. Condition (2.1) reads $h(a)+h(b) \leq h(b+c)+h(a-c)$. Morevoer, the assumption $y < y$ implies $b + c < a$. We also need to recall that $c > 0$ $(x < x')$: this implies that $b + c$ and $a - c$ are locates in the segment between b and a (and $b < a$). More precisely, we have

$$b + c = (1 - t)b + ta, \quad a - c = tb + (1 - t)a, \quad \text{for } t = \frac{c}{a - b} \in]0, 1[.$$

Thus, convexity yields

$$\begin{aligned} h(a)+h(b) &\leq h(b+c)+h(a-c) \\ &< (1-t)h(b)+th(a)+th(b)+(1-t)h(a)=h(a)+h(b). \end{aligned}$$

This gives a contradiction and proves the thesis in the strictly convex case.

The statement when h is only convex is obtained by approximation through the transport problem for the cost $c_\varepsilon(x,y) := h(y-x)+\varepsilon|y-x|^2$. In this case we can say that $\gamma_{T_{mon}}$ optimizes the cost $\int c_\varepsilon\, d\gamma$ and hence

$$\int \left(h(y-x)+\varepsilon|y-x|^2\right) d\gamma_{T_{mon}} \leq \int \left(h(y-x)+\varepsilon|y-x|^2\right) d\gamma,$$

for all $\gamma \in \Pi(\mu,\nu)$. Passing to the limit as $\varepsilon \to$, since $\int |x-y|^2 d\gamma, \int |x-y|^2 d\gamma_{T_{mon}} < +\infty$, we get that $\gamma_{T_{mon}}$ also optimizes the cost c. $\square$

Notice that the assumptions on the compactness of the support has exactly been put so as to guarantee the finiteness of the integral for the quadratic cost. Yet, it is possible to choose other strictly convex approximation in much more general situations, but we will not enter into details about these technicalities.

Easy examples where c is not strictly convex and T_{mon} is not the unique optimal tranport map may be built as follows.

Example 1 (Linear costs)**.** Suppose that $c(x,y)=a(x-y)$, the map $a : \mathbb{R}^d \to \mathbb{R}$ being linear. In this case any transport plan γ is optimal, and any transport map as well. This can be easily seen if one writes

$$\int a(x-y)\, d\gamma = \int a(x)\, d\gamma - \int a(y)\, d\gamma = \int a(x)\, d\mu - \int a(y)\, d\nu,$$

which shows that the result does not depend on γ but only on its marginals. This general example works for μ,ν compactly supported (so that we do not have any problem of integrability of $a(x)$ and $a(y)$), and in any dimension. Hence, also in dimension one.

Example 2 (Distance cost on the line)**.** Suppose that $c(x,y)=|x-y|$, and that $\mu,\nu \in \mathcal{P}(\mathbb{R})$ are such that $\sup \operatorname{spt}(\mu) < \inf \operatorname{spt}(\nu)$. In this case as well any transport plan γ is optimal, and any transport map as well. This can be seen by noticing that for every $(x,y) \in \operatorname{spt}(\mu) \times \operatorname{spt}(\nu)$ we have $c(x,y)=y-x$, which is again a linear cost.

Example 3 (Book-shifting)**.** Consider $c(x, y) = |x-y|$, $\mu = \frac{1}{2}\mathcal{L}^1_{[0,2]}$ and $\nu = \frac{1}{3}\mathcal{L}^1_{[1,3]}$. Then $T_{mon}(x) = x + 1$ is the monotone transport plan transporting μ onto ν. Its cost is $M(T) = \int |T_{mon}(x) - x| \, d\mu = 1$. Yet, the transport map T given by

$$T(x) = \begin{cases} x+2 & \text{if } x \leq 1, \\ x & \text{if } x > 1, \end{cases}$$

also satisfies $T_{\#}\mu = \nu$ and $\int |T(x) - x| \, d\mu = \frac{1}{2}\int_0^1 2 \, dx = 1$, and is optimal as well.

Starting from the fact that the optimal transport for all these costs is the monotone one we can express the cost for sending a given measure μ onto another measure ν in terms of their cumulative distribution functions F_μ and F_ν. We will do it in the easier case where both cumulative distribution functions are continuous and strictly increasing (i.e. the two measures have no atoms and have full supports).

Proposition 2.2.2. *Suppose that I and J are subintervals of $\mathbb{R}$, and that $\mu \in \mathcal{P}(I)$ and $\nu \in \mathcal{P}(J)$ are non-atomic probability measures such that F_μ and F_ν are strictly increasing on I and J, respectively. Then, the minimal value of the transport problem (PK) from μ to ν with cost $c(x, y) = h(x-y)$ if h is a convex function is given by*

$$\int_0^1 h(F_\nu^{-1} - F_\mu^{-1}) d\mathcal{L}^1.$$

If $h(z) = |z|$ then this also coincides with $\int_0^1 |F_\mu - F_\nu|$.

Proof. First, notice that if μ is non-atomic and F_μ is strictly increasing, then $(F_\mu)_{\#}\mu = \mathcal{L}^1_{|[0,1]}$. Indeed, $F_\mu : I \to [0, 1]$ and for every $t \in [0, 1]$, we have $\{x : F_\mu(x) \leq t\} = \{x : x \leq x_t\}$, where x_t is such that $F_\mu(x_t) = t$. Then $(F_\mu)_{\#}\mu(I\cap]-\infty, t]) = \mu(\{x : F_\mu(x) \leq t\}) = F_\mu(x_t) = t$.

Then we consider the optimal transport problem for the cost $c(x, y) = h(x - y)$ from μ to ν: we know that $T = (F_\nu)^{-1} \circ F_\mu$ is optimal, and hence the optimal cost equals

$$\int_I h\big((F_\nu)^{-1}(F_\mu(x)) - x\big) d\mu(x) = \int_0^1 h(F_\nu^{-1} - F_\mu^{-1}) d\mathcal{L}^1,$$

where the last equality comes from the fact that $x = \big((F_\mu) \circ (F_\mu)^{-1}\big)(x)$ and from $(F_\mu)_{\#}\mu = \mathcal{L}^1_{|[0,1]}$.

Finally, the particular case of $h(z) = |z|$ may be treated by geometric consideration: indeed, the integral $\int_0^1 \left|(F_\nu)^{-1} - (F_\mu)^{-1}\right|$ equals the area of the part of the strip $[0,1] \times \mathbb{R}$ bounded by the graphs of $(F_\mu)^{-1}$ and $(F_\nu)^{-1}$. In order to pass from the inverse functions to the direct ones it is enough to turn the head and look at the same strip from the variable t instead of x. More precisely, if $f, g : [a,b] \to \mathbb{R}$ are continuous strictly increasing functions, we have

$$\begin{aligned}\int |f-g| &= \mathcal{L}^2\big(\{(x,t) \in [a,b] \times \mathbb{R} : f(x) < t < g(x) \text{ or } g(x) < t < f(x)\}\big) \\ &= \mathcal{L}^2\big(\{(x,t) \in [a,b] \times \mathbb{R} : f(x) < t < g(x)\}\big) \\ &\qquad + \mathcal{L}^2\big(\{(x,t) \in [a,b] \times \mathbb{R} : g(x) < t < f(x)\}\big).\end{aligned}$$

Then, by Fubini's theorem, we have

$$\begin{aligned}&\mathcal{L}^2\big(\{(x,t) \in [a,b] \times \mathbb{R} : f(x) < t < g(x)\}\big) \\ &\qquad = \int_{\mathbb{R}} \mathcal{L}^1\big(\{x \in [a,b] : x < f^{-1}(t) \text{ and } g^{-1}(t) < x\}\big) dt \\ &\qquad\qquad = \int_{\mathbb{R}} \mathcal{L}^1\big(\{x \in [a,b] : g^{-1}(t) < x < f^{-1}(t)\}\big) dt.\end{aligned}$$

Analogously,

$$\begin{aligned}&\mathcal{L}^2\big(\{(x,t) \in [a,b] \times \mathbb{R} : g(x) < t < f(x)\}\big) \\ &\qquad\qquad = \int_{\mathbb{R}} \mathcal{L}^1\big(\{x \in [a,b] : f^{-1}(t) < x < g^{-1}(t)\}\big) dt.\end{aligned}$$

and, summing up,

$$\begin{aligned}&\mathcal{L}^2\big(\{(x,t) \in [a,b] \times \mathbb{R} : f(x) < t < g(x) \text{ or } g(x) < t < f(x)\}\big) \\ &\quad = \int_{\mathbb{R}} \mathcal{L}^1\big(\{x \in [a,b] : g^{-1}(t) < x < f^{-1}(t) \text{ or } f^{-1}(t) < x < g^{-1}(t)\}\big) dt \\ &\qquad\qquad = \int_{\mathbb{R}} |f^{-1}(t) - g^{-1}(t)| dt\end{aligned}$$

which allows to conclude when applied to $f = (F_\mu)^{-1}$ and $g = (F_\nu)^{-1}$. □

2.3 The Knothe transport

The Knothe transport, also known as Knothe-Rosenblatt rearrangement is a special transport map, which has a priori nothing to do with optimal transport, that may be associated to two given measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ and built by only using one-dimensional monotone maps. It was independently proposed by Rosenblatt [95] for statistical purposes and by Knothe [81]. The main interesting point of such a map is its computational simplicity. We will explain the principle of this transport map in the simpler case where the two measures are absolutely continuous: $\mu = f(x)dx$, $\nu = g(y)dy$.

Let us first define the densities f^d and g^d via

$$\hat{f}^d(x_d) = \int f(t_1, t_2, \dots, t_{d-1}, x_d)\, dt_1 dt_2 \dots dt_{d-1},$$
$$\hat{g}^d(y_d) = \int g(s_1, s_2, \dots, s_{d-1}, y_d)\, ds_1 ds_2 \dots ds_{d-1}$$

as well as, for $k < d$

$$\hat{f}^k(x_k, x_{k+1}, \dots, x_d) = \int f(t_1, \dots, t_{k-1}, x_k, \dots, x_d)\, dt_1 \dots dt_{k-1},$$
$$\hat{g}^k(y_k, y_{k+1}, \dots, y_d) = \int f(s_1, \dots, s_{k-1}, y_k, \dots, y_d)\, ds_1 \dots ds_{k-1}.$$

It is easy to check that $\hat{f}^d$ and $\hat{g}^d$ are the densities of $(\pi_d)_\# \mu$ and $(\pi_d)_\# \nu$, where $\pi_d : \mathbb{R}^d \to \mathbb{R}$ is the projection on the last variable. More generally, $\hat{f}^k$ and $\hat{g}^k$ are the densities of $\mu^k = (\pi_{k,d})_\# \mu$ and $\nu^k = (\pi_{k,d})_\# \nu$, where $\pi_{k,d} : \mathbb{R}^d \to \mathbb{R}^{d-k+1}$ is the map given by the projection onto the variables from k on: $\pi_{k,d}(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_d) = (x_k, x_{k+1}, \dots, x_d)$.

Then, we define

$$f^k(x_k, x_{k+1}, \dots, x_d) = \frac{\hat{f}^k(x_k, x_{k+1}, \dots, x_d)}{\hat{f}^{k+1}(x_{k+1}, \dots, x_d)},$$
$$g^k(y_k, y_{k+1}, \dots, y_d) = \frac{\hat{g}^k(y_k, y_{k+1}, \dots, y_d)}{\hat{g}^{k+1}(y_{k+1}, \dots, y_d)},$$

(notice that all these functions will only be used on the set of points where the denominator does not vanish).

The function f^k, considered as a function of x_k with parameters $(x_{k+1}, \dots, x_d)$, can be actually seen as the density of the disintegration of μ according to the variables $(x_{k+1}, \dots, x_d)$, as well as g^k is, in the same terms, the density of the disintegration of ν. We briefly sketch below the main notions about the disintegration of measures. With this language, the Knothe transport that we are going to define could be defined even in the case of non-absolutely continuous measures, under some assumptions on the absence of atoms.

Anyway, for the sake of simplicity, we prefer to stick here to the case of absolutely continuous measures.

Important notion – *Disintegrations of Measures*

Definition Consider a measurer space X endowed with a Borel measure μ and a map $f : X \to Y$ valued in a topological space Y. We say that a family $(\mu_y)_{y\in Y}$ is a disintegration of μ according to f if every μ_y is a probability measure concentrated on $f^{-1}(\{y\})$ and for every test function $\phi \in C(X)$ the map $y \mapsto \int_X \phi \, d\mu_y$ is measurable and

$$\int_X \phi \, d\mu = \int_Y \nu(dy) \int_X \phi \, d\mu_y,$$

where $\nu = f_{\#}\mu$.

In the particular case where $X = Y \times Z$ and f is the projection on the Y factor, we usually identify the measures μ_y, which are officially defined as measures on $Y \times Z$ concentrated on $\{y\} \times Z$, with measures on Z, so that we get

$$\int_{Y\times Z} \phi(y, z) \, d\mu(y, z) = \int_Y \nu(dy) \int_Z \phi(y, z) \, d\mu_y(z).$$

The disintegration of a measure μ exactly corresponds to the conditional law in probability. The reader is invited to read [63] to find proofs about conditional probabilities and then to translate them into the disintegration language. In probabilities, we usually speak of the conditional law of a random variable X knowing $Y = y$. This means that the probability $\mathbb{P}$ on the probability space Ω is disintegrated according to the map $Y : \Omega \to E$ (where E is the image space of Y) into probabilities $\mathbb{P}_y$ and that we take the law of X under $\mathbb{P}_y$.

The existence and the uniqueness of the disintegration depend on some assumptions on the spaces, but is true if $X = \mathbb{R}^d$.

In order to define the Knothe rearrangement, let us start from $k = d$ and define the transport map $T^d : \mathbb{R} \to \mathbb{R}$ as the monotone non-decreasing transport map sending f^d onto g^d (these functions being considered as probability densities). We know that this map is well-defined and we know how to compute it in terms of cumulative distribution functions. We will now define a family of maps $T^k : \mathbb{R}^{d-k+1} \to \mathbb{R}$, where the variables of T^k will be $(x_k, x_{k+1}, \ldots, x_d)$. The first one is the map T^d that we just defined. For the sake of notations, we also define some maps $\hat{\mathrm{T}}^k : \mathbb{R}^{d-k+1} \to \mathbb{R}^{d-k+1}$, given by

$$\hat{\mathrm{T}}^k(x_k, x_{k+1}, \ldots, x_d) = (T^k(x_k, x_{k+1}, \ldots, x_d), T^{k+1}(x_{k+1}, \ldots, x_d), \ldots, T^d(x_d)).$$

Obviously T^d and $\hat{\mathrm{T}}^d$ coincide.

Now, if we write $f^k_{(x_{k+1},\dots,x_d)}$ and $g^k_{(y_{k+1},\dots,y_d)}$ for the functions $x_k \mapsto f^k(x_k, x_{k+1}, \dots, x_d)$ and $y_k \mapsto g^k(y_k, y_{k+1}, \dots, y_d)$ and we interpret them as densities of probability measures, we can define the map T^k, if we suppose that we have already defined T^j for $j > k$, in the following way: take $T^k_{(x_{k+1},\dots,x_d)}$ to be the monotone non-decreasing transport map sending $f^k_{(x_{k+1},\dots,x_d)}$ onto $g^k_{\hat{\mathrm{T}}^{k+1}(x_{k+1},\dots,x_d)}$.

Finally, the Knothe-Rosenblatt rearrangement T is defined by $T = \hat{\mathrm{T}}^1$.

We want to check that this map T is a transport map from μ to ν.

Proposition 2.3.1. *The Knothe-Rosenblatt rearrangement defined above is such that $T_\#\mu = \nu$.*

Proof. We will prove by induction that $\hat{\mathrm{T}}^k$ satisfies $\hat{\mathrm{T}}^k_\#\mu^k = \nu^k$ (let us recall that $\mu^k = (\pi_{k,d})_\#\mu$ and $\nu^k = (\pi_{k,d})_\#\nu$, are the marginals of μ and ν onto the last variables from k on). This fact is evident by construction for $k = d$, and if we get it for $k = 1$ we have proven $T_\#\mu = \nu$.

We only need to prove that, if the claim is true for a given $k+1$, then it will be true for k. To check the equality $\hat{\mathrm{T}}^k_\#\mu^k = \nu^k$, we just need to use test functions $\phi(y_k, \dots, y_d)$, and check

$$\int \phi(T^k(x_k, x_{k+1}, \dots, x_d), T^{k+1}(x_{k+1}, \dots, x_d), \dots, T^d(x_d))d\mu = \int \phi(y_k, \dots, y_d)\, d\nu.$$

Moreover, by density (see below for Stone-Weierstrass theorem), it is enough to check the equality on functions ϕ which are separable, i.e. of the form $\phi(y_k, \dots, y_d) = \varphi(y_k)\psi(y_{k+1}, \dots, y_d)$. To ease the notations, for fixed k, we will denote by $\bar{x}$ the vector $\pi_{k+1,d}(x)$ and $\bar{y} = \pi_{k+1,d}(y)$. In this case we should check

$$\int \psi \circ \hat{\mathrm{T}}^{k+1} \left(\int \varphi \circ T^k\, f^k\, dx_k \right) \hat{f}^{k+1}\, d\bar{x} = \int \psi(\bar{y}) \left(\int \varphi(y_k)\, g^k\, dy_k \right) \hat{g}^{k+1}\, d\bar{y}.$$

In order to get this equality, we just need to use the definition of T^k, so that we get, for every x,

$$\int \varphi \circ T^k\, f^k\, dx_k = \int \varphi(y_k)\, g^k_{\hat{\mathrm{T}}^{k+1}(\bar{x})}(y_k)\, dy_k.$$

If we define $G : \mathbb{R}^{d-k} \to \mathbb{R}$ the function given by $G(\bar{y}) = \int \varphi(y_k)\, g^k_{\bar{y}}(y_k)\, dy_k$, we have now

$$\int \psi \circ \hat{\mathrm{T}}^{k+1} \left(\int \varphi \circ T^k\, f^k\, dx_k \right) \hat{f}^{k+1}\, d\bar{x} = \int \psi \circ \hat{\mathrm{T}}^{k+1}\, G \circ \hat{\mathrm{T}}^{k+1}\, \hat{f}^{k+1}\, d\bar{x}.$$

By taking the image measure of $\hat{f}^{k+1}$ under $\hat{\mathrm{T}}^{k+1}$, the last expression is equal to $\int \psi(\bar{y})G(\bar{y})\hat{g}^{k+1}(\bar{y})d\bar{y}$, which is in turn equal to

$$\int \psi(\bar{y}) \left(\int \varphi(y_k)\, g^k_{\bar{y}}(y_k)\, dy_k \right) \hat{g}^{k+1}(\bar{y})d\bar{y}$$

and proves the thesis. $\square$

Memo – *Stone-Weierstrass Theorem*

Theorem. Suppose that X is a compact space and that $E \subset C(X)$ is a subset of the space of continuous functions on X satisfying i) constant functions belong to E ii) E is an algebra, i.e. it is stable by sum and product : $f, g \in E \Rightarrow f+g, fg \in E$ iii) E separates the points of X, i.e. for all $x \neq y \in X$ there is $f \in E$ such that $f(x) \neq f(y)$. Then E is dense in $C(X)$ for the uniform convergence.

This theorem is usually evoked to justify the density of polynomials when $X \subset \mathbb{R}^d$. Trigonometric polynomials are also dense for the same reason (and, by the way, this may be used to prove the L^2 convergence of the Fourier series). More generally, in product spaces $X = Y \times Z$, linear combinations of separable functions of the form $\sum_i \phi_i(y)\psi_i(z)$ are dense because of this very same theorem.

We just proved that the map T is a transport map between μ and ν. Moreover, by construction, should it be a regular map, the Knothe transport T has a triangular Jacobian matrix with nonnegative entries on its diagonal. This is a common point with the Brenier optimal map for the quadratic cost, since its Jacobian matrix is the Hessian matrix of a convex function, which is symmetric and has positive eigenvalues (on the contrary, the Jacobian of the Knothe map is far from being symmetric). Also, we underline that under the absolute continuity assumption that we took all the monotone optimal transports that we use in defining T are invertible, sine not only the starting measures are non-atomic, but also the arrival ones are non-atomic (they are all absolutely continuous). As a consequence, each map $\hat{\mathrm{T}}^k$ is also invertible.

regularity

Finally, note that the computation of the Knothe transport only involves one-dimensional monotone rearrangements, which is the reason for its computational interest.

2.4 Knothe transport as a limit of quadratic optimal transports

Let us slightly modify the quadratic cost that we discussed in Chapter 1 and replace it with the weighted quadratic cost

$$c_\varepsilon(x, y) := \sum_{i=1}^{d} \lambda_i(\varepsilon)(x_i - y_i)^2$$

where the $\lambda_i(\varepsilon)$'s are positive scalars depending on a parameter $\varepsilon > 0$. If μ is absolutely continuous with respect to the Lebesgue measure, the corresponding optimal transportation problem admits a unique solution T_ε, provided that the minimal cost is finite. This happens if the support of μ and ν are compact, or, more generally, if their second moments are finite:

$$m_2(\mu) := \int |x|^2\, d\mu < +\infty, \quad m_2(\nu) := \int |x|^2\, d\nu < +\infty.$$

When in addition, for all $k \in \{1, ..., d-1\}$, $\lambda_k(\varepsilon)/\lambda_{k+1}(\varepsilon) \to 0$ as $\varepsilon \to 0$, it is natural to expect the convergence of T_ε to the Knothe transport T. We will show that this convergence holds under the absolute continuity assumption of the previous section. On the other hand, [46] proves the same result under a slightly different assumption, namely the absence of atoms in all the disintegrated measures. The strange point is that absence of atoms is also needed on the target measure ν, and that a counter-example to the convergence result exists if it is not satisfied. This convergence result was conjectured by Y. Brenier as a very natural one, and actually its proof is not hard. Yet, it was not known before that extra assumptions on ν were needed.

An example. To illustrate the problem in a particular case where explicit solutions are available, take $d = 2$, and μ and ν two Gaussian measures where $\mu = N(0, I_2)$ and $\nu = N\left(0, \begin{pmatrix} a & b \\ b & c \end{pmatrix}\right)$ (with $ac > b^2$, $a > 0$). Take $\lambda_1(\varepsilon) = \varepsilon$ and $\lambda_2(\varepsilon) = 1$. Then it can be verified that T_ε is linear, and that its matrix in the canonical basis of $\mathbb{R}^2$ is

$$T_\varepsilon = \frac{1}{\sqrt{a\varepsilon^2 + c + 2\varepsilon\sqrt{ac - b^2}}} \begin{pmatrix} a\varepsilon + \sqrt{ac - b^2} & b \\ b\varepsilon & c + \varepsilon\sqrt{ac - b^2} \end{pmatrix}$$

which converges as $\varepsilon \to 0$ to $T = \begin{pmatrix} \sqrt{a - b^2/c} & b/\sqrt{c} \\ 0 & \sqrt{c} \end{pmatrix}$, which is precisely the matrix of the Knothe transport from μ to ν.

We directly state our first result, whose proof, in the spirit of $\Gamma-$convergence developments (see [57]), will require several steps.

Theorem 2.4.1. *Let μ and ν be two absolutely continuous probability measures on $\mathbb{R}^d$ with compact supports and γ_ε be an optimal transport plan for the costs $c_\varepsilon(x,y) = \sum_{i=1}^d \lambda_i(\varepsilon)(x_i - y_i)^2$, for some weights $\lambda_k(\varepsilon) > 0$. Suppose that for all $k \in \{1, ..., d-1\}$, $\lambda_k(\varepsilon)/\lambda_{k+1}(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let T be the Knothe-Rosenblatt map between μ and ν and $\gamma_K \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ the associated transport plan (i.e. $\gamma_K := (id \times T)\sharp\mu$). Then $\gamma_\varepsilon \rightharpoonup \gamma_K$ as $\varepsilon \to 0$.*

Moreover, should the plans γ_ε be induced by transport maps T_ε, then these maps would converge to T in $L^2(\mu)$ as $\varepsilon \to 0$.

The proof will roughly stick to the following strategy. We will take a limit point of γ_ε and show that it shares the same (x_d, y_d)-marginal as γ_K. Then, we will disintegrate with respect to (x_d, y_d) and prove that the conditional (x_{d-1}, y_{d-1})-marginals coincide. We will end the proof by iterating this argument.

Proof. Take the plans γ_ε that are optimal for the Brenier-like cost c_ε given by

$$c_\varepsilon(x,y) = \sum_{i=1}^d \lambda_i(\varepsilon)(x_i - y_i)^2$$

(it is sufficient to consider the case $\lambda_d(\varepsilon) = 1$ and $\lambda_i(\varepsilon)/\lambda_{i+1}(\varepsilon) \to 0$). Suppose (which is possible, up to subsequences) $\gamma_\varepsilon \rightharpoonup \gamma$. We want to prove $\gamma = \gamma_K$.

By comparing γ_ε to γ_K and using optimality we first get

$$\int c_\varepsilon \, d\gamma_\varepsilon \leq \int c_\varepsilon \, d\gamma_K \tag{4.2}$$

and, passing to the limit as $\varepsilon \to 0$, since c_ε converges locally uniformly to $c^{(d)}(x,y) = (x_d - y_d)^2$, we get

$$\int c^{(d)} d\gamma \leq \int c^{(d)} d\gamma_K.$$

Yet, the function $c^{(d)}$ only depends on the variables x_d and y_d and this shows that the measure $(\pi_d)_\sharp\gamma$ gets a result at least as good as $(\pi_d)_\sharp\gamma_K$ with respect to the quadratic cost (π_d being the projection onto the last coordinates, i.e. $\pi_d(x,y) = (x_d, y_d)$). Yet, the measure γ_K has been chosen on purpose to get optimality from μ^d to ν^d with respect to this cost, and the two measures share the same marginals. Moreover, thanks to the fact that μ^d is absolutely continuous, this optimal transport plan (which is actually induced by a transport map) is unique. This implies $(\pi_d)_\sharp\gamma = (\pi_d)_\sharp\gamma_K$. Let us call γ^d this common measure.

We go back to (4.2) and go on by noticing that all the measures γ_ε have the same marginals as γ_K and hence their (separate) projection onto x_d and y_d are μ^d and ν^d, respectively. This implies that $(\pi_d)_\sharp\gamma_\varepsilon$ must realize a result which is no better than $(\pi_d)_\sharp\gamma_K$ as far as the quadratic cost is concerned and consequently we have

$$\begin{aligned}
&\int |x_d - y_d|^2 d(\pi_d)_\sharp\gamma_K(x_d, y_d) + \sum_{i=1}^{d-1} \lambda_i(\varepsilon) \int (x_i - y_i)^2 d\gamma_\varepsilon \\
&\le \int c_\varepsilon \, d\gamma_\varepsilon \le \int c_\varepsilon \, d\gamma_K \\
&= \int |x_d - y_d|^2 d(\pi_d)_\sharp\gamma_K(x_d, y_d) + \sum_{i=1}^{d-1} \lambda_i(\varepsilon) \int (x_i - y_i)^2 d\gamma_K,
\end{aligned}$$

which implies, by simplifying the common term in $d(\pi_d)_\sharp\gamma_K$, dividing by $\lambda_{d-1}(\varepsilon)$ and passing to the limit,

$$\int c^{(d-1)} d\gamma \le \int c^{(d-1)} d\gamma_K \tag{4.3}$$

(we use the general notation $c^{(k)}(x, y) = |x_k - y_k|^2$). We can notice that both integrals depend on the variables x_{d-1} and y_{d-1} only. Anyway, we can project onto the variables (x_{d-1}, x_d) and (y_{d-1}, y_d) (obtaining measures $(\pi_{d-1})_\sharp\gamma$ and $(\pi_{d-1})_\sharp\gamma_K$) so that we disintegrate with respect to the measure γ^d. We have

$$\begin{aligned}
&\int d\gamma^d(x_d, y_d) \int |x_{d-1} - y_{d-1}|^2 d\gamma^{d-1}_{(x_d,y_d)}(x_{d-1}, y_{d-1}) \\
&\qquad\qquad \le \int d\gamma^d(x_d, y_d) \int |x_{d-1} - y_{d-1}|^2 d\gamma^{d-1}_{(x_d,y_d),K}(x_{d-1}, y_{d-1}).
\end{aligned}$$

It is is sufficient to prove that the measures $\gamma^{d-1}_{(x_d,y_d)}$ share the same marginals on x_{d-1} and y_{d-1} as the corresponding $\gamma^{d-1}_{(x_d,y_d),K}$ to get that their quadratic performance should be no better than the corresponding performance of $\gamma^{d-1}_{(x_d,y_d),K}$ (this is because the Knothe measure has been chosen exactly with the intention of being quadratically optimal on (x_{d-1}, y_{d-1}) once x_d and

y_d are fixed). Yet, (**??**) shows that, on average, the result given by those measures is not worse than the results of the optimal ones. Thus, the two results coincide for almost any pair (x_d, y_d) and, by uniqueness of the optimal transports (this relies on the assumptions on the measures), we get $\gamma^{d-1}_{(x_d,y_d)} = \gamma^{d-1}_{(x_d,y_d),K}$. To let this proof work it is sufficient to prove that the projections of the two measures coincide for γ^d-a.e. pair (x_d, y_d). For fixed (x_d, y_d) we would like to prove, for any ϕ

$$\int \phi(x_{d-1}) d\gamma^{d-1}_{(x_d,y_d)} = \int \phi(x_{d-1}) d\gamma^{d-1}_{(x_d,y_d),K}$$

(and to prove an analogous equality for functions of y_{d-1}). Since we accept to prove it for a.e. pair (x_d, y_d), it is sufficient to prove this equality:

$$\begin{aligned} &\int d\gamma^d(x_d, y_d)\psi(x_d, y_d) \int \phi(x_{d-1}) d\gamma^{d-1}_{(x_d,y_d)} \\ &= \int d\gamma^d(x_d, y_d)\psi(x_d, y_d) \int \phi(x_{d-1}) d\gamma^{d-1}_{(x_d,y_d),K} \end{aligned}$$

for any ϕ and any ψ. This means proving

$$\int \psi(x_d, y_d)\phi(x_{d-1}) d\gamma^{d-1} = \int \psi(x_d, y_d)\phi(x_{d-1}) d\gamma^{d-1}_K,$$

which is not trivial since we only know that the two measures γ^{d-1} and γ^{d-1}_K have the same marginals with respect to the pairs (x_{d-1}, x_d), (y_{d-1}, y_d) (since they have the same projections onto x and onto y) and (x_d, y_d) (since we just proved it). But here there is a function of the three variables (x_{d-1}, x_d, y_d). Yet, we know that the measure γ^d is concentrated on the set $y_d = T_d(x_d)$ for a certain map T_d, and this allows to replace the expression of y_d, thus getting rid of one variable. This proves that the function $\psi(x_d, y_d)\phi(x_{d-1})$ is actually a function of (x_{d-1}, x_d) only, and that equality holds when passing from γ to γ_K.The same can be performed on functions $\psi(x_d, y_d)\phi(y_{d-1})$ but we have in this case to ensure that we can replace x_d with a function of y_d, i.e. that we can invert T_d. This is possible thanks to the absolute continuity of ν^d, since T_d is the optimal transport from μ^d to ν^d, but an optimal transport exists in the other direction as well and it gives the same optimal plan (thanks to uniqueness). These facts prove that the measures $\gamma^{d-1}_{(x_d,y_d)}$ and $\gamma^{d-1}_{(x_d,y_d),K}$ have the same marginals and hence, since they are both optimal, they coincide for a.e. pair (x_d, y_d). This implies $\gamma^{d-1} = \gamma^{d-1}_K$.

Now, it is possible to go on by steps: once we have proven that $\gamma^k = \gamma^k_K$, i.e. that we have convergence in what concerns the components from the $k-$th to the $d-$th, we go on by induction.

Take $k \leq d$ and write $x = (x^-, x^+)$ and $y = (y^-, y^+)$ with $x^-, y^- \in \mathbb{R}^{k-1}$ and $x^+, y^+ \in \mathbb{R}^{d-k+1}$ (decomposing into the components up to $k-1$ and after k). Then, define a competitor γ_K^ε in the following way: γ_K^ε is a measure on $\mathbb{R}^d \times \mathbb{R}^d$ with the following marginals

$$(\pi_{x^-,x^+})_\#\gamma_K^\varepsilon = \mu,\ (\pi_{y^-,y^+})_\#\gamma_K^\varepsilon = \nu,\ (\pi_{x^+,y^+})_\#\gamma_K^\varepsilon = \eta_{\varepsilon,h},$$

where $\eta_{\varepsilon,k}$, is the measure which optimizes the transport cost $\sum_{i=k}^d \lambda_i(\varepsilon)c^{(i)}$ between the marginals $(\pi_{x^+})_\#\mu$ and $(\pi_{y^+})_\#\nu$. Thanks to the recursive structure of the proof, we know that $\eta_{\varepsilon,k}$ converges to $(\pi_{x^+,y^+})_\#\gamma_K$.

The definition we did of the measure γ_K^ε does not give rise to a unique measure (and actually it is not even evident that one such a measure exists). Yet, one easy way to produce one is the following: disintegrate μ and ν according to π_{x^+} and π_{y^+}, respectively, thus getting two family of measures μ^{x^+} and ν^{y^+}. Then, for every pair (x^+, y^+) pick a transport plan γ^{x^+,y^+} between them. The measure $\gamma^{x^+,y^+} \otimes \eta_{\varepsilon,k}$ satisfies the required properties. Moreover, a good choice for γ^{x^+,y^+} is the Knothe transport between those two measures. Yet, for the sake of this step of the proof, the only important point is that, in what concerns the $(k-1)-$th component, we choose for any (x^+, y^+) the monotone mapping from $\mu_{k-1}^{x^+} := (\pi_{x_{k-1}})_\#\mu^{x^+}$ and $\nu_{k-1}^{y^+} := (\pi_{y_{k-1}})_\#\nu^{y^+}$, so that $\int c^{(k-1)} d\gamma^{x^+,y^+} = W_2^2(\mu_{k-1}^{x^+}, \nu_{k-1}^{y^+})$.

Spoiler – *The distance* W_2

Definition Given $\mu, \nu \in \mathcal{P}(X)$ two probability measures on a metric space X, let us define some quantities $W_p(\mu, \nu)$ through

$$W_p^p(\mu,\nu) := \inf\left\{ \int d(x,y)^p \, d\gamma(x,y) \ : \ \gamma \in \Pi(\mu,\nu) \right\},$$

so that W_p is the $p-$th root of the minimal transport cost $\mathcal{T}_c$ for cost c equal to the distance (d being the distance on X) to the power p.

We will see later (Chapter 5) that these quantities are actually distances (they are called Wasserstein distances) – up to some assumptions and technicalities – on the space of measures and they metrize the weak convergence. Here we are only interested in the following property, which is a particular case of Theorem 1.6.11.
Proposition If X is compact, then W_p^p is continuous on $\mathcal{P}(X) \times \mathcal{P}(X)$ for the weak convergence.

With this choice in mind, we write

$$\begin{aligned}
&\sum_{i\geq h}\lambda_i(\varepsilon)\int |x_i-y_i|^2 d\gamma_K^\varepsilon + \sum_{i=1}^{k-1}\lambda_i(\varepsilon)\int (x_i-y_i)^2 d\gamma_\varepsilon \\
&\leq \int c_\varepsilon\, d\gamma_\varepsilon \leq \int c_\varepsilon\, d\gamma_K^\varepsilon \\
&= \sum_{i\geq h}\lambda_i(\varepsilon)\int |x_i-y_i|^2 d\gamma_K^\varepsilon + \sum_{i=1}^{k-1}\lambda_i(\varepsilon)\int (x_i-y_i)^2 d\gamma_K^\varepsilon,
\end{aligned}$$

and consequently, by erasing the common terms and dividing by $\lambda_{k-1}(\varepsilon)$, we get

$$\int c^{(k-1)} d\gamma_\varepsilon \leq \int c^{(k-1)} d\gamma_K^\varepsilon = \int W_2^2(\mu_{k-1}^{x^+},\nu_{k-1}^{y^+}) d\eta_{\varepsilon,h}.$$

Then we pass to the limit as $\varepsilon \to 0$. The left-hand side of the last inequality tends to $\int c^{(k-1)} d\gamma$, while the right-hand side converges to

$$\int W_2^2(\mu_{k-1}^{x^+},\nu_{k-1}^{y^+}) d(\pi_{x^+,y^+})_\# \gamma_K = \int c^{(k-1)} d\gamma_K.$$

This convergence result is justified by the following Corollary 2.4.2 of Lemma 1.1.5.

Anyway we get in the end

$$\int c^{(k-1)} d\gamma \leq \int c^{(k-1)} d\gamma_K$$

and we can go on similarly to the previous proof. As the reader may easily see, getting this last inequality was obtained thanks to a procedure which was similar to what we didi for getting (4.3), but it required some more work since γ_K is not exactly optimal for all the components $i \geq k$, but is approximated by the optimal couplings.

We disintegrate with respect to γ^k and we act exactly as before: proving that the marginals of the disintegrations coincide is sufficient to prove equality of the measures. Here we will use test-functions of the form

$$\psi(x_h, x_{k+1}, \ldots, x_d, y_h, y_{k+1}, \ldots, y_d)\phi(x_{k-1})$$

and

$$\psi(x_h, x_{k+1}, \ldots, x_d, y_h, y_{k+1}, \ldots, y_d)\phi(y_{k-1}).$$

The same trick as before, i.e. replacing the variables y with functions of the variables x is again possible. To invert the trick and replace x with y

one needs to invert part of Knothe's transport. This is possible since, as we noticed, our assumptions implied that all the monotone transports we get are invertible. In the end we get, as before, $\gamma^{k-1} = \gamma_K^{k-1}$. This procedure may go on up to $k = 2$, thus arriving at $\gamma = \gamma_K$.

We have now proven $\gamma_\varepsilon \rightharpoonup \gamma_K$. Yet, if all these transport plans come from transport maps, it is well known that $(T_\varepsilon \times id)_\sharp\mu \rightharpoonup (T \times id)_\sharp\mu$ implies $T_\varepsilon \to T$ in $L^p(\mu)$, for any $p > 1$, as far as T_ε is bounded in $L^p(\mu)$. Actually, weak convergence is a simple consequence of boundedness: to go on, we can look at Young's measures. The assumption (the limit is a transport map as well) exactly means that all the Young measures are dirac masses, which implies strong convergence. In particular we get $L^2(\mu)$ convergence and μ-a.e. convergence for a subsequence. □

In the proof we used the following result, which is a corollary of Lemma 1.1.5.

Corollary 2.4.2. *If μ and ν are two probabilities over $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ with finite second momentum which disintegrate as $\mu = \mu^+ \otimes \mu^{x^+}$ and $\nu = \nu^+ \otimes \nu^{y^+}$, then the functional*

$$\eta \mapsto \int W_2^2(\mu^{x^+}, \nu^{x^+}) d\eta(x^+, y^+)$$

is continuous over the transport plans between μ^+ and ν^+.

Proof. It is sufficient to apply the previous lemma to the following setting: $\Omega = \mathbb{R}^k$, $X = Y = \mathcal{P}_2(\mathbb{R}^{n-k})$, the set of probability measures with finite second momentum on $\mathbb{R}^{n-k}$ endowed with any distance metrizing weak convergence and making this space a separable metric space (for instance we can use the Wasserstein distance W_2, but we will not stress this fact). Take $c(a, b) = W_2^2(a, b)$, and the two functions f and g given by twice the second momentum. Indeed, from the elementary inequality $|x - y|^2 \leq 2|x|^2 + 2|y|^2$, we get $W_2^2(a, b) \leq 2m_2(a) + 2m_2(b)$. Moreover, the integral $\int m_2(\mu^{x^+})d\mu^+$ is bounded by the momentum of μ and so happens for ν, thus proving that f and g are integrable. □

Let us remark here that if instead of considering the quadratic cost c_ε, one considers the more general separable cost

$$c_\varepsilon(x, y) := \sum_{i=1}^{d} \lambda_i(\varepsilon) c_i(x_i - y_i)$$

where each c_i is a smooth strictly convex function (with suitable growth), then the previous convergence proof carries over.

A counterexample when the measures have atoms We now show that interestingly, the absolute continuity hypothesis in theorem 2.4.1 is necessary not only for μ, but also for ν. We propose a very simple example in $\mathbb{R}^2$ where μ is absolutely continuous with respect to the Lebesgue measure but ν is not, and we show that the conclusion of theorem 2.4.1 fails to hold. On the square $\Omega := [-1,1] \times [-1,1]$, define μ such that $\mu(dx) = 1_{\{x_1 x_2 < 0\}} dx/2$ so that the measure μ is uniformly spread on the upper left and the lower right quadrants, and $\nu = \mathcal{H}^1_{|S}/2$, being S the segment $[-1,1] \times \{0\}$.

The Knothe-Rosenblatt map is easily computed as $(y_1, y_2) = T(x) := (2x_1 + sgn(x_2)), 0)$. The solution of any symmetric transportation problem with $\lambda^\varepsilon = (\varepsilon, 1)$ is $(y_1, y_2) = T^0(x) := (x_1, 0)$ (no transport may do better than this one, which projects on the support of ν). Therefore, in this example the optimal transportation maps fail to tend to the Knothe-Rosenblatt map. The reason is the atom in the measure $\nu^2 = \delta_0$.

2.5 Discussion

2.5.1 Histogram equalization

2.5.2 The projected transport distance and its transport

An interesting idea based on the fact "one-dimensional transports are easier to compute, let us use them for building multi-dimensional one" is the following, due to M. Bernot.

Consider two measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ and project them onto any one-dimensional direction. This means taking $e \in S^{d-1}$ (the unit sphere of $\mathbb{R}^d$), considering the map $\pi_e : \mathbb{R}^d \to \mathbb{R}$ given by $\pi_e(x) = x \cdot e$ and looking at the image measures $(\pi_e)_\# \mu$ and $(\pi_e)_\# \nu$. They are measures on the real line, call $T_e : \mathbb{R} \to \mathbb{R}$ the monotone optimal transport between them. Let us consider that, in order to be efficient for the direction e, every point x of $\mathbb{R}^d$ should be displaced of a vector $S_e(x) := (T_e(\pi_e(x)) - \pi_e(x))e$. To do a global displacement, consider $S(x) = \int_{S^{d-1}} S_e(x)\, d\sigma(e)$, where σ is the uniform measure on the sphere, normalized so as to be a probability measure.

Obviously there is no reason to guarantee that $id + S$ is a transport map from μ to ν. But if one fixes a small time step $\tau > 0$ and uses a displacement τS getting a measure $\mu_1 = (id + \tau S)_\# \mu$, then it is possible to iterate the construction (re-computing the one-dimensional maps, the displacement S and use a small time-step) to build μ_2, and then μ_3.... The composition of all the maps $id + \tau S$ is then a transport map between μ and μ_n, which is close to ν if n is large enough and τ small enough. It is not guaranteed to be optimal, but it is a very efficient transport and keeps many of the properties of optimal transports (monotonicity...). Compared to the Knothe transport, it is not anisotropic and has a much better transport cost.

It is also possible to use this same idea to define a new quantity measuring the distance between two measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$. This quantity can be called "projected Wasserstein distance" (in the original works it was rather called "sliced Wasserstein distance", but we prefer to insist on the fact that it is obtained through projections, the slices being rather the parallel hyperplanes that are identified through each projection). This quantity is defined in the following way:

$$PW_2(\mu,\nu) := \left(⨍_{S^{d-1}} W_2^2((\pi_e)_\#\mu, (\pi_e)_\#\nu)\, d\sigma(e) \right)^{1/2}.$$

The fact that it is a distance comes from W_2 being a distance. The triangle inequality may be proven using the triangle inequality for W_2 (see Chapter 5) and for the L^2 norm. Positivity and symmetry are evident. The equality $PW_2(\mu,\nu) = 0$ implies $W_2^2((\pi_e)_\#\mu, (\pi_e)_\#\nu)$ for all $e \in S^{d-1}$. This means $(\pi_e)_\#\mu = (\pi_e)_\#\nu)$ for all e and is enough (see below) to prove $\mu = \nu$

Good to know ! – *The Radon transform*

Definition Given a measure μ on $\mathbb{R}^d$, we define the radon transform $R\mu$ of μ as the family of measures (μ_e), parametrized by $e \in S^{d-1}$, and defined as $(\pi_e)_\#\mu$. When μ is absolutely continuous, these measures are also absolutely continuous, and we can define $R\mu$ as a function over $S^{d-1} \times \mathbb{R}$, through $R\mu(e,x) = f_e(x)$, where $\mu_e = f_e(x)dx$.

Proposition Given two measures μ, ν on $\mathbb{R}^d$, if their Radon transforms coincide, then $\mu = \nu$.

it Proof Notice that $\int_{\mathbb{R}^d} e^{i\xi\cdot x} d\mu = \int_{\mathbb{R}} e^{it} d\mu_\xi(t)dt$. This means that the Radon transform uniquely determines the Fourier transform of a measure, so that $R\mu = R\nu \Rightarrow \hat{\mu} = \hat{\nu}$. The equality $\mu = \nu$ follows from standard results about the Fourier transform.

The Radon transform is a very well known object in applications such as the tomography. Indeed, suppose to have a three dimensional unknown distribution of mass, that you can not directly observe. Yet, it is possible to fix a plane in $\mathbb{R}^3$ and, by using appropriate tools, to deduce the projection of μ onto this plane, getting the amount of mass "behind" every point of the plane. This can be done for every plane, and gives even more information than $R\mu$, since two-dimensional projections instead of projections onto $\mathbb{R}$ are used. TAC devices are based on the inversion of the Radon transform.

2.5.3 Isoperimetric inequality via Knothe or Brenier maps

It is interesting to learn that one f the first motivations to use Knothe transport was first introduced in order to deal with some geometric inequalities (see [81]).

Consider for instance the isoperimetric inequality, stating that every set E has a larger perimeter than the ball B with the same volume $|B| = |E|$. As usual in these discussion sessions, we will not give full details on the ideas that we mention, and we will be quite sloppy concerning regularity. Anyway, think that E is a smooth domain in $\mathbb{R}^d$. By scaling, we can also suppose that E has the same volume of the unit ball $B = B(0,1)$

One idea to prove this result is the following: consider the densities $\mu = \frac{1}{|E|}I_E$ and $\nu = \frac{1}{|B|}I_B$, and use the Knothe transport between these densities, denoted by T. Due to the considerations on the Jacobian matrix of T, it has only positive eigenvalues and it satisfies the condition $\det(DT) = |B|/|E| = 1$ (with no absolute value at the determinant). Hence we can write

$$\omega_d = |B| = |E| = \int_E (\det(DT))^{\frac{1}{d}} \leq \frac{1}{d}\int_E \nabla\cdot T = \frac{1}{d}\int_{\partial E} T\cdot n \leq \frac{1}{d}Per(E), \tag{5.4}$$

where the inequalities are obtained thanks to the arithmetic-geometric inequality $\frac{1}{n}\sum_i \lambda_i \geq (\lambda_1 \dots \lambda_n)^{\frac{1}{n}}$ applied to the eigenvalues of DT (which gives an inequality between the determinant and the trace, i.e. the divergence of T), and to the fact that $T : E \to B$, whence $T \cdot n \leq 1$.This shows $Per(E) \geq d\omega_d = Per(B)$.

Notice that we only used on T the fact that its Jacobian has positive eigenvalues and that it transports one measure onto the other. This is also satisfied by the Brenier map, simply this map was not known at the time this proof was first performed.

The same proof can also be used to prove the uniqueness of the optimal sets (i.e., if a set E is such that $Per(E) = d\omega_d$, then it is a unit ball).

Suppose indeed to have equality in the above inequalities. This implies $d(\det(DT))^{\frac{1}{d}} = \nabla\cdot T$ a.e. and, knowing the cases of equality in the arithmetic-geometric inequality, this means that the eigenvalues of DT should all be equal and, thanks to the condition on the determinant, equal to 1. If the matrix was known to be symmetric (which is the case for Brenier's map, but not for Knothe's one), then we would get $DT = I$ and

$T(x) = x + x_0$. Since T would be a translation, then we could conclude $E = B$, up to translations.

It is slightly harder for the Knothe transport, since DT is not symmetric. Yet, if all the eignevalues are 1, one deduces that T_d is a translation. If we suppose from the beginning that (up to a fixed translation) E and B share the same barycenter, then this translation is the identity. This means that $(\pi_d)_\#\mu = (\pi_d)_\#\nu$. This is not yet enough to get $\mu = \nu$, and hence $E = B$, unless one uses the fact that the direction for this Kntohe transport are arbitrary. This means that, if E is a set giving equality in the isoperimetric inequality, then its projections onto any direction coincide with those of the ball. In particular, $\pi_d(E) \subset [-1, 1]$ whatever the direction of the $d-$th axis. This implies $E \subset B$ and, by equality of the volumes, $E = B$ (the argument may also be done through the measures and the notion of Radon transform, since $(\pi_d)_\#\mu = (\pi_d)_\#\nu$ for arbitrary direction of the $d-$th axis implies $\mu = \nu$).

Chapter 3

L^1 and L^∞ theory

Chapter one gave, in the framework of the general theory of optimal transportation based on duality methods, an existence result of the optimal transport ma when the cost is of the form $c(x, y) = h(x - y)$ and h is strictly convex. In particular, this applies to the minimization problems

$$\min \left\{ ||T - id||_{L^p(\gamma)} \ : \ \gamma \in \Pi(\mu, \nu) \right\},$$

for $p \in]1, +\infty[$, which correspond to the cost functions $|x - y|^p$. We look in this chapter at the two limit cases $p = 1$ and $p = \infty$, which require additional techniques.

3.1 The Monge case, with cost $|x - y|$

This section will prove that if $\mu \ll \mathcal{L}^d$ and $\nu \in \mathcal{P}(\Omega)$ is any probability measure, then there exists an optimal transport map T for the cost $|x - y|$.

3.1.1 Duality for distance costs

Let us spend some times on the case where the cost $c(x, y)$ is actually a distance d (thus satisfying triangle inequality, vanishing if $x = y \ldots$). Since the cost is symmetric, we will avoid the distinction between the $\bar{c}$ and the c transform.

Proposition 3.1.1. *If $c : X \times X \to \mathbb{R}$ is a distance then a function $u : X \to \mathbb{R}$ is $c-$concave if and only if it Lipschitz continuous with Lipschitz constant less than 1 w.r.t. the distance c. We will denote by Lip_1 the set of those functions.*

Moreover, for every $u \in \mathrm{Lip}_1$ we have $u^c = -u$.

Proof. First take a $c-$concave function u. It can be written as

$$u(x) = v^c(x) = \inf_y d(x,y) - v(y)$$

for some function $v : X \to \mathbb{R} \cup \{-\infty\}$. One can ignore the points y such that $v(y) = -\infty$ and, for the others, notice that $x \mapsto d(x,y) - v(y)$ is a Lip_1 function, since $x \mapsto d(x,y)$ is Lip_1 as a consequence of the triangle inequality. Hence, $u \in \mathrm{Lip}_1$ since the infimum of a family of Lipschitz continuous functions with the same constant shares the same modulus of continuity.

Take now a function $u \in \mathrm{Lip}_1$. We claim that one can write

$$u(x) = \inf_y d(x,y) + u(y).$$

Actually, it is clear that the infimum at the r.h.s. is not larger than $u(x)$, since one can always use $y = x$. On the other hand, $u \in \mathrm{Lip}_1$ implies $u(x) - u(y) \le d(x,y)$, i.e. $u(x) \le u(y) + d(x,y)$ for every y, and hence the infimum is not smaller than $u(x)$. This expression shows that $u = (-u)^c$ and that u is $c-$concave.

Applying this last formula to $-u$, which is also Lip_1, proves that $u^c = -u$ and the last part of the claim follows. □

As a consequence the duality formula, in the case of a distance cost function, gives

$$\min\left\{\int d(x,y)\, d\gamma,\ \gamma \in \Pi(\mu,\nu)\right\} = \max\left\{\int u\, d(\mu - \nu)\ :\ u \in \mathrm{Lip}_1\right\}. \tag{1.1}$$

3.1.2 Secondary variational problem

We can now pick a maximizer for the dual problem, which is a Lip_1 function u called *Kantorovitch potential*, that we will consider as fixed from now on. Let us call $O(\mu,\nu)$ the set of optimal transport plans for the cost $|x-y|$. For notational simplicity we will also denote by c_p the functional associating to $\gamma \in \mathcal{P}(\Omega\times\Omega)$ the quantity $\int |x-y|^p d\gamma$, and m_p its minimal value on $\Pi(\mu,\nu)$. In this language $O(\mu,\nu) = \mathrm{argmin}_{\gamma\in\Pi(\mu,\nu)}\, c_1(\gamma) = \{\gamma \in \Pi(\mu,\nu)\ :\ c_1(\gamma) \le m_1\}$. Notice that $O(\mu,\nu)$ is a closed subset (w.r.t. the weak convergence of measures) of $\Gamma(\mu,\nu)$, which is compact. This is a general fact whenever we minimize a semicontinuous functional of γ. In this case c_1 is also continuous (w.r.t. the same convergence), and hence the set $\{\gamma \in \Pi(\mu,\nu)\ :\ c_1(\gamma) \le m_1\}$

is closed, but lower semicontinuituy would have been enough. We will use this fact several times.

First, let us notice that it holds

$$\gamma \in O(\mu,\nu) \Leftrightarrow \mathrm{spt}(\gamma) \subset \{(x,y) :\, u(x)-u(y)=|x-y|\}.$$

This is true because optimality implies $\int (u(x)-u(y))d\gamma = \int |x-y|d\gamma$ and the global inequality $u(x)-u(y) \leq |x-y|$ gives equality $\gamma-$a.e. All these functions being continuous, the equality finally holds on the whole support. Viceversa, equality on the support allows to integrate it and prove that $c_1(\gamma)$ equals the value of the dual problem, which is the same of the primal, hence one gets optimality.

Since we know that in general there is no uniqueness for the minimizers of c_1 and that they are not all induced by transport maps, we will select a special minimizer, better than the others, and prove that it is actually induced by a map.

Let us consider the problem

$$\min\{c_2(\gamma) \,:\, \gamma \in O(\mu,\nu)\}.$$

This problem has a solution $\bar\gamma$ since c_2 is continuous for the weak convergence and $O(\mu,\nu)$ is compact for the same convergence. We do not know a priori about the uniqueness of such a minimizer. It is interesting to notice that the solution of this problem may also be obtained as the limits of solutions γ_ε of the transport problem

$$\min\{c_1(\gamma)+\varepsilon c_2(\gamma), :\, \gamma \in \Pi(\mu,\nu)\},$$

but we will not exploit this fact here (its proof can be done in the spirit of Section 2.4).

The goal is now to characterize this plan $\bar\gamma$ and prove that it is induced by a transport map.

The fact that the condition $\gamma \in O(\mu,\nu)$ may be rewritten as a condition on the support of γ is really useful since it allows to state that $\bar\gamma$ also solves

$$\min \int c\, d\gamma \,:\, \gamma \in \Pi(\mu,\nu), \qquad \text{where } c(x,y) = \begin{cases} |x-y|^2 & \text{if } u(x)-u(y)=|x-y| \\ +\infty & \text{otherwise.} \end{cases}$$

Actually, minimizing this new cost implies being concentrated on the set where $u(x)-u(y)=|x-y|$ (i.e. belonging to $O(\mu,\nu)$) and minimizing the quadratic cost among those plans γ concentrated on the same sets (i.e. among those $\gamma \in O(\mu,\nu)$).

Let us spend some words more in general on costs of the form

$$c(x,y) = \begin{cases} |x-y|^2 & \text{if } (x,y) \in A \\ +\infty & \text{otherwise,} \end{cases}$$

where A is a given closed subset of $\Omega \times \Omega$. First of all we notice that such a cost is l.s.c. on $\Omega \times \Omega$. To prove it, just take a sequence $(x_k, y_k) \to (x, y)$. We want to prove $\liminf_k c(x_k, y_k) \geq c(x, y)$, and there is nothing to prove if the liminf on the left is $+\infty$. We can suppose hence that the liminf is finite, which means that, up to a subsequence, $(x_{k_j}, y_{k_j}) \in A$. We can choose such a subsequence so that $\lim_j c(x_{k_j}, y_{k_j}) = \liminf_k c(x_k, y_k)$. Since A is closed we also have $(x, y) \in A$. Then, we can replace every occurrence of c with the expression it has on A (i.e. the quadratic cost, which is continuous), and since we have $|x_{k_j} - y_{k_j}|^2 \to |x-y|^2$ we have $\liminf_k c(x_k, y_k) = c(x, y)$ (the fact that c is only l.s.c. and not continuous comes from the possibility of getting $+\infty$ on the left).

Semi-continuity of the cost implies that optimal plans are concentrated on a set which is $c - CM$. What does it mean in such a case? $c-$cyclical monotonicity is a condition which imposes an inequality for every k, every σ and every family $(x_1, y_1), \ldots, (x_k, y_k)$; here we only need to use the condition for $k = 2$. This means that, if $\Gamma \subset \Omega \times \Omega$ is $c - CM$ we have

$$(x_1, y_1), (x_2, y_2) \in \Gamma \Rightarrow c(x_1, y_1) + c(x_2, y_2) \leq c(x_1, y_2) + c(x_2, y_1).$$

For costs c of this form, this is only useful when both (x_1, y_2) and (x_2, y_1) belong to A (otherwise we have $+\infty$ at the right had side of the inequality). If we also use the fact that

$$|x_1 - y_1|^2 + |x_2 - y_2|^2 \leq |x_1 - y_2|^2 + |x_2 - y_1|^2 \Leftrightarrow (x_1 - x_2) \cdot (y_1 - y_2) \geq 0,$$

(which is easy to get by expanding the squares), this means that if Γ is $c-$CM, then

$$(x_1, y_1), (x_2, y_2) \in \Gamma, (x_1, y_2), (x_2, y_1) \in A \Rightarrow (x_1 - x_2) \cdot (y_1 - y_2) \geq 0.$$

3.1.3 Geometric properties of transport rays.

Let us consider for a while the role played by the function u. We collect some properties.

Lemma 3.1.2. *If $x, y \in \Omega$ are such that $u(x) - u(y) = |x - y|$, then u is affine on the whole segment $[x, y] := \{z = (1-t)x + ty,\ t \in [0, 1]\}$.*

Proof. Take $z=(1-t)x+ty$. Just consider that the Lip_1 condition implies

$$u(x)-u(z)\le|x-z|=t|x-y],\quad u(z)-u(y)\le|z-y|=(1-t)|x-y].$$

Summing up the two inequalities we get $u(x)-u(y)\le|x-y|$, but the assumption is that this should be an equality. Hence we can infer that both inequalities are equalities, and in particular $u(z)=u(x)-t|x-y|$. □

Lemma 3.1.3. *If $z\in]x,y[:=\{z=(1-t)x+ty,\ t\in]0,1[\}$, for a pair of points $x\ne y$ such that $u(x)-u(y)=|x-y|$, then u is differentiable at z and $\nabla u(z)=e:=\frac{x-y}{|x-y|}$.*

Proof. First of all, let us give an estimate on the increment of u in directions orthogonal to e. Consider a unit vector h with $h\cdot e=0$ and a point $z'\in]x,y[$, and let A be such that $z'\pm Ae\in[x,y]$. Set $\delta:=u(z'+th)-u(z')$. By using $u\in\mathrm{Lip}_1$ we can say

$$u(z'+th)-u(z'-Ae)=u(z'+th)-u(z')+u(z')-u(z'-Ae)=\delta+A\le\sqrt{t^2+A^2}$$

as well as

$$u(z'+th)-u(z'+Ae)=u(z'+th)-u(z')+u(z')-u(z'+Ae)=\delta-A\le\sqrt{t^2+A^2}.$$

By raising these inequalities to power 2 we get

$$\delta^2+A^2\pm2A\delta\le t^2+A^2.$$

These two inequalities give $\pm2A\delta\le\delta^2\pm2A\delta\le t^2$, and hence $2A|\delta|\le t^2$, i.e. $|\delta|\le t^2/2A$.

Consider now a point $z\in]x,y[$ and a number $A<\min\{|z-x],|z-y|\}$. Any point w sufficiently close to z may be written as $w=z'+th$ with h a unit vector orthogonal to e, $t\ll1$, $z'\in]x,y[$ such that $z'\pm Ae\in[x,y]$. This allows to write

$$u(w)-u(z)=u(z'+th)-u(z')+u(z')-u(z)=(z'-z)\cdot e+O(t^2)=(w-z)\cdot e+O(t^2).$$

Using $O(t^2)=o(t)=o(|w-z|)$, we get $\nabla u(z)=e$. □

Definition 9. We call transport ray any non-trivial (i.e. different from a singleton) segment $[x,y]$ such that $u(x)-u(y)=|x-y|$ which is maximal for the inclusion among segments of this form. The corresponding open segment $]x,y[$ is called the interior of the transport ray and x and y its boundary points. We call T_u the union of all non degenerate transport rays,

$T_u^{(b)}$ the union of their boundary points and $T_u^{(i)}$ the union of their interiors. Moreover, let $T_u^{(b+)}$ be the set of upper boundary points of non-degenerate transport rays (i.e. those where u is minimal on the transport ray, say the points y in the definition $u(x)-u(y)=|x-y|$) and $T_u^{(b-)}$ the set of lower boundary points of non-degenerate transport rays (where u is maximal, i.e. the points x).

Corollary 3.1.4. *Two different transport ray can only meet in a point in a point z which is a boundary point for both of them, and in such a case u is not differentiable at z. In particular, if one removes the negligibel set $S(u)$ of non-differentiability points of u, the transport rays are disjoint.*

Proof. Suppose that two transport rays meet at a point z which is internal for both rays. In such a case u must have two different gradients at z, following both the directions of the rays, which is impossible (recall that the different rays meeting at one point must have different directions, since otherwise they are the same transport ray, by maximality).

Suppose now that they meet at z, which is in the interior of a transport ray whose direction is $e = \frac{x-y}{|x-y|}$ but is a boundary point for another ray, with direction $e' \neq e$. In such a case u should be differentiable at z and $\nabla u(z) = e$. Hence, we should have

$$t = u(z+te') - u(z) = e \cdot te' + o(t)$$

(the first equality coming from the behavior of u on the ray stemming from z with direction e' and the second from the definition of $\nabla u(z)$). Yet, this implies $e \cdot e' = 1$ and hence $e = e'$ (since $|e| = |e'| = 1$), which is a contradiction.

Suppose now that the intersection point z is a boundary point for both segments, one with direction e and the other one with direction $e' \neq e$. In this case there is no contradiction but if one supposes that there exists $\nabla u(z) = w$ one gets

$$t = u(z+te) - u(z) = w \cdot te + o(t), \quad t = u(z+te') - u(z) = w \cdot te' + o(t)$$

This implies $w \cdot e = w \cdot e' = 1$ but, since $|w| \leq 1$ (due to $u \in \mathrm{Lip}_1$) and $|e| = |e'| = 1$, we should get $w = e = e'$, which is, again, a contradiction. □

We will see later on that we need to say something more on the direction of the transport rays.

From this section on we fix a transport plan $\bar\gamma$, optimal for the secondary variational problem, and we will try to prove that it is actually induced by a transport map. We use here that $\bar\gamma$ is actually concentrated on a set Γ which is $c-$CM and see how this interacts with transport rays. We can suppose $\Gamma \subset A = \{(x,y) : u(x) - u(y) = |x-y|\}$, since anyway $\bar\gamma$ must be concentrated on such a set, so as to have a finite value for $\int c\, d\gamma$. We want to say that $\bar\gamma$ behaves, on each transport ray, as the monotone increasing transport. More precisely, the following is true.

Lemma 3.1.5. *Suppose that x_1, x_2, y_1 and y_2 are all points of a transport ray $[x,y]$ and that $(x_1,y_1),(x_2,y_2) \in \Gamma$. Define an order relation on such a transport ray through $x \le x' \Leftrightarrow u(x) \ge u(x')$. Then if $x_1 < x_2$ we also have $y_1 \le y_2$.*

Proof. We already know that $x_1 \ge y_1$ and $x_2 \ge y_2$ (thanks to $\Gamma \subset A$). Hence, the only case to be considered is the case where we have $x_2 > x_1 \ge y_1 > y_2$. If we prove that this is not possible than we have proven the thesis. And this is not possible due to the fat that Γ is $c-$CM, since on this transport ray, due to the order relationship we have supposed and to the behavior of u on such a segment, the condition $(x_1,y_2),(x_2,y_1) \in A$ is guaranteed. This implies $(x_1 - x_2)\cdot(y_1 - y_2) \ge 0$. But this is the scalar product of two vectors parallel to e, and on the segment this simply means that y_1 and y_2 must be ordered exactly as x_1 and x_2 are. $\square$

From what we have seen when we discussed the one-dimensional situation, we know that when s is a segment and $\Gamma \subset s \times s$ is such that $(x_1,y_1),(x_2,y_2) \in \Gamma$ and $x_1 < x_2$ imply $y_1 \le y_2$ (for the order relation on s), then Γ is contained in the graph of a monotone increasing multivalued function, which associates to every point either a point or a segment. Yet, the interiors of these segments being disjoints, there is at most a countable number of points where the image is not a singleton. This means that Γ is contained in a graph over s, up to a countable number of points of s.

If we combine what we get on every transport ray, we have obtained the following:

Proposition 3.1.6. *The optimal transport plan $\bar\gamma$ is concentrated on a set Γ with the following properties:*

- *if $(x,y) \in \Gamma$, then*
  - *either $x \in S(u)$ (but this set of points is Lebesgue-negligible),*
  - *or $x \notin T_u$, i.e. it does not belong to a non-degenerate transport ray (and in this case we necessarily have $y = x$, since otherwise $[x,y]$ would be contained in a non-degenerate transport ray),*

  - *or* $x \in T_u^{(b+)} \setminus S(u)$ *(and in this case we necessarily have* $y = x$*, since* x *is contained in a unique transport ray* s *and it cannot have other images* $y \in s$*, due to the order relation* $x \leq y$*,*
  - *or* $x \in T_u \setminus (T_u^{(b+)} \cup S(u))$ *and* y *belongs to the same transport ray* s *of* x *(which is unique);*

- *on each transport ray* s*,* $\Gamma \cap (s \times s)$ *is contained in the graph of a monotone increasing multivalued function;*

- *on each transport ray* s*, the set* $N_s = \{x \in s \setminus T_u^{(b+)} : \#(\{y : (x,y) \in \Gamma\} > 1\}$ *is countable.*

It is clear that $\bar{\gamma}$ *is induced by a transport map if* $\mu(\bigcup_s N_s) = 0$*, i.e. if we can get rid of a countable number of points on every transport ray.*

This could also be expressed in terms of disintegration of measures (if μ is absolutely continuous, then all the measures μ_s given by the disintegrations of μ along the rays s are atomless, see Section 2.3 for the notion of disintegration), but we will try to avoid such an argument for the sake of simplicity. The only point that we need is the following (*Property N*, for negligibility): if $B \subset \Omega$ is such that

- $B \subset T_u \setminus T_u^{(b+)}$,

- $B \cap s$ is at most countable for every transport ray s,

then $\mu(B) = 0$. Since we assume $\mu \ll \mathcal{L}^d$, it is sufficient to guarantee $\mathcal{L}^d(B) = 0$.

This property is not always satisfied by any disjoint family of segments in $\mathbb{R}^d$ and there is an example (by Alberti, Kirchheim and Preiss, later improved by Ambrosio and Pratelli, see [7]) where a disjoint family of segments contained in a cube is such that the collection of their middle points has positive measure. We will prove that the direction of the transport rays satisfy additional properties, which guarantee *property N*.

Just a last remark: we are ignoring here measurability issues of the transport map T that we are constructing. Actually, such map is obtained by gluing the monotone maps on every segment, but this should be done in a measurable way. It is possible to prove that this is the case, either by restricting to a $\sigma-$compact set Γ or by considering the disintegrations μ_s and ν_s and using the fact that, on each s, T is the monotone map sending μ_s onto ν_s (and hence it inherits some measurability properties of the dependence of μ_s and ν_s w.r.t. s, which are guaranteed by abstract disintegration theorems).

It appears that the main tool to prove Property N is the Lipschitz regularity of the directions of the transport rays (which is the same as the direction of ∇u).

Theorem 3.1.7. *Property N holds if ∇u is Lipschitz continuous or if there exists a countable family of sets E_h such that ∇u is Lipschitz continuous when restricted to each E_h and $\mathcal{L}^d(Tu \setminus \bigcup_h E_h) = 0$.*

Proof. First, suppose that ∇u is Lipschitz. Consider all the hyperplanes parallel to $d-1$ coordinate axes and with rational coordinates on the last coordinate. Since B is only made of points belonging to non-degenerate transport rays, every point of B belongs to a transport ray that meets at least one of these hyperplanes at exactly one point of its interior. Since these hyperplanes are a countable quantity, we can suppose that B is included in a collection S_Y of transport rays all meeting the same hyperplane Y. If we can prove that B is negligible under this assumption, then it will be negligible even if we withdraw it, by countable union. Not only, we can also suppose that B does not contain any point which is a boundary point of two different transport rays, since we already know that those points are negligible. Now, let us fix such an hyperplane Y and let us consider a map $f : Y \times \mathbb{R} \to \mathbb{R}^d$ of the following form: for $y \in Y$ and $t \in \mathbb{R}$ the point $f(y,t)$ is defined as $y + t\nabla u(y)$. This map is well-defined and injective on a set $\omega \subset Y \times \mathbb{R}$ which is the one we are interested in. This set ω is defined as those pairs (y,t) where y is in the interior of a transport ray, which gives that u is differentiable at such a point, and $y + t\nabla u(y)$ belongs to a transport ray of S_Y and it is not the boundary point of more than one transport ray. f is injective, since getting the same point as the image of (y,t) and of (y',t') would mean that two different transport rays cross at such a point. B is contained in the image of f by construction, so that f is a bijection. The map f is also Lipschitz continuous, as a consequence of the Lipschitz behavior of ∇u. We can hence consider $B' := f^{-1}(B)$. This set is a subset of $Y \times \mathbb{R}$ containing at most countably many points on every line $\{y\} \times \mathbb{R}$. By Fubini's theorem, this implies $\mathcal{L}^d(B') = 0$. Then we have also $\mathcal{L}^d(B) = \mathcal{L}^d(f(B')) \leq \mathrm{Lip}(f)^d \mathcal{L}^d(B')$, which implies $\mathcal{L}^d(B) = 0$.

It is clear that the property is also true when ∇u is not Lipschitz but is Lipschitz continuous on each set E_h of a partition covering almost all the points of T_u, since one can apply the same kind of arguments to all the sets $B \cap E_h$ and then use countable unions. □

We now need to prove that ∇u is Lipschitz continuous, at least on a countable decomposition.

Definition 10. A function $f:\omega\to\mathbb{R}^d$ is said to be countably Lipschitz if there exist a family of sets E_h such that f is Lipschitz continuous on each E_h and $\mathcal{L}^d(\omega\setminus\bigcup_h E_h)=0$. Notice that "being Lipschitz continuous on a set E" or "being the restriction to E of a Lipschitz continuous function defined on the whole $\mathbb{R}^d$" are actually the same property, due to Lipschitz extension theorems.

We want to prove that ∇u is countably Lipschitz. We will first prove that it coincides with some $\lambda-$convex or $\lambda-$concave functions on a sequence of sets covering almost everything. This requires a definition.

Definition 11. A function $f:\Omega\to\mathbb{R}$ is said to be $\lambda-$convex if $x\mapsto f(x)-\frac{\lambda}{2}|x|^2$ is convex, and $\lambda-$concave if $x\mapsto f(x)+\frac{\lambda}{2}|x|^2$ is concave. Notice that the number λ is not required to be positive, so that $\lambda-$convex functions for $\lambda>0$ are strictly convex, and if $\lambda<0$ they just have second derivatives bounded from below. Analogous considerations for $\lambda-$concave functions.

Proposition 3.1.8. *There exist some sets E_h such that*

- *u coincides with a $\lambda-$concave function on each E_h (for a value of λ depending on h),*
- $\bigcup_h E_h = T_u^{(i)}\cup T_u^{(b-)}$,
- *on each set E_h the function u is the restriction of a λ concave function.*

Proof. Let us define

$$E_h=\left\{x\in T_u\ :\ \exists z\in T_u \text{ with } |x-z|>\frac{1}{h},\ u(x)-u(z)=|x-z|\right\},$$

which is roughly speaking made of those points in the transport rays that are at least at a distance $\frac{1}{h}$ apart from the upper boundary point of the ray. It is clear that $E_h\subset T_u\setminus T_u^{(b+)}$. Moreover, we easily have $\bigcup_h E_h=T_u\setminus T_u^{(b+)}$.

Let us fix a function $c_h:\mathbb{R}^d\to\mathbb{R}$ with the following properties: $c_h\in C^2(\mathbb{R}^d)$, $|\nabla c_h|\le 1$, $c_h(z)\ge|z|$ for all $z\in\mathbb{R}^n$, $c_h(z)=|z|$ for all $z\notin B(0,1/h)$. It is easy to check that, if $x\in E_h$, one has

$$u(x)=\inf_{y\in\mathbb{R}^d}|x-y|+u(y)\le\inf_{y\in\mathbb{R}^d}c_h(x-y)+u(y)\le\inf_{y\notin B(x,1/h)}|x-y|+u(y)=u(x),$$

where the first inequality is a consequence of $|z|\le c_h(z)$ and the second is due to the restriction to $y\notin B(x,1/h)$. The last equality is justified by the definition of E_h. This implies that all the inequalities are actually equalities, and that $u(x) = u_h(x)$ for all $x \in E_h$, where

$$u_h(x) := \inf_{y \in \mathbb{R}^d} c_h(x - y) + u(y).$$

It is important to notice that u_h is a $\lambda -$ *concave* function.

Let us justify that u_h is $\lambda-$concave, for $\lambda \approx -\frac{1}{h}$. Actually, it is possible to choose c_h so that $D^2 c_h \leq -\lambda I$, for $\lambda = -\frac{2}{h}$ (and anyway the C^2 regularity is enough to bound the derivatives of c_h on bounded sets). This means that c_h is $\lambda-$concave. Consider

$$u_h(x) - \frac{\lambda}{2}|x|^2 = \inf_{y \in \mathbb{R}^d} c_h(x-y) - \frac{\lambda}{2}|x|^2 + u(y) = \inf_{y \in \mathbb{R}^d} c_h(x-y) - \frac{\lambda}{2}|x-y|^2 - \lambda x \cdot y + \frac{\lambda}{2}|y|^2 + u(y).$$

This last expression shows that $u_h(x) - \frac{\lambda}{2}|x|^2$ is concave in x, since it is expressed as the infimum of concave functions. Indeed $c_h(x - y) - \frac{\lambda}{2}|x - y|^2$ is concave and $\lambda x \cdot y$ is linear, and the other terms are constant in x. Hence $u_h(x)$ is $\lambda-$concave. $\square$

The previous theorem allows to replace the function u with the functions u_h, which are more regular (since they are $\lambda-$concave, they share the same regularity of convex functions). Yet, this is not enough yet, since convex functions in general are not even differentiable. A new countable decomposition is needed, and can be obtained from the following theorem.

Theorem 3.1.9. *If f is a convex function, then ∇f is countably Lipschitz.*

The proof of this theorem may be found in [4], in Theorem 5.34. It is also true when we replace ∇f with an arbitrary BV function, and this is the framework that one finds in [4].

Memo – *BV functions in $\mathbb{R}^d$*

An L^1 function $f : \mathbb{R}^d \to \mathbb{R}$ is said to belong to to be a BV function if its distributional derivatives are finite radon measures. This means that we require

$$\int f \, \nabla \cdot \xi \leq C ||\xi||_{L^\infty},$$

for every C^1 vector field $\xi : \mathbb{R}^d \to \mathbb{R}^d$. The space BV is hence a wider class than the Sobolev spaces $W^{1,p}$, where the distributional derivatives are supposed to belong to L^p (i.e., they are absolutely continuous measures with integrability properties).

It happens that the distributional derivative ∇f cannot be any measure, but must be of the form $\nabla f = \nabla^a f(x)dx + D^s f + D^j f$, where $\nabla^a f(x)dx$ is the absolutely continuous part of ∇f (with respect to the Lebesgue measure), $D^j f$ is the so-called "jump part" and has a density with respect to $\mathcal{H}^{d-1}$ (it is concentrated on the set J_f of those points where there is an hyperplane such that the function f has two different limits, in a suitable measure-theoretical sense, on the two sides of the hyperplane, and the density of $D^j u$ with respect to $\mathcal{H}^{d-1}_{|J_f}$ is exactly the difference of these two limit values times the direction of the normal vector to the hyperplane: $D^j f = (f^+ - f^-)\nu_{J_f} \cdot \mathcal{H}^{d-1}_{|J_f}$), and D^c is the so-called Cantor part, which is singular to the Lebesgue measure but vanishes on any $(d-1)-$dimensional set.

We denote by $BV_{loc}(\mathbb{R}^d)$ the space of functions which are locally BV in the sense that their derivatives are Radon measures, i.e. measures which are locally finite (in the definition with test functions ξ, this means that we only use $\xi \in C^1_c$ and the constant C may depend on the support of ξ. Vector-valued BV functions are just defined as functions $g = (g_1, \dots, g_k) : \mathbb{R}^d \to \mathbb{R}^k$ which are componentwise BV, i.e. $f_i \in BV$ for each $i = 1, \dots, k$. It is interesting that gradients $g = \nabla f$ of convex functions are always locally BV. This depends on the fact that the Jacobian matrix of the gradient of a convex function is indeed the Hessian of such a function, and is positive-defined. This means that the matrix-valued distribution $\nabla(\nabla f) = D^2 f$ is a positive distribution, and we know that positive distributions are necessarily positive measures (warning: this requires precise definitions when we work with matrix- and vector-valued functions).

BV functions satisfy several fine regularity properties almost everywhere for the Lebesgue of the $\mathcal{H}^{d-1}-$measure, and the reader may refer to [4] or [66]. In particular we cite the following (not at all easy) result, which implies Theorem 3.1.9.

Theorem If $f \in BV_{loc}(\mathbb{R}^d)$, then f is countably Lipschitz.

As a consequence of Proposition 3.1.8 and Theorem 3.1.9 one has.

Proposition 3.1.10. *If u is a Kantorovitch potential, then $\nabla u : T_u \to \mathbb{R}^d$ is countably Lipschitz.*

Proof. It is clear that the countably Lipschitz regularity of Theorem 3.1.9 is also true for the gradients of $\lambda-$convex and $\lambda-$concave functions. This means that this is true for ∇u_h and, by countable union, for ∇u. □

Finally, we get

Theorem 3.1.11. *Under the usual assumption $\mu \ll \mathcal{L}^d$, the secondary variational problem admits a unique solution $\bar{\gamma}$, which is induced by a transport map T, monotone on every transport ray.*

Proof. Proposition 3.1.10 together with Theorem 3.1.7 guarantee that Property N holds. Hence, Proposition 3.1.6 may be applied to get $\bar{\gamma} = \gamma_T$. The uniqueness follows in the usual way: if two plans $\bar{\gamma}' = \gamma_{T'}$ and $\bar{\gamma}'' = \gamma_{T''}$ optimize the secondary variational problem, the same should be true for $\frac{1}{2}\gamma_{T'} + \frac{1}{2}\gamma_{T''}$. Yet, for this measure to be induced by a ma, it is necessary to have $T' = T''$ a.e. $\square$

Notice that as a byproduct of this analysis we also obtain $\mathcal{L}^d(T_u^{(b)}) = 0$, since $T_u^{(b)}$ is a set meeting every transport ray in two points, and it is hence negligible by Property N. But we could not have proven it before, so unfortunately every strategy based on a decomposition of $T_u^{(i)}$ is not complete.

3.1.4 Approximation issues for the ray-monotone optimal transport

The present section deals with some natural questions concerning the optimal transport plan $\bar{\gamma}$ which is monotone along each transport ray, and in particular approximation questions. Indeed, we have seen (Theorem 1.6.10) that for any weakly converging sequence $\gamma_n \rightharpoonup \gamma$, if γ_n is optimal for a continuous cost between its marginal, then so is γ. Yet, we know that for $c(x, y) = |x - y|$ the optimal transport plan is not in general unique, and for many applications one prefers to stick to the one which is monotone on transport rays. Hence, the question is how to select this special transport plan by approximation, when the measures and/or the cost vary.

For many applications (see in particular Chapter 4), it is interesting to approximate transport plans through transport maps sending a given measure to an atomic one. This is because this kind of transport maps is actually composed of different homotheties defined on a partition of the domain.

Here is a useful approximation lemma in the spirit of $\Gamma-$convergence developments (see [57] and Section 2.4).

For fixed measures $\mu, \nu \in \mathcal{P}(\Omega)$, consider the following family of minimization problems (P_ε):

$$(P_\varepsilon) = \min\Big\{W_1((\pi_1)_\#\gamma, \nu) + \varepsilon C_1(\gamma) + \varepsilon^2 C_2(\gamma) + \varepsilon^{3d+3}\#((\pi_1)_\#\gamma) \ : \\ \gamma \in \mathcal{P}(\Omega \times \Omega),\ (\pi_0)_\#\gamma = \mu\Big\},$$

where W_1 is the usual Wasserstein distance, i.e. the minimum value of the transport problem for the cost $c(x, y) = |x - y|$, $C_p(\gamma) = \int |x - y|^p \gamma(dx, dy)$

for $p = 1, 2$ and the symbol $\#$ denotes the cardinality of the support of a measure.

Spoiler – *The distance* W_1

The Wasserstein distances W_p will be treated in details in Chapter 5 (and have been briefly defined in Chapter 2), but we just want to develop here the few tools that we need about W_1 at this point.

Definition Given $\mu, \nu \in \mathcal{P}(X)$ two probability measures on a metric space X, we define $W_1(\mu, \nu)$ through

$$W_1(\mu,\nu) := \inf\left\{\int d(x,y)\, d\gamma(x,y) \,:\, \gamma \in \Pi(\mu,\nu)\right\} = \sup\left\{\int u\, d(\mu-\nu) \,:\, u \in \mathrm{Lip}_1\right\}.$$

Checking that it is a distance From the definition via the minimization of the primal problem it is clear that we have $W_1(\mu,\nu) = 0 \iff \mu = \nu$ (since if the cost vanishes, then there is a plan $\gamma \in \Pi(\mu,\nu)$ supported on the diagonal, which implies that its two marginals coincide). Symmetry and positivity are obvious. The triangle inequality, on the other hand, comes from the dual definition. Indeed, if one takes a potential u optimal for (μ,ν) and a third measure ρ, one has $W_1(\mu,\nu) = \int u\, d(\mu-\nu) = \int u\, d(\mu-\rho) + \int u\, d(\rho-\nu) \le W_1(\mu,\rho) + W_1(\rho,\nu)$.

This distance W_1 is the only one among Wasserstein distances (defined with other exponents $p \ge 1$) to be a dual Sobolev norm, and it is indeed the dual norm in the space X', if X is the space of zero-mean Lipschitz functions with norm $||u|| := \mathrm{Lip}(u)$.

Semicontinuity The distance W_1 is l.s.c. for the weak convergence of probability measures. If the space is compact this is easily seen through the fact that for each $u \in \mathrm{Lip}_1$ the map $(\mu,\nu) \mapsto \int u\, d(\mu-\nu)$ is continuous and, passing to the supremum, we get l.s.c. (compactness is needed to guarantee boundedness of these functions). Anyway, the result stays true on non-compact spaces (and it is enough to look at the primal formulation).

Actually, the minimization of (P_ε) consists in looking for a transport plan with first marginal equal to μ satisfying the following criteria with decreasing degree of importance: the second marginal must be close to ν, the C_1 cost of transportation should be small, the C_2 as well, and, finally, the second marginal must be atomic with not too many atoms.

This minimization problem has obviously at least a solution (by the direct method, being Ω compact). We call γ_ε such a solution and $\nu_\varepsilon := (\pi_1)_\# \gamma_\varepsilon$ its second marginal. It is straightforward that ν_ε is an atomic measure and that γ_ε is the (unique, if $\mu \ll \mathcal{L}^d$, since the cost is strictly convex) optimal transport from μ to ν_ε for the cost $C_1+\varepsilon C_2$. Set

$$\bar\gamma = \operatorname{argmin}\left\{C_2(\gamma)\ :\ \gamma \text{ is a } C_1\text{−optimal transport plan from } \mu \text{ to } \nu\right\}. \tag{1.2}$$

This transport plan $\bar\gamma$ is unique and it is known to be the unique optimal transport plan from μ to ν which is monotone on transport rays (this is a consequence of Section 3.1.4); notice that the functional C_2 could have been replaced by any functional $\gamma\mapsto\int\phi(x-y)d\gamma$ for a strictly convex function ϕ).

Lemma 3.1.12. *As $\varepsilon\to 0$ we have $\nu_\varepsilon\rightharpoonup\nu$ and $\gamma_\varepsilon\rightharpoonup\bar\gamma$.*

Proof. It is sufficient to prove that any possible limit of subsequences coincide with ν or $\bar\gamma$, respectively. Let γ_0 be one such a limit and $\nu_0=(\pi_1)_\#\gamma_0$ the limit of the corresponding subsequence of ν_ε. Moreover, let p_n be any measurable map from Ω to a grid $G^n\subset\Omega$ composed of Cn^d points, with the property $|p_n(x)-x|\le 1/n$. Set $\nu^n:=(p_n)_\#\nu$ and notice $\#\nu^n\le Cn^d$, as well as $\nu^n\rightharpoonup\nu$.

First step: $\nu_0=\nu$. Take γ^n any transport plan from μ to ν^n. By optimality of γ_ε we have

$$W_1(\nu_\varepsilon,\nu)\le W_1(\nu^n,\nu)+\varepsilon C_1(\gamma^n)+\varepsilon^2C_2(\gamma^n)+C\varepsilon^{3d+3}n^d.$$

Fix n, let ε go to 0 and get

$$W_1(\nu_0,\nu)\le W_1(\nu^n,\nu)\le\frac{1}{n}.$$

Then let $n\to\infty$ and get $W_1(\nu_0,\nu)=0$, which implies $\nu_0=\nu$.

Second step: γ_0 is optimal for C_1 from μ to ν. Take any optimal transport plan γ^n (for the C_1 cost) from μ to ν^n. These plans converge to a certain optimal plan $\tilde\gamma$ from μ to ν. Then, by optimality, we have

$$\varepsilon C_1(\gamma_\varepsilon)\le W_1(\nu^n,\nu)+\varepsilon C_1(\gamma^n)+\varepsilon^2C_2(\gamma^n)+C\varepsilon^{3d+3}n^d\le\frac{1}{n}+\varepsilon C_1(\gamma^n)+C\varepsilon^2+C\varepsilon^{3d+3}n^d.$$

Then take $n\approx\varepsilon^{-2}$ and divide by ε :

$$C_1(\gamma_\varepsilon)\le\varepsilon+C_1(\gamma^n)+C\varepsilon+C\varepsilon^{d+2}.$$

Passing to the limit we get

$$C_1(\gamma_0)\le C_1(\tilde\gamma)=W_1(\mu,\nu),$$

which implies that γ_0 is optimal.

Third step: $\gamma_0 = \bar\gamma$. Take any optimal transport plan γ (for the cost C_1) from μ to ν. Set $\gamma^n = (id \times p_n)_\# \gamma$. We have $(\pi_1)_\# \gamma^n = \nu^n$. Then we have

$$W_1(\nu_\varepsilon, \nu) + \varepsilon C_1(\gamma_\varepsilon) + \varepsilon^2 C_2(\gamma_\varepsilon) \le W_1(\nu^n, \nu) + \varepsilon C_1(\gamma^n) + \varepsilon^2 C_2(\gamma^n) + C\varepsilon^{3d+3} n^d.$$

Moreover we have

$$C_1(\gamma_\varepsilon) \ge W_1(\mu, \nu_\varepsilon) \ge W_1(\mu, \nu) - W_1(\nu_\varepsilon, \nu),$$
$$C_1(\gamma^n) \le C_1(\gamma) + \int |p_n(y) - y| \gamma(dx, dy) \le C_1(\gamma) + \frac{1}{n} = W_1(\mu, \nu) + \frac{1}{n}.$$

Hence we have

$$(1-\varepsilon) W_1(\nu_\varepsilon, \nu) + \varepsilon W_1(\mu, \nu) + \varepsilon^2 C_2(\gamma_\varepsilon) \le \frac{1}{n} + \varepsilon W_1(\mu, \nu) + \frac{\varepsilon}{n} + \varepsilon^2 C_2(\gamma^n) + C\varepsilon^{3d+3} n^d.$$

Getting rid of the first term (which is positive) in the left hand side, simplifying $\varepsilon W_1(\mu, \nu)$, and dividing by ε^2, we get

$$C_2(\gamma_\varepsilon) \le \frac{1+\varepsilon}{n\varepsilon^2} + C_2(\gamma^n) + C\varepsilon^{3d+1} n^d.$$

Here it is sufficient to take $n \approx \varepsilon^{-3}$ and pass to the limit to get

$$C_2(\gamma_0) \le C_2(\gamma),$$

which is the condition characterizing $\bar\gamma$ (C_2-optimality among C_1-minimizers). □

This approximation result will be useful in Chapter 4. It is mainly based on the fact that when we minimize the transport cost $|x-y| + \varepsilon |x-y|^2$ we converge to the solution of the secondary variational problem, i.e. to the monotone transport map. As we said, any kind of strictly convex perturbation should do the same job as the quadratic one.

Yet, there are other approximation that are as natural as this one, but are still open. We list two of them.

Open Problem (stability of the monotone transport): take γ_n to be the optimal transport plan for the cost $|x-y|$ between μ_n and ν_n, monotone on transport rays. Suppose $\mu_n \rightharpoonup \mu, \nu_n \rightharpoonup \nu$ and $\gamma_n \rightharpoonup \gamma$. Is γ the ray-monotone optimal transport plan between μ and ν?

Open Problem (limit of $\sqrt{\varepsilon^2 + |x-y|^2}$): take γ_ε to be the optimal transport plan for the cost $\sqrt{\varepsilon^2 + |x-y|^2}$ between two given measures μ and

ν. If $\mathrm{spt}(\mu)\cap\mathrm{spt}(\nu)=\emptyset$, then γ_ε can be easily proven to converge to the ray-monotone optimal transport plan (because, at the first non-vanishing order in ε, the perturbation of the cost is of the form $|x-y|+\varepsilon^2/|x-y|+o(\varepsilon^2)$, and the function $h(t)=1/t$ is strictly convex on $\{t>0\}$). Is it true if the measures have intersecting (or identical) supports as well?

This last approximation is very useful when proving (or trying to prove) regularity results (see [83]).

3.2 The supremal case, L^∞

We consider now a different problem: instead of minimizing $c_p(\gamma)=\int|x-y|^p d\gamma$, we want to minimize the maximal displacement, i.e. its L^∞ norm.

Let us define

$$\begin{aligned} c_\infty(\gamma) := ||x-y||_{L^\infty(\gamma)} &= \inf\{m\in\mathbb{R}\,:\,|x-y|\le m \text{ for } \gamma-a.e.(x,y)\}\\ &= \max\{|x-y|\,:\,(x,y)\in\mathrm{spt}(\gamma)\}\end{aligned}$$

(where the last equality, between an L^∞ norm and a maximum on the support, is justified by the continuity of the function $|x-y|$).

Lemma 3.2.1. *For every $\gamma\in\mathcal{P}(\Omega\times\Omega)$ the quantities $c_p(\gamma)^{1/p}$ increasingly converge to $c_\infty(\gamma)$ as $p\to+\infty$. In particular $c_\infty(\gamma)=\sup_{p\ge1}c_p(\gamma)^{1/p}$ and c_∞ is l.s.c. for the weak convergence in $\Pi(\mu,\nu)$. Thus, it admits a minimizer over $\Pi(\mu,\nu)$, which is compact.*

Proof. It is well known that, on any finite measure space, L^p norms converge to the L^∞ norm, and we will not reprove it here. This may be applied to the function $(x,y)\mapsto|x-y|$ on $\Omega\times\Omega$, endowed with the measure γ, thus getting $c_p(\gamma)^{1/p}\to c_\infty(\gamma)$. Yet, it is important that this convergence is monotone here, and this is true when the measure is a probability. In such a case, we have for $p<q$, using Hölder (or Jensen) inequality

$$\int|f|^p d\gamma\le\left(\int|f|^q d\gamma\right)^{p/q}\left(\int 1\,d\gamma\right)^{1-p/q}=\left(\int|f|^q d\gamma\right)^{p/q},$$

for every $f\in L^q(\gamma)$. This implies, by taking the $p-$th root, $||f||_{L^p}\le||f||_{L^q}$. Applied to $f(x,y)=|x-y|$ this gives the desired monotonicity.

From that we infer that c_∞ is the supremum of a family of functional which are continuous for the weak convergence (since c_p is the integral of a bounded continuous function, Ω being compact, and taking the $p-$th root does not break continuity). As a supremum of continuous functionals, it is l.s.c. and the conclusion follows. $\square$

The goal now is to analyze the solution of

$$\min c_\infty(\gamma) \ : \ \gamma \in \Pi(\mu, \nu)$$

ant to prove that there is at least a minimizer γ induced by a transport map. This map would minimize

$$\min ||T(x) - x||_{L^\infty(\mu)} \ , \ T_\# \mu = \nu.$$

Here as well there will be no uniqueness (it is almost always the case when we minimize an L^∞ criterion), hence we define $O_\infty(\mu,\nu) = \mathrm{argmin}_{\gamma\in\Pi(\mu,\nu)} c_\infty(\gamma)$, the set of optimal transport plans for this L^∞ cost. Notice that $O_\infty(\mu,\nu)$, since c_∞ is l.s.c. (as for $O(\mu,\nu)$). Suppose now that $\min\{c_\infty(\gamma) \ : \ \gamma \in \Pi(\mu,\nu)\} = L$: notice also that we have

$$\gamma \in O_\infty(\mu,\nu) \Leftrightarrow \mathrm{spt}(\gamma) \subset \{(x,y) : |x-y| \leq L\}$$

(since any transport plan γ concentrated on the pairs where $|x - y| \leq L$ satisfies $c_\infty(\gamma) \leq L$ and is hence optimal). We will suppose $L > 0$ otherwise this means that it is possible to obtain ν from μ with no displacement, i.e. $\mu = \nu$ and the optimal displacement is the identity.

Consequently, exactly as for the L^1 case, we can define a secondary variational problem:

$$\min\{c_2(\gamma) \ : \ \gamma \in O_\infty(\mu,\nu)\}.$$

This problem has a solution $\bar\gamma$ since c_2 is continuous for the weak convergence and $O_\infty(\mu,\nu)$ is compact. Also for this minimizer, we do not know a priori any uniqueness. Again, it is possible to say that $\bar\gamma$ also solves

$$\min \int c\, d\gamma \ : \ \gamma \in \Pi(\mu,\nu), \qquad \text{where } c(x,y) = \begin{cases} |x-y|^2 & \text{if } |x-y| \leq L \\ +\infty & \text{otherwise.} \end{cases}$$

The arguments are the same as in the L^1 case. Moreover, also the form of the cost c is similar, and this cost is l.s.c. as well. Hence, $\bar\gamma$ is concentrated on a set $\Gamma \subset \Omega \times \Omega$ which is $c-$CM. This means

$$(x_1,y_1),(x_2,y_2) \in \Gamma, |x_1-y_2|, |x_2-y_1| \leq L \Rightarrow (x_1-x_2)\cdot(y_1-y_2) \geq 0. \quad (2.3)$$

We can also suppose $\Gamma \subset \{(x,y) \ : \ |x-y| \leq L\}$. We will try to improve a little bit the set Γ, by removing negligible sets and getting better properties. Then, we will show that the remaining set $\tilde\Gamma$ will be contained in the graph of a map T, thus obtaining the result.

First, we need to recall the notion of Lebesgue points:

Memo – *Density points*

Definition. For a measurable set $E \subset \mathbb{R}^d$ we call Lebesgue point of E a point $x \in E$ such that

$$\lim_{r\to 0} \frac{\mathcal{L}^d(E \cap B(x,r))}{\mathcal{L}^d(B(x,r))} = 1.$$

The set of Lebesgue points of E is denoted by $\mathrm{Leb}(E)$ and it is well-known that $\mathcal{L}^d(E \setminus \mathrm{Leb}(E)) = 0$. This is actually a consequence of a more general fact, the fact that, given a function $f \in L^1_{loc}(\mathbb{R}^d)$, a.e. point x is a Lebesgue point for f, in the sense that $\lim_{r\to 0} ⨍_{B(x,r)} |f(y) - f(x)|dy = 0$, which also implies $f(x) = \lim_{r\to 0} ⨍_{B(x,r)} f(y)\, dy$. If this is applied to $f = I_E$, then one recovers the notion of Lebesgue points of a set (also called density points).

Let B_i be a countable family of balls in $\mathbb{R}^d$, generating the topology of $\mathbb{R}^d$ (for instance, all the balls with rational radius and rational center). Consider now

$$A_i := (\pi_x)(\Gamma \cap (\Omega \times B_i)),$$

i.e. the set of all points x such that at least a point y with $(x,y) \in \Gamma$ belongs to B_i (the points that have at least one "image" in B_i). Then set $N_i := A_i \setminus \mathrm{Leb}(A_i)$. This set has zero Lebesgue measure, and hence it is $\mu-$negligible. Also $\mu(\bigcup_i N_i) = 0$. As a consequence, one can define $\tilde{\Gamma} := \Gamma \setminus ((\bigcup_i N_i) \times \Omega)$. The plan $\bar{\gamma}$ is still concentrated on $\tilde{\Gamma}$, since we only removed $\mu-$negligible points. Moreover, $\tilde{\Gamma}$ has the following property: if $(x_0, y_0) \in \tilde{\Gamma}$, then, for every $\varepsilon, \delta > 0$, every unit vector ξ and every sufficiently small $r > 0$, there are a point $x \in \big(B(x_0, r) \setminus B(x_0, \frac{r}{2})\big) \cap C(x_0, \xi, \delta)$ and a point $y \in B(y_0, \varepsilon)$ such that $(x,y) \in \tilde{\Gamma}$, where $C(x_0, \xi, \delta)$ is the following convex cone:

$$\{x : (x - x_0) \cdot \xi > (1 - \delta)|x - x_0|\}.$$

This means that one can choose a point x close to x_0, which is sent to at least a point y close to y_0, also imposing the direction $x - x_0$ up to an error δ as small as we want. This is true since there is at least one of the ball B_i containing y_0 and contained in $B(y_0, \varepsilon)$. Since $x_0 \in A_i$ and we have removed N_i, this means that x_0 is a Lebesgue point for A_i. Since the region $\big(B(x_0, r) \setminus B(x_0, \frac{r}{2})\big) \cap C(x_0, \xi, \delta)$ is a portion of the ball $B(x_0, r)$ which takes a fixed proportion (depending on δ) of volume of the whole ball, for $r \to 0$ it is clear that A_i (and also $\mathrm{Leb}(A_i)$) must meet it (otherwise

x_0 would not be a Lebesgue point). It is hen sufficient to pick a point in $\mathrm{Leb}(A_i) \cap \big(B(x_0,r) \setminus B(x_0,\frac{r}{2})\big) \cap C(x_0,\xi,\delta)$ and we are done.

Moreover, $\tilde\Gamma$ stays $c-$CM and enjoys property (2.3).

Lemma 3.2.2. *If (x_0,y_0) and (x_0,z_0) belong to $\tilde\Gamma$, then $y_0 = z_0$.*

Proof. Suppose by contradiction $y_0 \neq z_0$. In order to fix the ideas, let us suppose $|x_0 - z_0| \le |x_0 - y_0|$ (and in particular $y_0 \neq x_0$, since otherwise $|x_0 - z_0| = |x_0 - y_0| = 0$ and $z_0 = y_0$).

Now, use the property of $\tilde\Gamma$ and find $(x,y) \in \tilde\Gamma$ with $y \in B(y_0,\varepsilon)$ and $x \in \big(B(x_0,r) \setminus B(x_0,\frac{r}{2})\big) \cap C(x_0,\xi,\delta)$, for a vector ξ to be determined later. Use now the fact that $\tilde\Gamma$ is $c-$CM applied to (x_0,z_0) and (x,y).

If we can prove that $|x - z_0|, |x_0 - y| \le L$, then we should have

$$(x - x_0)\cdot(y - z_0) \ge 0.$$

Yet, the direction of $x - x_0$ is almost that of ξ (up to an error of δ) and that of $y - z_0$ is almost that of $y_0 - z_0$ (up to an error of the order of $\varepsilon/|z_0 - y_0|$). If we choose ξ such that $\xi\cdot(y_0 - z_0) < 0$, this means that for small δ and ε we would get a contradiction.

Moreover, we need to guarantee $|x - z_0|, |x_0 - y| \le L$, in order to prove the thesis. Let us compute $|x - z_0|^2$: we have

$$|x - z_0|^2 = |x_0 - z_0|^2 + |x - x_0|^2 + 2(x - x_0)\cdot(x_0 - z_0).$$

In this sum we have

$$|x_0 - z_0|^2 \le L^2; \quad |x - x_0|^2 \le r^2; \quad 2(x - x_0)\cdot(x_0 - z_0) = |x - x_0|(\xi\cdot(x_0 - z_0) + O(\delta)).$$

Suppose now that we are in one of the following situations: either choose ξ such that $\xi\cdot(x_0 - z_0) < 0$ or $|x_0 - z_0|^2 < L^2$. In both cases we get $|x - z_0| \le L$ for r and δ small enough. In the fist case, since $|x - x_0| \ge \frac{r}{2}$, we have a negative term of the order of r and a positive one of the order of r^2; in the second we add to $|x_0 - z_0|^2 < L^2$ some terms of the order of r or r^2.

Analogously, for $|x_0 - y|$ we have

$$|x_0 - y|^2 = |x_0 - x|^2 + |x - y|^2 + 2(x_0 - x)\cdot(x - y).$$

The three terms satisfy

$$|x_0 - x|^2 \le r^2; \quad |x - y|^2 \le L^2; \quad 2(x_0 - x)\cdot(x - y) = |x - x_0|(-\xi\cdot(x_0 - y_0) + O(\delta + \varepsilon + r)).$$

In this case, either we have $|x_0 - y_0| < L$ (which would guarantee $|x_0 - y| < L$ for ε, δ small enough), or we need to impose $\xi\cdot(y_0 - x_0) < 0$.

Notice that imposing $\xi \cdot (y_0 - x_0) < 0$ and $\xi \cdot (x_0 - z_0) < 0$ automatically gives $\xi \cdot (y_0 - z_0) < 0$. Set $v = y_0 - x_0$ and $w = x_0 - z_0$. If it is possible to find ξ with $\xi \cdot v < 0$ and $\xi \cdot w < 0$ we are done. When is it the case that two vectors v and w do not admit the existence of a vector ξ with both scalar products that are negative? the only case is when they go in opposite directions (notice that it is enough to check this fact in $\mathbb{R}^2$, and the only case is when the angle between them is exactly 180°). But this would mean that x_0, y_0 and z_0 are colinear, with z_0 between x_0 and y_0 (since we supposed $|x_0 - z_0| \leq |x_0 - y_0|$). If we want z_0 and y_0 to be distinct points, we should $|x_0 - z_0| < L$. Hence in this case we do not need to check $\xi \cdot (x_0 - z_0) < 0$. We only need a vector ξ satisfying $\xi \cdot (y_0 - x_0) < 0$ and $\xi \cdot (y_0 - z_0) < 0$, but the directions of $y_0 - x_0$ and $y_0 - z_0$ are the same, so that this can be guaranteed by many choices of ξ, since we only need the scalar product with one only direction to be negative. Take for instance $\xi = -(y_0 - x_0)/|y_0 - x_0|$. □

Theorem 3.2.3. *The secondary variational problem* $\min c_2(\gamma) \,:\, \gamma \in O_\infty(\mu, \nu)$ *admits a unique solution $\bar{\gamma}$, it is induced by a transport map T, and such a map is an optimal transport for the problem*

$$\min ||T(x) - x||_{L^\infty(\mu)} \ , \ T_\# \mu = \nu.$$

Proof. We have already seen that $\bar{\gamma}$ is concentrated on a set $\tilde{\Gamma}$ satisfying some useful properties. Lemma 3.2.2 shows that $\tilde{\gamma}$ is contained in a graph, since for any x_0 there is no more than one possible point y_0 such that $(x_0; y_0) \in \tilde{\Gamma}$. Let us consider such a point y_0 as the image of x_0 and call it $T(x_0)$. Then $\bar{\gamma} = \gamma_T$. The optimality of T in the "Monge" version of this L^∞ problem comes from the usual comparison with the Kantorovitch version on plans γ. The uniqueness comes from the standard arguments (since it is true as well that convex combinations of minimizers should be minimizers, and this allows to perform the usual proof). □

3.3 Discussion

3.3.1 Different norms and more general convex costs

The attentive reader will have notice that the proof in Section 3.1 about the case of the distance cost function was specific to the case of the distance induced by the Euclidean norm. Starting from the original problem by Monge, and lying on the proof strategy by Sudakov [103] (which had a problem later solved by Ambrosio in [2, 9], but was meant to treat the case of an arbitrary norm), the case of different norms has been extensively studied in the last year. Notice that the case of uniformly convex norms (i.e. those such that the Hessian of the square of the norm is bounded from below by a matrix which is positively definite) is the one which is most similar to the Euclidean case, and can be treated in a similar way.

One of the first extension was the case studied by Ambrosio, Kirchheim and Pratelli about crystalline norms ([7]), i.e. norms such that their unit balls are convex polyhedra. One can guess that the fact that we have a finite number of faces makes the task easier, but it already required a huge approximation work. Indeed, if one uses duality, the gradient $\nabla u(x)$ of a Kantorovich potential is not enough to determine the direction of the displacement $y-x$ for a pair $(x,y)\in \mathrm{spt}(\gamma)$, because a crystalline norm has the same gradient on all the points of a same face (and not only on points on the same ray, as is the case for the Euclidean norm, or for other strictly convex norms). To handle this problem, [7] develops a strategy with a double approximation, where $||z||$ is replaced by something like $||z||+\varepsilon|z|+\varepsilon^2|z|^2$, where $||z||$ is the norm we consider, $|z|$ is the Euclidean norm which is added so as to select a direction, and its square is added so as to get strict convexity and monotonicity on each ray (exactly as we saw in Section 3.1). In this way the authors prove the existence of an optimal map for crystalline norms in any dimensions, and for arbitrary norms in $\mathbb{R}^2$.

Later, the way was open to the generalization to other norms. The main tool is always a secondary variational problem, since for norm costs it is in general false that optimal plans all come from a map, and one needs to select a special one. The problem with secondary variational problems, is that they correspond to transport problem with new costs c which are not finitely valued. This prevents from using duality on the secondary problem. Indeed, Proposition 1.6.6 proves that we have equality between the primal and the dual problem for all costs which are l.s.c., but does not guarantee the existence of an optimizer in the dual; existence is usually guaranteed either via Ascoli-Arzelà arguments, or through Theorem 1.6.2, which requires finiteness of the cost. This is why some methods avoiding duality, and concentrating on $c-$cyclical monotonicity of sets where the transport plan which is optimal for the secondary variational problem is concentrated have been developed. The first time they appeared is in [50], for a different problem (an L^∞ case) and it corresponds to the idea that we presented in Section 3.2 (looking for density points of this concentration set). Later, Champion and DePascale managed to use the same tools to prove the existence of an optimal map first for arbitrary strictly convex norms (in [51], the same result being obtained differently at almost the same time by Caravenna in [43]), and then for general norms in [52]. This last result was more difficult to obtain and required some extra approximation tools developed in [98], that we will see in Chapter 4.

But the history of optimal transport did not look only at norms. Many studies have been done for different distance cost functions (distances or squared distances on manifolds, geodesic distances on $\mathbb{R}^d$ if obstacle are present…). In this book we prefer to stick to the Euclidean case, and in this section we only consider costs of the form $c(x, y) = h(x - y)$ for h convex. Even this case is far from being completely understood. In a paper that the author write with Carlier and De Pascale a general (straightforward) strategy of decomposition according to the "faces" of the cost function h.

The decomposition is based on the following steps:

- Consider an optimal plan γ and look at the optimality conditions as in Section 1.3. For all $(x_0, y_0) \in \mathrm{spt}(\gamma)$, if x_0 is a differentiability point for the potential ϕ (we write $x_0 \in \mathrm{Diff}(\phi)$), one gets $\nabla\phi(x_0) \in \partial h(x_0 - y_0)$, which is equivalent to

$$x_0 - y_0 \in \partial h^*(\nabla\phi(x_0)). \tag{3.4}$$

  Let us define

$$\mathcal{F}_h := \{\partial h^*(p) \ : \ p \in \mathbb{R}^d\},$$

  which is the set of all values of the subdifferential multi-map of h^*. These values are those convex sets where the function h is affine, and they will be called *faces* of h. The fact that ϕ is differentiable $\mu-$a.e. in enforced by supposing $\mu \ll \mathcal{L}^d$ and h to be Lipschitz, so that ϕ is also Lipschitz.

  Thanks to (3.4), for every fixed x, all the points y such that (x, y) belongs to the support of an optimal transport plan are such that the difference $x - y$ belong to a same face of c. Classically, when these faces are singleton (i.e. when c^* is differentiable, which is the same as c being strictly convex), this is the way to obtain a transport map, since only one y is admitted for every x.

  Equation (3.4) also enables one to classify the points x as follows. For every $K \in \mathcal{F}_h$ we define the set

$$X_K := \{x \in \mathrm{Diff}(\phi) \ : \ \partial c^*(\nabla\phi(x)) = K\}.$$

  Hence γ may be decomposed into several subplans γ_K according to the criterion $x \in X_K$. If K varies among all possible faces this decomposition covers $\gamma-$almost all pairs (x, y). Moreover, if (x, y) belongs to $\mathrm{spt}\,\gamma$ and x to $\mathrm{Diff}(\phi)$, then $x \in X_K$ implies $x - y \in K$.

If the set $\mathcal{F}_h$ is finite or countable, we can define

$$\gamma_K := \gamma_{|X_K \times \mathbb{R}^d},$$

which is the simpler case. Actually, in this case, the marginal measures μ_K and ν_K of γ_K (i.e. its images under the maps $(x, y) \mapsto x$ and $(x, y) \mapsto y$, respectively) are submeasures of μ and ν, respectively. In particular μ_K inherits the absolute continuity from μ. This is often useful for proving existence of transport maps.

If $\mathcal{F}_h$ is uncountable, in some cases one can still rely on a countable decomposition by considering the set $\mathcal{F}_h^{multi} := \{K \in \mathcal{F}_h \ : K \text{ is not a singleton } \}$. If $\mathcal{F}_h^{multi}$ is countable, then one can separate those x such that $\partial h^*(\nabla(\phi(x))$ is a singleton (where a transport already exists) and look at a decomposition for $K \in \mathcal{F}_h^{multi}$ only.

- This decomposition reduces the transport problem to a superposition of transport problems of the type

$$\min\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} h(x-y)\,\gamma(dx,dy) \ : \gamma \in \Pi(\mu_K, \nu_K),\ spt\gamma \subset \{x - y \in K\}\right\}.$$

The advantage is that the cost c restricted to K is easier to study. If K is a face of h, then h is affine on K and in this case the transport cost does not depend any more on the transport plan.

- The problem is reduced to find a transport map from μ_K to ν_K satisfying the constraint $x - T(x) \in K$, knowing a priori that a transport plan satisfying the same constraint exists.

In some cases (for example if K is a convex compact set containing 0 in its interior) this problem may be reduced to an L^∞ transport problem. In fact if one denotes by $||\cdot||_K$ the (gauge-like) “norm” such that $K = \{x \, : \, ||x||_K \le 1\}$, one has

$$\min\left\{\max\{||x-y||_K,\ (x,y) \in \mathrm{spt}(\gamma)\}, \gamma \in \Pi(\mu,\nu)\right\} \le 1 \qquad (3.5)$$

and the question is whether the same result would be true if one restricted the admissible set to transport maps only (passing from Kantorovich to Monge, say). The answer would be positive if a solution of (3.5) were induced by a transport map T. This is what we presented in Section 3.2 in the case $K = \overline{B(0,1)}$, and which was first proven in [50] via a different strategy (instead of selecting a minimizer via a secondary variational problem, selecting the limit of the minimizers for the L^p norms or $p \to \infty$). Notice also that this issue is easy in dimension one where the monotone transport solves all the L^p problems, and hence the L^∞ as well (and this does not need the measure to be absolutely continuous, but just atomless).

Let us notice that the assumption that the number of faces is countable is quite restrictive, and is essentially used to guarantee the absolute continuity of μ_K, with no need of a disintegration argument (which could lead to difficulties at least as hard as that faced by Sudakov). However, an interesting example that could be approached by finite decomposition is that of crystalline norms. In this case the faces of the cost c are polyhedral cones but, if the support of the two measures are bounded, we can suppose that they are compact convex polyhedra. This means, thanks to the considerations above, that it is possible to perform a finite decomposition and to reduce the problem to some L^∞ minimizations for norms whose unit balls are polyhedra (the faces of the cone). In particular the L^1 problem for crystalline norms is solved if we can solve L^∞ optimal transport problem for other crystalline norms.

It becomes then interesting to solve the L^∞ problem as in Section 3.2, replacing the unit ball constraint $|x-y| \leq 1$ with a more general constraint $x - y \in C$, the set C being a generic convex set, for instance a polyhedron. This is studied in [78] by minimizing the quadratic cost

$$c(x,y) = \begin{cases} |x-y|^2 & \text{if } x-y \in C, \\ +\infty & \text{otherwise,} \end{cases}$$

which is also of the form $c(x,y) = h(x-y)$ for h strictly convex (but not real valued). The existence of an optimal map (better: the fact that any optimal γ for this problem is induced by a map) is proven when μ is absolutely continuous and C is either strictly convex or has a countable quantity of faces. This can be achieved by adapting the arguments of Section 3.2 and proving that, if (x_0, y_0) and (x_0, y_1) belong to $\mathrm{spt}(\gamma)$ (and hence $y_0 - x_0, y_1 - x_0 \in C$), then the middle point $\frac{1}{2}(y_0 + y_1) - x_0$ cannot lie in the interior of C. If C is strictly convex this proves $y_0 = y_1$. If not, this proves that the points $y_0 - x_0$ and $y_1 - x_0$ are on a same face of C and one can perform a dimensional reduction and proceed by induction.

In particular, this argument applies to the case of polyhedra and of arbitrary convex sets in $\mathbb{R}^2$ (since no more than a countable quantity of segments may be contained in ∂C), and provides an alternative proof for the result of [7].

Many generalizations of this last result have been performed: Bertrand and Puel studied the "relativistic" cost $h(z) := 1 - \sqrt{1 - |z|^2}$, which is naturally endowed with the constraint $|z| \leq 1$ and is a strictly convex function (but not finite-valued since the constraint means $h(z) = +\infty$ outside the unit ball). They proved existence of an optimal map by adapting the arguments of [50, 78] to the case of a strictly convex function h with a strictly convex constraint C (adapting to the case of a countable number of faces does not seem to be impossible).

Finally, Chen, Jiang and Yang combined the arguments of [78] with those of [52], thus getting existence for the cost function

$$c(x,y) = \begin{cases} |x-y| & \text{if } x - y \in C, \\ +\infty & \text{otherwise,} \end{cases}$$

under the usual assumptions on C (strictly convex or countable number of faces). these results can be found in [53, 54] and are the first which combine lack of strict convexity and infinite values.

All these subsequent improvements pave the way to an obvious conjecture, i.e. the fact that a mixture of duality, variational approximation and density points techniques could finally prove the existence of an optimal transport map in a very general case:

Open Problem (existence for every convex cost): suppose $\mu \ll \mathcal{L}^d$ and take a cost $c(x,y)$ of the form $h(x-y)$ with $h : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ convex; prove that there exists an optimal map.

Yet, we are from a complete proof. Even the case where h is finite-valued but not strictly convex, or the strictly convex case with arbitrary constraints are not yet complete.

3.3.2 Concave costs (L^p, with $p < 1$)

Another class of transport costs which is very reasonable for applications, rather than convex functions of the Euclidean distance, is that of concave costs, more precisely $c(x,y) = \ell(|x-y|)$ where $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly concave and increasing function. From the economical and modelization point of view, this is the most natural choice: moving a mass has a cost which is proportionally less if the distance increases, as everybody can notice from travel fares. In many practical cases, moving two masses on a distance d each is more expensive than moving one at distance $2d$ and keeping at rest the other. The typical example is the power cost $|x-y|^\alpha$, $\alpha < 1$.

Notice that all these costs satisfy the triangle inequality and are thus distances on $\mathbb{R}^d$. Moreover, under strict convexity assumptions, these costs satisfy a strict triangle inequality. This last fact implies that the common mass between μ and ν must stay at rest.

Theorem 3.3.1. *Let γ be an optimal transport plan for the cost $c(x,y) = \ell(|x-y|)$ with $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ strictly concave, increasing, and such that $\ell(0) = 0$. Let $\gamma = \gamma_D + \gamma_O$, where γ_D is the restriction of γ to the diagonal $D = \{(x,x) : x \in \mathbb{R}^d\}$ and γ_O is the part outside the diagonal, i.e. the restriction to $D^c = (\mathbb{R}^d \times \mathbb{R}^d) \setminus D$. Then this decomposition is such that $(\pi_x)_\#\gamma_O$ and $(\pi_y)_\#\gamma_O$ are mutually singular measures.*

Proof. It is clear that γ_O is concentrated on $\operatorname{spt}\gamma \setminus D$ and hence $(\pi_x)_\#\gamma_O$ is concentrated on $\pi_x(\operatorname{spt}\gamma \setminus D)$ and $(\pi_y)_\#\gamma_O$ is concentrated on $\pi_y(\operatorname{spt}\gamma \setminus D)$. We claim that these two sets are disjoint. Indeed suppose that a common point z belongs to both. Then, by definition, there exists y such that $(z,y) \in \operatorname{spt}\gamma \setminus D$ and x such that $(x,z) \in \operatorname{spt}\gamma \setminus D$. This means that we can apply $c-$cyclical monotonicity to the points (x,z) and (z,y) and get

$$\ell(|x-z|)+\ell(|z-y|) \le \ell(|x-y|)+\ell(|z-z|) = \ell(|x-y|) < \ell(|x-z|)+\ell(|z-y|),$$

where the last strict inequality gives a contradiction. □

This gives a first constraint on how to build optimal plans γ: look at μ and ν, take the common part $\mu \wedge \nu$, leave it on place, subtract it from the rest, and then build an optimal transport between the two remainders, which will have no mass in common. Notice that when the cost c is linear in the Euclidean distance, then the common mass *may* stay at rest but is not forced to do so (the very well known example is the transport from $\mu = \mathcal{L}^1 \llcorner [0,1]$ and $\nu = \mathcal{L}^1 \llcorner [\frac{1}{2}, \frac{3}{2}]$, where both $T(x) = x + \frac{1}{2}$ and $T(x) = x+1$ on $[0, \frac{1}{2}]$ and $T(x) = x$ on $]\frac{1}{2}, 1]$ are optimal); on the contrary, when the cost is strictly convex in the Euclidean distance, in general the common mass *does not* stay at rest (in the previous example only the translation is optimal for $c(x,y) = |x-y|^p$, $p > 1$). Notice that the fact that the common mass stays at rest implies that in general there is no optimal map T, since whenever there is a set A with $\mu(A) > (\mu \wedge \nu)(A) = \nu(A)$ then almost all the points of A must have two images: themselves, and another point outside A.

Yet, this suggests to study the case where μ and ν are mutually singular, and the best one can do would be proving the existence of an optimal map in this case. In particular, this allows to avoid the singularity of the function $(x,y) \mapsto \ell(|x-y|)$ is mainly concentrated on the diagonal $\{x = y\}$ (look at the example $|x-y|^\alpha$), since when the two measures have no common mass almost no point x is transported to $y = x$.

Yet, exploiting this fact needs some attention. The easiest case is when μ and ν have disjoint supports, since in this case there is a lower bound on $|x-y|$ and this allow to stay away from the singularity. Yet, $\mathrm{spt}\,\mu\cap\mathrm{spt}\,\nu=\emptyset$ is too restrictive, since even in the case where μ and ν have smooth densities f and g it may happen that, after subtracting the common mass, the two supports meet on the region $\{f=g\}$.

The problem has been solved in one of the first papers about optimal transportation, written by Gangbo and McCann in 1996, [73], where they choose the slightly less restrictive assumption $\mu(\mathrm{spt}(\nu))=0$. This assumption covers the example above of two continuous densities, but does not cover other cases such as μ being the Lebesgue measure on a bounded domain Ω and ν being an atomic measure with an atom at each rational point, or other examples that one can build with fully supported absolutely continuous measures concentrated on disjoint sets A and $\Omega\setminus A$ (see below). In the present section we show how to prove the existence of a transport map when μ and ν are singular to each other and $\mu\ll\mathcal{L}^d$.

For the curious reader – *Disjoint sets with full support*

Producing dense closed sets with empty interior but arbitrary measure is a standard trick in real analysis, but one could wonder if it is easy to see two disjoint Borel sets $A,B\subset[0,1]^d$ such that the support of the Lebesgue measures $\mathcal{L}^d\llcorner A$ and $\mathcal{L}^d\llcorner B$ are the full cube $[0,1]^d$. This is a non-trivial exercise that we present (in dimension one; then, by product, the very same example could be easily transported to higher dimension) for the interest reader.

Proposition There exists a Borel set $A\subset[0,1]$ such that for every interval $I\subset[0,1]$ one has $0<\mathcal{L}^1(I\cap A)<\mathcal{L}^1(I)$. In particular, A and $B:=A^c$ are two disjoint sets such that $\mathrm{spt}(\mathcal{L}^1\llcorner A)=[0,1]$ and $\mathrm{spt}(\mathcal{L}^1\llcorner B)=[0,1]$.

Proof Through a very well-know construction, i.e. numbering rational points q_n and taking $\bigcup_n]q_n-\varepsilon 2^{-n},q_n+\varepsilon 2^{-n}[$, we are able to build an open set, containing all rational points, with arbitrary small measure. Its complement in $[0,1]$ is a closed set with empty interior and positive measure. By scaling, we can produce a closed set with empty interior and positive measure contained in an arbitrarily small interval, and with arbitrarily small positive measure.

Let us now take a sequence of open intervals I_n which is a base of the topology of $]0,1[$, and more precisely choose the following sequence: $I_0=]0,1[$, I_1 and I_2 are the two dyadic intervals of length $1/2$, I_3,I_4,I_5 and I_6 are the two dyadic intervals of length $1/4$... We need the following property in the choice of the sequence: if $n_1\geq n_2$, either $I_{n_1}\cap I_{n_2}=\emptyset$, or $I_{n_1}\subset I_{n_2}$. Now define recursively a sequence of closed sets F_n in this way:

- F_0 is a closed set with empty interior and positive measure contained in I_0. Due to its empty interior, it cannot be of full measure in I_0.

- for each $n \geq 0$, F_{n+1} is a closed set with empty interior contained in I_{n+1} such that its measure is positive but smaller than $2^{-n}\mathcal{L}^1(I_{n+1} \setminus \bigcup_{i\leq n} F_i)$. Notice that $\mathcal{L}^1(I_{n+1} \setminus \bigcup_{i\leq n} F_i) > 0$ since $\bigcup_{i\leq n} F_i$ is a closed set with empty interior (as a finite union of closed sets of $\mathbb{R}$ with empty interior) and hence it cannot be of full measure in the interval I_{n+1}.

We claim that $A := \bigcup_{n\geq 0} F_n$ is the set that we look for. Indeed, if we take an arbitrary interval I it contains a certain I_{n_0} and $\mathcal{L}^1(I \cap A) \geq \mathcal{L}^1(I_{n_0} \cap F_{n_0}) > 0$. We are only left to prove $\mathcal{L}^1(I \setminus A) > 0$, and it is enough to prove $\mathcal{L}^1(I_{n_0} \setminus A) > 0$. Notice that $\mathcal{L}^1(I_{n_0} \setminus A) = \lim_n \mathcal{L}^1(I_{n_0} \setminus \bigcup_{i\leq n} F_i)$. At every step $n \geq n_0$, when we add the set F_n, either we do not touch the interval I_{n_0}, or we reduce the remaining measure by a factor not worse than $(1 - 2^{-n})$ (in the sense that the remaining measure is at least $(1 - 2^{-n})$ times the measure at the previous step; here is where we use the assumption on the sequence of intervals). In this way, we get $\mathcal{L}^1(I_{n_0} \setminus \bigcup_{i\leq n} F_i) \geq \mathcal{L}^1(I_{n_0} \setminus \bigcup_{i\leq n_0} F_i) \times \prod_{n_0<i\leq n}(1 - 2^{-n})$. Passing to the logarithms, using $\mathcal{L}^1(I_{n_0} \setminus \bigcup_{i\leq n_0} F_i) > 0$ and $\sum_{n_0<i} \ln(1 - 2^{-n}) > -\infty$ as a consequence of the fact that the series of 2^{-n} converges, the limit is a positive number, and this proves $\mathcal{L}^1(I_{n_0} \setminus A) > 0$.

We will prove the following result.

Theorem 3.3.2. *Suppose that μ and ν are two mutually singular probability measures on $\mathbb{R}^d$ such that $\mu \ll \mathcal{L}^d$, and take the cost $c(x, y) = \ell(|x - y|)$, for $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ strictly concave, C^1 and increasing. Then there exists a unique optimal transport plan γ and it is induced by a transport map.*

Proof. We will use the standard procedure explained in Chapter 1 for costs of the form $c(x, y) = h(x - y)$. In particular we get the existence of a Kantorovich potential ϕ such that, if $(x_0, y_0) \in \operatorname{spt}\gamma$, then

$$x_0 - y_0 = (\nabla h)^{-1}(\nabla\phi(x_0)). \tag{3.6}$$

This allows to express y_0 as a function of x_0, thus proving that there is only one point $(x_0, y) \in \operatorname{spt}\gamma$ and hence that γ comes from a transport $T(x) = x - (\nabla h)^{-1}(\nabla\phi(x))$. This approach also proves uniqueness in the same way. In Chapter 1 we presented it under strict convexity of h, so that ∇h is injective and $(\nabla h)^{-1} = \nabla h^*$. But the injectivity is also true if $h(z) = \ell(|z|)$ since $\nabla h(z) = \ell'(|z|)\frac{z}{|z|}$ and the modulus of this vector identifies the modulus $|z|$ (since ℓ' is strictly increasing) and the direction gives also the direction of z.

The main difficulty is the fact that we need to guarantee that ϕ is differentiable a.e. with respect to μ. Since $\mu \ll \mathcal{L}^d$, it would be enough to have ϕ ipschitz continuous, which is usually proven using the fact that

$\phi(x) = \inf_y h(|x-y|) - \psi(y)$. Yet, concave functions on $\mathbb{R}^+$ may have an infinite slope at 0 and be non-Lipschitz, and this could be the case for ϕ as well. This suggests the use of an alternative notion of differentiability.

Important notion – *Approximate gradient*

We recall here some notions about a measure-theoretical notion replacing the gradient for less regular functions. The interested reader can find many details in [66].

Let us start from the following observation: given a function $f : \Omega \to \mathbb{R}$ and a point $x_0 \in \Omega$, we say that f is differentiable at $x_0 \in \Omega$ and that its gradient is $\nabla f(x_0) \in \mathbb{R}^d$ if for every $\epsilon > 0$ the set

$$\{x \in \Omega \,:\, |f(x) - f(x_0) - \nabla f(x_0) \cdot (x - x_0)| > \epsilon\, |x - x_0|\}$$

is at positive distance from x_0, i.e. if there exist a ball around x_0 which does not meet it. Instead of this requirement, we could ask for a weaker condition, namely that x_0 is a zero-density point for the same set (i.e. a Lebesgue point of its complement). More precisely, if there exists a vector v such that

$$\lim_{\delta\to 0} \frac{|\{x \in \Omega \,:\, |f(x) - f(x_0) - v \cdot (x - x_0)| > \epsilon\, |x - x_0|\}|}{|B(x_0, \delta)|} = 0$$

then we say that f is approximately differentiable at x_0 and its approximate gradient is v. The approximate gradient will be denoted by $\nabla_{app} f(x_0)$. As one can expect, it enjoys several of the properties of $v(x_0)$, that we list here.

- The approximate gradient, provided it exists, is unique.
- The approximate gradient is nothing but the usual gradient if f is differentiable.
- The approximate gradient shares the usual algebraic properties of gradients, in particular $\nabla_{app}(f+g)(x_0) = \nabla_{app} f(x_0) + \nabla_{app} g(x_0)$.
- If x_0 is a local minimum or local maximum for f, and if $\nabla_{app} f(x_0)$ exists, then $\nabla_{app} f(x_0) = 0$.

Another very important property is a consequence of Rademacher theorem.

Proposition Let $f, g : \Omega \to \mathbb{R}$ be two functions defined on a same domain Ω with g Lipschitz continuous. Let $A \subset \Omega$ be a Borel set such that $f = g$ on A. Then f is approximately differentiable almost everywhere on A and $\nabla_{app} f(x) = \nabla g(x)$ for a.e. $x \in A$.

Proof It is enough to consider all the points in A which are Lebesgue points of A and at the same time differentiability points of g. These points cover almost all A. It is easy to check that the definition of approximate gradient of f at a point x_0 is satisfied if we take $v = \nabla g(x_0)$

We just need to prove that ϕ admits an approximate gradient Lebesgue-a.e.: this would imply that Equation (3.6) is satisfied if we replace the gradient with the approximate gradient.

Recall that we may suppose

$$\phi(x) = \phi^{cc}(x) = \inf_{y\in\mathbb{R}^d} \ell(|x-y|) - \phi^c(y) \ .$$

Now consider a countable family of closed balls B_i generating the topology of $\mathbb{R}^d$, and for every i consider the function defined as

$$\phi_i(x) := \inf_{y\in B_i} \ell(|x-y|) - \phi^c(y) \ .$$

for $x \in \mathbb{R}^n$. One cannot provide straight Lipschitz properties for ϕ_i, since a priori y is arbitrarily close to x and in general ℓ is not Lipschitz close to 0. However ϕ_i is Lipschitz on every B_j such that $\operatorname{dist}(B_i, B_j) > 0$. Indeed if $x \in B_j$, $y \in B_i$ one has $|x-y| \geq d > 0$, therefore the Lipschitz constant of $\ell(|\cdot - y|) - \phi^c(y)$ does not exceed $\ell'(d)$ (or the right derivative of ℓ at d). It follows that ϕ_i is Lipschitz on B_j, and its constant does not exceed $\ell'(d)$.

Then ϕ has an approximate gradient almost everywhere on $\{\phi = \phi_i\}\cap B_j$. By countable union, ϕ admits an approximate gradient a.e. on

$$\bigcup_{\substack{i,j\\ d(B_i,B_j)>0}} [\{\phi_i = \phi\} \cap B_j] \ .$$

In order to prove that ϕ has an approximate gradient μ-almost everywhere, it is enough to prove that

$$\mu\left(\bigcup_{\substack{i,j\\ d(B_i,B_j)>0}} \{\phi_i = \phi\} \cap B_j\right) = 1.$$

In order to do this, notice that for every i and j we have

$$\pi_x(\operatorname{spt}\gamma \cap (B_j \times B_i)) \subset \{\phi = \phi_i\} \cap B_j \ .$$

Indeed, let $(x, y) \in \operatorname{spt}\gamma \cap (B_j \times B_i)$. Then $\phi(x) + \phi^c(y) = l(|x-y|)$. It follows that

$$\phi_i(x) = \inf_{y'\in B_i} \ell(\left|x-y'\right|) - \phi^c(y') \leq \ell(|x-y|) - \phi^c(y) = \phi(x) \ .$$

On the other hand, for every $x \in \mathbb{R}^n$

$$\phi_i(x) = \inf_{y\in B_i} \ell(|x-y|) - \phi^c(y) \geq \inf_{y\in\mathbb{R}^n} \ell(|x-y|) - \phi^c(y) = \phi(x) \ .$$

As a consequence of this,

$$
\begin{aligned}
\mu\left(\bigcup_{\substack{i,j\\ d(B_i,B_j)>0}}\{\phi_i=\phi\}\cap B_j\right) &\geq \mu\left(\bigcup_{\substack{i,j\\ d(B_i,B_j)>0}}\pi^x(\operatorname{spt}\gamma\cap(B_j\times B_i))\right)\\
&=\mu\left(\pi^x\left(\operatorname{spt}\gamma\cap\bigcup_{\substack{i,j\\ d(B_i,B_j)>0}}B_j\times B_i\right)\right)\\
&=\mu(\pi^x(\operatorname{spt}\gamma\setminus D))\\
&=\gamma\left[(\pi^x)^{-1}(\pi^x(\operatorname{spt}\gamma\setminus D))\right]\\
&\geq\gamma(\operatorname{spt}\gamma\setminus D)=1,
\end{aligned}
$$

since the diagonal is γ-negligible. The proof is completed. $\square$

From the previous theorem, we can also easily deduce the following extension. Define $\mu\wedge\nu$ as the maximal positive measure which is both less or equal than μ and than ν, and $(\mu-\nu)_+=\mu-\mu\wedge\nu$, so that the two measures μ and ν uniquely decompose into a common part $\mu\wedge\nu$ and two mutually singular parts $(\mu-\nu)_+$ and $(\nu-\mu)_+$.

Theorem 3.3.3. *Suppose that μ and ν are probability measures on $\mathbb{R}^d$ such that $(\mu-\nu)_+$ gives no mass to all $(d-1)-$rectifiable sets, and take the cost $c(x,y)=\ell(|x-y|)$, for $\ell:\mathbb{R}_+\to\mathbb{R}_+$ strictly concave, C^1 and increasing. Then there exists a unique optimal transport plan γ, and it has the form $(id,id)_\#(\mu\wedge\nu)+(id,T)_\#(\mu-\nu)_+$.*

As we said, the above results can be extended to more general situations: differentiability of ℓ is not really necessary, and the assumption $\mu\ll\mathcal{L}^d$ ca be weakened. The sharp assumption is that μ does not give mass small sets, i.e. suppose that, for every $A\subset\mathbb{R}^d$ which is $\mathcal{H}^{d-1}$-rectifiable, we have $\mu(A)=0$. These extensions are detailed in [91], and the key tool is the following lemma is an interesting result from Geometric Measure Theory that can be used instead of Lebesgue points-type results when we face a measure which is not absolutely continuous but "does not give mass to small sets". It states that, in such a case, $\mu-$a.e. point x is such that every cone exiting from x, even if very small, has positive μ mass. In particular it means that we can find points of $\operatorname{spt}\mu$ in almost arbitrary directions close to x. This lemma, difficult to find in the literature, is known in a part of the optimal transport community and is also part of the folklore of some branches of the GMT community. It can be considered as a part of Lemma 3.3.5 in [67], which proves the $(d-1)-$rectifiability of any set such that every point admits the existence of a small two-sided cone containing no other point of the set, up to the fact that in our case we only consider one-sided cones.

Lemma 3.3.4. *Let μ be a Borel measure on $\mathbb{R}^d$, and suppose that μ does not charge small sets. Then μ is concentrated on the set*

$$\{x : \forall\epsilon > 0, \forall\delta > 0, \forall u \in \mathbb{S}^{d-1},\ \mu(C(x, u, \delta, \epsilon)) > 0\}\ ,$$

where

$$C(x, u, \delta, \epsilon) = C(x, u, \delta) \cap \overline{B(x, \epsilon)} := \{y : \langle y-x, u\rangle \geq (1-\delta)\,|y - x|\} \cap \overline{B(x, \epsilon)}$$

Chapter 4

Divergence-constrained problems and transport density

4.1 Eulerian and Lagrangian points of view

4.1.1 Statical and Dynamical models

This section presents a very informal introduction to the physical interpretation of dynamical models in optimal transport.

In fluid mechanics - and in many other topics with similar modelizations - it is classical to consider two complementary ways of describing motions, which are called Lagrangian and Eulerian.

When we describe a motion via Lagrangian formalism we give "names" to particles (using either a specific label, or the initial position they had, for instance) and then describe, for every time t and every label, what happens to that particle. "What happens" means providing its position and/or its speed. Hence we could for instance give some functions $y_x(t)$ standing for the position at time t of particle originally located at x. Other possibility, instead of giving names we could consider bundles of particles with the same behavior and indicate how many are they. This amounts to giving a measure on possible behaviors.

The description may be more or less refined. For instance if one only considers two different times $t = 0$ and $t = 1$, the behavior of a particle is only given by its initial and final positions. A measure on those pairs (x, y) is exactly a transport plan. This explains how the Kantorovitch problem is expressed in Lagrangian coordinates. The Monge problem is also Lagrangian, where particles are labelled by their initial position.

More refined models can be easily conceived, since it quite evident that reducing a movement to the initial and final positions is embarrassingly poor. Measures on the set of paths (curves $\omega : [0,1] \to \Omega$, with possible assumptions on their regularity) have been used in many modelizations, and in particolar in traffic issues, branched transport (see the Discussion Section for both these subjects), or in Brenier's variational formulation of the incompressible Euler equations for fluids (see Section 1.7.4 and [38, 39, 25]).

On the other hand, in the Eulerian formalism we describe, for every time t and every point x, what happens at such a point at such a time. "What happens" usually means what are the velocity, the density and/or the flow rate (both in intensity and in direction) of particles located at time t at point x.

Eulerian models may be distinguished into statical and dynamical ones. In a dynamical model we usually use two variables, i.e. the density $\rho(t,x)$ and the velocity $v(t,x)$. It is possible to write the equation satisfied by the density of a family of particles moving according to the velocity field v. This means that we prescribe the initial density ρ_0, and that the position of the particle originally located at x will be given by the solution of the ODE

$$\begin{cases} y_x'(t) = v(t, y_x(t)) \\ y_x(0) = x \end{cases}, \tag{1.1}$$

we define the map T_t through $T_t(x) = y_x(t)$, and we look for the measure $\rho_t := (T_t)_\# \rho_0$. It is well known that ρ_t and v_t solve together the so-called continuity equation

$$\partial_t \rho_t + \nabla \cdot (\rho_t v_t) = 0$$

that is briefly addressed here below.

The statical framework is a bit harder to understand, since it is maybe not clear what "statical" means when we want to describe movement. One has to think to a permanent, cyclical movement, where some mass is constantly injected into the motion at some points and constantly withdrawn somewhere else. We can also think at a time average of some dynamical model: suppose that you observe the traffic in a city and you wonder what happens at each point, but you do not want an answer depending on time. You could for instance consider as a traffic intensity at every point the average traffic intensity at such a point on the whole day. In this case we usually use a unique variable v standing for the mass flow rate (which equals density times speed), and the divergence $\nabla \cdot v$ stands for the excess of mass which is injected into the motion at every point. More precisely, if particles are injected into the motion according to a density μ and then exit with density ν, the vector fields v standing for flows connecting these two measures must satisfy

$$\nabla \cdot v = \mu - \nu.$$

4.1.2 The continuity equation

This section is devoted to the equation

$$\partial_t \rho_t + \nabla \cdot (\rho_t v_t) = 0,$$

its meaning, formulations, and uniqueness results. Even if most of the Chapter will be devoted to the statical divergence equation, we will see later that the dynamical case can be useful to produce transport plans given a vector field, and we need to develop some tools.

First, let us spend some time on the notion of solution for this equation.

Definition 12. We say that a family of pairs measures/vector fields (ρ_t, v_t) with $v_t \in L^1(\rho_t; \mathbb{R}^d)$ and $\int_0^1 ||v_t||_{L^1(\rho_t)} dt = \int_0^1 \int_\Omega |v_t| d\rho_t \, dt < +\infty$ solves the continuity equation in the distributional sense if for any test function $\phi \in C^1_c(]0,1[\times \overline{\Omega})$, compactly supported in time but not necessarily in space, we have

$$\int_0^1 \int_\Omega \partial_t \phi \, d\rho_t \, dt + \int_0^1 \int_\Omega \nabla \phi \cdot v_t \, d\rho_t \, dt = 0. \tag{1.2}$$

Obviously this formulation includes Neumann boundary conditions on $\partial\Omega$ for v_t. If we want to impose the initial and final measures we can say that (ρ_t, v_t) solves the same equation, in the sense of distribution, with initial and final data ρ_0 and ρ_1, respectively, if for any test function $\phi \in C^1([0,1] \times \overline{\Omega})$ (no compact support assumptions), we have

$$\int_0^1 \int_\Omega \partial_t \phi \, d\rho_t \, dt + \int_0^1 \int_\Omega \nabla \phi \cdot v_t \, d\rho_t \, dt = \int_\Omega \phi(1,x) \, d\rho_1(x) - \int_\Omega \phi(0,x) \, d\rho_0(x). \tag{1.3}$$

On the other hand we can define a weak solution of the continuity equation through the following condition: we say that (ρ_t, v_t) solves the continuity equation in the weak sense if for any test function $\psi \in C^1([0,1] \times \Omega)$, the function $t \mapsto \int \psi \, d\rho_t$ is absolutely continuous and, for a.e. t, we have

$$\frac{d}{dt} \int_\Omega \psi \, d\rho_t = \int_\Omega \nabla \psi \cdot v_t \, d\rho_t.$$

Notice that in this case $t \mapsto \rho_t$ is automatically continuous for the weak convergence, and imposing the values of ρ_0 and ρ_1 may be done pointwisely.

Proposition 4.1.1. *The two notions of solutions are actually equivalent: every weak solution is actually a distributional solution and every distributional solution admits a representative (another family $\tilde{\mu}_t = \mu_t$ for a.e. t) which is weakly continuous and is a weak solution.*

Proof. To prove the equivalence, take a distributional solution, and test it against functions ϕ of the form $\phi(t,x) = a(t)\psi(x)$. We get

$$\int_0^1 a'(t) \int_\Omega \psi(x)\, d\rho_t\, dt + \int_0^1 a(t) \int_\Omega \nabla\psi \cdot v_t\, d\rho_t\, dt = 0.$$

The arbitrariness of a shows that the distributional derivative (in time) of $\int_\Omega \psi(x)\, d\rho_t$ is $\int_\Omega \nabla\psi \cdot v_t\, d\rho_t$. This last function is L^1 in time since $\int_0^1 \left| \int_\Omega \nabla\psi \cdot v_t\, d\rho_t \right| dt \leq \operatorname{Lip}\psi \int_0^1 ||v_t||_{L^1(\rho_t)} dt < +\infty$. This implies that (ρ, v) is a weak solution.

Conversely, the same computations shows that weak solution satisfy (1.2) for any ϕ of the form $\phi(t,x) = a(t)\psi(x)$. It is then enough to prove that finite linear combination of these functions are dense in $C^1([0,1]\times\mathbb{R}^n)$ (this is true, but is a non-trivial exercise!). $\square$

It is also evident that smooth functions satisfy the equation in the classical sense if and only if they are weak (or distributional) solutions.

The main way to produce solutions to the continuity equation is to use the flow of the vector field v_t. Let us check the validity of the equation when ρ_t is obtained from such a flow through (1.1). Let us suppose that $\operatorname{spt}(\rho_t) \subset \Omega$ (which is satisfied if ρ_0 is concentrated on Ω and v satisfies suitable Neumann boundary conditions). We will check that we have a weak solution. Fix a test function $\phi : \Omega \to \mathbb{R}$ and compute

$$\begin{aligned}\frac{d}{dt}\int \phi\, d\rho_t = \frac{d}{dt}\int \phi(y_x(t))\, d\rho_0(x) &= \int \nabla\phi(y_x(t)) \cdot y_x'(t)\, d\rho_0(x) \\ = \int \nabla\phi(y_x(t)) \cdot v(t, y_x(t))\, d\rho_0(x) &= \int \nabla\phi(y) \cdot v(t,y)\, d\rho_t(y),\end{aligned}$$

which proves that we have $\partial_t\rho_t = -\nabla\cdot(\rho_t v_t)$, in the weak sense.

Then, we would like to give at least a uniqueness result on ρ if v is Lipschitz continuous. This is true in a very general framework (see [5], Proposition 8.2.7) for a proof of the fact that the solution in the space of measures is unique for given v), but we prefer to give an easier proof which requires to consider smooth (Lipschitz in time and space, which is enough to consider the equation in an a.e. pointwise sense) solutions.

Theorem 4.1.2. *Suppose that Ω is a compact domain and $v : [0,T] \times \Omega \to \mathbb{R}^d$ is Lipschitz continuous in x uniformly in t, and consider two Lipschitz solutions $\rho^{(1)}$ and $\rho^{(2)}$ of $\partial_t \rho_t + \nabla \cdot (\rho_t v_t) = 0$, with $\rho^{(1)}(0,x) = \rho^{(2)}(0,x)$. Then $\rho^{(1)} = \rho^{(2)}$.*

Proof. The equation being linear, we only need to consider a solution $\rho = \rho^{(1)} - \rho^{(2)}$ with $\rho(0,x) = 0$ and prove that it vanishes for every time. Consider $E(t) = \frac{1}{2}\int_\Omega \rho(t,x)^2 dx$. We have

$$E'(t) = \int \rho_t(\partial_t \rho_t) = \int \nabla \rho_t \cdot v_t \rho_t = \int \nabla\big(\frac{1}{2}\rho_t^2\big)\cdot v_t = -\int \frac{1}{2}\rho_t^2 \nabla \cdot v_t \leq CE(t),$$

where we used $-\nabla \cdot v_t \leq C$ as a consequence of the Lipschitz continuity assumption. A simple application fo Gronwall's lemma allows to prove $E(t) = 0$ for every t, since $E(0) = 0$, and gives the thesis. □

We also give a variant of this theorem to adapt to the case of unbounded domains. The result is not sharp, as global bounds on v and on its Lipschitz constants are not really necessary, and we refer to [5] **THEOREM** for a more general proof.

Theorem 4.1.3. *Suppose thatv $: [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous and bounded in x ,uniformly in t, and consider two locally Lipschitz solutions $\rho^{(1)}$ and $\rho^{(2)}$ of $\partial_t \rho_t + \nabla \cdot (\rho_t v_t) = 0$, with $\rho^{(1)}(0,x) = \rho^{(2)}(0,x)$. Then $\rho^{(1)} = \rho^{(2)}$.*

Proof. Again, we only need to consider a solution ρ with $\rho(0,x) = 0$ and prove that it vanishes for every time. Consider $E(t,R) = \frac{1}{2}\int_{B(0,R)} \rho(t,x)^2 dx$. We have

$$\begin{aligned}\partial_t E(t) &= \int_{B(0,R)} \rho_t(\partial_t \rho_t) = -\int_{B(0,R)} \nabla \rho_t \cdot v_t \rho_t + \int_{\partial B(0,R)} \rho_t^2 v_t \cdot n \\ &= \int \frac{1}{2}\rho_t^2 \nabla \cdot v_t + \int_{\partial B(0,R)} \rho_t^2 v_t \cdot n \leq CE(t,R) + C\partial_R E(t,R),\end{aligned}$$

where we used uniform bounds on both $\nabla \cdot v_t$ and v_t. Then we fix (t_0, R_0) and we apply Gronwall Lemma to $t \mapsto f(t) = E(t, R_0 + Ct_0 - Ct)$, thus getting

$$E(t_0, R_0) = f(t_0) \leq f(0)e^{Ct_0} = E(0, R_0 + Ct_0)e^{Ct_0} = 0. \quad \square$$

We finish this section by proving that this result may be applied to the solution produced by the flow, which is actually smooth thanks to change-of variable formula allowing to reconstruct its density.

Proposition 4.1.4. *If ρ_0 is smooth and v is smooth, then ρ_t is smooth in t and x. If ρ_0, v and $\nabla \cdot v$ are Lipschitz continuous, then ρ_t is Lipschitz continuous in t and x.*

Proof. If v_t is Lipschitz, the flow map is injective (as a well-known consequence of the uniqueness of the solution of the ODE). Hence, the density of the image measure is obtained from the initial density through a simple change-of-variable involving the Jacobian factor. This means that the regularity of $\rho(t,x)$ only depends on the regularity of the Jacobian $a(t,x) = \det(A(t,x))$ and $A(t,x) = D_x y(t,x)$ where $y(t,x) = y_x(t)$ is defined through (1.1).

Notice that we have $A(0,x) = Id$, $a(0,x) = 1$ and

$$A'(t,x) = \partial_t D_x y(t,x)) = D_x\big(\partial_t y(t,x)\big) = D_x\big(v_t(y(t,x))\big) = Dv_t(y(t,x)) \cdot A(t,x),$$

which implies, thanks to usual matrix calculus

$$\begin{aligned} a'(t,x) = a(t,x)\mathrm{trace}(A(t,x)^{-1}A'(t,x)) \\ = a(t,x)\mathrm{trace}(A(t,x)^{-1}Dv_t(y(t,x))A(t,x)) \\ = a(t,x)\mathrm{trace}(Dv_t(y(t,x)))) = a(t,x)\nabla \cdot v_t(y(t,x)). \end{aligned}$$

This means that if $\nabla \cdot v_t$ is bounded from below, then $a(t,x)$ never vanishes, and if $\nabla \cdot v_t$ is smooth in x, so is $a(t,x)$. The considerations below allow to deduce the regularity of $\rho(t,x)$. $\square$

Memo – *Change-of-variable and image measures*

Proposition. Suppose that $\rho \in L^1(\Omega)$ is a positive density on $\Omega \subset \mathbb{R}^d$ and $T : \Omega \to \mathbb{R}^d$ is a Lipschitz injective map, which is thus differentiable a.e.. We suppose that $\det(DT) \neq 0$a.e. on $\{\rho > 0\}$. Then the image measure $T_\#\rho$ is absolutely continuous and its density u is given by

$$u(y) = \frac{\rho(T^{-1}(y))}{\det(DT(T^{-1}(y)))}.$$

If T is non-injective, the formula becomes $T_\#\rho = u \cdot \mathcal{L}^d$ with u given by

$$u(y) = \sum_{x:T(x)=y} \frac{\rho(x)}{\det(DT(x))}.$$

The same formulae stay true id T is only countably Lipschitz, with the differential DT which is actually the differential of the restriction of T to each set where it is Lipschitz continuous (and coincides thus with the approximate differential of T).

The conclusion of this section on the continuity equation is the following corollary, which can be proven by putting together the results of this section.

Corollary 4.1.5. *Suppose that ρ_0, v and $\nabla \cdot v$ are Lipschitz continuous functions of $x \in \mathbb{R}^d$ uniformly in $t \in [0,T]$, and that v is uniformly bounded. Then there exists a unique Lipschitz continuous solution to the continuity equation $\partial_t \rho_t + \nabla \cdot (\rho_t v_t)$ (with Neumann condition and initial datum ρ_0) and it is obtained through the flow of v_t.*

4.2 Beckmann's problem

4.2.1 Introduction, formal equivalences and variants

The problem that has been proposed by Beckmann as a model for optimal transport in the '50s, without knowing Kantorovitch's works and the possible links between the two theories, is the following.

Beckmann's minimal flow problem Consider the minimization

$$(PB) \quad \min \left\{ \int |v(x)|\, dx \;\middle|\; v : \Omega \to \mathbb{R}^n,\, \nabla \cdot v = \mu - \nu \right\}, \tag{2.4}$$

where the divergence condition is to be read in the weak sense, with Neumann boundary conditions, i.e. $-\int \nabla\phi d\lambda = \int \phi\, d(\mu - \nu)$ for any $\phi \in C^1(\overline{\Omega})$.

This proposition links the Monge-Kantorovich problem to the minimal flow problem first proposed by Beckmann in [11], under the name of *continuous transportation model*. He did not know this link, as Kantorovich's theory was being developed independently almost in the same years.

We will see now that an equivalence between (PB) and (PK) holds true. To do that, we can look at the following considerations and formal computations.

We take the problem (PB) and re-write the constraint on v by means of the quantity

$$\sup_\phi \int -\nabla\phi \cdot v\, dx + \int \phi\, d(\mu - \nu) = \begin{cases} 0 & \text{if } \nabla \cdot v = \mu - \nu \\ +\infty & \text{otherwise} \end{cases}.$$

Hence one can write (PB) as

$$\min_v \int |v(x)|\, dx + \sup_\phi \int -\nabla\phi \cdot\, v\, dx + \int \phi\, d(\mu - \nu)$$
$$= \sup_\phi \int \phi\, d(\mu - \nu) + \inf_v \int |v(x)|\, dx - \int \nabla\phi \cdot\, v\, dx, \tag{2.5}$$

where inf and sup have been exchanged formally as in the previous computations. After that one notices that

$$\inf_v \int |v(x)|\,dx - \int \nabla\phi\cdot\, v\,dx = \begin{cases} 0 & \text{if } |\nabla\phi|\le 1 \\ -\infty & \text{otherwise} \end{cases}$$

and this leads to the dual formulation for (PB) which gives

$$\sup_{\phi\,:\,|\nabla\phi|\le 1} \int_\Omega \phi\; d(\mu-\nu).$$

Since this problem is exactly the same as (PD) (a consequence of the fact that Lip_1 functions are exactly those functions whose gradient is smaller than 1), this a formal equivalence between (PB) and (PK). The reason for saying that it is only formal lies in the fact that we did not prove the equality in (2.5). Notice that we need to suppose that Ω is convex, otherwise functions with gradient smaller than 1 are only Lip_1 according to the geodesic distance in Ω.

Most of the considerations above, especially those on the problem (PB) do not hold for costs other than the distance $|x-y|$. The only possible generalizations which are known concern a cost c which comes from a Riemannian distance $k(x)$.

The simplest possible generalization of Problem (PB) is the following:

$$\min \int k(x)|v(x)|dx \;:\; \nabla\cdot v = \mu-\nu$$

that corresponds, by duality with the functions u such that $|\nabla u|\le k$, to

$$\min \int d_k(x,y)d\gamma \;:\; \gamma\in\Pi(\mu,\nu),$$

where $d_k(x,y)=\inf_{\omega(0)=x,\,\omega(1)=y} L_k(\omega) := \int_0^1 k(\omega(t))|\omega'(t)|dt$ is the distance associated to the Riemannian metric k.

This generalization above comes from the modelization of a non-uniform cost for the movement (due to geographical obstacles or configurations). It can be applied to several situation but in some other t is not satisfying, for instance in urban transport, where we want to consider the fact that the metric k is usually not a priori known, but it depends on the traffic distribution itself. We will develop this aspect in the discussion section at the end of this chapter, together with a completely different problem which is somehow "opposite": instead of looking at transport problems where concentration of the mass is penalized because it stands for traffic congestion, looking at problemns where it is encouraged because of the so-called "economy of scale" (i.e. the biggest the mass you transport, the cheapest the individual cost).

4.2.2 Producing a minimizer for PB

The first remark on Problem (PB) is that it is probably not well-posed, in the sense that there could not exist an L^1 vector field minimizing the L^1 norm under divergence constraints. This is easy to understand if we think at using the direct method in Calculus of Variations to prove existence : we take a minimizing sequence v_n and we would like to extract a converging subsequence. If we could, and we had $v_n \rightharpoonup v$, then it would be easy to prove that v still satisfies $\nabla \cdot v = \mu - \nu$, since the relation

$$-\int \nabla\phi \cdot v_n \, dx = \int \phi \, d(\mu - \nu)$$

would pass to the limit as $n \to \infty$. Yet, the information that $\int |v(x)| dx \leq C$ is not enough to extract a converging sequence, even weakly. Indeed, the space L^1 being non-reflexive, bounded sequences are not guaranteed to have weakly converging subsequences. This is on the contrary the case for dual spaces (and for reflexive spaces, which are roughly speaking the dual of their dual).

Notice that the strictly convex version that is proposed for traffic purposes in the discussion section is much better to handle: if for instance we minimize $\int |v|^2 \, dx$ then we can use compactness in L^2, which is a Hilbert space, and hence reflexive.

To avoid this difficulty, one needs to set (PB) in the framework of vector measures.

Memo – *Vector measures*

Definition: A finite vector measure λ on a set Ω is a map associating to every Borel subset $A \subset \Omega$ a value $\lambda(A) \in \mathbb{R}^d$ such that, for every disjoint union $A = \bigcup_i A_i$ with $A_i \cap A_j = \emptyset$ for $i \neq j$, we have

$$\sum_i |\lambda(A_i)| < +\infty \quad \text{and} \quad \lambda(A) = \sum_i \lambda(A_i).$$

Here the norm that we use on $\mathbb{R}^d$ to evaluate the series above is arbitrary, the finiteness of the result does not depend on this choice, since all norms on $\mathbb{R}^d$ are equivalent.

We denote by $\mathcal{M}^d(\Omega)$ the set of finite vector measures on Ω. To such measures we can associate a positive scalar measure $\lambda \in \mathcal{M}_+(\Omega)$ through

$$||\lambda||(A) := \sup \left\{ \sum_i ||\lambda(A_i)|| \ : \ A = \bigcup_i A_i \ \text{ with } A_i \cap A_j = \emptyset \text{ for } i \neq j \right\}.$$

This measure depends on the choice of the norm $||\cdot||$ on $\mathbb{R}^d$. Let us suppose for simplicity that it is the Euclidean norm (in such a case, we will often write $|\lambda|$.

The integral of a Borel function $f:\Omega\to\mathbb{R}^d$ w.r.t. λ is well-defined if $|f|\in L^1(\Omega,||\lambda||)$ (again, this does not depend on the choice of the norm), is denoted $\int f\cdot d\lambda$ and can be computed as $\sum_{i=1}^d\int f_i\,d\lambda_i$, thus reducing to integrals of scalar functions according to scalar measures. It could also be defined as a limit of integral of piecewise constant functions.

Functional analysis facts The quantity $||\lambda||(\Omega)$ is a norm on $\mathcal{M}^d(\Omega)$, and this normed space is the dual of $C_0(\Omega;\mathbb{R}^d)$, the space of continuous function on Ω vanishing at infinity,through the duality $(f,\lambda)\mapsto\int f\cdot d\lambda$. This gives a notion of $\overset{*}{\rightharpoonup}$ convergence for which bounded sets in $\mathcal{M}^d(\Omega)$ are compact.

A last clarifying fact is the following.

Proposition : For every $\lambda\in\mathcal{M}^d(\Omega)$ and every norm $||\cdot||$ there exists a Borel function $\xi:\Omega\to\mathbb{R}^d$ such that $\lambda=\xi\cdot||\lambda||$ and $||\xi||=1$ a.e. (for the measure $||\lambda||$).

Sketch of proof: The existence of a function ξ is a consequence, via Radon-Nikodym Theorem, of $\lambda\ll||\lambda||$ (every A set such that $||\lambda(A)||=0$ obviously satisfies $\lambda(A)=0$), possibly applied componentwise. The condition $||\xi||=1$ may be proven by considering the sets $\{||\xi||<1-\varepsilon\}$ and $\{\xi\cdot e>a+\varepsilon\}$ for all hyperplane such that the unit ball $B_1:=\{x\in\mathbb{R}^d\,:\,||x||\le 1\}$ is contained in $\{x\in\mathbb{R}^d\,:\,x\cdot e\le a\}$ (and, actually, we have $B_1=\bigcap_{e,a}\{x\in\mathbb{R}^d\,:\,x\cdot e\le a\}$, the intersection being possibly reduced to a countable intersection). These sets must be negligible otherwise we have a contradiction on the definition of $||\lambda||$.

Theorem 4.2.1. *Suppose that Ω is a compact convex domain in $\mathbb{R}^d$. Then, the problem*

$$(PB)\quad\min\{|v|(\Omega)\,:\,v\in\mathcal{M}^n(\Omega)\,,\,\nabla\cdot v=\mu-\nu\}$$

(with divergence imposed in the weak sense, i.e. for every $\phi\in C^1(\overline{\Omega})$ we impose $-\int\nabla\phi\cdot dv=\int\phi\,d(\mu-\nu)$, which also includes Neumann boundary conditions) admits a solution. Moreover, its minimal value equals the minimal value of (PK) and a solution of (PB) can be built from a solution of (PK). The two problems are hence equivalent.

Proof. The first point that we want to prove is the equality of the minimal values $(PB)=(PK)$ and we start from $(PB)\ge(PK)$. In order to do so, take an arbitrary function $\phi\in\mathrm{Lip}_1\cap C^1$ and consider that for any v with $\nabla\cdot v=\mu-\nu$, we have

$$|v|(\Omega)=\int 1\,d|v|\ge\int(-\nabla\phi)\cdot dv=\int\phi\,d(\mu-\nu)$$

(where we used the fact that $\phi \in \mathrm{Lip}_1 \Rightarrow |\nabla\phi| \leq 1$). If one takes a sequence of $\mathrm{Lip}_1 \cap C^1$ functions converging to the Kantorovitch potential u such that $\int u\, d(\mu - \nu) = \max(PD) = \min(PK)$ (for instance take convolutions $\phi_k = \rho_k * u$) then he gets

$$\int d|v| \geq (PK)$$

for any admissible v, i.e. $(PB) \geq (PK)$.

We will show at the same time the reverse inequality and how to construct an optimal v from an optimal γ for (PK).

Actually, one way to produce a solution to this divergence-constrained problem, is the following: take an optimal transport plan γ and build a vector measure v_γ defined through

$$< v_\gamma, \phi >:= \int_{\Omega\times\Omega} \int_0^1 \omega'_{x,y}(t) \cdot \phi(\omega_{x,y}(t)) dt\, d\gamma,$$

for every $\phi \in C^0(\Omega; \mathbb{R}^d)$, $\omega_{x,y}$ being a parametrization of the segment $[x, y]$. Even if for this proof it would not be important, we will fix the constant speed parametrization, i.e. $\omega_{x,y}(t) = (1-t)x + ty$. It is clear that this is the point where convexity of Ω is needed.

It is not difficult to check that this measure satisfies the divergence constraint, since if one takes $\phi = \nabla\psi$ then

$$\int_0^1 \omega'_{x,y}(t) \cdot \phi(\omega_{x,y}(t)) = \int_0^1 \frac{d}{dt} (\psi(\omega_{x,y}(t))\, dt = \psi(y) - \psi(x)$$

and hence $< v_\gamma, \nabla\psi >= \int \psi\, d(\nu - \mu)$.

To estimate its mass we can see that $|v_\gamma| \leq \sigma_\gamma$, where the scalar measure σ_γ is defined through

$$< \sigma_\gamma, \phi >:= \int_{\Omega\times\Omega} \int_0^1 |\omega'_{x,y}(t)|\phi(\omega_{x,y}(t)) dt\, d\gamma, \quad \forall \phi \in C^0(\Omega; \mathbb{R})$$

and it is called *transport density*. Actually, we can even say more, since we can use

$$\omega'_{x,y}(t) = -|x - y| \frac{x - y}{|x - y|} = -|x - y| \nabla u(\omega_{x,y}(t)),$$

which is valid for every $t \in]0, 1[$ and every $x, y \in \mathrm{spt}(\gamma)$ (so that $\omega_{x,y}(t)$ is in the interior of the transport ray $[x, y]$, if $x \neq y$; anyway for $x = y$, both expression vanish).

This allow to write, for every $\phi \in C^0(\Omega;\mathbb{R}^d)$

$$< v_\gamma, \phi >= \int_{\Omega\times\Omega}\int_0^1 -|x-y|\nabla u(\omega_{x,y}(t))\cdot\phi(\omega_{x,y}(t))dt\,d\gamma$$
$$= -\int_0^1 dt\int \nabla u(\omega_{x,y}(t))\cdot\phi(\omega_{x,y}(t))|x-y|d\gamma$$

If we introduce the function $\pi_t : \Omega\times\Omega\to\Omega$ given by $\pi_t(x,y) = \omega_{x,y}(t)$, we get

$$< v_\gamma, \phi >= -\int_0^1 dt\int \nabla u(z)\cdot\phi(z)d\big((\pi_t)_\#(c\cdot\gamma)\big),$$

where $c\cdot\gamma$ is the measure on $\Omega\times\Omega$ with density $c(x,y) = |x-y|$ w.r.t. γ.

Since on the other hand the same kind of computations give

$$< \sigma_\gamma, \psi >= \int_0^1 dt\int \psi(z)d\big((\pi_t)_\#(c\cdot\gamma)\big),$$

we get $< v_\gamma, \phi >=< \sigma_\gamma, -\phi\cdot\nabla u >$, which shows

$$v_\gamma = -\nabla u\cdot\sigma_\gamma.$$

This gives the density of v_γ with respect to σ_γ and proves $|v_\gamma|\le\sigma_\gamma$.

The mass of σ_γ is obviously

$$\int d\sigma_\gamma = \int\int_0^1 |\omega'_{x,y}(t)|dt\,d\gamma = \int |x-y|d\gamma = \min(PK),$$

which proves the optimality of v_γ since no other v may do better than this, and also proves $\min(PB) = \min(PK)$. □

It is interesting to investigate whether $\sigma_\gamma \ll \mathcal{L}^d$, since this would imply that Problem (B) is well-posed in L^1 instead of the space of vector measure. For the sake of the variants that we will see later on, it would be interesting to give conditions so that $\sigma_\gamma\in L^p$ as well. All these subjects have been widely studied by De Pascale, Pratelli (see [60, 61, 62]) but there is a more recent (and shorter) proof of the same estimates in [98]. It is in particular true that $\mu,\nu\in L^p$ implies that $\sigma_\gamma\in L^p$ and that it is sufficient that one of the two measures is absolutely continuous in order to get the same on σ_γ.

Notice that it would be possible to prove, at least under some absolute continuity assumptions on μ or ν, (see Theorem 7.3 in [2]) that

- any minimizer of (PB) is given by v_γ for a suitable optimal transport plan ;

- all the optimal transport plans γ provide the same v_γ.

This induces in particular a uniqueness results for (PB) which is not obvious, since it is a convex but not strictly convex problem.

4.2.3 Traffic intensity and traffic flows for measures on curves

We introduce in this section some objects that generalize both v_γ and σ_γ and hat will be useful both for proving the characterization of the optimal v as coming from an optimal plan γ and for the modelization issues of the Discussion Section.

Let us introduce some notations.

Given an absolutely curve ω ; $[0,1] \mapsto \Omega$ and a continuous function φ, let us set

$$L_\varphi(\sigma) := \int_0^1 \varphi(\omega(t))|\dot{\omega}(t)|dt. \tag{2.6}$$

This quantity is the length of the curve weighted with the weight φ. When we take $\varphi = 1$ we get the usual length of ω and we denote it by $L(\omega)$ instead of $L_1(\omega)$.

We consider probability measures Q on $C := \mathrm{Lip}([0,1],\Omega)$. The convergence that we use on C is the uniform convergence with bounds on the Lipschitz constants, i.e. we say that $\omega_n \to \omega$ if $\mathrm{Lip}(\omega_n)$ is bounded and $\omega_n \to \omega$ uniformly. This is the same as the weak convergence in the space of Lipschitz curves. Notice that Ascoli-Arzelà's theorem guarantees that the sets $\{\omega \in C \,:\, \mathrm{Lip}(\omega) \le c\}$ are compact for this convergence for every c. We will associate two measures on Ω to such a Q. The first is a scalar one, called traffic intensity and denoted by $i_Q \in \mathcal{M}(\Omega)$; it is defined by

$$\int \varphi\, di_Q := \int_C \Big(\int_0^1 \varphi(\omega(t))|\dot{\omega}(t)|dt\Big)dQ(\omega) = \int_C L_\varphi(\omega)dQ(\omega).$$

for all $\varphi \in C(\Omega,\mathbb{R}_+)$. This definition is a generalization of the notion of transport density and the interpretation is the following: for a subregion A, $i_Q(A)$ represents the total cumulated traffic in A induced by Q, it is indeed the average over all paths of the length of this path intersected with A.

We also associate a vector measure to this probability Q, in the same spirit as what we did in order to define v_γ. Let us consider the vector-field θ_Q defined through

$$\forall X \in C(\Omega,\mathbb{R}^d) \quad \int_\Omega X \cdot d\theta_Q := \int_C \left(\int_0^1 X(\omega(t)) \cdot \dot{\omega}(t)dt\right) dQ(\omega).$$

Since this is a kind of vectorial traffic intensity, we will call it *traffic flow*. Taking a gradient field $X = \nabla\psi$ in the previous definition yields

$$\int_\Omega \nabla\psi \cdot d\theta_Q = \int_{C([0,1],\Omega)} [\psi(\theta(1)) - \psi(\theta(0))] dQ(\gamma) = \int_\Omega \psi(\mu_1 - \mu_0)$$

where $\mu_i := (e_i)_\# Q$ for $i = 0, 1$. This means that

$$\nabla \cdot \theta_Q = \mu_0 - \mu_1.$$

Moreover it is easy to check that

$$|\theta_Q| \leq i_Q.$$

This last inequality is not in general an equality, since the curves of Q could produce some cancellations (imagine a non-negligible amount of curves passing through the same point with opposite directions, so that $\theta_Q = 0$ and $i_Q > 0$).

It is straightforward that the constructions of v_γ and σ_γ given in the previous section are just a particular case of this one, more precisely they are obtained in the case where Q is the image through the map associating to every pair (x, y) the segment $\omega_{x,y}$ of the measure $\gamma \in \mathcal{P}(\Omega \times \Omega)$, optimal transport plan for the Euclidean cost.

We need some properties of the traffic intensity and traffic flow.

Proposition 4.2.2. *Both θ_Q and i_Q are invariant under reparametrization (i.e., if $T : C \to C$ is a map such that for every ω the curve $T(\omega)$ is just a reparametrization in time of ω, then $\theta_{T_\# Q} = \theta_Q$ and $i_{T_\# Q} = i_Q$).*

For every Q, the total mass $i_Q(\Omega)$ equals the average length of the curves according to Q, i.e. $\int_C L(\omega)\, dQ(\omega) = i_Q(\Omega)$.

If $Q_n \rightharpoonup Q$ and $i_{Q_n} \rightharpoonup i$, then $i \geq i_Q$.

If $Q_n \rightharpoonup Q$ and $i_{Q_n} \rightharpoonup i_Q$ (i.e. if there is equality above), then $\theta_{Q_n} \rightharpoonup \theta_Q$.

Proof. The invariance by reparametrization comes from the fact that both $L_\varphi(\omega)$ and $\int_0^1 X(\omega(t)) \cdot \omega'(t) dt$ do not change under reparametrization.

The formula $\int_C L(\omega)\, dQ(\omega) = i_Q(\Omega)$ is obtained from the definition of i_Q by testing with the function 1.

To check the inequality $i \geq i_Q$, fix a positive test function $\phi \in C(\Omega)$ and write

$$\int \phi\, di_{Q_n} = \int_C \left(\int_0^1 \phi(\omega(t)) |\dot\omega(t)| dt \right) dQ_n(\omega). \tag{2.7}$$

Notice that the function $C \ni \omega \mapsto \int_0^1 \phi(\omega(t))|\dot\omega(t)|dt$ is positive and lower-semi-continuous w.r.t. ω. Indeed, if $\omega_n \to \omega$, then $\omega_n' \rightharpoonup \omega$ weakly-* in L^∞, which implies, up to subsequences, the existence of an L^∞ function $\xi \geq |\omega'|$ such that $|\omega_n'| \rightharpoonup \xi$; moreover, $\phi(\omega_n(t)) \to \phi(\omega(t))$ uniformly, which gives $\int \phi(\omega_n(t))|\omega_n'(t)|dt \to \int \phi(\omega(t))\xi(t)dt \geq \int \phi(\omega(t))|\omega'(t)|dt$.

This allows to pass to the limit in (2.7), thus obtaining

$$\int \phi\, di = \lim_n \int \phi\, di_{Q_n} = \liminf_n \int_C \left(\int_0^1 \phi(\omega(t))|\dot\omega(t)|dt\right) dQ_n(\omega)$$
$$\geq \int_C \left(\int_0^1 \phi(\omega(t))|\dot\omega(t)|dt\right) dQ(\omega) = \int \phi\, di_Q,$$

which proves the claim.

To check the last property, fix a bounded vector test function X and look at

$$\int X \cdot d\theta_{Q_n} = \int_C \left(\int_0^1 X(\omega(t))\cdot\dot\omega(t)dt\right) dQ_n(\omega)$$
$$= \int_C \left(\int_0^1 X(\omega(t))\cdot\dot\omega(t)dt + ||X||_\infty L(\omega)\right) dQ_n(\omega) - ||X||_\infty i_{Q_n}(\Omega), \quad (2.8)$$

where we just added and subtracted the total mass of i_{Q_n}, which is equal to the average length of ω according to Q_n.

Now notice that $C \ni \omega \mapsto \int_0^1 X(\omega(t))\cdot\dot\omega(t)dt + ||X||_\infty L(\omega)$ is a positive quantity and it is l.s.c. in ω (it is a consequence of what we proved above, by taking $\phi = 1$). This means that if we pass to the limit in (2.8) we get

$$\liminf_n \int X \cdot d\theta_{Q_n}$$
$$\geq \int_C \left(\int_0^1 X(\omega(t))\cdot\dot\omega(t)dt + ||X||_\infty L(\omega)\right) dQ(\omega) - ||X||_\infty i_Q(\Omega)$$
$$= \int_C \left(\int_0^1 X(\omega(t))\cdot\dot\omega(t)dt\right) dQ(\omega) = \int X\cdot d\theta_Q.$$

By replacing X with $-X$ we also get the opposite inequality and we have proven $\theta_{Q_n} \rightharpoonup \theta_Q$. □

Good to know! – *Dacorogna-Moser transport*

A particular case of the construction in [56] (first used in optimal transport by [65]):

Construction : Suppose that $w : \Omega \to \mathbb{R}^d$ is a Lipschitz vector field parallel to the boundary (i.e.$w \cdot n_\Omega = 0$ on $\partial\Omega$) with $\nabla \cdot w = f_0 - f_1$, where f_0, f_1 are positive probability densities which are Lipschitz continuous and bounded from below. Then we can define the non-autonomous vector field $\tilde{w}(t,x)$ via

$$\tilde{w}(t,x) = \frac{w(x)}{f_t(x)} \quad \text{where } f_t = (1-t)f_0 + tf_1$$

and consider the Cauchy problem

$$\begin{cases} y_x'(t) = \tilde{w}(t, y_x(t)) \\ y_x(0) = x \end{cases},$$

We define the map $Y : \Omega \to C$ through $Y(x) = y_x(\cdot)$, and we look for the measure $Q = Y_\# f_0$ and $\rho_t := (e_t)_\# Q$. Thanks to the consideration in Section 4.1.2, ρ_t solves the continuity equation $\partial_t \rho_t + \nabla \cdot (\rho_t \tilde{w}_t) = 0$. Yet, it is easy to check that f_t also solves the same equation since $\partial_t f_t = f_1 - f_0$ and $\nabla \cdot (\tilde{w} f_t) = \nabla \cdot w = f_0 - f_1$. By the uniqueness result of Section 4.1.2, from $\rho_0 = f_0$ we infer $\rho_t = f_t$.

In particular, $x \mapsto y_x(1)$ is a transport map from f_0 to f_1.

It it interesting to check what are the traffic intensity and the traffic flow associated to the measure Q in Dacorogna-Moser construction. Fix a scalar test function φ:

$$\begin{aligned}\int_\Omega \varphi \, di_Q = \int_\Omega \int_0^1 \varphi(y_x(t)) |\tilde{w}(t, y_x(t))| dt f_0(x) dx \\ = \int_0^1 \int_\Omega \varphi(y) |\tilde{w}(t,y)| f_t(y) dy dt = \int_\Omega \varphi(y) |w(y)| dy\end{aligned}$$

so that $i_Q = |v|$. Analogously, fix a vector test function X

$$\begin{aligned}\int_\Omega X \cdot d\theta_Q = \int_\Omega \int_0^1 X(y_x(t)) \cdot \tilde{w}(t, y_x(t)) dt f_0(x) dx \\ = \int_0^1 \int_\Omega X(y) \cdot \tilde{w}(t,y) f_t(y) dy dt = \int_\Omega X(y) \cdot w(y) dy,\end{aligned}$$

which shows $\theta_Q = w$ (indeed, in this case we have $|\theta_Q| = i_Q$ and this is due to the fact that no cancellation is possible, since all the curves share the same direction at every given point).

With these tools it is possible to prove that every admissible vector field v in Beckmann problem is of the form $v = \theta_Q$.

Lemma 4.2.3. *Consider two probabilities $\mu, \nu \in \mathcal{P}(\Omega)$ and a vector measure v satisfying $\nabla \cdot v = \mu - \nu$ in distributional sense (with Neumann boundary conditions). Then, for every domain Ω' containing Ω is its interios, there exist a family of vector fields $w^\varepsilon \in C^\infty(\Omega')$ with $w^\varepsilon \cdot n_{\Omega'} = 0$, and two families of densities $\mu^\varepsilon, \nu^\varepsilon \in C^\infty(\Omega')$, with $\nabla \cdot w^\varepsilon = \mu^\varepsilon - \nu^\varepsilon$ and $\int_{\Omega'} \mu^\varepsilon = \int_{\Omega'} \nu^\varepsilon = 1$, weakly converging to w, μ and ν as measures, respectively and satisfying $|w^\varepsilon| \rightharpoonup |v|$.*

Proof. First, take convolutions (in the whole space $\mathbb{R}^d$) with a gaussian kernel η_ε, so that we get $\hat{v}^\varepsilon := v * \eta_\varepsilon$ and $\hat{\mu}^\varepsilon := \mu * \eta_\varepsilon$ $\hat{\nu}^\varepsilon := \nu * \eta_\varepsilon$, still satisfying $\nabla \cdot \hat{v}^\varepsilon = \mu^\varepsilon - \nu^\varepsilon$. Since the Gaussian Kernel is strictly positive, we also have strictly positive densities for $\hat{\mu}^\varepsilon$ and $\hat{\nu}^\varepsilon$. These convolved densities and vector field would do the job required by the theorem, but we have to take care of the support (which is not Ω') and of the boundary behavior.

Let us set $\int_{\Omega'} \hat{\mu}^\varepsilon = 1 - a_\varepsilon$ and $\int_{\Omega'} \hat{\nu}^\varepsilon = 1 - b_\varepsilon$. It is clear that $a_\varepsilon, b_\varepsilon \to 0$ as $\varepsilon \to 0$. Consider also $\hat{v}^\varepsilon \cdot n_{\Omega'}$: due to $d(\Omega, \partial\Omega') > 0$ and to the fact that η_ε goes uniformly to 0 locally outside the origin, we also have $|\hat{v}^\varepsilon \cdot n_{\Omega'}| \leq c_\varepsilon$, with $c_\varepsilon \to 0$.

Consider u^ε the solution to

$$\begin{cases} \Delta u^\varepsilon = \frac{a_\varepsilon - b_\varepsilon}{|\Omega'|} & \text{inside } \Omega' \\ \frac{\partial u^\varepsilon}{\partial n} = -\hat{v}^\varepsilon \cdot n_{\Omega'} & \text{on } \partial\Omega', \\ \int_{\Omega'} u = 0 \end{cases}$$

and the vector field $\delta^\varepsilon = \nabla u^\varepsilon$. Notice that a solution exists thanks to $\int_{\partial\Omega'} \hat{v}^\varepsilon \cdot n_{\Omega'} = a_\varepsilon - b_\varepsilon$. Notice also that an integration by parts shows

$$\int_{\Omega'} |\nabla u^\varepsilon|^2 = -\int_{\partial\Omega'} u^\varepsilon (\hat{v}^\varepsilon \cdot n_{\Omega'}) - \int_{\Omega'} u^\varepsilon \left(\frac{a_\varepsilon - b_\varepsilon}{|\Omega'|} \right) \leq C ||\nabla u^\varepsilon||_{L^2} (c_\varepsilon + a_\varepsilon + b_\varepsilon),$$

and provides an estimate on $\int_{\Omega'} |\nabla u^\varepsilon|^2 \to 0$ (since we have a dependence of order two at the left hand side and of order one at the right hand side). This shows $||\delta^\varepsilon||_{L^2} \to 0$.

Now take

$$\mu^\varepsilon = \hat{\mu}^\varepsilon + \frac{a_\varepsilon}{|\Omega'|}; \quad \nu^\varepsilon = \hat{\nu}^\varepsilon + \frac{b_\varepsilon}{|\Omega'|}; \quad w^\varepsilon = \hat{v}^\varepsilon + \delta^\varepsilon,$$

and check that all the requirements are satisfied. In particular, the last one is satisfied since $||\delta^\varepsilon||_{L^1} \to 0$ and $|\hat{v}^\varepsilon| \rightharpoonup |v|$ by general properties of the convolutions. $\square$

Remark 10. Notice that considering explicitly the dependence on Ω' it is also possible to obtain the same statement with a sequence of domains Ω'_ε converging to Ω (for instance in the Hausdorff topology). It is just necessary to choose them so that, setting $t_\varepsilon := d(\Omega, \partial\Omega'_\varepsilon)$, we have $||\eta_\varepsilon||_{L^\infty(B(0,t_\varepsilon)^c)} \to 0$.

With these tools we can now prove

Proposition 4.2.4. *For every finite vector measure $v \in \mathcal{M}^d(\Omega)$ and $\mu, \nu \in \mathcal{P}(\Omega)$ with $\nabla \cdot v = \mu - \nu$ there exist a measure $Q \in \mathcal{P}(C)$ with $(e_0)_\# Q = \mu$ and $(e_1)_\# Q = \nu$ such that $|\theta_Q| \le i_Q \le |v|$, with $|\theta_Q| \ne |v|$ unless $\theta_Q = v$.*

Proof. By means of Lemma 4.2.3 and Remark 10 we can produce an approximating sequence $(w^\varepsilon, \mu^\varepsilon, \nu^\varepsilon) \rightharpoonup (w, \mu, \nu)$ of C^∞ functions supported on domains Ω_ε converging to Ω. We apply Dacorogna-Moser's construction to this sequence of vector fields, thus obtaining a sequence of measures Q_ε. We can consider these measures as probability measures on $\mathrm{Lip}([0,1];\Omega')$, where $\Omega \subset \Omega_\varepsilon \subset \Omega'$ which are, each, concentrated on curves valued in Ω_ε. They satisfy $i_{Q_\varepsilon} = |w^\varepsilon|$ and $\theta_{Q_\varepsilon} = w^\varepsilon$. We can reparametrize (without changing their names) by constant speed the curves on which Q_ε is supported, without changing traffic intensities and traffic flows. This means using curves ω such that $L(\omega) = \mathrm{Lip}(\omega)$. The equalities

$$\int_C \mathrm{Lip}(\omega)\, dQ_\varepsilon(\omega) = \int_C L(\omega)\, dQ_\varepsilon(\omega) = \int_{\Omega'} i_{Q_\varepsilon} = \int_{\omega'} |w^\varepsilon| \to |v|(\Omega') = |v|(\Omega)$$

show that $\int_C \mathrm{Lip}(\omega)\, dQ_\varepsilon(\omega)$ is bounded and hence Q_ε is tight. Hence, up to subsequences, we can assume $Q_\varepsilon \to Q$. The measure Q is obviously concentrated on curves valued in Ω. The measures Q_ε were constructed so that $(e_0)_\# Q_\varepsilon = \mu^\varepsilon$ and $(e_1)_\# Q_\varepsilon = \nu^\varepsilon$, which implies, at the limit, $(e_0)_\# Q = \mu$ and $(e_1)_\# Q = \nu$. Moreover, thanks to Proposition 4.2.2, since $i_{Q_\varepsilon} = |w_\varepsilon| \rightharpoonup |v|$, we get $|v| \ge i_Q \ge |\theta_Q|$. The same Proposition 4.2.2 also states that, if $|v| = i_Q$, then $\theta_{Q_\varepsilon} \rightharpoonup \theta_Q$. Yet, we also know $\theta_{Q_\varepsilon} = w_\varepsilon \rightharpoonup v$ and we deduce $v = \theta_Q$. □

CYCLES

4.2.4 Beckman problem in one dimension

The one-dimensional case is very easy in what concerns Beckmann's formulation of the optimal transport problem, but it is interesting to analyze it both for checking the consistency with the Monge's formulaltion and for using the results throughout next sections. We will take $\Omega = [a, b] \subset \mathbb{R}$.

First of all, notice that the condition $\nabla \cdot v = \mu - \nu$ is much stronger in dimension one than in higher dimension. Indeed, the divergence is the trace of the Jacobian matrix, and hence prescribing it only gives one constraint on a matrix which has a priori $d \times d$ degrees of freedom. On the contrary, in dimension one there is only one partial derivative for the vector field v (which is actually a scalar), and this completely prescribes the behavior of v. Indeed, the condition $\nabla \cdot v = \mu - \nu$ with Neumann boundary conditions implies that v must be the primitive of $\mu - \nu$ with $v(a) = 0$ (the fact that μ and ν have the same mass also implies $v(b) = 0$). Notice that the fact that its derivative is a measure gives $v \in BV([a, b])$.

Memo – *Bounded variation functions in one variable*

BV functions are generally defined as L^1 functions whose distributional derivatives are measures. In dimension one this has a lot of consequences. In particular these functions coincide a.e. with functions which have bounded total variation in a pointwise sense: for each $f : [a, b] \to \mathbb{R}$ define

$$TV(f; [a, b]) := \sup\{\sum_{i=0}^{N-1} |f(t_{i+1} - f_i| \ : \ a = t_0 < t_1 < t_2 < \cdots < t_N = b\}.$$

Functions of bounded total variation are defined as those f such that $TV(f; [a, b]) < \infty$. It is easy to check that BV functions are a vector space, and that monotone functions are BV (indeed, if f is monotone we have $TV(f; [a, b]) = |f(b)-f(a)|$.Lipschitz functions are also BV and $TV(f; [a, b]) \le \mathrm{Lip}(f)(b - a)$. On the other hand, continuous functions are not necessarily BV, neither it is the case for differentiable functions (obviously, C^1 functions, which are Lipschitz on bounded intervals, are BV). As an example one can consider

$$f(x) = \begin{cases} \frac{x}{\log x} \sin\left(\frac{1}{x}\right) & \text{if } 0 < x \le \frac{1}{2} \\ 0 & \text{for } x = 0, \end{cases}$$

which is differentiable everywhere but not BV (consider a partition using points of the form $x = 2/(k\pi)$ and use $\sum_k \frac{1}{k \log k} = +\infty$).

On the other hand, BV functions have several properties:

Properties of BV functions in $\mathbb{R}$ If $TV(f; [a, b]) < \infty$ then f is the difference of two monotone functions (in particular we can write $f(x) = TV(f; [a, x]) - (TV(f; [a, x]) - f(x))$, both terms being non-decreasing functions); it is a bounded function and $\sup f - \inf f \le TV(f; [a, b])$; it has the same continuity and differentiability properties of monotone functions (it admits left and right limits at every point, it is continuous up to a countable set of points and differentiable a.e.).

In particular in dimension one we have $BV \subset L^\infty$ which is not the case in higher dimension (in general, we have $BV \subset L^{d/(d-1)}$).

We finish by stressing the connections with measures: for every positive measure μ on $[a, b]$ we can build a monotone function by taking its cumulative distribution function, i.e. $F(x) = \mu([a, x])$ and the distributional derivative of this function is exactly the measure μ. Conversely, every monotone increasing function on a compact interval is the cumulative distribution function of a (uique) positive measure, and every BV function is the cumulative distribution function of a (unique) signed measure.

As a consequence, we have the following facts:

- In dimension one, there is only one competitor v which is given by $v(x) = F(x) - G(x)$ with $F(x) = \mu([a, x])$ and $G(x) = \nu([a, x])$.
- This field v belongs to $BV([a, b])$ and hence to every L^p space, including L^∞.
- The minimal cost in Beckmann's problem is given by $||F - G||_{L^1}$, which is consistent with Proposition 2.2.2.
- The transport density σ, characterized by $v = -u' \cdot \sigma$ is given by $\sigma = |v|$ and shares the same summability properties of v; it also belongs to BV as a composition of a BV function with the absolute value function.

4.2.5 Characterization and uniqueness of the optimal v

In this section we will show two facts: first we prove that the optimal v in the Beckmann's problem always comes from an optimal transport plan γ and then we prove that all the optimal γs give the same v_γ and the same σ_γ, provided one of the two measures is absolutely continuous.

Theorem 4.2.5. *Let v be optimal in (PB): then there is an optimal transport plan γ such that $v = v_\gamma$.*

Proof. Thanks to Proposition 4.2.4, we can find a measure $Q \in \mathcal{P}(C)$ with $(e_0)_\# Q = \mu$ and $(e_1)_\# Q = \nu$ such that $|v| \geq |\theta_Q|$ Yet, the optimality of v implies the equality $|v| = |\theta_Q|$ and the same Proposition 4.2.4 gives in such a case $v = \theta_Q$, as well as $|v| = i_Q$. We assume Q to be concentrated on curves parametrized by constant speed. Define $S : \Omega \times \Omega \to C$ the map associating to every pair (x, y) the segment $\omega_{x,y}$ parametrized with constant speed: $\omega_{x,y}(t) = (1 - t)x + ty$. The statement is proven if we can prove that $Q = S_\# \gamma$ wit γ an optimal transport plan.

Indeed, using again the optimality of v and Proposition 4.2.4, we get

$$\min(PB) = |v|(\Omega) = i_Q(\Omega) = \int_C L(\omega)\, dQ(\omega \geq \int_C |\omega(0) - \omega(1)| dQ(\omega)$$
$$= \int_{\Omega\times\Omega} |x-y|\, d((e_0, e_1)_\# Q)(x,y) \geq \min(PK).$$

The equality $\min(PB) = \min(PK)$ implies that all these inequalities are equalities. In particular Q must be concentrated on curves such that $L(\omega) = |\omega(0) - \omega(1)|$, i.e. segments. Also, the measure $(e_0, e_1)_\# Q$, which belongs to $\Pi(\mu, \nu)$, must be optimal in (PK). This concludes the proof. □

The proof of the following result is essentially taken from [2].

Theorem 4.2.6. *If $\mu \ll \mathcal{L}^d$, then the vector field v_γ does not depend on the choice of the optimal plan γ.*

Proof. Let us fix a Kantorovich potential u for the transport between μ and ν. This potential does not depend on the choice of γ. It determines a partition into transport rays: Corollary 3.1.4 reminds us that the only points of Ω which belong to several transport rays are non-differentiability points for u, and are hence Lebesgue-negligible. Let us call S the set of points which belong to several transport rays: we have $\mu(S) = 0$ but we do not suppose $\nu(S) = 0$ (ν is not supposed to be absolutely continuous). However, γ is concentrated on $(\pi_x)^{-1}(S^c)$. We can then disintegrate (see Section 2.3) γ according to the transport ray containing the point x. More precisely, we define a map $R : \Omega\times\Omega \to \mathcal{R}$, valued in the set $\mathcal{R}$ of all transport rays, sending each pair (x, y) into the ray containing x. This is well-defined $\gamma-$a.e. and we can write $\gamma = \gamma^r \otimes \lambda$, where $\lambda = R_\# \gamma$ and we denote by r the variable related to transport rays. Notice that, for a.e. $r \in \mathcal{R}$, the plan γ_r is optimal between its own marginals (otherwise we could replace it with an optimal plan, do it in a measurable way, and improve the cost of γ).

The measure v_γ may also be obtained through this disintegration, and we have $v_\gamma = v_{\gamma^r} \otimes \lambda$. This means that, in order to prove that v_γ does not depend on γ, we just need to prove that each v_{γ^r} and the measure λ do not depend on it. For the measure λ this is easy: it has been obtained as an image measure through a map only depending on x, and hence only depends on μ. Concerning v_{γ^r}, notice that it is obtained in the standard Beckmann way from an optimal plan, γ^r. Hence, thanks to the considerations in Section 4.2.4, it uniquely depends on the marginal measures of this plan.

This means that we only need to prove that $(\pi_x)_\# \gamma^r$ and $(\pi_y)_\# \gamma^r$ do not depend on γ. Again, this is easy for $(\pi_x)_\# \gamma^r$, since it must coincide with the disintegration of μ according to the map R (by uniqueness of the disintegration). It is more delicate for the second marginal.

The second marginal $\nu^r := (\pi_y)_\#\gamma^r$ will be decomposed in two parts: $(\pi_y)_\#(\gamma^r_{|\Omega\times S})$ and $(\pi_y)_\#(\gamma^r_{|\Omega\times S^c})$. This second part coincides with the disintegration of $\nu_{|S^c}$, which obviously does not depend on γ (since it only depends on the set S, which is built upon u).

We need now to prove that $\nu^r_{|S} = (\pi_y)_\#(\gamma^r_{|\Omega\times S})$ does not depend on γ. Yet, this measure can only be concentrated on the two endpoints of the transport ray r, since these are the only points where different transport rays can meet. This means that this measure is purely atomic and composed by at most two Dirac masses. Not only, the endpoint where u is maximal cannot contain some mass of ν: indeed the transport must follow a precise direction on each transport ray (as a consequence of $u(x)-u(y) = |x-y|$ on $\mathrm{spt}(\gamma)$), and the only way to have some mass of the target measure at the "beginning" of the transport ray would be to have an atom for the source measure as well. Yet, μ is absolutely continuous and Property N holds (see Section 3.1.4 and Theorem 3.1.7, which means that the set of rays r where μ^r has an atom is negligible. Hence $\nu^r_{|S}$ is a single Dirac mass. The mass equilibrium condition between μ^r and ν^r implies that the value of this mass must be equal to the difference $1-\nu^r_{|S^c}(r)$, and this last quantity does not depend on γ but only on μ and ν.

Finally, this proves that each v_{γ^r} does not depend on the choice of γ. □

Corollary 4.2.7. *If $\mu \ll \mathcal{L}^d$, then the optimal solution of (PB) is unique.*

Proof. We have seen in Theorem 4.2.5 that any optimal v is of the form v_γ and in Theorem 4.2.6 that all the fields v_γ coincide. □

4.3 Summability of the transport density

The analysis of Beckmann problem performed in the previous sections was mainly made in a measure setting, and the optimal v, as well as the transport density σ, where just measures on Ω. We investigate here the question whether they have extra regularity properties supposing extra assumptions on μ and/or ν.

We will give summability results, proving that σ is in some cases absolutely continuous and proving L^p estimates. The proofs are essentially taken from [98]: previous results, through very different techniques, were first presented in [60, 61, 62]. In these papers, different estimates on the "dimension" of σ are aslo presented, thus giving interesting information should σ fail to be absolutely continuous.

Notice that higher order questions, such as whether σ is continuous or Lipschitz or more regular provided μ and ν have smooth densities are completely open up to now (with the exception of a partial result in dimension 2, see [**?**], where a continuity result is given if μ and ν have Lipschitz densities on disjoint convex domains).

In all that follows Ω is a compact and convex domain in $\mathbb{R}^d$, and two probability measures are given on it. Since we will need to interpolate between them, we will rather call them μ_0 and μ_1 (and the interpolation will be called μ_t). At least one of them will be absolutely continuous, which implies uniqueness for σ (see Theorem 4.2.6).

Theorem 4.3.1. *Suppose $\mu_0 \ll \mathcal{L}^d$ and let σ be the transport density associated to the transport of μ_0 onto μ_1. Then $\sigma \ll \mathcal{L}^d$.*

Proof. Let γ be an optimal transport from μ_0 to μ_1 and take $\sigma = \sigma_\gamma$; call μ_t the standard interpolation between the two measures: $\mu_t = (\pi_t)_\# \gamma$ where $\pi_t(x, y) = (1-t)x + ty$.

We have already seen that the transport density σ may be written as

$$\sigma = \int_0^1 (\pi_t)_\sharp (c \cdot \gamma) dt,$$

where $c : \Omega \times \Omega \to \mathbb{R}$ is the cost function $c(x, y) = |x - y|$ (hence $c \cdot \gamma$ is a positive measure on $\Omega \times \Omega$).

Since Ω is bounded it is evident that we have

$$\sigma \leq C \int_0^1 \mu_t \, dt. \tag{3.9}$$

To prove that σ is absolutely continuous, it is sufficient to prove that almost every measure μ_t is absolutely continuous, so that, whenever $|A| = 0$, we have $\sigma(A) \leq C \int_0^1 \mu_t(A) dt = 0$.

We will prove $\mu_t \ll \mathcal{L}^d$ for $t < 1$. First, we will suppose that μ_1 is finitely atomic (the point $(x_i)_{i=1,\dots,N}$ being its atoms). In this case we will choose γ to be any optimal transport plan induced by a transport map T (which exists, since $\mu_0 \ll \mathcal{L}^d$). Notice that the absolute continuity of σ is an easy consequence of the behavior of the optimal transport from μ_0 to μ_1 (which is composed by N homothecies), but we also want to quantify this absolute continuity, in order to go on with an approximation procedure.

Remember that μ_0 is absolutely continuous and hence there exists a correspondence $\varepsilon \mapsto \delta = \delta(\varepsilon)$ such that

$$|A| < \delta(\varepsilon) \Rightarrow \mu_0(A) < \varepsilon. \tag{3.10}$$

Take now a Borel set A and look at $\mu_t(A)$. The domain Ω is the disjoint union of a finite number of sets $\Omega_i = T^{-1}(\{x_i\})$. We call $\Omega_i(t)$ the images of Ω_i through the map $x \mapsto (1-t)x + tT(x)$. These sets are essentially disjoint. Why? because if a point z belongs to $\Omega_i(t)$ and $\Omega_j(t)$, then two transport rays cross at z, the one going from $x_i' \in \Omega_i$ to x_i and the one from $x_j' \in \Omega_j$ to x_j. The only possibility is that these two rays are actually the same, i.e. that the five points x_i', x_j', z, x_i, x_j are aligned. But this implies that z belongs to one of the lines connecting two atoms x_i and x_j. Since we have finitely many of these lines this set is negligible. Notice that this argument only works for $d > 1$ (we will not waste time on the case $d = 1$, since the transport density is always a BV and hence bounded function). Moreover, if we sticked to the optimal transport which is monotone on transport rays, we could have actually proved that these sets are truly disjoint, with no negligible intersection.

Hence we have

$$\mu_t(A) = \sum_i \mu_t(A \cap \Omega_i(t)) = \sum_i \mu_0\left(\frac{A \cap \Omega_i(t) - tx_i}{1-t}\right) = \mu_0\left(\bigcup_i \frac{A \cap \Omega_i(t) - tx_i}{1-t}\right).$$

Since for every i we have

$$\left|\frac{A \cap \Omega_i(t) - tx_i}{1-t}\right| = \frac{1}{(1-t)^d}\,|A \cap \Omega_i(t)|$$

we have

$$\left|\bigcup_i \frac{A \cap \Omega_i(t) - tx_i}{1-t}\right| \le \frac{1}{(1-t)^d}|A|.$$

Hence it is sufficient to suppose $|A| < (1-t)^d\delta(\varepsilon)$ to get $\mu_t(A) < \varepsilon$. This confirms $\mu_t \ll \mathcal{L}^d$ and gives an estimate that may pass to the limit.

Take a sequence $(\mu_1^n)_n$ of atomic measures converging to μ_1. The corresponding optimal transport plans γ^n converge to an optimal transport plan γ and μ_t^n converge to the corresponding μ_t (see Theorem 1.6.10 in Chapter 1). Hence, to prove absolute continuity for the transport density σ associated to such a γ it is sufficient to prove that these μ_t are absolutely continuous.

Take a set A such that $|A| < (1-t)^d\delta(\varepsilon)$. Since the Lebesgue measure is regular, A is included in an open set B such that $|B| < (1-t)^d\delta(\varepsilon)$.

Hence $\mu_t^n(B) < \varepsilon$. Passing to the limit, thanks to weak convergence and semicontinuity on open sets, we have

$$\mu_t(A) \leq \mu_t(B) \leq \liminf_n \mu_t^n(B) \leq \varepsilon.$$

This proves $\mu_t \ll \mathcal{L}^d$ and hence $\sigma \ll \mathcal{L}^d$. □

Remark 11. Where did we use the optimality of γ? we did it when we said that the $\Omega_i(t)$ are disjoint. For a discrete measure μ_1, it is always true that the measures μ_t corresponding to any transport plan γ are absolutely continuous for $t < 1$, but their absolute continuity may degenerate at the limit if we allow the sets $\Omega_i(t)$ to superpose (since in this case densities sum up and the estimates may depend on the number of atoms).

Remark 12. Notice that we strongly used the equivalence between the two different definitions of absolute continuity, i.e. the $\varepsilon \leftrightarrow \delta$ correspondence on the one hand and the condition on negligible sets on the other. Indeed, to prove that the condition $\mu_t \ll \mathcal{L}^d$ passes to the limit we need the first one, while to deduce $\sigma \ll \mathcal{L}^d$ we need the second one, since if we deal with non-negligible sets we have some $(1-t)^d$ factor to deal with...

Remark 13. As a byproduct of the proof we can see that any optimal transport pla from μ_0 to μ_1 which is approximable through optimal transport plans from μ_0 to atomic measures must be such that all the interpolating measures μ_t are absolutely continuous. This property is not satisfied by any optimal transport plan, since for instance the plan γ which sends $\mu_0 = \mathcal{L}^2_{|[-2,-1]\times[0,1]}$ onto $\mu_1 = \mathcal{L}^2_{|[1,2]\times[0,1]}$ moving (x, y) to $(-x, y)$ is optimal but is such that $\mu_{1/2} = \mathcal{H}^1_{|\{0\}\times[0,1]}$. Hence, this plan cannot be approximated by optimal plans sending μ_0 onto atomic measures. On the other hand, we proved in Lemma 3.1.12 that the monotone optimal transport can indeed be approximated in a similar way.

In the previous theorem we did not treat the one dimensional case, which is highly detailed in Section 4.2.4.

From now on we will often confuse absolutely continuous measures with their densities and write $||\mu||_p$ for $||f||_{L^p(\Omega)}$ when $\mu = f \cdot \mathcal{L}$.

Theorem 4.3.2. *Suppose $\mu_0 = f \cdot \mathcal{L}^d$, with $f \in L^p(\Omega)$. The, if $p < d' := d/(d-1)$, the unique transport density σ associated to the transport of μ_0 onto μ_1 belongs to $L^p(\Omega)$ as well, and if $p \geq d'$ it belongs to any space $L^q(\Omega)$ for $q < d'$.*

Proof. Start from the case $p < d'$: following the same strategy (and the same notations) as before, it is sufficient to prove that each measure μ_t (for $t \in [0,1[$) is in L^p and to estimate their L^p norm. Then we will use

$$||\sigma||_p \leq C \int_0^1 ||\mu_t||_p dt,$$

(which is a consequence of (3.9) and of Minkowski inequality), the conditions on p being chosen exactly so that this integral converges.

Consider first the discrete case: we know that μ_t is absolutely continuous and that its density coincides on each set $\Omega_i(t)$ with the density of an homothetic image of μ_0 on Ω_i, the homothecy ratio being $(1-t)$. Hence, if f_t is the density of μ_t, we have

$$\begin{aligned}\int_\Omega f_t(x)^p dx &= \sum_i \int_{\Omega_i(t)} f_t(x)^p dx = \sum_i \int_{\Omega_i} \left(\frac{f(x)}{(1-t)^d}\right)^p (1-t)^d dx \\ &= (1-t)^{d(1-p)} \sum_i \int_{\Omega_i} f(x)^p dx = (1-t)^{d(1-p)} \int_\Omega f(x)^p dx.\end{aligned}$$

We get $||\mu_t||_p = (1-t)^{-d/p'}||\mu_0||_p$, where $p' = p/(p-1)$ is the conjugate exponent of p.

This inequality, which is true in the discrete case, stays true at the limit as well. If μ_1 is not atomic, approximate it through a sequence μ_1^n and take optimal plans γ^n and interpolating measures μ_t^n. Up to subsequences we have $\gamma^n \rightharpoonup \gamma$ (for an optimal transport plan γ) and $\mu_t^n \rightharpoonup \mu_t$ (for the corresponding interpolation); by semicontinuity we have

$$||\mu_t||_p \leq \liminf_n ||\mu_t^n||_p \leq (1-t)^{-d/p'}||\mu_0||_p$$

and we deduce

$$||\sigma||_p \leq C \int_0^1 ||\mu_t||_p dt \leq C||\mu_0||_p \int_0^1 (1-t)^{-d/p'} dt.$$

The last integral is finite whenever $p' > d$, i.e. $p < d' = d/(d-1)$.

The second part of the statement (the case $p \geq d'$) is straightforward once one considers that any density in L^p also belongs to any L^q space for $q < p$. □

EXAMPLE

What we just saw in the previous theorems is that the measures μ_t inherit some regularity (absolute continuity or L^p summability) from μ_0 exactly as it happens for homotheties of ratio $1-t$. This regularity degenerates as $t \to 1$, but we saw two cases where this degeneracy produced no problem: for proving absolute continuity, where the separate absolute continuous behavior of almost all the μ_t was sufficient, and for L^p estimates, provided the degeneracy stays integrable.

It is natural to try to exploit another strategy: suppose both μ_0 and μ_1 share some regularity assumption (e.g., they belong to L^p). Then we can give estimate on μ_t for $t \leq 1/2$ starting from μ_0 and for $t \geq 1/2$ starting from μ_1. In this way we have no degeneracy!

This strategy works quite well, but it has an extra difficulty: in our previous estimates we didn't know a priori that μ_t shared the same behavior of piecewise homotheties of μ_0, we got it as a limit from discrete approximations. And, when we pass to the limit, we do not know which optimal transport γ will be selected as a limit of the optimal plans γ^n. This was not important in the previous section, since any optimal γ induces the same transport density σ. Yet, here we would like to glue together estimates on μ_t for $t \leq 1/2$ which have been obtained by approximating μ_1, and estimates on μ_t for $t \geq 1/2$ which come from the approximation of μ_0. Should the two approximations converge to two different transport plans, we could not put together the two estimates and deduce anything on σ.

Hence, the main technical issue which we need to consider is proving that one particular optimal transport plan, i.e. the one which is monotone on transport rays, will be approximable in both directions. Lemma 3.1.12 exactly does the job (and, indeed, it was proven in [98] exactly for this purpose). Yet, the transport plans γ_ε we build in the approximation are not optimal for the cost $\int |x-y| d\gamma$ but for some costs $\int (|x-y|+\varepsilon|x-y|^2) d\gamma$. We need to do this in order to force the selected limit optimal transport to be the monotone one (through a secondary variational problem, say). Anyway, this will not be an issue since these approximating optimal transport will share the same geometric properties that will imply disjointness for the sets $\Omega_i(t)$ will allow for density estimates.

The first tool we need is a uniform L^p estimates of the measures μ_t in terms of the norm of μ_0, when μ_t is an interpolation from μ_0 to μ_1 corresponding to a transport plan γ which is optimal for another cost, different from $|x-y|$. In this case we do not have any transport ray argument, but the result is somehow even stronger under strict convexity assumptions.

Even if not precisely stated, the reader will be easily be able to check that all the results of this section stay true for $p = +\infty$ as well.

Lemma 4.3.3. *Let γ be an optimal transport plan between μ_0 and an atomic* *measure μ_1 for a transport cost $c(x,y) = \phi(y-x)$ where $\phi : \mathbb{R}^d \to \mathbb{R}$ is a strictly convex function. Set as usual $\mu_t = (\pi_t)_\#\gamma$. Then we have $||\mu_t||_p \le (1-t)^{-d/p'}||\mu_0||_p$.*

Proof. The result is exactly the same as in Theorem 4.3.2, where the key tool is the fact that μ_t coincides on every set $\Omega_i(t)$ with a homothety of μ_0. The only fact that must be checked again is the disjointness of the sets $\Omega_i(t)$.

To do so, take a point $x \in \Omega_i(t) \cap \Omega_j(t)$. Hence there exist x_i, x_j belonging to Ω_i and Ω_j, respectively, so that $x = (1-t)x_i + ty_i = (1-t)x_j + ty_j$, being $y_i = T(x_i)$ and $y_j = T(x_j)$ atoms of μ_1. But this would mean that $T_t := (1-t)id + tT$ is not injective, which is a contradiction to the following Lemma /refinjectivity Tt. Hence the sets $\Omega_i(t)$ are disjoint and this implies the bound on μ_t. □

Lemma 4.3.4. *Let γ be an optimal transport plan between μ_0 and μ_1 for a transport cost $c(x,y) = \phi(y-x)$ where $\phi : \mathbb{R}^d \to \mathbb{R}$ is a strictly convex function, and suppose that it is induced by a transport map T. To avoid problems with negligible sets, suppose that $(x, T(x)) \in \operatorname{spt}\gamma$ for all x. Then the map $x \mapsto (1-t)x + tT(x)$ is injective for $t \in]0,1[$.*

Proof. Suppose that there exist $x \neq x'$ such that $(1-t)x + tT(x) = (1-t)x' + tT(x') = X$. Set $a = T(x) - x$ and $b = T(x') - x'$. This also means $x = X - ta$ and $x' = X - tb$. In particular, $x \neq x'$ implies $a \neq b$.

The $c-$cyclical monotonicity of the support of the optimal γ implies

$$\phi(a) + \phi(b) \le \phi(T(x') - x) + \phi(T(x) - x') = \phi(tb + (1-t)a) + \phi(ta + (1-t)b).$$

Yet, $a \neq b$, and strict convexity imply

$$\phi(tb+(1-t)a)+\phi(ta+(1-t)b) < t\phi(b)+(1-t)\phi(a)+t\phi(a)+(1-t)\phi(b) = \phi(a)+\phi(b),$$

which is a contradiction. □

Remark 14. Disjointness of the sets $\Omega_i(t)$ is easier in this strictly convex setting. If the cost is $|x-y|$ this is no more true, but it is anyway true that the two vector a and b should be parallel, i.e. all the points should be aligned, as we pointed out in Theorem 4.3.1. If μ does not give mass to lines, than the sets are essentially disjoint. Otherwise one can say that they are truly disjoint if one only looks at the optimal transport which is monotone on transport rays.

Theorem 4.3.5. *Suppose that μ_0 and μ_1 are probability measures on Ω, both belonging to $L^p(\Omega)$, and σ the unique transport density associated to the transport of μ_0 onto μ_1. Then σ belongs to $L^p(\Omega)$ as well.*

Proof. Let us consider the optimal transport plan $\bar\gamma$ from μ_0 to μ_1 defined by (1.2). We know that this transport plan may be approximated by plans γ_ε which are optimal for the cost $|x-y|+\varepsilon|x-y|^2$ from μ_0 to some discrete atomic measures ν_ε. The corresponding interpolation measures $\mu_t(\varepsilon)$ satisfy the L^p estimate from Lemma 4.3.3 and, at the limit, we have

$$||\mu_t||_p \le \liminf_{\varepsilon\to 0} ||\mu_t(\varepsilon)||_p \le (1-t)^{-d/p'}||\mu_0||_p.$$

The same estimate may be performed from the other direction, since the same transport plan $\bar\gamma$ may be approximated by optimal plans for the cost $|x-y|+\varepsilon|x-y|^2$ from atomic measures to μ_1. Putting together the two estimates we have

$$||\mu_t||_p \le \min\left\{(1-t)^{-d/p'}||\mu_0||_p,\, t^{-d/p'}||\mu_1||_p\right\} \le 2^{d/p'}\max\left\{||\mu_0||_p, ||\mu_1||_p\right\}.$$

Integrating these L^p norms we get the bound on $||\sigma||_p$. $\square$

EXAMPLE

Theorem 4.3.6. *Suppose $\mu_0 \in L^p(\Omega)$ and $\mu_1 \in L^q(\Omega)$. For notational simplicity take $p > q$. Then, if $p < d/(d-1)$, the transport density σ belongs to L^p and, if $p \ge d/(d-1)$, it belongs to $L^r(\Omega)$ for all the exponents r satisfying*

$$r < r(p,q,d) := \frac{dq(p-1)}{d(p-1)-(p-q)}.$$

Proof. The first part of the statement (the case $p < d/(d-1)$ is a consequence of Theorem 4.3.2. For the second one, using exactly the same argument as before (Theorem 4.3.5) we get

$$||\mu_t||_p \le (1-t)^{-d/p'}||\mu_0||_p; \quad ||\mu_t||_q \le t^{-d/q'}||\mu_1||_q.$$

We then apply standard Hölder inequality to derive the usual interpolation estimate for any exponent $q < r < p$:

$$||f||_r \le ||f||_p^\alpha ||f||_q^{1-\alpha} \quad \text{with } \alpha = \frac{p(r-q)}{r(p-q)}, \text{ and } 1-\alpha = \frac{q(p-r)}{r(p-q)}.$$

This implies

$$||\mu_t||_r \le C||\mu_t||_p \le C||\mu_0||_p \text{ for } t < \frac{1}{2}; \quad ||\mu_t||_r \le C(1-t)^{-\alpha d/p'}||\mu_0||_p^\alpha ||\mu_1||_p^{1-\alpha} \text{ for } t > \frac{1}{2}.$$

Then, take $r < r(p,q,d)$, so that $\alpha d/p' < 1$ is ensured and hence the L^r norm is integrable, thus giving a bound on $||\sigma||_r$. $\square$

Remark 15. We do not know whether this exponent $r(p,q,d)$ is sharp or not and whether σ belongs or not to $L^{r(p,q,d)}$.

On the contrary, Example 4.15 in [60] shows the sharpness of the bound on p that we set in Theorem 4.3.2.

4.4 Discussion

4.4.1 Congested transport

As we saw in Section 4.2, Beckmann's problem can admit an easy variant if we prescribe a positive function $k : \Omega \to \mathbb{R}_+$, where $k(x)$ stands for the local cost at x per unit length of a path passing through x. This models the possibility that the metric is non-homogeneous, due to geographical obstacles given a priori. Yet, it happens in many situation, in particular in urban traffic as everybody knows, that this metric k is indeed non-homogeneous, but is not given a priori: it depends on the traffic, i.e. it depends on the choice of all the commuters. In Beckmann's language, we must look for a vector field v optimizing a transport cost depending on v itself!

The easiest modelization, chosen by Beckmann [11] and later in [48] is to consider the same framework as (PB) but supposing that $k(x) = g(|v(x|)$ is a function of the modulus of the vector field v. This is quite formal for the moment (for instance it is not meaningful if v is a measure, but we will not set this problem in the class of measures, indeed). In this case we would like to solve

$$\min \int \mathcal{H}(v(x))dx \quad : \nabla \cdot v = \mu - \nu, \tag{4.11}$$

where $\mathcal{H}(v) = H(|v|)$ and $H(t) = g(t)t$. Notice that if H is superlinear (if $g(t) \to \infty$ as $t/to\infty$, i.e. if the congestion effect becomes bigger and bigger when the traffic increases) this problem is well posed in the class of vector fields $v \in L^1$ (or of absolutely continuous vector measures). For instance, if $g(t) = t$, which is the easiest case one can imagine, we must minimize the L^2 norm under divergence constraints:

$$\min \int |v(x)|^2 dx \quad : v \in L^2(\Omega; \mathbb{R}^d),\ \nabla \cdot v = \mu - \nu.$$

This problem is easily solvable since one can see that the optimal v must be a gradient (we will develop this computation in a more general framework below), and setting $v = \nabla u$ one gets $\Delta u = \mu - \nu$. This is complemented with Neumann boundary conditions and allow to find u, and then v.

We want now to discuss the meaning and pertinence of this model, keeping into account the following natural questions:

- is this the good modelization, or the coefficient k should rather depend on other traffic quantities, in particular a traffic intensity like i_Q? (notice that v can have cancellations);

- what is the connection with equilibrium issues? in traffic congestion, typically every agent decides alone which path to choose, and the final traffic intensity is rather the output of a collection of individual choices, rather than the result of a global optimization made by a single planner;

- is the example $g(t) = t$ a good choice in the modelization ? this implies $g(0) = 0$, i.e. no cost where there is not traffic, but we know that we cannot move at infinite speed even if there is no traffic.

To start our analysis we would like to present first an equilibrium model developed by Wardrop, [106], on a discrete network.

Traffic equilibria on a finite network The main data of the model are a finite oriented connected graph $G = (N, E)$ modeling the network, and edge travel times functions $g_e : w \in \mathbb{R}_+ \mapsto g_e(w)$ giving, for each edge $e \in E$, the travel time on arc e when the flow on this edge is w. The functions g_e are all nonnegative, continuous, nondecreasing and they are meant to capture the congestion effects (which may be different on the different edges, since some roads may be longer or wider and may have different responses to congestion). The last ingredient of the problem is a transport plan on pairs of nodes $(x, y) \in N^2$ interpreted as pairs of sources/destinations. We denote by $(\gamma_{x,y})_{(x,y)\in N^2}$ this transport plan: $\gamma_{x,y}$ represents the "mass" to be sent from x to y. We denote by $C_{x,y}$ the set of simple paths connecting x to y, so that $C := \cup_{(x,y)\in N^2} C_{x,y}$ is the set of all simple paths. A generic path will be denoted by ω and we will use the notation $e \in \omega$ to indicate that the path ω uses the edge e.

The unknown of the problem is the flow configuration. The edge flows are denoted by $i = (i_e)_{e\in E}$ and the path flows are denoted by $q = (q_\omega)_{\omega\in C}$: this means that i_e is the total flow on edge e and q_ω is the mass traveling on the path ω. Of course the i_e's and q_ω's are nonnegative and constrained by the mass conservation conditions:

$$\gamma_{x,y} = \sum_{\omega \in C_{x,y}} q_\omega, \quad \forall (x,y) \in N^2 \tag{4.12}$$

and

$$i_e = \sum_{\omega \in C \,:\, e \in \omega} q_\omega, \quad \forall e \in E, \tag{4.13}$$

which means that i is a function of q. Given the edge flows $i = (i_e)_{e \in E}$, the total travel-time of the path $\omega \in C$ is

$$T_i(\omega) = \sum_{e \in \omega} g_e(i_e). \tag{4.14}$$

In [106], Wardrop defined a notion of noncooperative equilibrium that has been very popular since among engineers working in the field of congested transport and that may be described as follows. Roughly speaking, a Wardrop equilibrium is a flow configuration such that every actually used path should be a shortest path taking into account the congestion effect i.e. formula (4.14). This leads to

Definition 13. A Wardrop equlibrium is a flow configuration $i = (i_e)_{e \in E}$, $q = (q_\omega)_{\omega \in C}$ (all nonnegative of course), satisfying the mass conservation constraints (4.12) and (4.13), such that, in addition, for every $(x,y) \in N^2$ and every $\omega \in C_{x,y}$, if $q_\omega > 0$ then

$$T_i(\omega) = \min_{\omega' \in C_{x,y}} T_i(\omega').$$

A few years after Wardrop introduced his equilibrium concept, Beckmann, McGuire and Winsten [12] realized that Wardrop equilibria can be characterized by the following variational principle:

Theorem 4.4.1. *The flow configuration $i = (i_e)_{e \in E}$, $q = (q_\omega)_{\omega \in C}$ is a Wardrop equilibrium if and only if it solves the convex minimization problem*

$$\inf_{(i,q)} \sum_{e \in E} H_e(i_e) \text{ s.t. nonnegativity and } (4.12)\ (4.13) \tag{4.15}$$

where, for each e, we take H_e to be the primitive of g_e.

Proof. Assume that $q = (q_\omega)_{\omega \in C}$ (with associated edge flows $(i_e)_{e\in E}$) is optimal for (4.15) then for every admissible $\eta = (\eta_\omega)_{\omega\in C}$ with associated (through (4.13)) edge-flows $(u_e)_{e\in E}$, one has

$$\begin{aligned} 0 &\le \sum_{e\in E} H_e'(i_e)(u_e - i_e) = \sum_{e\in E} g_e(i_e) \sum_{\omega\in C\,:\, e\in\omega} (\eta_\omega - q_\omega) \\ &= \sum_{\omega\in C} (\eta_\omega - q_\omega) \sum_{e\in\omega} g_e(i_e) \end{aligned}$$

so that

$$\sum_{\omega\in C} q_\omega T_i(\omega) \le \sum_{\omega\in C} \eta_\omega T_i(\omega)$$

minimizing the right-hand side thus yields

$$\sum_{(x,y)\in N^2} \sum_{\omega\in C_{x,y}} q_\omega T_i(\omega) = \sum_{(x,y)\in N^2} \gamma_{x,y} \min_{\omega'\in C_{x,y}} T_i(\omega')$$

which exactly says that (q, i) is a Wardrop equilibrium. To prove the converse, it is enough to see that problem (4.15) is convex so that the inequality above is indeed sufficient for a global minimum. □

The previous characterization actually is the reason why Wardrop equilibria became so popular. Not only, one deduces for free existence results, but also uniqueness for w (not for q) as soon as the functions g_e are increasing (so that H_e is strictly convex).

Remark 16. It would be very tempting to deduce from theorem 4.4.1 that equilibria are efficient since they are minimizers of (4.15). One has to be cautious with this quick interpretation since the quantity $\sum_{e\in E} H_e(i_e)$ does not represent the natural total social cost measured by the total time lost in commuting which reads as

$$\sum_{e\in E} i_e g_e(i_e). \tag{4.16}$$

The efficient transport patterns are minimizers of (4.16) and thus are different from equilibria in general. Efficient and equilibria configurations coincide in the special case of power functions where $g_e(w) = a_e w^\alpha$, but this case is not realistic since it implies that traveling times vanish if there is no traffic... Moreover, a famous counter-example due to Braess shows that it may be the case that adding an extra road on which the travelling time is aways zero leads to an equilibrium where the total commuting time is increased!

This illustrates the striking difference between efficiency and equilibrium, a topic which is very well-documented in the finite-dimensional network setting where it is frequently associated to the literature on the so-called *price of anarchy* (see [**?**]).

Remark 17. In the problem presented in this paragraph, the transport plan γ is fixed, this may be interpreted as a *short-term problem.* Instead, we could consider the *long-term* problem where only the distribution of sources μ_0 and the distribution of destinations μ_1 are fixed. In this case, one requires in addition, in the definition of an equilibrium that γ is efficient in the sense that it minimizes among transport plans between μ_0 and μ_1 the total cost

$$\sum \gamma_{x,y} d_i(x,y) \text{ with } d_i(x,y) := \min_{\omega \in C_{x,y}} T_i(\omega).$$

In the long-term problem where one is allowed to change the assignment as well, equilibria still are characterized by a convex minimization problem where one also optimizes over γ.

Optimization and equlibrium in a continuous framework We want now to generalize the previous analysis to a continuous framework. In the continuous setting, there will be no network, all paths in a certain given region will therefore be admissible. The first idea is to formulate the whole path-dependent transport pattern in terms of a probability measure Q on the set of paths (this is the continuous analogue of the path flows $(q_\sigma)_\sigma$ of the previous paragraph). The second one is to measure the intensity traffic generated by Q in a similar way as one defines transport density in the Monge's problem (this is the continuous analogue of the arc flows $(i_e)_e$ of the previous paragraph). The last and main idea will be in modelling the congestion effect through a metric that is monotone increasing in the traffic intensity (the analogue of $g_e(i_e)$).

We will deliberately avoid to enter into technicalities so the following description will be pretty informal (see [47] for details). From now on, Ω denotes an open bounded connected subset of $\mathbb{R}^2$ (a city, say), and we are also given :

- either two probability measures μ and ν (distribution of sources and destinations) on Ω in the case of the long-term problem,
- or a transport plan γ (joint distribution of sources and destinations) that is a joint probability on $\Omega \times \Omega$) in the short-term case,

- or more generally a convex and closed subset $\Gamma \subset \Pi(\mu,\nu)$ and we accept any $\gamma \in \Gamma$ (this is just a common mathematical framework for the two previous cases, where we can take $\Gamma = \{\gamma\}$ or $\Gamma = \Pi(\mu,\nu)$.

We will use the notations of Section 4.2.3, and use probability measures Q on $C := \mathrm{Lip}([0,1],\Omega)$, compatible with mass conservation, i.e. such that

$$(e_0, e_1)_\# Q \in \Gamma, \text{ with } e_t(\sigma) := \sigma(t), \ \forall t \in [0,1].$$

We shall denote by $\mathcal{Q}(\Gamma)$ the set of admissible transport patterns. We are interested in finding an equilibrium i.e. a $Q \in \mathcal{Q}(\Gamma)$ that is supported on geodesics for a metric ξ_Q depending on Q itself (congestion).

The intensity of traffic associated to $Q \in \mathcal{Q}(\Gamma)$ is by definition the measure i_Q defined in Section 4.2.3.

The congestion effect is then captured by the *metric* associated to Q: suppose $i_Q \ll \mathcal{L}^2$ and set

$$\xi_Q(x) := g(x, i_Q(x))$$

for a given increasing function $g(x,.) : \mathbb{R}_+ \to \mathbb{R}_+$. The fact that there exists at least one $Q \in \mathcal{Q}(\Gamma)$ such that $i_Q \ll \mathcal{L}^2$ is not always true and depends on Γ but, for instance, it is true when $\Gamma = \Pi(\mu,\nu)$ and μ and ν are such that the transport density is absolutely continuous. Notice also that, for $\Gamma = \{\gamma\}$ (which is the most restrictive case) and $\mu,\nu \in L^\infty$, considerations from incompressible fluid mechanics in [38] allow to build a Q such that $i_Q \in L^\infty$. Let us now describe what a reasonable definition of an equilibrium should look like. If the overall transport pattern is Q, an agent commuting from x to y choosing a path $\omega \in C_{x,y}$ (i.e. an absolutely continuous curve ω such that $\omega(0) = x$ and $\omega(1) = y$) spends time

$$L_{\xi_Q}(\omega) = \int_0^1 g(\omega(t), i_Q(\omega(t))|\dot\omega(t)|dt.$$

She will then try to minimize this time i.e. to achieve the corresponding geodesic distance

$$c_{\xi_Q}(x,y) := \inf_{\omega \in C_{x,y}} L_{\xi_Q}(\omega).$$

Paths in $C_{x,y}$ such that $c_{\xi_Q}(x,y) = L_{\xi_Q}(\omega)$ are called geodesics (for the metric induced by the congestion effect generated by Q).

We can define

Definition 14. A Wardrop equilbrium is a $Q \in \mathcal{Q}(\Gamma)$ such that

$$Q(\{\omega \ : \ L_{\xi_Q}(\omega) = c_{\xi_Q}(\omega(0), \omega(1)) = 1. \tag{4.17}$$

Existence, and even well-posedness (what does it mean $L_\xi(\omega)$ if ξ is only measurable and ω is a Lipschitz curve?) of these equilibria are not straightforward. Again, we will characterize equilibria as solutions of a given minimal traffic problem.

Let us consider the (convex) variational problem

$$\inf_{Q \in \mathcal{Q}(\Gamma)} \int_\Omega H(x, i_Q(x))dx \tag{4.18}$$

where $H'(x,.) = g(x,.)$, $H(x,0) = 0$. We shall refer to (4.18) as the congested optimal mass transportation problem for reasons that will be clarified later. Under some technical assumptions that we do not reproduce here, the main results of [47] can be summarized by:

Theorem 4.4.2. *Problem* (4.18) *admits at least one minimizer. Moreover* $\overline{Q} \in \mathcal{Q}(\Gamma)$ *solves* (4.18) *if and only if it is a Wardrop equilibrium and* $\gamma_Q := (e_0, e_1)_\# Q$ *solves the optimization problem*

$$\min \int_{\Omega \times \Omega} c_{\xi_Q}(x,y)\, d\gamma(x,y) \ : \ \gamma \in \Gamma.$$

In particular, if $\Gamma = \{\gamma\}$ *this last condition does not play any role (there is only one competitor) and we show existence of a Wardrop equilibrium corresponding to any given transport plan* γ. *If, on the contrary,* $\Gamma = \Pi(\mu, \nu)$, *then the second condition means that* γ *solves a Monge-Kantorovich problem for a distance cost depending on* Q *itself, which is a new equilibrium condition.*

The full proof is quite involved since it requires to take care of some regularity issues in details. In particular, the use of the weighted length functional $L_{\overline{\xi}}$ and thus also the geodesic distance $c_{\overline{\xi}}$ require some attention since defining these quantities actually makes sense only if $\overline{\xi}$ is continuous or at least l.s.c.. In [47] a possible construction when $\overline{\xi}$ is just an L^q function is given. Let us also mention that recent regularity results (see below) actually prove that $\overline{\xi}$ is in fact a continuous function, under reasonable assumptions on the data.

We have proved that, as in the finite-dimensional network case, Wardrop equilibria have a variational characterization which is in principle easier to deal with than the definition. Unfortunately, the convex problems (4.18) and (**??**) may be difficult to solve since they involve measures on sets of curves that is two layers of infinite dimensions! We will not deal here with the numerical strategies, bases on convex optimization duality, and on the so-called Fast Marching Method to compute c_ξ for given ξ (and later to compute variations of c_ξ when ξ varies), and we refer to [17, 18]. These numerical methods are quite efficient and generalize what already done on finite networks, and are better suited for the short-term case.

On the contrary, in the next paragraph we develop an interesting feature of the long-term problem.

Beckmann-like reformulation of the long-term problem In the long-term problem (4.18), we have one more degree of freedom since the transport plan is not fixed. This will enable us to reformulate the problem as a variational divergence constrained problem à la Beckmann and ultimately to reduce the equilibrium problem to solving some nonlinear PDE.

As we already did in Section 4.2.3, for any $Q \in \mathcal{Q}(\Gamma)$ we can take the vector-field θ_Q.

If we consider the scalar problem (4.18), it is easy to see that its value is larger than that of the minimal flow problem à la Beckmann:

$$\min_{\sigma\,:\,\nabla\cdot\sigma=\mu-\nu} \int_\Omega \mathcal{H}(\sigma(x))dx \tag{4.19}$$

where $\mathcal{H}(\sigma) = H(|\sigma|)$ and H is taken independent of x only for simplicity. The inequality is justified by two facts: minimizing over all vector fields v with prescribed divergence gives a smaller result than minimizing over the vector fields θ_Q, and then we use $|\theta_Q| \leq i_Q$ and the fact that H is increasing.

We would like to understand if the two problems are equivalent.

Proposition 4.2.4 does the job: if we take a minimizer v for this minimal flow problem, then we are able to build a measure Q and, as we did in Theorem 4.2.5, the optimality of v gives $v = \theta_Q$ and $|\theta_Q| = i_Q$, thus proving that the minimal values are the same and that we can build a minimizer v from a minimizer Q (just take $v = \theta_Q$) and conversely a minimizer Q from v (use Proposition 4.2.4).

The connection between the two problems would be stronger should the Q that we build from v be somehow canonical and unique, instead of being obtained through an approximation and compactness argument. This means that we would like to have regularity results on the minimizer v, so that we can directly apply to it the construction by Dacorogna and Moser, without approximating and extracting a subsequence. Notice that, if H is strictly convex, the minimizer v is unique.

To be able to solve the Cauchy problem

$$\begin{cases} y_x'(t) = \tilde{v}(t, y_x(t)) \\ y_x(0) = x \end{cases},$$

with

$$\tilde{v}(t,x) = \frac{v(x)}{f_t(x)} \quad \text{where } f_t = (1-t)\mu + t\nu$$

one would need $\tilde{v}$ to be regular enough (say, Lipschitz continuous). Obviously, we can decide to add some assumptions on μ and ν, which will be supposed to be absolutely continuous with regular densities (at least Lipschitz continuous and bounded from below).

However, one needs to prove regularity for the optimal v, and for this one needs to look at the optimality conditions satisfied by v as a minimizer of (4.19). **PROOF OPTIMALITY** By duality, the solution of (4.19) is $v = \nabla\mathcal{H}^*(\nabla u)$ where $\mathcal{H}^*$ is the Legendre transform of $\mathcal{H}$ and u solves the PDE:

$$\begin{cases} \nabla\cdot(\nabla\mathcal{H}^*(\nabla u)) &= \mu_0 - \mu_1, & \text{in } \Omega, \\ \nabla\mathcal{H}^*(\nabla u)\cdot n_\Omega &= 0, & \text{on } \partial\Omega, \end{cases} \tag{4.20}$$

This equation turns out to be a standard Laplace equation if $\mathcal{H}$ is quadratic, or it becomes a $p-$Laplace equation for other power functions. In these cases, regularity results are well-known, under regularity assumptions on μ_0 and μ_1. Yet, let us recall that $H' = g$ where g is the congestion function, so it is natural to have $g(0) > 0$: the metric is positive even if there is no traffic! This means that the radial function $\mathcal{H}$ is not differentiable at 0 and then its subdifferential at 0 contains a ball. By duality, this implies $\nabla\mathcal{H}^* = 0$ on this ball which makes (4.20) very degenerate, even worse than the $p-$Laplacian. For instance, a reasonable model of congestion is $g(t) = 1 + t^{p-1}$ for $t \geq 0$, with $p > 1$, so that

$$\mathcal{H}(\sigma) = \frac{1}{p}|\sigma|^p + |\sigma|,\ \mathcal{H}^*(z) = \frac{1}{q}(|z|-1)_+^q,\ \text{with } q = \frac{p}{p-1} \tag{4.21}$$

so that the optimal σ is

$$\sigma = \Big(|\nabla u| - 1\Big)_+^{q-1}\frac{\nabla u}{|\nabla u|},$$

where u solves the very degenerate PDE:

$$\nabla\cdot\left(\Big(|\nabla u| - 1\Big)_+^{q-1}\frac{\nabla u}{|\nabla u|}\right) = \mu_0 - \mu_1, \tag{4.22}$$

with Neumann boundary condition

$$\left(|\nabla u| - 1\right)_+^{q-1} \frac{\nabla u}{|\nabla u|} \cdot n_\Omega = 0.$$

Note that there is no uniqueness for u but there is for v.

For this degenerate equation (more degenerate than the p-laplacian since the diffusion coefficient identically vanishes in the zone where $|\nabla u| \leq 1$), getting Lipschitz continuity on v is not reasonable. Yet, Sobolev regularity of v and Lipschitz regularity results for solutions of this PDE can be found in [34]. This enables one to build a flow *à la* DiPerna-Lions [64] and then to justify rigorously the construction above, even without a Cauchy-Lipschitz flow. Interestingly, recent continuity results are also available (see [102] in dimension 2, and then [55], with a different technique in arbitrary dimension), obtained as a consequence of a fine analysis of this degenerate elliptic PDE. Besides the interest for this regularity result in itself, we also stress that continuity for v implies continuity for the optimal i_Q, and this exactly gives the regularity which is required in the proof of Theorem 4.4.2 (the main difficulty being defining $c_{\bar{\xi}}$ for a non-continuous $\bar{\xi}$, and this is the reason why our proof in Section 3 is only formal).

4.4.2 Branched transport

Opposite from what we saw in the previous section about congested transport, in many other practical issues we would like to look for a way of transporting the mass so that it moves as much jointly as possible, favoring particles to share the same displacement instead of spreading all around the domain and using as many different paths as possible. This comes from a very different modelization, which is more suitable for other purposes than studying traffic congestion: supose for instance that you have to build the network system to transport the mass; in this case you do not want to build infinitely many small roads, each one meant to transport a unique particle from its starting point to its destination, but you prefer to build one unique bigger road. This is usually due to "economy of scale" principles, something that we can experience everyday (exactly as it happens for traffic congestion, but on different phenomena): the idea is that buy, or building, something bigger will cost more, but proportionally less. In particular costs are supposed to be sub-additive (the cost of the sum of two objects must be less than the sum of the two costs), and in many cases in economy they have "decreasing marginal costs" (i.e. the cost for adding a unit to a given background quantity is a decreasing function of the background, which means that the cost is actually concave).

Notice that modeling this kind of effects require, either in Lagrangian or Eulerian language, to look at the paths actually followed by each particles, and it could not be done with the only use of a transport plan $\gamma \in \Pi(\mu, \nu)$. But, once we choose the good formulation via the tools developed in this chapter, we ca guess the shape of the optimal solution for this kind of problem: particles are collected at some points, move together as much as possible, and then branch towards their different destinations. This is why this class of problems is nowadays known as "branched transport".

As we did for congested transport, we start from the discrete framework to give a presentation of the problem and then move to the continuous models. Notice that also in this case the discrete framework is somehow classical in optimization and operational research, and the continuous one is much more recent. Anyway, it has been investigated by different schools during the years 2000 and it precedes the study of congested transport.

The first model taking into account subadditive capacities for routes was proposed by Gilbert [75] (and then [76]), where it is presented as an extension of Steiner's minimal length problems. The main applications that Gilbert referred to were in the field of communication networks. Given two atomic probability measures $\mu = \sum_{i=1}^m a_i \delta_{x_i}$ and $\nu = \sum_{j=1}^n b_j \delta_{y_j}$, consider

$$(P_G) \qquad \min\ E(G) := \sum_h w_h^\alpha \mathcal{H}^1(e_h), \tag{4.23}$$

where the infimum is among all weighted oriented graphs $G = (e_h,\, \hat{e}_h,\, w_h)_h$ (where e_h are the edges, $\hat{e}_h$ represent their orientations and w_h the weights) satisfying Kirchhoff's Law: in each segment vertex which is not one of the x_i's or y_j's the total incoming mass equals the outcoming, while in each x_i we have

$$a_i + \text{ incoming mass } = \text{ outcoming mass}$$

and, conversely, in each y_j we have

$$\text{incoming mass } = \text{ outcoming mass} + b_j.$$

These conditions correspond exactly to the well known Kirchhoff Law for electric circuits. The orientations $\hat{e}_h$ do not appear in the energy E but appear in fact in Kirchhoff constraints. The exponent α is a fixed parameter $0 < \alpha < 1$ so that the function $t \mapsto t^\alpha$ is concave and subadditive. In this way larger links bringing the mass from μ to ν are preferred to several smaller links transporting the same total mass. It is not difficult to check that the energy of any finite graph may be improved if we remove cycles from the graph. In this way we can minimize among finite graphs which are actually trees. This implies a bound on the number of edges and hence ensure a suitable compactness which is enough to prove existence of a minimizer.

Lots of branching structures transporting different kind of fluids, such as road systems, communication networks, river basins, blood vessels, leaves and trees and so on, may be easily thought of as coming from a variational principle. They appear when transport costs encourage joint transportation. Recently these problems received a lot of attention by mathematicians, but in fact a mathematical formalization for them is very classical and has been performed first for atomic measures and then generalized. We briefly present here the problem introduced by Gilbert in [75] and [76], where it is presented as an extension of Steiner's minimal length problems. The main applications that Gilbert referred to were in the field of communication networks and the energy to be minimized represents the costs for building the network.

Given two finitely atomic probability measures $\mu = \sum_{i=1}^{m} a_i \delta_{x_i}$ and $\nu = \sum_{j=1}^{n} b_j \delta_{y_j}$, consider

$$(P_G) \qquad \min\ E(G) := \sum_h w_h^\alpha \mathcal{H}^1(e_h), \tag{4.24}$$

where the infimum is among all weighted oriented graphs $G = (e_h,\, \hat{e}_h,\, w_h)_h$ (where e_h are the edges, $\hat{e}_h$ represent their orientations and w_h the weights) satisfying Kirchhoff's Law: in each segment vertex which is not one of the x_i's or y_j's the total incoming mass equals the outcoming, while in each x_i we have

$$a_i + \text{ incoming mass } = \text{ outcoming mass}$$

and, conversely, in each y_j we have

$$\text{incoming mass } = \text{ outcoming mass} + b_j.$$

These conditions correspond exactly to the well known Kirchhoff Law for electric circuits. The orientations $\hat{e}_h$ do not appear in the energy E but appear in fact in Kirchhoff constraints. The exponent α is a fixed parameter $0 < \alpha < 1$ so that the function $t \mapsto t^\alpha$ is concave and subadditive. In this way larger links bringing the mass from μ to ν are preferred to several smaller links transporting the same total mass. It is not difficult to check that the energy of any finite graph may be improved if we remove cycles from the graph. In this way we can minimize among finite graphs which are actually trees. This implies a bound on the number of edges and hence ensures a suitable compactness which is enough to prove existence of a minimizer.

More recently Xia, in [107], has proposed a new formalization leading to generalizations of this problem to arbitrary probability measures μ and ν. In this case the interest of the author of [107] is to view this problem as an extension of Monge-Kantorovich optimal transport theory. Actually Steiner and Monge's problems represent the limit cases $\alpha = 0$ and $\alpha = 1$, respectively.

Let us briefly see how Xia extended to the continuous case the discrete irrigation model proposed by Gilbert. The key point is formalizing the problem by using measures (or currents), since the constraint on the incoming and outcoming masses in each vertex (Kirchoff Law) may be easily written as $\nabla \cdot \lambda_G = \mu - \nu$, where $\lambda_G = \sum_h w_h [[e_h]]$ is a vector measure ($[[e]]$ being the integration measure measure on the segment e: $[[e]] = \hat{e} \cdot \mathcal{H}^1 \llcorner e$). This consideration lead Xia in [107] to extend the problem by relaxation to generic probabilities μ and ν. The problems becomes

$$(P_X) \qquad\qquad \min\ \bar{E}(\lambda):\ \ \nabla \cdot \lambda = \mu - \nu$$

where

$$\bar{E}(\lambda) := \inf \left\{ \liminf_n E(\lambda_{G_n}) \ :\ G_n \text{ are finite graphs and } \lambda_{G_n} \rightharpoonup \lambda \right\}.$$

It is also possible to prove a representation formula for the relaxed energy $\bar{E}$: we have

$$\bar{E}(\lambda) = \begin{cases} \int_M \theta^\alpha \, d\mathcal{H}^1, & \text{if } \lambda = (M, \theta, \xi), \\ +\infty & \text{otherwise,} \end{cases} \tag{4.25}$$

where the equality $\lambda = (M, \theta, \xi)$ means that M is a $1-$rectifiable set, θ a real multiplicity, ξ a measurable unit vector field on M tangent to M itself and λ is the vector measure $\theta\xi \cdot \mathcal{H}^1 \llcorner M$.

Notice that (P_X) means minimizing an energy $\bar{E}$ under a divergence constraint, exactly as in the minimal flow problem (Proposition **??**). The difference is that, instead of minimizing the total mass of the vector measure whose divergence is prescribed, we minimize what is sometimes called its $\alpha-$mass M^α (see [**?**] and [**?**]).

It should be proven that, when μ and ν are both actually atomic measures, we retrieve the problem by Gilbert. This is not trivial, as we admitted lots of new competitors. Moreover, as our relaxation process did not keep fixed the marginal measures μ and ν, it is not even a priori clear that the infimum value has not changed. To deal with this problem we need some necessary optimality conditions: we would like to state that, once we minimize over vector measures Xia's functional, if μ and ν are themselves finitely atomic, then any minimizer must actually be a finite graph. The problem of regularity is addressed to in [?] and [?], but here we will not be concerned with it.

Another non trivial issue is understanding when the minimum value, which is always finite in the discrete case, is finite in the general case. This leads to some conditions on α and the measures μ and ν. We will resume them in next section.

This section is an informal summary of the models in [?] and [23] and their properties. Languages and approaches have been sometimes simplified to present them in a more concise way.

Let Ω be a fixed domain in $\mathbb{R}^d$. Let us denote by Γ the set of $1-$Lipschitz curves $\gamma : [0, +\infty[\to \Omega$ that are eventually constant. It means that, if we define the stopping time of a curve γ by

$$\sigma(\gamma) = \inf \{s \,:\, \gamma \text{ is constant on } [s, +\infty[\} \,,$$

these are curves with $\sigma(\gamma) < +\infty$. Let us also denote by Γ_{arc} the set of those curves in Γ which are parametrized by arc length and by Γ_{inj} the set of curves in Γ which are injective on $[0, \sigma(\gamma)[$. In the sequel we will often identify a curve with its image, in the sense that sometimes we will write γ instead of $\gamma([0, \sigma(\gamma)]) = \gamma([0, +\infty[)$.

Given a probability measure η on the space Γ, for any point $x \in \mathbb{R}^d$ the $\eta-$multiplicity of x is defined by

$$[x]_\eta := \eta \{\gamma \in \Gamma \,:\, x \in \gamma([0, \sigma(\gamma)])\} \,. \tag{4.26}$$

Then we can define

$$Z_\eta(\gamma) = \int_0^{\sigma(\gamma)} [\gamma(t)]_\eta^{\alpha-1} dt \quad \text{and} \quad J(\eta) = \int_\Gamma Z_\eta \, d\eta. \tag{4.27}$$

Notice that, for simplicity, here Z_η is defined without the term $|\gamma'|(t)$ which appears in the original definition in [23]. As a consequence, it will be deduced later that minimizers are actually parametrized by arc length.

Finally, we consider the maps π_0, $\pi_\infty : \Gamma \to \Omega$, given by $\pi_0(\gamma) = \gamma(0)$, and $\pi_\infty(\gamma) = \gamma(\sigma(\gamma))$. The two image measures $(\pi_0)_\sharp\eta$ and $(\pi_\infty)_\sharp\eta$, which belong to $\mathcal{P}(\Omega)$, will be called the starting and the terminal measure of η, respectively. Following the notation of [23] we may define a *traffic plan* as a measure $\eta \in \mathcal{P}(\Gamma)$ such that $\int_\Gamma \sigma(\gamma)\eta(d\gamma) < +\infty$. We will also call *pattern* a traffic plan η such that $(\pi_0)_\sharp\eta = \delta_0$. In the case of a pattern the terminal measure will also be called the measure irrigated by η

The minimization problem proposed in [23] is

$$(P) \quad \min \quad J(\eta) \ : \ \eta \text{ is a traffic plan}, (\pi_\infty)_\sharp \eta = \mu,\ (\pi_0)_\sharp \eta = \nu,$$

where μ and ν are given measures in $\mathcal{P}(\Omega)$. We also denote the set of admissible traffic plans by $TP(\nu, \mu)$. As $[\gamma(t)]_\eta \leq 1$, we have $Z_\eta(\gamma) \geq \sigma(\gamma)$. Hence it is straightforward that any η such that $J(\eta) < +\infty$ is actually a traffic plan.

Definition 15. A traffic plan η which minimizes J among all the traffic plans with the same starting and terminal measures, with $J(\eta) < +\infty$, will be called an *optimal traffic plan.* In the case $\nu = \delta_0$ it will be called *optimal pattern.*

A useful tool developed in [23] (see also [22]) is the following: if η is concentrated on $\Gamma_{arc} \cap \Gamma_{inj}$ then the following remarkable formula holds:

$$J(\eta) = \int_{\mathbb{R}^d} [x]_\eta^\alpha \, \mathcal{H}^1(dx). \tag{4.28}$$

This formula gives an evident link with Gilbert and Xia's models.

In the next chapter we will mainly deal with the problem of optimal patterns, i.e. with the case $\nu = \delta_0$. This problem requires some extra tools and concepts that we will present in a while. Before that, let us introduce another concept which is very typical of the general traffic plan case.

Definition 16. A curve $\gamma_0 : [s_0, t_0] \to \Omega$ is said to be an *arc of* η if

$$\eta\left(\{\gamma \in \Gamma \ : \ \gamma_0([s_0, t_0]) \subset \gamma\}\right) > 0.$$

We move now to the concepts we need to specifically deal with the case $\nu = \delta_0$.

For any $t \geq 0$ consider an equivalence relation on Γ given by "the two curves γ_1 and γ_2 are in relation at time t if they agree on the interval $[0, t]$", and denote the equivalence classes by $[\cdot]_t$, so that

$$[\gamma]_t = \{\tilde{\gamma} \ : \ \tilde{\gamma}(s) = \gamma(s) \text{ for any } s \leq t\}.$$

For notational simplicity, let us set $|\gamma|_{t,\eta} := \eta([\gamma]_t)$.

Definition 17. Given $\eta \in \mathcal{P}(\Gamma)$, a curve $\gamma \in \Gamma$ is said to be $\eta-$good if

$$Z_\eta^0(\gamma) := \int_0^{\sigma(\gamma)} |\gamma|_{t,\eta}^{\alpha-1} dt < +\infty.$$

Remark 18. When $\nu = \delta_0$, the problem of minimizing the functional J^0 given by $J^0(\eta) = \int_\Gamma Z^0_\eta d\eta$, is exactly the problem adressed in [**?**]. Its equivalence with the traffic plan model we are presenting here, proposed in [23], is proven in [**?**] and in [84] and relies on optimality conditions.

Remark 19. Other intermediate models may be introduced, all differing in the definition of the multiplicity of the curve γ at time t. See for instance [**?**] or [84].

Here are now the most important optimality results that can be found in [**?**], [23], [22], [**?**] and [84] or easily deduced from them.

1. Problem (P) admits a solution, provided the infimum is finite (i.e. there is at least a solution with finite energy).

2. If η is an optimal traffic plan, then η is concentrated on $\Gamma_{arc} \cap \Gamma_{inj}$. In particular, we may apply formula (4.28) for J.

3. Suppose that η is an optimal traffic plan, that two curves $\gamma_0, \gamma_1 \in \Gamma_{arc} \cap \Gamma_{inj}$ meet twice (i.e. $\gamma_0(s_0) = \gamma_1(s_1)$, $\gamma_0(t_0) = \gamma_1(t_1)$ and $s_i \neq t_i$) and that γ_0 on the interval $[s_0, t_0]$ ia an arc of η. Then either both curves coincide in the trajectory between the two common points or we have $\int_{s_0}^{t_0} [\gamma_0(t)]_\eta^{\alpha-1} dt < \int_{s_1}^{t_1} [\gamma_1(t)]_\eta^{\alpha-1} dt$. In particular two different arcs of η cannot part and then meet again.

4. If η is an optimal pattern (in particular $\nu = \delta_0$), then for $\eta-$a.e. curve γ and a.e. $t < \sigma(\gamma)$ we have $[\gamma(t)]_\eta = |\gamma|_{t,\eta}$. Roughly speaking this means that if all the mass starts from a common point then there is no parting-and-meeting-again-later (this is the single path property described in [**?**].

5. As a consequence, any optimal pattern η is concentrated on the set of $\eta-$good curves, and any $\eta-$good curve γ belongs to $\Gamma_{arc} \cap \Gamma_{inj}$ and satisfies $[\gamma(t)]_\eta = \eta([\gamma]_t)$ for any $t < \sigma(\gamma)$.

6. Last but not least $\min(P) = \min(P_X)$, which means that the minima of the Lagrangian and of the Eulerian model coincide.

For the whole set of equivalences between the different models, see [**?**].

Remark 20. Notice that an optimal traffic plan η is concentrated on the set of $\eta-$good curves, but this does not mean that this set is linked to the support of η. In fact any restriction of an $\eta-$good curve is itself an $\eta-$good curve and hence, for instance, in the discrete case, we have plenty of $\eta-$good curves but the support of η is finite. In particular the set of $\eta-$good curves may be very different from the set of fibers of a traffic plan that we find in [23] or [**?**] and does not depend on any parametrization χ, but it is more intrinsic.

Remark 21. These Lagrangian models may be useful to understand differences and similarities with the concentration case of Chapter 4. In fact it is easy to realize that, even in simple cases such as discrete ones, the way the two models combine length and masses are different. In fact, in the case where some masses $(m_i)_i$ are transported each one on a segment whose length is l_i, in the Xia (or traffic plan or pattern) model the cost is $\sum_i m_i^\alpha l_i$ while in the concentration case of Chapter 4 is $\left(\sum_i m_i^\alpha\right)\left(\sum_i m_i l_i^p\right)^{1/p}$. But the situation changes if we take $p = \infty$ and this is the reason why we insisted on the case of the space $\mathcal{W}_\infty$ in Section 0.2 and in Chapter 4. In fact, if we take a Lipschitz curve μ in $\mathcal{W}_\infty$ (and we can think a $1-$Lipschitz curve up to reparameterization on a different interval), in analogy to Theorem **??**, we may think that there is a velocity field v with $||v|| \leq 1$ and that there is a measure η on Γ (concentrated on solutions of the ODE associated to the vector field v, i.e. on $1-$Lipschitz curves) such that $\mu_t = (\pi_t)_\sharp \eta$ (this is suggested by some results in [**?**], but it has to be proven). For simplicty let us have a look at the pattern case, i.e. $\mu_0 = \delta_0$. In terms of η the two models give a cost at time t which is $\int_\Gamma |\gamma|_{t,\eta}^{\alpha-1} I_{t<\sigma(\gamma)}\eta(d\gamma) = \sum_{i\in I(t)} m_i^\alpha$ for one and $G_\alpha(\mu_t) = \sum_i m_i^\alpha$ for the other. Here $m_i = \eta([\gamma_i]_t)$ and the curves γ_i are representatives of the equivalence classes of time t, the set $I(t)$ denoting those indexes such that the corresponding classes have not yet stopped. Due to the optimality condition 3 these masses correspond to the masses of the atoms of μ_t (in the sense that two $\eta-$good curves arrive at time t at the same point if and only if they have stayed together from time 0). This shows that the only difference between the two models is the fact that in the model concerning curves in $\mathcal{W}_\infty$ we take into account in the cost also the masses that have stopped. This is in fact the main difference, which is due to the fact that the cost at time t is chosen to depend only on the configuration of masses at time t. It is the price to be paid, having a less accurate and less realistic model, in order to have it mathematically simpler (as a particular case of an abstract geodesic problem).

The minimum value of (P_X) (or of (P)), which obviously depends on μ and ν, will be denoted by $d_\alpha(\mu,\nu)$. About its finiteness, there are results on α ensuring $d_\alpha(\mu,\nu) < +\infty$ for any pair of probabilities (μ,ν) and results concerning the two measures as well, and in particular sort of their dimension.

We know that in the case $\alpha = 1$ any pair of compactly supported measures may be linked with finite energy, because we are actually facing the Monge-Kantorovich problem. It is proven in [107] that, when α is sufficiently close to 1, namely $\alpha > 1 - 1/d$, the minimum stays finite for any pair (μ, ν). This is obtained by means of a dyadic construction which is very similar to the one we did in Chapter 4 (actually, its our construction which is very similar to the one performed by Xia). Moreover the following uniform estimate (see [107]) holds

$$d_\alpha(\mu, \nu) \le C_{\alpha,d} \operatorname{diam}(\Omega). \tag{4.29}$$

It is not difficult to extend the whole model to the case of finite measures instead of probabilities, thus getting, when μ and ν are two measures with the same mass m,

$$d_\alpha(\mu, \nu) \le C_{\alpha,d} m^\alpha \operatorname{diam}(\Omega). \tag{4.30}$$

From (4.30) and the fact that the distance d_α depends only on $\mu - \nu$ we can deduce a sharper estimate which refines (4.29), namely

$$d_\alpha(\mu, \nu) \le C_{\alpha,d}\, \delta^\alpha \operatorname{diam}(\omega), \tag{4.31}$$

whenever $\mu - \nu = \delta(\mu' - \nu')$ and μ' and ν' are probability measures on $\omega \subset \Omega$ (i.e. we have taken into account the possibility that the two measures differ only on a small set and the mass of the difference is small).

In dimension one this means that for $\alpha > 0$ there is finiteness of the minimum. For $\alpha = 0$ the problem reduces to a length minimization and in the particular case of $d = 1$ this has always a finite solution.

In larger dimensions, however, when α is below this threshold there are pairs of measures which are not linkable by a finite energy configuration. Since in order to link μ to ν and estimate $d_\alpha(\mu, \nu)$, we can always decide to link μ to δ_0 and then δ_0 to ν, we will give the following definition.

Definition 18. A measure μ is called $\alpha-$*irrigable* if $d_\alpha(\mu, \delta_0) < +\infty$. The quantity $d_\alpha(\mu, \delta_0)$ will also be denoted by $X_\alpha(\mu)$.

In the case $d > 1$ and $\alpha < 1 - 1/d$, for a measure μ being $\alpha-$irrigable is a fact somehow linked to its "dimension". The proofs are in [**?**] and [**?**] and give both irrigability and non-irrigability results. In view of the fact that, for lots of applications, it is very interesting to deal with the case of the Lebesgue measure on Ω, we will here presents only the results which are relevant for such a case.

Proposition 4.4.3. *If μ is $\alpha-$irrigable, then μ is concentrated on a set which is $\mathcal{H}^{d(\alpha)}-$negligible, where $d(\alpha) = 1/(1-\alpha)$. In particular the Lebesgue measure is not $\alpha-$irrigable for $\alpha \leq 1 - 1/d$.*

We do not provide here the complete proof of this fact, but we want to give a proof of the fact that a measure whose density with respect to the Lebesgue measure is bounded away from zero may not be irrigated for $\alpha < 1 - 1/d$. It is consequently a very weak result, as it requires the strict inequality on α and very strong assumptions on the measure, but it has the advantage of using only the formulation of the problem given by Xia. This proof comes from some conversations with P. Tilli.

Theorem 4.4.4. *Suppose $\alpha < 1 - 1/d$ and that $\mu \in \mathcal{P}(\Omega)$ is such that $\mu(Q) \geq c|Q|$ for a certain $c > 0$ and any cube $Q \subset \Omega$. Then μ is not $\alpha-$irrigable.*

Proof. Let us divide Ω into small cubes Q_i of side ε, thus having approximatively $C\varepsilon^{-d}$ cubes. Inside any cube we place a subcube Q_i', with side $c\varepsilon$ ($c < 1$). We fix now two sequence of discrete probability measure μ_n and ν_n, converging to μ and δ_0 respectively, such that $d_\alpha(\mu, \nu) = \liminf_n d_\alpha(\mu_n, \nu_n)$. Once we fix the sequence and the cubes, we will eventually have $\mu_n(Q_i') \geq C_1\varepsilon^d$ and $\nu_n(Q_i) \leq C_2\varepsilon^d$, for $C_1 > C_2$ and any index i up to the one for which we have $0 \in Q_i$. Hence we may deduce that, in the optimal discrete graph linking μ_n to ν_n, for all the indexes i but one, there should be at least a mass $(C_2 - C_1)\varepsilon^d$ passing through the region $Q_i \setminus Q_i'$. Since the distance to be covered is at least $(1 - c)\varepsilon$, the energy of the part of the graph contained in $Q_i \setminus Q_i'$ must be at least $C\varepsilon^{1+d\alpha}$. The total energy is hence at least $C\varepsilon^{1+d(\alpha-1)}$. We can deduce $d_\alpha(\mu, \nu) \geq C\varepsilon^{1+d(\alpha-1)}$ and, being ε arbitrary and $1 + d(\alpha - 1) < 0$, we get $d_\alpha(\mu, \nu) = +\infty$. □

Remark 22. In the previous proof, in the case $\alpha = 1-1/d$ we could not get the result. Anyway, notice that the energy has been hugely underestimated, as a consequence of the fact that in any cube Q_i only the contribution of the mass coming from Q_i' has been considered, while for most of the cubes this could be negligible with respect to the mass arriving from other cubes.

Remark 23. Notice that the threshold $1 - 1/d$ is the same which appears in Chapter 4 for the concentration case.

In [107] it is proven that, for $\alpha > 1 - 1/d$, the quantity d_α defines a new distance over the space of probability measures $\mathcal{P}(\Omega)$, which induces the weak topology and endows $\mathcal{P}(\Omega)$ with a structure of length space.

It is natural, as the branching transport problem (P_X) comes from a variant of Monge's problem, to compare the distance arising here (d_α) and the one coming from Monge-Kantorovich theory (W_1). As far as now we know that the two distances induce the same topology on $\mathcal{P}(\Omega)$, which is the same induced by the weak convergence, and it is easily checked ([107]) that $W_1 \leq d_\alpha$. The purpose of this Section is to give a sharp quantitative estimate of the kind $d_\alpha \leq C(W_1)^\beta$. This question was raised as a conjecture by Cedric Villani while reviewing the PhD Thesis [22]. Such an inequality would give an a priori estimate on d^α which is, by the way, numerically relevant. Indeed W_1 is much easier to compute by linear programming than d_α, which involves a non-convex optimization problem.

Three continuous extensions of the Gilbert-Steiner problem In analogy with the Monge-Kantorovich problem, the discrete Gilbert-Steiner model has been recently set in a continuous framework where the wells and sources are arbitrary measures, instead of a finite sum of Dirac masses. There were at least three approaches to this generalization, which we shall review briefly.

Xia's relaxation

Xia, in [107], has proposed a new formalization leading to generalizations of this problem to arbitrary probability measures μ and ν.

In fact Steiner and Monge's problems represent the limit cases $\alpha = 0$ and $\alpha = 1$, respectively.

The important modelization idea by Xia is that if one looks at Gilbert's problem with $\mu = \sum_{i=1}^m a_i \delta_{x_i}$ and $\nu = \sum_{j=1}^n b_j \delta_{y_j}$, then any irrigation graph G, which - we recall it - is a weighted directed graph satisfying Kirchhoff's law, can be identified with a vector measure

$$G = \sum_{e \in E(G)} w(e) \mathcal{H}^1|_e \vec{e} \tag{4.32}$$

where $\vec{e}$ denotes the unit vector in the direction of e and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. It turns out that G satisfies Kirchhoff's law if and only if

$$\operatorname{div} G = \mu - \nu \tag{4.33}$$

in the distributional sense.

If we take now μ, ν two probability measures on a domain $\Omega \subset \mathbb{R}^N$, a vector measure u on Ω with values in $\mathbb{R}^N$ is called by Xia [**?**] a *transport path* from μ^+ to μ^- if there exist two sequences μ_i^-, μ_i^+ of finite atomic measures with equal mass and a sequence of finite graphs G_i irrigating (μ_i^+, μ_i^-) such that $\mu_i^+ \to \mu^+$, $\mu_i^- \to \mu^-$ as measures and $G_i \to T$ as vector measures. The energy of T is defined by

$$M^\alpha(T) := \inf \liminf_{i\to\infty} M^\alpha(G_i)$$

where the infimum is taken over the set of all possible approximating sequences $\{\mu_i^+, \mu_i^-, G_i\}$ to T. Denote

$$M^\alpha(\mu^+, \mu^-) := \inf\{M^\alpha(T) : T \text{ is a transport path from } \mu^+ \text{ to } \mu^-\}.$$

If $\alpha \in (1 - \frac{1}{N}, 1]$, by Theorem 3.1 in [**?**], the above infimum is finite and attained for any pair (μ^+, μ^-). Xia showed or conjectured in a series of papers several structure and regularity properties of optimal transport paths which we shall comment later on.

Maddalena-Solimini's patterns

Maddalena and Solimini [85] gave a different (Lagrangian) formulation in the case of a single source supply $\mu^+ = \delta_S$. They model the transportation network as a set of particle trajectories, or "fibers", $\chi(\omega, \cdot)$, where $\chi(\omega, t) \in \mathbb{R}^N$ represents the location of a particle $\omega \in \Omega$ at time t and $\chi(\omega, 0) = S$. The set Ω is an abstract probability space indexing all fibers ; it is endowed with a measure $|\cdot|$ (without loss of generality one could take $\Omega = [0, 1]$ endowed with the Lebesgue measure). All the fibers are required to stop at some time $T(\omega)$ and to satisfy $\chi(\omega, 0) = S$ for all ω, i.e. all fibers start at the same root S. The set of fibers is given a structure corresponding to the intuitive notion of branches. Two fibers ω and ω' belong to the same branch at time t if $\chi(\omega, s) = \chi(\omega', s)$ for $s \leq t$. Then the partition of Ω given by the branches at time t yields a time filtration. The branch of ω at time t is denoted by $[\omega]_t$ and its measure by $|[\omega]_t|$. The energy of the set of fibers, or "irrigation pattern" is defined by

$$\tilde{E}^\alpha(\chi) = \int_\Omega \int_0^{T(\omega)} |[\omega]_t|^{\alpha-1} d\omega dt$$

It is easily checked on discrete trees that this definition extends the Gilbert energy (**??**). The measure μ^- irrigated by a pattern is easily defined. For every Borel set A in $\mathbb{R}^N$, $\mu^-(A)$ is the measure of the set of fibers stopping in A, $\mu^-(A) = |\{\omega,\ \chi(\omega, T(\omega)) \in A\}|$.

Traffic plans

In [?] the pattern formalism was extended to the case where the source is any Radon measure. The authors of [?] called "traffic plan" any probability measure on the set of Lipschitz paths. The equivalence of all models is proven in [84] and [?]. More precisely:

1. When the irrigated measures μ^+ and μ^- are finite atomic, the traffic plan minimizers are the same as the Gilbert finite graph minimizers.

2. For two general probability measures μ^+ and μ^-, Xia's minimizers are also optimal traffic plans and conversely.

3. When $\mu^+ = \delta_S$ is a single source, optimal patterns and optimal traffic plans are equivalent notions.

Throughout the paper we shall refer to the formalism of traffic plans which is the slight extension of the pattern formalism as explained above. The next section formalizes all definitions and recalls all properties we shall need in the sequel. They refer mainly to [?], [?], [?], [85]. The used formalism and the form given to statements follow [?], [?] and [?].

Chapter 5

Wasserstein distances and curves in the Wasserstein spaces

5.1 Definition and triangle inequality

First, for $\Omega \subset \mathbb{R}^n$ and $p \geq 1$, let us set

$$\mathcal{P}_p(\Omega) := \{\mu \in \mathcal{P}(\Omega) \, : \, \int |x|^p d\mu < +\infty\}.$$

This subset of $\mathcal{P}(\Omega)$ will be the space where we define our distances. Obviously, if Ω is bounded then $\mathcal{P}_p(\Omega) = \mathcal{P}(\Omega)$.

For $\mu, \nu \in \mathcal{P}_p(\Omega)$, let us define

$$W_p(\mu,\nu) := \inf \left\{ \int |x-y|^p d\gamma \, \gamma \in \Pi(\mu,\nu) \right\}^{1/p},$$

i.e. the $p-$th root of the minimal transport cost for the cost $|x-y|^p$. The assumption $\mu, \nu \in \mathcal{P}_p(\Omega)$ guarantees finiteness of this value, since $|x-y|^p \leq C(|x|^p + |y|^p)$ and hence $W_p(\mu,\nu)^p \leq C(\int |x|^p d\mu + \int |x|^p d\nu)$.

Notice that, due to Jensen inequality, since for any $\gamma \in \Pi(\mu,\nu)$ we have $\gamma(\Omega \times \Omega) = 1$, for $p \leq q$ we can infer

$$\left(\int |x-y|^p d\gamma \right)^{1/p} = ||x-y||_{L^p(\gamma)} \leq ||x-y||_{L^q(\gamma)} = \left(\int |x-y|^q d\gamma \right)^{1/q},$$

which implies $W_p(\mu,\nu) \leq W_q(\mu,\nu)$. In particular $W_1(\mu,\nu) \leq W_p(\mu,\nu)$ for every $p \geq 1$. We will not define here W_∞ (as a limit for $p \to \infty$, or, which is the same, as the minimal value of the supremal problem $\min_{\gamma\in\Pi(\mu,\nu)} ||x-y||_{L^\infty(\gamma)})$.

On the other hand, for bounded Ω an opposite inequality holds, since

$$\left(\int |x-y|^p d\gamma\right)^{1/p} \le \operatorname{diam}(\Omega)^{\frac{p}{p-1}} \left(\int |x-y|\, d\gamma\right)^{1/p},$$

which implies $W_p(\mu,\nu) \le C W_1(\mu,\nu)^{1/p}$, for $C = \operatorname{diam}(\Omega)^{p'}$ and $p' = \frac{p}{p-1}$.

Proposition 5.1.1. *The quantity W_p defined above is actually a distance over $\mathcal{P}_p(\Omega)$.*

Proof. First, let us notice that $W_p \ge 0$. Then, we also notice that $W_p(\mu,\nu) = 0$ implies, as a consequence that the minimum in the definition of W_p is attained, that there exists $\gamma \in \pi(\mu,\nu)$ such that $\int |x-y|^p d\gamma = 0$, which means that γ is concentrated on $\{x = y\}$. This implies $\mu = \nu$ since, for any test function ϕ we have

$$\int \phi\, d\mu = \int \phi(x) d\gamma = \int \phi(y) d\gamma = \int \phi\, d\nu.$$

We need now to prove the triangle inequality. For that, we give two different proofs. The first (Lemma **??**) is easier and uses transport maps and approximation; the second (Lemma 5.1.4) is more general but requires a trickier object, i.e. disintegrations of measures. □

Lemma 5.1.2. *Given $\mu,\nu \in \mathcal{P}_p(\mathbb{R}^n)$ and χ_ε any usual regularizing kernel in L^1 with $\int \chi_\varepsilon = 1$, we have*

$$\lim_{\varepsilon\to 0} W_p(\mu * \chi_\varepsilon, \nu * \chi_\varepsilon) = W_p(\mu,\nu).$$

Proof. Take an optimal transport plan $\gamma \in \Pi(\mu,\nu)$ and define a measure $\gamma_\varepsilon \in \Pi(\mu * \chi_\varepsilon, \nu)$ through

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \psi(x,y) d\gamma_\varepsilon := \int_{\mathbb{R}^n\times\mathbb{R}^n}\int_{\mathbb{R}^n} \psi(x+z, y+z)\chi_\varepsilon(z) dz\, d\gamma(x,y).$$

We need to check that its marginals are actually $\mu * \chi_\varepsilon$ and $\nu * \chi_\varepsilon$. For that just consider

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n} \psi(x) d\gamma_\varepsilon &= \int_{\mathbb{R}^n\times\mathbb{R}^n}\int_{\mathbb{R}^n} \psi(x-z)\chi_\varepsilon(z) dz\, d\gamma(x,y) = \int_{\mathbb{R}^n} dz \int_{\mathbb{R}^n\times\mathbb{R}^n} \psi(x-z) d\gamma(x,y)\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\psi * \chi_\varepsilon)(x) d\mu(x) = \int \psi\, d\mu * \chi_\varepsilon.\end{aligned}$$

The computation for the second marginal is the same. It is then easy to show that $\int |x-y|^p d\gamma_\varepsilon = \int |x-y|^p d\gamma$, since

$$\int |x-y|^p d\gamma_\varepsilon = \int_{\mathbb{R}^n\times\mathbb{R}^n}\int_{\mathbb{R}^n} |(x+z)-(y+z)|\chi_\varepsilon(z)dz\, d\gamma(x,y) = \int |x-y|^p d\gamma.$$

This shows

$$\limsup_{\varepsilon\to 0} W_p(\mu*\chi_\varepsilon,\nu)^p \leq \int |x-y|^p d\gamma_\varepsilon = \int |x-y|^p d\gamma.$$

One cans also obtain the opposite inequality with a standard semicontinuity argument. Consider $\mu^\varepsilon := \mu*\chi_\varepsilon$ and $\nu^\varepsilon := \nu*\chi_\varepsilon$, and use only the weak convergences $\mu_\varepsilon \rightharpoonup \mu$ and $\nu_\varepsilon \rightharpoonup \nu$. The sequence γ_ε of optimal plans in $\Pi(\mu_\varepsilon,\nu_\varepsilon)$ is tight, since its marginals are tight (look at the proof of Theorem 1.1.4). First fix a sequence $\varepsilon_k\to 0$ such that $\lim_k W_p(\mu*\chi_{\varepsilon_k},\nu) = \liminf_{\varepsilon\to 0} W_p(\mu*\chi_\varepsilon,\nu)$. Then extract a subsequence ε_{k_j} so as to guarantee that the optimal transport plans $\gamma^{\varepsilon_{k_j}}$ have a weak limit γ^0. This weak limit must belong to $\Pi(\mu,\nu)$ (the fact that the marginals of γ_0 are μ and ν follows by the properties of composition with continuous functions of the weak convergence). Then we have

$$W_p(\mu,\nu)^p \leq \int |x-y|^p d\gamma^0 \leq \liminf_j \int |x-y|^p d\gamma^{\varepsilon_{k_j}} = \liminf_{\varepsilon\to 0} W_p(\mu_\varepsilon,\nu_\varepsilon),$$

where the first inequality follows from the fact that γ^0 is not necessarily optimal but is admissible and the second by semicontinuity (Lemma 1.1.3). $\square$

Then, we can perform a proof of the triangle inequality based on the use of optimal transport maps.

Lemma 5.1.3. *triangle with maps The quantity W_p satisfies the triangle inequality (with transport maps and approximation).*

Proof. First consider the case where μ and ρ are absolutely continuous and ν is arbitrary. Let T be the optimal transport from μ to ρ and S from ρ to ν. Then $S\circ T$ is an admissible transport from μ to ν, since $(S\circ T)_\#\mu = S_\#(T_\#\mu) = S_\#\rho = \nu$. Then we have

$$W_p(\mu,\nu) \leq \left(\int |S(T(x))-x|^p d\mu\right)^{1/p} = ||S\circ T - id||_{L^p(\mu)} \leq ||S\circ T - T||_{L^p(\mu)} + ||T-id||_{L^p(\mu)}.$$

Yet,

$$||S\circ T - T||_{L^p(\mu)} = \left(\int |S(T(x)) - T(x)|^p d\mu\right)^{1/p} = \left(\int |S(y) - y|^p d\rho\right)^{1/p} = W_p(\rho,\nu)$$

and $||T - id||_{L^p(\mu)} = W_p(\mu,\rho)$, hence

$$W_p(\mu,\nu) \leq W_p(\mu,\rho) + W_p(\rho,\nu).$$

This gives the proof when $\mu, \rho \ll \mathcal{L}^d$. For the general case, first write the triangle inequality for $\mu * \chi_\varepsilon$, $\rho * \chi_\varepsilon$ and $\nu * \chi_\varepsilon$, then pass to the limit as $\varepsilon \to 0$ using Lemma 5.1.2. □

The proof above strongly uses the results about transport maps from Chapter 1, and is somehow specific to $\mathbb{R}^n$ (in a general metric space the tricky part would be to approximate arbitrary measures with measures such that the Wasserstein distance can be computed with maps instead of plans; absolutely continuous measures are not necessary to use, but we need non-atomic ones). To give a more general result, we provide a different proof, which may happen to be more difficult for the reader who is not accustomed to disintegrations of measures (see Chapter 2).

Lemma 5.1.4. *The quantity W_p satisfies the triangle inequality (with disintegrations).*

Proof. Let us take μ, ρ and $\nu \in \mathcal{P}_p(\Omega)$, $\gamma^+ \in \Pi(\mu,\rho)$ and $\gamma^- \in \Pi(\mu,\rho)$. We can also choose $\gamma^\pm$ to be optimal. Let us use the Lemma 5.1.5 below to say that there exists a measure $\sigma\mathcal{P}(\Omega\times\Omega\times\Omega)$ such that $(\pi_{x,y})_\#\sigma = \gamma^+$ and $(\pi_{y,z})_\#\sigma = \gamma^-$, where $\pi_{x,y}$ and $\pi_{y,z}$ denote the projections on the two first and two last variables, respectively. Let us take $\gamma := (\pi_{x,z})_\#\sigma$. By composition of the projections, it is easy to see that $(\pi_x)_\#\gamma = (\pi_x)_\#\sigma = (\pi_x)_\#\gamma^+ = \mu$ and, analogously, $(\pi_z)_\#\gamma = \nu$. This means $\gamma \in \Pi(\mu,\nu)$ and

$$\begin{aligned}
W_p(\mu,\nu) &\leq \left(\int |x-z|^p d\gamma\right)^{1/p} = \left(\int |x-z|^p d\sigma\right)^{1/p} = ||x-z||_{L^p(\sigma)} \\
&\leq ||x-y||_{L^p(\sigma)} + ||y-z||_{L^p(\sigma)} = \left(\int |x-z|^p d\sigma\right)^{1/p} + \left(\int |x-z|^p d\sigma\right)^{1/p} \\
&= \left(\int |x-z|^p d\gamma^+\right)^{1/p} + \left(\int |x-z|^p d\gamma^-\right)^{1/p} = W_p(\mu,\rho) + W_p(\rho,\nu). \quad \square
\end{aligned}$$

Lemma 5.1.5. *Given two measures $\gamma^+ \in \Pi(\mu,\rho)$ and $\gamma^- \in \Pi(\mu,\rho)$ there exists at least a measure $\sigma \in \mathcal{P}(\Omega\times\Omega\times\Omega)$ such that $(\pi_{x,y})_\#\sigma = \gamma^+$ and $(\pi_{y,z})_\#\sigma = \gamma^-$, where $\pi_{x,y}$ and $\pi_{y,z}$ denote the projections on the two first and two last variables, respectively.*

Proof. Start by taking γ^+ and disintegrate it w.r.t. the projection π_y. We get a family of measures $\gamma_y^+ \in \mathcal{P}(\Omega)$ (we can think of them as measures over Ω, instead of viewing them as measures over $\Omega\times\{y\}\subset\Omega\times\Omega$)/ They satisfy (and they are defined by)

$$\int_{\Omega\times\Omega}\phi(x,y)d\gamma^+(x,y) = \int_\Omega d\rho(y)\int_\Omega \phi(x,y)\,d\gamma_y^+(x),$$

for every measurable function ϕ of two variables. In the same way, one has a family of measures $\gamma_y^- \in \mathcal{P}(\Omega)$ such that for every ψ we have

$$\int_{\Omega\times\Omega}\psi(y,z)d\gamma^-(y,z) = \int_\Omega d\rho(y)\int_\Omega \psi(y,z)\,d\gamma_y^-(z).$$

For every y take now $\gamma_y^+\otimes\gamma_y^-$, which is a measure over $\Omega\times\Omega$. Define σ through

$$\int_{\Omega^3}\zeta(x,y,z)d\sigma(x,y,z) := \int_\Omega d\rho(y)\int_{\Omega\times\Omega}\zeta(x,y,z)\,d\left(\gamma_y^+\otimes\gamma_y^-\right)(x,z).$$

It is easy to check that, for ϕ depending only on x and y, we have

$$\int_{\Omega^3}\phi(x,y)d\sigma = \int_\Omega d\rho(y)\int_{\Omega\times\Omega}\phi(x,y)\,d\left(\gamma_y^+\otimes\gamma_y^-\right)(x,z) = \int_\Omega d\rho(y)\int_\Omega\phi(x,y)\,d\gamma_y^+(x) = \int\phi\,d\gamma^+.$$

This proves $(\pi_{x,y})_\#\sigma = \gamma^+$ and the proof of $(\pi_{y,z})_\#\sigma = \gamma^-$ is completely analogous. □

5.2 Topology induced by W_p

First of all, let us clarify that we often use the term "weak convergence", when speaking of probability measures, to denote the convergence in the duality with bounded continuous functions (which is often referred to as narrow convergence), and write $\mu_n \rightharpoonup \mu$ to say that μ_n converges in such a sense to μ. Notice also that, when both μ_n and μ are probability measures, this convergence coincides with the convergence in the duality with functions $\phi \in C_0(\Omega)$, vanishing at infinity. To convince of such a fact, we only need to show that if we take $\phi \in C_b(\Omega)$, $\mu_n, \mu \in \mathcal{P}(\Omega)$ and we suppose $\int \psi d\mu_n \to \int \psi d\mu$ for every $\psi \in C_0(\Omega)$, then we also have $\int \phi d\mu_n \to \int \phi d\mu$. If all the measures are probability, we can add for free a constant C to ϕ and, since ϕ is bounded, we can choose C so that $\phi + C \geq 0$. Hence $\phi + C$ is the sup of an increasing family of functions in C_0 (take $(\phi + C)\chi_n$, χ_n being an increasing family of cut-off functions with $\chi_n = 1$ on $B(0, n)$. Hence, by semicontinuity we have $\int(\phi + C)d\mu \leq \liminf_n \int(\phi + C)d\mu_n$, which implies $\int \phi d\mu \leq \liminf_n \int \phi d\mu_n$. If the same argument is performed with $-\phi$ we have te desired convergence of the integrals.

Once the weak convergence is understood, we can start from the following result.

Theorem 5.2.1. *If Ω is compact, then $\mu_n \rightharpoonup \mu$ if and only if $W_1(\mu_n, \mu) \to 0$.*

Proof. Let us recall the duality formula, which gives for arbitrary $\mu, \nu \in \mathcal{P}(\Omega)$

$$W_1(\mu, \nu) = \min\left\{\int |x - y|\, d\gamma,\ \gamma \in \Pi(\mu, \nu)\right\} = \max\left\{\int \phi\, d(\mu - \nu)\ :\ \phi \in \mathrm{Lip}_1\right\}.$$

Let us start from a sequence μ_n such that $W_1(\mu_n, \mu) \to 0$. Thanks to the duality formula, for every $\phi \in \mathrm{Lip}_1(\Omega)$ we have $\int \phi\, d(\mu_n - \mu) \to 0$. By linearity, the same will be true for any Lipschitz function. By density, for any function in $C_b(\Omega)$. This shows that the Wasserstein convergence implies the weak convergence.

To prove the opposite implication, let us first fix a subsequence μ_{n_k} such that $\lim_k W_1(\mu_{n_k}, \mu) = \limsup_n W_1(\mu_n, \mu)$. For every k, pick a function $\phi_{n_k} \in Lip_1(\Omega)$ such that $\int \phi_{n_k}\, d(\mu_{n_k} - \mu) = W_1(\mu_{n_k}, \mu)$. Up to adding a constant, which does not affect the integral, we can suppose that ϕ_{n_k} all vanish on a same point, and they are hence uniformly bounded and equicontinuous. By Ascoli's theorem we can extract a sub-subsequence uniformly converging to a certain $\phi \in \mathrm{Lip}_1(\Omega)$. By replacing the original subsequence with this new one we can avoid relabeling. We have now

$$W_1(\mu_{n_k}, \mu) = \int \phi_{n_k}\, d(\mu_{n_k} - \mu) \to \int \phi\, d(\mu - \mu) = 0,$$

where the convergence of the integral is justified by the weak convergence $\mu_{n_k} \rightharpoonup \mu$ together with the strong convergence (in $C(\Omega)$) $\phi_{n_k} \to \phi$. This shows that $\limsup_n W_1(\mu_n, \mu) \leq 0$ and concludes the proof. □

Theorem 5.2.2. *If Ω is compact and $p \geq 1$, then $\mu_n \rightharpoonup \mu$ if and only if $W_p(\mu_n, \mu) \to 0$.*

Proof. We have already proved this equivalence for $p = 1$. For the other values of p, just use the inequalities

$$W_1(\mu,\nu) \le W_p(\mu,\nu) \le C W_1(\mu,\nu)^{1/p},$$

that give the equivalence between the convergence for W_p and for W_1. □

We can now pass to the case of unbounded domains.

Theorem 5.2.3. *Consider any $\Omega \subset \mathbb{R}^d$ and $p \ge 1$, then $W_p(\mu_n,\mu) \to 0$ if and only if $\mu_n \rightharpoonup \mu$ and $\int |x|^p d\mu_n \to \int |x|^p d\mu$.*

Proof. Consider first a sequence $\mu_n \in \mathcal{P}_p(\Omega)$ which is converging to μ for the W_p distance. It is still true in this case that

$$\sup \left\{ \int \phi \, d(\mu_n - \mu) \, : \, \phi \in \mathrm{Lip}_1 \right\} \to 0,$$

which gives the weak convergence testing against any Lipschitz function. Notice that Lipschitz functions are dense (for the uniform convergence) in the space $C_0(\Omega)$ (while it is not necessarily the case for $C_b(\Omega)$) and that this is enough to prove $\mu_n \to \mu$.

To obtain the other condition, namely $\int |x|^p d\mu_n \to \int |x|^p d\mu$ (which is not a consequence of the weak convergence, since $|x|^p$ is not bounded), it is sufficient to notice that

$$\int |x|^p d\mu_n = W_p^p(\mu_n, \delta_0) \to W_p^p(\mu, \delta_0) = \int |x|^p d\mu.$$

We need now to prove the opposite implication. Consider a sequence a $\mu_n \rightharpoonup \mu$ satisfying also $\int |x|^p d\mu_n \to \int |x|^p d\mu$. Fix $R > 0$ and consider the function $\phi(x) := (|x| \wedge R)^p$, which is continuous and bounded. We have

$$\int (|x|^p - (|x| \wedge R)^p) \, d\mu_n = \int |x|^p d\mu_n - \int \phi \, d\mu_n \to \int |x|^p d\mu - \int \phi \, d\mu = \int (|x|^p - (|x| \wedge R)^p) \, d\mu.$$

Since $\int (|x|^p - (|x| \wedge R)^p) \, d\mu \le \int_{B(0,R)^c} |x|^p d\mu$ it is possible to choose R so that

$$\int (|x|^p - (|x| \wedge R)^p) \, d\mu < \varepsilon/2$$

and hence one can also guarantee that $\int (|x|^p - (|x| \wedge R)^p) \, d\mu_n < \varepsilon$ for all n large enough.

We use now the inequality $(|x|-R)^p \leq |x|^p - R^p = |x|^p - (|x| \wedge R)^p$ which is valid for $|x| \geq R$ (see Lemma 5.2.4 below) to get

$$\int (|x|-R)^p d\mu_n < \varepsilon \text{ for } n \text{ large enough and } \quad \int (|x|-R)^p d\mu < \varepsilon.$$

Consider now $\pi_R : \mathbb{R}^d \to \overline{B(0,R)}$ defined as the projection over $\overline{B(0,R)}$. This map is well defined and continuous and is the identity on $\overline{B(0,R)}$. Moreover, for every $x \notin \overline{B(0,R)}$ we have $|x - \pi_R(x)| = |x| - R$. We can deduce

$$W_p(\mu, (\pi_R)_\# \mu) \leq \left(\int (|x|-R)^p d\mu \right)^{1/p} \leq \varepsilon^{1/p}, \quad W_p(\mu_n, (\pi_R)_\# \mu_n) \leq \left(\int (|x|-R)^p d\mu_n \right)^{1/p} \leq \varepsilon^{1/p}$$

Notice also that, due to the usual composition of the weak convergence with continuous functions, from $\mu_n \rightharpoonup \mu$ we also infer $(\pi_R)_\# \mu_n \rightharpoonup (\pi_R)_\# \mu$. Yet, these measures are all concentrated on the compact set $\overline{B(0,R)}$ and here we can use the equivalence between weak convergence and W_p convergence. Hence, we get

$$\begin{aligned} \limsup_n W_p(\mu_n, \mu) &\leq \limsup_n \left(W_p(\mu_n, (\pi_R)_\# \mu_n) + W_p((\pi_R)_\# \mu_n, (\pi_R)_\# \mu) + W_p(\mu, (\pi_R)_\# \mu) \right) \\ &\leq 2\varepsilon^{1/p} + \lim_n W_p((\pi_R)_\# \mu_n, (\pi_R)_\# \mu) = 2\varepsilon^{1/p}. \end{aligned}$$

The parameter $\varepsilon > 0$ being arbitrary, we get $\limsup_n W_p(\mu_n, \mu) = 0$ and the proof is concluded. □

Lemma 5.2.4. *For $a, b \in \mathbb{R}_+$ and $p \geq 1$ we have $a^p + b^p \leq (a+b)^p$.*

Proof. Suppose without loss of generality that $a \geq b$. Then we can write $(a+b)^p = a^p + p\xi^{p-1}b$, for a point $\xi \in [a, a+b]$. Use now $p \geq 1$ and $\xi \geq a \geq b$ to get $(a+b)^p \geq a^p + b^p$. □

5.3 Lipschitz curves in W_p and the continuity equation

In this section we analyze some properties about Lipschitz and absolutely continuous curves in the space W_p. In order to do that, we need some simple elements from analysis in metric spaces. The reader can have a look at [10] in order to know more.

Digression – *Curves and speed in metric spaces*

Let us recall here some properties about Lipschitz curves in metric spaces.

A curve ω is just a continuous function defined on a interval, say $[0,1]$ and valued in a metric space (X,d). As a map between metric spaces, saying that it is Lipschitz continuous or not is well-defined. on the contrary, its speed (i.e. $\omega'(t)$) has no meaning, unless X is a vector space.

Surprisingly, it is possible to give a meaning to $|\omega'|(t)$ (i.e. to the modulus of the velocity).

Definition If $\omega : [0,1] \to X$ is a curve valued in the metric space (X,d) we define the metric derivative of ω at time t, denoted by $|\omega'|(t)$ through

$$|\omega'|(t) := \lim_{h\to 0} \frac{d(\omega(t+h),\omega(t))}{|h|},$$

provided this limit exists.

The following theorem guarantees the existence of the metric derivative for Lipschitz curves. **PUT REFERENCE OR GIVE PROOF**

Theorem Suppose that $\omega : [0,1] \to X$ is Lipschitz continuous, then the metric derivative $|\omega'|(t)$ exists for a.e. $t \in [0,1]$. Moreover we have, for $t < s$,

$$d(\omega(t),\omega(s)) \le \int_t^s |\omega'|(\tau)d\tau.$$

The goal of this session is to identify the Lipschitz curves in the space $\mathcal{P}_p(\Omega)$ endowed with the distance W_p with the solution of the continuity equation $\partial_t\mu_t + \nabla\cdot(v_t\mu_t) = 0$ with L^p vector fields v_t, and to connect the L^p norm of v_t with the metric derivative $|\mu'|(t)$.

We recall (see Section 4.1) that the continuity equation may be interpreted as the equation ruling the evolution of the density μ_t of a family of particles initially distributed according to μ_0 and each of them following the flow

$$\begin{cases} y_x'(t) = v(t, y_x(t)) \\ y_x(0) = x. \end{cases}$$

The following is the main theorem relating Lipschitz curves for the distance W_p with solutions of the continuity equation. For simplicity, we will only state it in the framework of a compact, convex domain Ω. Convexity is not an important issue, as one can always enlarge the reference domain and take its convex envelop. As far as boundedness is concerned, we will explain where we use this assumption and how to get rid of it.

Theorem 5.3.1. *Let $(\mu_t)_{t\in[0,1]}$ be a Lipschitz curve for the distance W_p ($p > 1$). Then for a.e. $t \in [0,1]$ there exists a vector field $v_t \in L^p(\mu_t;\mathbb{R}^n)$ such that*

- *the continuity equation $\partial_t\mu_t + \nabla\cdot(v_t\mu_t) = 0$ is satisfied in the weak sense (see above),*
- *for a.e. t we have $||v_t||_{L^p(\mu_t)} \le |\mu'|(t)$.*

On the contrary, if $(\mu_t)_{t\in[0,1]}$ is a family of measures in $\mathcal{P}_p(\Omega)$ and for each t we have a vector field $v_t \in L^p(\mu_t;\mathbb{R}^n)$ with $||v_t||_{L^p(\mu_t)} \le C$, then $(\mu_t)_t$ is actually a Lipschitz curve for the W_p distance and for a.e. t we have $|\mu'|(t) \le ||v_t||_{L^p(\mu_t)}$.

Proof. Let us start from the first part of the statement, i.e. the existence of vector fields v_t satisfying the continuity equation with μ_t. Fist fix a function $\psi \in C^1 \cap \mathrm{Lip} \cap C_b$ and compute

$$\int \psi\, d\mu_{t+h} - \int \psi\, d\mu_t = \int (\psi(y) - \psi(x))\, d\gamma \le \mathrm{Lip}(\psi) \int |x-y| d\gamma$$
$$\le \mathrm{Lip}(\psi) \left(\int |x-y|^p d\gamma\right)^{1/p} = \mathrm{Lip}(\psi) W_p(\mu_t, \mu_{t+h}) \le \mathrm{Lip}(\psi)\,\mathrm{Lip}(\mu)|h|,$$

where γ is an optimal transport plan from μ_t to μ_{t+h}.

This proves that $t \mapsto \int \psi\, d\mu_t$ is Lipschitz continuous. We want now to estimate its derivative, so that we will be able to express it as $\int \nabla\psi \cdot v_t\, d\mu_t$ for a suitable v_t.

Let us define a function $H : \Omega\times\Omega \to \mathbb{R}$ through

$$H(x,y) = \sup_{s\in[0,1]} |\nabla\psi((1-s)x + sy)|.$$

This function has several useful properties: first, it satisfies $|\psi(x) - \psi(y)| \le |x-y|H(x,y)$, second it is continuous, third we have $H(x,x) = |\nabla\psi(x)|$. The continuity is proven in a subsequent lemma (Lemma 5.3.2). Let us admit it: we may write

$$\int \psi\, d\mu_{t+h} - \int \psi\, d\mu_t = \int (\psi(y) - \psi(x))\, d\gamma_h \le \int |x-y| H(x,y) d\gamma_h$$
$$\le \left(\int |x-y|^p d\gamma_h\right)^{1/p} \left(\int H(x,y)^q d\gamma_h\right)^{1/q} = W_p(\mu_t,\mu_{t+h}) \left(\int H(x,y)^q d\gamma_h\right)^{1/q} \tag{3.1}$$

Here γ_h is an optimal plan from μ_t to μ_{t+h}. As $h \to 0$, it converges to an optimal plan γ_0 from μ_t to μ_t, which is nothing but $(id, id)_\# \mu_t$. Moreover, thanks to H being continuous, it also holds

$$\lim_{h\to 0} \int H(x,y)^q d\gamma_h = \int H(x,y)^q d\gamma_0 = \int |\nabla\psi(x)|^q d\mu_t.$$

Hence, if we divide by h and pass to the limit in (3.1), we get

$$\frac{d}{dt}\int \psi\, d\mu_t \le |\mu'|(t) ||\nabla\psi||_{L^q(\mu_t)}.$$

The main idea now is the following: for fixed t, consider the application $\psi \mapsto \frac{d}{dt}\int \psi\, d\mu_t$ as a function of $\nabla\psi$. This is possible since this application is invariant under the addition of constants to ψ. Hence, if we define $X = \{\nabla\psi \,:\, \psi \in C^1 \cap \mathrm{Lip} \cap C_b\} \subset L^q(\mu_t)$, we have a linear operator L defined on X through $L(\xi) := \frac{d}{dt}\int \psi\, d\mu_t$. It satisfies $L(\xi) \le C||\xi||_{L^q(\mu_t)}$. By Hahn-Banach it can be extended to the whole $L^q(\mu_t)$ and by the identification of $(L^q)'$ with L^p there exists a vector field $v_t \in L^p(\mu_t)$ satisfying $||v_t||_{L^p(\mu_t)} \le C = |\mu'|(t)$ and $L(\xi) = \int \xi \cdot v_t\, d\mu_t$. This would conclude, since we have built the desired vector fields v_t and they satisfy for every $\psi \in C^1 \cap \mathrm{Lip} \cap C_b$

$$\frac{d}{dt}\int \psi\, d\mu_t = \int \nabla\psi \cdot v_t\, d\mu_t.$$

Yet, there is a subtlety in what we have done that has to be fixed: if we fix ψ, the estimates above are true for almost any t, but the (negligible) set of t where, for instance, the derivative does not exist, depends on ψ. To bypass this difficulty, one can fix a countable dense set of functions ψ, dense for the C^1 convergence (for instance all polynomials with rational coefficients). Let us now call D this set, X_D the vector space generated by $\nabla\psi$ for $\psi \in D$, and define v_v by extending the functional L defined on X_D. We have the existence of a negligible set N such that, for $t \not{i}nN$ and $\psi \in D$, we have

$$\frac{d}{dt}\int \psi\, d\mu_t = L(\nabla\psi) = \int \nabla\psi \cdot v_t\, d\mu_t.$$

The only delicate point is to prove that the formula stays true for $\psi \notin t$, at least for a.e. t. This is equivalent to proving

$$\int \psi\, d\mu_{t+h} - \int \psi\, d\mu_t = \int_t^{t+h} d\tau \int \nabla\psi \cdot v_\tau\, d\mu_\tau$$

and it can be obviously be obtained by starting from the fact that it is true on a sequence $\psi_n \in D$ and passing to the limit $\psi_n \to \psi$.

Let us now prove the converse implication. Suppose that we have

$$\partial_t \mu_t + \nabla \cdot E_t = 0 \quad \text{where } E_t = v_t \cdot \mu_t.$$

The goal is to prove an estimate on $W_p(\mu_t, \mu_{t+h})$ so as to prove Lipschitz continuity of $t \mapsto \mu_t$.

To be able to do something we first regularize by convolution with a C^∞ strictly positive kernel η_ε (as a function of space only). We define $\mu_t^\varepsilon := \eta_\varepsilon * \mu_t$ and $E_t^\varepsilon := \eta_\varepsilon * E_t$, and $v_t^\varepsilon := E_t^\varepsilon / \mu_t^\varepsilon$. The vector field v_t^ε is well defined and regular, since $\mu_t^\varepsilon > 0$. Since we want to apply the uniqueness result for the continuity equation that we summarized in Corollary 4.1.5, we also need to check uniform global bounds on v^ε and on μ^ε. To do this, we need to perform a particular choice of η_ε, and in particular we take of the form $\eta^\varepsilon(z) = \varepsilon^- d\eta(|z|/\varepsilon)$, where $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is a smooth function such that $\eta(t) = Ce^{-t}$ for $t \geq 1$. In Lemma 5.3.3 we check that this guarantees that μ^ε and v^ε satisfy the required global bounds.

Thanks to our definitions, we have

$$\partial_t \mu_t^\varepsilon + \nabla \cdot (v_t^\varepsilon \mu_t^\varepsilon) = 0.$$

Yet, v^ε satisfies the assumptions of Corollary 4.1.5, which states that the only solution of $\partial_t \rho_t + \nabla \cdot (v_t^\varepsilon \rho_t) = 0$ is given by the flow of v^ε. Hence, by uniqueness, we get

$$\mu_t^\varepsilon = (T_t)_\# \mu_0^\varepsilon, \quad \text{where } T_t(x) = y_x(t), \quad \text{and} \quad \begin{cases} y_x'(t) = v_t^\varepsilon(y_x(t)) \\ y_x(0) = x \end{cases}.$$

This provides a useful transport plan between μ_t and μ_{t+h}, taking $\gamma = (T_t, T_{t+h})_\# \mu_0^\varepsilon \in \Pi(\mu_t^\varepsilon, \mu_{t+h}^\varepsilon)$. We obtain

$$\begin{aligned} W_p(\mu_t^\varepsilon, \mu_{t+h}^\varepsilon) &\leq \left(\int |x-y|^p d\gamma \right)^{1/p} = \left(\int |T_t(x) - T_{t+h}(x)|^p d\mu_0^\varepsilon \right)^{1/p} \\ \leq |h|^{1/q} \left(\int \int_t^{t+h} \left| \frac{d}{d\tau} T_\tau(x) \right|^p d\tau \, d\mu_0^\varepsilon \right)^{1/p} &= |h|^{1/q} \left(\int_t^{t+h} d\tau \int |v_\tau^\varepsilon(y_x(\tau))|^p \, d\mu_0^\varepsilon \right)^{1/p} \\ &= |h|^{1/q} \left(\int_t^{t+h} d\tau \int |v_\tau^\varepsilon(y)|^p \, d\mu_\tau^\varepsilon(y) \right)^{1/p}. \end{aligned}$$

We prove in Lemma 5.3.4 that $\int |v_\tau^\varepsilon(y)|^p \, d\mu_\tau^\varepsilon(y) \leq ||v_\tau||^p_{L^p(\mu_\tau)}$. If we accept this inequality, we can go on. First we give a Lipschitz estimate on μ^ε: if we use $||v_\tau||_{L^p(\mu_\tau)} \leq C$, then we have

$$W_p(\mu_t^\varepsilon, \mu_{t+h}^\varepsilon) \leq |h|^{1/q} (|h| C^p)^{1/p} = C|h|,$$

which proves that $t \mapsto \mu_t^\varepsilon$ is Lipschitz with constant C. In particular, we can pass to the limit as $\varepsilon \to 0$ and get the same Lipschitz constant for $t \mapsto \mu_t$.

After that, we can refine the estimate

$$W_p(\mu_t^\varepsilon, \mu_{t+h}^\varepsilon) \leq |h|^{1/q} \left(\int_t^{t+h} d\tau ||v_\tau||^p_{L^p(\mu_\tau)} \right)^{1/p} = |h| \left(\frac{1}{|h|} \int_t^{t+h} d\tau ||v_\tau||^p_{L^p(\mu_t)} \right)^{1/p}.$$

First we can pass to the limit $\varepsilon \to 0$, thus obtaining

$$\frac{W_p(\mu_t, \mu_{t+h})}{|h|} \leq \left(\frac{1}{|h|} \int_t^{t+h} d\tau ||v_\tau||^p_{L^p(\mu_\tau)} \right)^{1/p}$$

and then as $h \to 0$ at points t which are Lebesgue points of $t \mapsto ||v_\tau||^p_{L^p(\mu_t)}$:

$$|\mu'|(t) \leq ||v_t||_{L^p(\mu_t)}. \quad \square$$

Lemma 5.3.2. *If $\psi \in C^1(\Omega)$ and Ω is convex, the function*

$$H(x,y) = \sup_{s \in [0,1]} |\nabla \psi((1-s)x + sy)|$$

is continuous on $\Omega \times \Omega$ and satisfies $H(x,x) = |\nabla \psi(x)|$ and $|\psi(x) - \psi(y)| \leq |x-y| H(x,y)$.

Proof. Since H is defined as a supremum over the parameter s of the functions $(x,y) \mapsto |\nabla \psi((1-s)x + sy)|$, which are obviously continuous from ψ being C^1, we infer that H is lower semicontinuous. Moreover, take a sequence $(x_n, y_n) \to (x,y)$ and suppose $\limsup_n H(x_n, y_n) > H(x,y)$, and let s_n be such that $H(x_n,y_n) = |\nabla \psi((1-s_n)x_n + s_n y_n)|$. Up to extracting subsequences, we can suppose that the limsup is a limit, and $s_n \to s$. Hence we get $H(x_n, y_n) = |\nabla \psi((1-s_n)x_n + s_n y_n)| \to |\nabla \psi((1-s)x + sy)| \leq H(x,y)$, which is a contradiction. Thus, we proved that for any sequence $(x_n, y_n) \to (x,y)$ we have $\limsup_n H(x_n, y_n) \leq H(x,y)$, which also proves upper semicontinuity, and H is continuous.

The behavior on the diagonal is evident since $(1-s)x + sy = x$ when $y = x$. The inequality $|\psi(x) - \psi(y)| \leq |x-y| H(x,y)$ is a consequence of the intermediate value theorem. $\square$

Lemma 5.3.3. *With the above choice of the convolution kernel η^ε, the function μ^ε is Lipschitz in (t,x) and v^ε and $\nabla \cdot v^\varepsilon$ are Lipschitz in x and bounded, uniformly bounded in t (for fixed $\varepsilon > 0$) provided Ω is bounded.*

Proof. We can fix $\varepsilon = 1$ as the computations for other values of ε are similar. From $\mu^\varepsilon(t,x) = \int \eta(|y-x|)\mu_t(dx)$ we have a Lipschitz bound in x from standard convolution properties, but also

$$\partial_t \mu^\varepsilon(t,x) = \frac{d}{dt}\int \eta(|y-x|)\mu_t(dx) \leq \int \eta'(|y-x|)\frac{x-y}{|x-y|}\cdot v_t\mu_t(dx),$$

which is bounded by $\mathrm{Lip}(\eta)||v_t||_{L^1(\mu_t)}$.

As far as v^ε is concerned, it is trickier since it is defined as $E^\varepsilon/\mu^\varepsilon$. From $\eta > 0$ we get $\mu^\varepsilon > 0$ which guarantees that v^ε is smooth as the ratio between two smooth functions, with non-vanishing denominator. Yet, we want to compute explicit and global bounds for large $|x|$. If $R_0 = \mathrm{diam}(\Omega)$ and $0 \in \Omega$ we have for sure, for $|x| = R$, $\mu^\varepsilon(x) \geq \eta(R+R_0)$. On the other hand we have, for $R \geq 2R_0 \wedge 1$, $|E^\varepsilon| \leq C\eta(R-R_0)$, $|\nabla E^\varepsilon| \leq C\eta'(R-R_0)$, $|D^2E^\varepsilon| \leq C\eta'`(R-R_0)$ and, similarly, $|\mu^\varepsilon| \leq C\eta(R-R_0)$ $|\nabla\mu^\varepsilon| \leq C\eta'(R-R_0)$, $|D^2\mu^\varepsilon| \leq C\eta'`(R-R_0)$. From

$$|E^\varepsilon| = \frac{|E^\varepsilon|}{\mu^\varepsilon},\ |\nabla\varepsilon^\varepsilon| \leq \frac{|\nabla E^\varepsilon|}{\mu^\varepsilon} + \frac{|E^\varepsilon|\,|\nabla\mu^\varepsilon|}{(\mu^\varepsilon)^2}$$

and

$$|D^2\varepsilon^\varepsilon| \leq \frac{|D^2E^\varepsilon|}{\mu^\varepsilon} + 2\frac{|\nabla E^\varepsilon|\,|\nabla\mu^\varepsilon|}{(\mu^\varepsilon)^2} + + \frac{|E^\varepsilon|\,|D^2\mu^\varepsilon|}{(\mu^\varepsilon)^2} + 2\frac{|E^\varepsilon|\,|\nabla\mu^\varepsilon|^2}{(\mu^\varepsilon)^3},$$

using $\eta = \eta' = \eta''$ on $[1,+\infty[$ and $\eta(R+R_0) = e^{-2R_0}\eta(R-R_0)$, we get uniform bounds. □

We notice that the application of the above Lemma 5.3.3 is the only point where we needed Ω to be bounded. Without this assumption, we cannot apply the uniqueness result for the continuity equation that we presented in the previous chapter. On the other hand, we stress that more general uniqueness results exist, and we simply chose not to present them for the sake of simplicity.

Lemma 5.3.4. *For* $(t,x) \in \mathbb{R}\times\mathbb{R}^n$ *we have*

$$\sup_{a\in\mathbb{R},\, b\in\mathbb{R}^n,\, a+\frac{1}{q}|b|^q\leq 0} at+b\cdot x = f_p(t,x) := \begin{cases} \frac{1}{p}\frac{|x|^p}{t^{p-1}} & \text{if } t>0, \\ 0 & \text{if } t=0, x=0 \\ +\infty & \text{if } t=0, x\neq 0, \quad \text{or } t<0. \end{cases} \tag{3.2}$$

In particular f_p is convex. Moreover, if $E = v\cdot\mu$, $E^\varepsilon = \eta_\varepsilon * E$ and $\mu^\varepsilon = \eta_\varepsilon * \mu$, $v^\varepsilon = E^\varepsilon/\mu^\varepsilon$, we have

$$\int f_p(\mu^\varepsilon(x), E^\varepsilon(x))dx = \int \frac{1}{p}|v^\varepsilon(x)|^p \mu^\varepsilon(x)dx = \frac{1}{p}||v^\varepsilon||^p_{L^p(\mu^\varepsilon)} \le \frac{1}{p}||v||^p_{L^p(\mu)}. \tag{3.3}$$

Proof. First of all, we establish (3.2). Suppose $t > 0$: then it is clear that it is better to take the maximal possible value of a in the sup, and hence $a = -\frac{1}{q}|b|^q$. This gives

$$\sup_b \left(-\frac{1}{q} t|b|^q + b\cdot x \right) = t\,\left(\sup_b -\frac{1}{q}|b|^q + b\cdot (\frac{x}{t}) \right),$$

which can be computed if we remember that the Legendre transform of $p \mapsto \frac{1}{q}|p|^q$ is $x \mapsto \frac{1}{p}|x|^p$. This gives

$$\sup_b -\frac{1}{q}|b|^q + b\cdot y = \frac{1}{p}|y|^p \qquad \text{for all } y, \tag{3.4}$$

and hence

$$\sup\{at + b\cdot x \,:\, a\in\mathbb{R},\, b\in\mathbb{R}^n,\, a+\frac{1}{q}|b|^q \le 0\} = t\frac{1}{p}\left|\frac{x}{t}\right|^p = \frac{1}{p}\frac{|x|^p}{t^{p-1}}.$$

The case $t = 0$ and $x = 0$ is obvious; if $t = 0$ and $x \neq 0$ it is clear that any vector b may be compensated by a sufficiently negative value of a, which gives

$$\sup\{at + b\cdot x \,:\, a\in\mathbb{R},\, b\in\mathbb{R}^n,\, a+\frac{1}{q}|b|^q \le 0\} = \sup_b b\cdot x = +\infty.$$

Finally, in the case $t < 0$ it is clear that one a arbitrarily negative and $b = 0$ so that

$$\sup\{at + b\cdot x \,:\, a\in\mathbb{R},\, b\in\mathbb{R}^n,\, a+\frac{1}{q}|b|^q \le 0\} \ge \sup_{a<0} at = +\infty.$$

The fact that f_p is convex follows as a consequence of the fact that it is expressed as a supremum of linear functions.

Let us now prove (3.3). Take arbitrary measurable functions $a : \Omega \to \mathbb{R}$ and $b : \Omega \to \mathbb{R}^n$ satisfying $a(x) + \frac{1}{q}|b(x)|^q \le 0$ for all x. From standard properties of convolutions we have

$$\int a d\mu^\varepsilon + \int b\cdot dE^\varepsilon = \int a^\varepsilon\, d\mu + \int b^\varepsilon \cdot\, dE,$$

where $a^\varepsilon := a * \eta_\varepsilon$ and $b^\varepsilon := b * \eta_\varepsilon$. Notice that, by Jensen inequality

$$|b^\varepsilon(y)|^q = \left| \int b(x)\eta_\varepsilon(x-y)dx \right|^q \leq \int |b(x)|^q \eta_\varepsilon(x-y)dx$$

and hence

$$a^\varepsilon(y) + \frac{1}{q}|b^\varepsilon(y)|^q \leq \int a(x) + \left(\frac{1}{q}|b(x)|^q\right)\eta_\varepsilon(x-y)dx \leq 0.$$

This proves that we have

$$\int a^\varepsilon\, d\mu + \int b^\varepsilon \cdot\, dE \leq -\int \frac{1}{q}|b|^\varepsilon\, d\mu + \int b^\varepsilon \cdot v d\mu \leq \frac{1}{p}\int |v|^p d\mu,$$

where we used (again) (3.4).

This finally proves that for any $a(x)$ and $b(x)$ we have

$$\int a d\mu^\varepsilon + \int b \cdot dE^\varepsilon \leq \frac{1}{p}\int |v|^p d\mu,$$

and, passing to the sup in a and b, one gets

$$\int f_p(\mu^\varepsilon(x), E^\varepsilon(x))dx \leq \frac{1}{p}\int |v|^p d\mu. \quad \square$$

5.4 Constant speed geodesics in W_p

We will see in this section how constant speed geodesics in W_p are related to optimal transports. Before, we need to recall the main facts about geodesics in metric space.

Memo – *Constant speed geodesics in general metric spaces*

First of all, let us define the length of a curve ω in a general metric space (X, d).

Definition The length Length(ω) of a curve $\omega : [0,1] \to X$ is defined as

$$\text{Length}(\omega) := \sup\left\{ \sum_{k=0}^{n-1} d(\omega(t_k), \omega(t_{k+1})\ :\ n \geq 1,\, 0 = t_0 < t_1 < \cdots < t_n = 1 \right\}.$$

It is interesting to notice the following result

Proposition For any Lipschitz curve $\omega : [0,1] \to X$ we have

$$\text{Length}(\omega) = \int_0^1 |\omega'|(t)dt.$$

We collect now some more definitions.

Definition A curve $\omega : [0,1] \to X$ is said to be a geodesic between x_0 and $x_1 \in X$ if it minimizes the length among all curves such that $\omega(0) = x_0$ and $\omega(1) = x_1$.

A space (X, d) is said to be a *length space* if it holds

$$d(x,y) = \inf\{\text{Length}(\omega) \,:\, \omega \in \text{Lip},\, \omega(0) = x, \omega(1) = y\}.$$

A space (X, d) is said to be a *geodesic space* if it holds

$$d(x,y) = \min\{\text{Length}(\omega) \,:\, \omega \in \text{Lip},\, \omega(0) = x, \omega(1) = y\},$$

i.e. if it is a length space and there exist geodesics between arbitrary points.

We will not enter into details here about the conditions for the existence of geodesics. It is clear that the fact that Length(ω) is defined as a sup is crucial so to establish semicontinuity with respect to pointwise (or uniform) convergence.

After that, we define constant speed geodesics.

Definition In a length space, a curve $\omega : [0,1] \to X$ is said to be a *constant speed geodesic* between $\omega(0)$ and $\omega(1) \in X$ if it satisfies

$$d(\omega(t), \omega(s)) = |t-s| d(\omega(0), \omega(1)) \quad \text{for all } t, s \in [0,1].$$

A curve with this property is automatically a geodesic.

The following characterization is useful

Proposition Fix an exponent $p > 1$ and consider curves connecting x_0 to x_1. The three following facts are equivalent

1. ω is a constant speed geodesic,
2. $|\omega'|(t) = d(\omega(0), \omega(1))$ a.e.,
3. ω solves $\min\{\int_0^1 |\omega'|(t)^p dt \,:\, \omega(0) = x_0, \omega(1) = x_1\}$.

First, we prove that the space $\mathcal{P}_p(\Omega)$ endowed with the Wasserstein distance is indeed a length space, provided Ω is convex.

Theorem 5.4.1. *Suppose that Ω is convex, take $\mu, \nu \in \mathcal{P}_p(\Omega)$ and γ an optimal transport plan in $\Pi(\mu,\nu)$ for the cost $|x-y|^p$ ($p \geq 1$). Define $\pi_t : \Omega \times \Omega \to \Omega$ through $\pi_t(x,y) = (1-t)x + ty$. Then the curve $\mu_t := (\pi_t)_\# \gamma$ is a constant speed geodesic in $\mathcal{P}_p(\Omega)$ connecting $\mu_0 = \mu$ to $\mu_1 := \nu$.*

In the particular case where μ is absolutely continuous, or in general that $\gamma = \gamma_T$, this very curve is obtained as $((1-t)id + tT)_\# \mu$.

As a consequence, the space $\mathcal{P}_p(\Omega)$ endowed with the Wasserstein distance is a geodesic space (i.e. a length space where the infimum of the length of the curves connecting two given points is a minimum).

Proof. It is sufficient to prove that $W_p(\mu_t, \mu_s) \le W_p(\mu, \nu)|t-s|$. To do that, take $\gamma_t^s := (\pi_t, \pi_s)_\# \gamma \in \Pi(\mu_t, \mu_s)$ and compute

$$W_p(\mu_t,\mu_s) \le \left(\int |x-y|^p d\gamma_t^s\right)^{\frac{1}{p}} = \left(\int |\pi_t(x,y) - \pi_s(x,y)|^p d\gamma\right)^{\frac{1}{p}} = |t-s| \left(\int |x-y|^p d\gamma\right)^{\frac{1}{p}} = |t-s| W_p(\mu,$$

where we used that $|(1-t)x + ty - (1-s)x - sy| = |(t-s)(x-y)|$. □

On can wonder what is the velocity field associated to a geodesic curve μ_t defined as above, since it is a Lipschitz curve and hence it must admit the existence of a velocity field v_t (at least if $p > 1$) satisfying the continuity equation $\partial_t \mu_t + \nabla \cdot (\mu_t v_t) = 0$.

In rough terms this means: take $y \in \operatorname{spt}(\mu_t) \subset \Omega$, for $t \in]0,1[$, and try to find the speed of the particle(s) passing at y at time t. This should be the value of $v_t(y)$. It would be easier to answer this question if we had uniqueness of "the particle" passing through y at time t. We will use the following theorem.

Lemma 5.4.2. *Let γ be an optimal transport plan for a cost $c(x,y) = h(y-x)$ with h strictly convex. between two probabilities μ and ν and $t \in]0,1[$. Define $\mu_t = (\pi_t)_\# \gamma$ and take $y \in \operatorname{spt}(\mu_t)$. Then there exists a unique pair $(x,z) \in \operatorname{spt}(\gamma)$ such that $y = (1-t)x + tz$. These values of x and z will be denoted by $X_t(y)$ and $Z_t(y)$. If Ω is compact, the two maps X_t and Z_t are also continuous.*

In particular, if $\gamma = \gamma_T$ comes from a transport map, then the map $T_t := (1-t)id + tT$ is invertible and $T_t^{-1} = X_t$.

Proof. The claim is essentially the same as in Lemma 4.3.4, and the uniqueness of $(x,z) \in \operatorname{spt}(\gamma)$ comes from $c-$cyclical monotonicity of $\operatorname{spt}(\gamma)$. The continuity of X_t and Z_t is obtained by compactness. Take $y_n \to y$ and suppose (up to subsequences) $(X_t(y_n), Z_t(y_n)) \to (X, Z)$. Since $\operatorname{spt}(\gamma)$ is closed then $(X,Z) \in \operatorname{spt}(\gamma)$. Yet, the previous uniqueness proof implies $(X,Z) = (X_t(y), Z_t(y))$. And since any limit of converging subsequences must coincide with the value at y, in a compact space this gives continuity. □

We can now identify the velocity field of the geodesic μ_t: we know that every particle initially located at x moves on a straight line with constant speed $T(x) - x$, which implies $v_t(y) = (T - id)(T_t^{-1}(y)$. More generally, if γ is not induced by a map, we have $v_t(y) = Z_t(y) - X_t(y)$.

Proposition 5.4.3. *Let $\mu_t = (\pi_t)_\#\gamma$ be the geodesic connecting μ to ν introduced above. Then the velocity field $v_t := Z_t - X_t$ is well defined on* $\mathrm{spt}(\mu_t)$ *for each $t \in]0,1[$ and satisfies*

$$\partial_t \mu_t + \nabla \cdot (\mu_t v_t) = 0, \quad ||v_t||_{L^p(\mu_t)} = |\mu'|(t) = W_p(\mu,\nu).$$

Proof. We already saw that X_t and Z_t are well-defined, so we only need to check the continuity equation and the L^p estimate. For the continuity equation, take $\phi \in C^1$ and compute

$$\begin{aligned} \frac{d}{dt}\int \phi \, d\mu_t &= \frac{d}{dt}\int \phi((1-t)x + tz)\, d\gamma(x,z) = \int \nabla\phi((1-t)x+tz)\cdot(z-x)\, d\gamma(x,z) \\ &= \int \nabla\phi(\pi_t(x,z)\cdot(Z_t(\pi_t(x,z)) - X_t(\pi_t(x,z))d\gamma(x,z) = \int \nabla\phi(y)\cdot(Z_t(y)-X_t(y))\, d\mu_t(y). \end{aligned}$$

To compute the L^p norm we have

$$\int |v_t|^p d\mu_t = \int |Z_t(y) - X_t(y)|^p d\mu_t(y) = \int |z-x|^p d\gamma(x,z) = W_p(\mu,\nu)^p,$$

and we used in both the computations $Z_t(\pi_t(x,z)) - X_t(\pi_t(x,z) = z - x$ for every $(x,z) \in \mathrm{spt}(\gamma)$. □

We now want to prove that, at least for $p > 1$, all the geodesics for the distance W_p have this form, i.e. they are given by $\mu_t = (\pi_t)_\#\gamma$ for an optimal γ. Non-uniqueness would stay true in case γ is not unique. The proof that we provide here is different from that of [5].

Proposition 5.4.4. *Let $(\mu_t)_t$ be a constant speed geodesic between μ and ν for the distance W_p and suppose $p > 1$. Then there exists an optimal $\gamma \in \Pi(\mu,\nu)$ for the transport cost $c(x,y) = |x-y|^p$ such that, for every $t \in [0,1]$, we have $\mu_t = (\pi_t)_\#\gamma$.*

Proof. Since the curve is a constant speed geodesic, we have $|\mu'|(t) = W_p(\mu,\nu)$ for a.e. t (the metric derivative being computed according to W_p). It is Lipschitz, and for a.e. t there exists, thanks to Theorem 5.3.1, a vector field $v_t \in L^p(\mu_t)$ solving $\partial_t\mu_t + \nabla\cdot(v_t\mu_t) = 0$, and with $||v_t||_{L^p(\mu_t)} \le W_p(\mu,\nu)$. The same approximation argument (with the same convolution kernels) as in Theorem 5.3.1 provides a regularized curve of measures μ_t^ε and a regularized vector field v_t^ε. This vector field has a flow $Y : \Omega \to \mathrm{Lip}([0,1];\Omega)$ where $Y(x)$ is the curve $y_x(\cdot)$, and we know $\mu_t^\varepsilon = (Y_t)_\#\mu_0^\varepsilon$, where $Y_t(x) = Y(x)(t) = y_x(t)$. Let us also define the measure $Q^\varepsilon := Y_\#\mu_0^\varepsilon$. If we denote by $K_p : \mathrm{Lip}([0,1];\Omega) \to \mathbb{R}$ the $p-$Kinetic energy, given by $K_p(\omega) :=$

$\int_0^1 |\omega'(t)|^p dt$, we have (from Jensen inequality) $|\omega(0)-\omega(1)|^p \le K_p(\omega)$, with equality if and only if ω is a segment parametrized with constant speed.

Hence we have

$$W_p^p(\mu_0^\varepsilon, \mu_1^\varepsilon) \le \int |\omega(0)-\omega(1)|^p dQ^\varepsilon(\omega) \le \int K_p(\omega) dQ^\varepsilon(\omega)$$
$$= \int_0^1 ||v_t^\varepsilon||^p_{L^p(\mu_t^\varepsilon)} dt \le W_p^p(\mu,\nu).$$

The first inequality comes from $(e_0, e_1)_\# Q^\varepsilon \in \Pi(\mu_0^\varepsilon, \mu_1^\varepsilon)$; the second is a consequence of the above inequality for K_p; the subsequent equality comes from this computation

$$\int K_p(\omega) dQ^\varepsilon(\omega) = \int_\Omega \left(\int_0^1 |y_x'(t)|^p dt \right) d\mu_0^\varepsilon(x) = \int_0^1 dt \int_\Omega |v_t(Y_t(x))|^p d\mu_0^\varepsilon(x),$$

and from the fact that the image measure of μ_0^ε through Y_t is μ_t^ε.

Now (as we saw many times in Chapter 4), consider that the measures Q^ε are tight, since there is a bound on $\int K_p dQ^\varepsilon$ and K_p is such that $\{\omega \in \mathrm{Lip}(\Omega) : K_p(\omega) \le L\}$ is compact in $C^0(\Omega)$ for the uniform convergence for every L. Also, K_p is also l.s.c. for the same convergence. Hence, we can obtain the existence of a subsequence such that $Q^\varepsilon \rightharpoonup Q$ and $\int K_p dQ \le \liminf_\varepsilon \int K_p dQ^\varepsilon$. From $\mu_t^\varepsilon = (e_t)_\# Q^\varepsilon$ we get $\mu_t = (e_t)_\# Q$. If we use $\mu^\varepsilon \rightharpoonup \mu$ and $\nu^\varepsilon \rightharpoonup \nu$ and set $\gamma := (e_0, e_1)_\# Q \in \Pi(\mu,\nu)$ we have therefore

$$W_p^p(\mu,\nu) \le \int |x-y|^p d\gamma = \int |\omega(0)-\omega(1)|^p dQ \le \int K_p dQ \le W_p^p(\mu,\nu).$$

This implies that all inequalities are equalities, and hence Q is concentrated on curves which are constant speed segments and γ is an optimal transport plan. This provides $\mu_t = (\pi_t)_\# \gamma$. □

Notice how the above proof recalls that of Theorem 4.2.5, with here a dynamical framework, compared to the statical framework in Chapter 4.

5.5 Discussion

5.5.1 The W_∞ distance

5.5.2 Wasserstein and branched transport distances

Chapter 6

Benamou-Brenier and other continuous numerical methods

6.1 The Benamou-Brenier formulation and its numerical applications

We combine here the results of the three last sections:

- looking for an optimal transport for the cost $c(x, y) = |x - y|^p$ is equivalent to looking for constant speed geodesic in W_p because from optimal plans we can reconstruct geodesics and from geodesics (via their velocity field) it is possible to reconstruct the optimal transport ;
- constant speed geodesics may be found by minimizing $\int |\mu'|(t)^p dt$;
- in the case of the Wasserstein spaces, we have $|\mu'|(t)^p = \int |v_t|^p d\mu_t$.

As a consequence of these last considerations, for $p > 1$ it happens that solving

$$\min \int_0^1 \int_\Omega |v_t|^p d\mu_t \, dt \ : \ \partial_t \mu_t + \nabla \cdot (v_t \mu_t) = 0, \ \mu_0 = \mu, \, \mu_1 = \nu$$

selects constant speed geodesics connecting μ to ν, and hence allows to find the optimal transport between these two measures.

On the other hand, this minimization problem in the variables (μ_t, v_t) has non-linear constraints (due to the product $v_t\mu_t$) and the functional is non-convex (since $(t,x) \mapsto t|x|^p$ is not convex). Yet, it is possible to transform it into a convex problem.

For this, it is sufficient to switch variables, from (μ_t, v_t) into (μ_t, E_t) where $E_t = v_t\mu_t$. The problem becomes

$$\min \int_0^1 \int_\Omega f_p(\mu_t, E_t)\, dt \ :\ \partial_t\mu_t + \nabla\cdot E_t = 0,\ \mu_0 = \mu,\ \mu_1 = \nu$$

and the constraints are now linear, as well as the functional convex. Yet, the function f_p is convex, but not that convex, since it is $1-$homogeneous, and hence non-strictly convex. This reduces the efficiency of any gradient descent algorithm in order to solve the problem.

In [15], the authors propose a numerical method, based on this convex change of varibales, on duality, and on what is called "augmented Lagrangian", in the framework of a Uzawa-type algorithm.

Here are the main steps to conceive the algorithm.

First of all, we will write the constraint in a weak form (actually, in the sense of distributions), thanks to (1.3). This means that we actually want to solve

$$\min_{\mu,E} \int_0^1 dt \int_\Omega f_p(\mu_t, E_t) + \sup_\phi - \int_0^1\int_\Omega \partial_t\phi\, d\mu_t\, dt + \int_0^1\int_\Omega \nabla\phi\cdot dE_t\, dt + \int_\Omega \phi(1,x)\, d\nu(x) - \int_\Omega \phi(0,x)\, d\mu$$

where the sup is computed over all functions defined on $[0,1]\times\Omega$ (we do not care about their regularity, since in a numerical methods they will be anyway represented by functions defined on the points on a grid in $[0,1]\times\mathbb{R}^n$). Let us denote by G the function defined through $G(\phi) = \int_\Omega \phi(1,x)\, d\nu(x) - \int_\Omega \phi(0,x)\, d\mu(x)$.

The quantity f_p in this variational problem may be expressed as a sup and hence we solve

$$\min_{\mu,E} \sup_{a,b,\phi,\, a+\frac{1}{q}|b|^q\le 0} \int_\Omega a(x) d\mu_t\, dt + \int_\Omega b(x)\cdot dE_t\, dt - \int_0^1\int_\Omega \partial_t\phi\, d\mu_t\, dt - \int_0^1\int_\Omega \nabla\phi\cdot dE_t\, dt + G(\phi).$$

Denote $m = (\mu, E)$. Here $m : [0,1]\times\Omega \to \mathbb{R}^{n+1}$ is a $(n+1)-$dimensional vector field defined on a $(n+1)-$dimensional space. We do not care here about m being a measure or a true function, since anyway we will work in a discretized setting, and m will be a function defined on every point of a grid in $[0,1]\times\mathbb{R}^n$. Analogously, we denote $q = (a,b)$ and $K = \{q(t,x) =$

$(a(t,x), b(t,x)) : a(t,x) + \frac{1}{q}|b(t,x)|^q \leq 0 \ \forall (t,x)\}$. We also denote by $\nabla_{t,x}\phi$ the space-time gradient of ϕ, i.e. $\nabla_{t,x}\phi = (\partial_t \phi, \nabla\phi)$.

The problem may be re-written as

$$\min_m \sup_{q,\phi\,:\,q\in K} < q - \nabla_{t,x}\phi, m > + G(\phi).$$

Here comes the idea of the augmented lagrangian: one could add in the optimization a term $\rho||q - \nabla_{t,x}\phi||^2$ (for a small step size ρ), because this would change nothing. Actually, it is clear that we have

$$\min_m \sup_{q,\phi\,:\,q\in K} < q-\nabla_{t,x}\phi, m > + G(\phi) \geq \min_m \sup_{q,\phi\,:\,q\in K} < q-\nabla_{t,x}\phi, m > + G(\phi) - \rho||q-\nabla_{t,x}\phi||^2.$$

Moreover, if one takes the optimal m for the "augmented" problem at the r.h.s., it is clear that, due to optimality, it should satisfy $\nabla F(m) = 0$, where

$$F(m) := \sup_{q,\phi\,:\,q\in K} < q - \nabla_{t,x}\phi, m > + G(\phi) - \rho||q - \nabla_{t,x}\phi||^2.$$

Define also $\tilde{F}(m) := \sup_{q,\phi\,:\,q\in K} < q - \nabla_{t,x}\phi, m > + G(\phi)$ and notice $F \leq \tilde{F}$. Yet, $\nabla F(m) = q - \nabla_{t,x}\phi$ for the optimal (q,ϕ) (usual rules for the gradient of functions defined through maximization, even if a precise statement should pass through sub-gradients...). This means that for this m we should have $q = \nabla_{t,x}\phi$ and hence $F(m) =< q - \nabla_{t,x}\phi, m > + G(\phi) - \rho||q - \nabla_{t,x}\phi||^2 =< q - \nabla_{t,x}\phi, m > + G(\phi) \leq \tilde{F}(m)$, thus finally $F(m) = \tilde{F}(m)$. This also shows that m, which minimized the augmented problem, also solves the original one.

Hence, we look for a solution of

$$\min_m \sup_{q,\phi\,:\,q\in K} < q - \nabla_{t,x}\phi, m > + G(\phi) - \rho||q - \nabla_{t,x}\phi||^2.$$

The algorithm that one can consider to find the optimal m is the following: since we know that the gradient $\nabla F(m)$ is obtained simply as $q - \nabla_{t,x}\phi$ for the optimal (q,ϕ), we only need to produce a sequence m_k and to find, for each of them, the optimal (q_k, ϕ_k). Yet, instead of finding exactly the optimal (q_k, ϕ_k) we will optimize in two steps (first the optimal ϕ for fixed q, then the optimal q for this ϕ).

The algorithm will work in three iterates steps. Suppose we have a triplet (m_k, q_k, ϕ_k)

- Given m_k and q_k, find the optimal ϕ_{k+1}, by solving

$$\max_\phi - < \nabla_{t,x}\phi, m_k > + G(\phi) - \rho||q_k - \nabla_{t,x}\phi||^2,$$

which amounts to minimizing a quadratic problem in $\nabla_{t,x}\phi$. The solution can be found as the solution of a Laplace equation $\Delta_{t,x}\phi = \nabla\cdot(q_k+m_k)$, with a space-time Laplacian, with Neumann boundary conditions prescribed on $t=0$ and $t=1$ by G. Most Laplace solvers can find this solution in time $O(N\ln N)$, where N is the number of points in the discretization.

- Given m_k and ϕ_{k+1}, find the optimal q_{k+1}, by solving

$$\max < q, m_k > -\rho||q-\nabla_{t,x}\phi_{k+1}||^2 \,:\, q\in K.$$

This problem is equivalent to the projection of $\nabla_{t,x}\phi_{k+1}-\frac{1}{\rho}m_k$ and no gradient appears in the minimization. This means that the minimization may be performed pointwisely, by selecting for each (t,x) the point $q=(a,b)$ which is the closest to $\nabla_{t,x}\phi_{k+1}(t,x)-\frac{1}{\rho}m_k(t,x)$ in the convex set (sort of a paraboloid) $\{a,b)\,:\, a+\frac{1}{q}|b|^q\leq 0\}$. If we have a method for this projection in $\mathbb{R}^{n+1}$, requiring a constant numnber of operations, then the cost for this pointwise step is $O(N)$.

- Finally we update m by setting $m_{k+1}=m_k-\rho(q_{k+1}-\nabla_{t,x}\phi_{k+1})$.

6.2 Angenent-Haker-Tannenbaum

6.3 Numerical solution of the Monge-Ampère equation

6.4 Discussion

6.4.1 Discrete numerical methods

Chapter 7

Functionals on the space of probabilities

We consider in this chapter some class of functionals on the space $\mathcal{P}(\Omega)$, which can be of interest in many variational problems, and are natural in many modeling issues. Indeed, in several applied models, we can face minimization problems where the unknown to be determined is the distribution of a certain amount of mass, and the criteria involve one of more of the following functionals.

- The integral of a given function (potential energy)

$$\mathcal{V}(\mu) = \int V \, d\mu.$$

- The double integral of a function on $\Omega \times \Omega$ according to the tensor product $\mu \otimes \mu$ (interaction energy)

$$\mathcal{W}(\mu) = \int W(x,y) \, d\mu(x) \, d\mu(y).$$

- The Wasserstein distance W_p (or a function of it) from a fixed measure; for simplicity, we consider rather the $p-$th power of W_p:

$$\mathcal{D}(\mu) = W_p^p(\mu, \nu).$$

- The norm in a dual functional space: given a Banach space of functions on Ω, define

$$||\mu||_{X'} = \sup_{\phi \in X, ||\phi|| \leq 1} \int \phi \, d\mu = \sup_{\phi \in X \setminus \{0\}} \frac{\int \phi \, d\mu}{||\phi||}.$$

- The integral of a function of the density

$$\mathcal{F}(\mu) : \begin{cases} \int f(\rho(x))dx & \text{if } \mu = \rho \cdot dx, \\ +\infty & \text{otherwise }. \end{cases}$$

- The sum of a function of the masses of the atoms

$$\mathcal{G}(\mu) : \begin{cases} \sum_i g(a_i) & \text{if } \mu = \sum_i a_i \delta_{x_i}, \\ +\infty & \text{if } \mu \text{ is not purely atomic.} \end{cases}$$

The goal of this chapter is to study some properties of these functionals. The first questions that we analyze are classical variational issues (semicontinuity, convexity, first variation...). Then, we also introduce and analyze a new notion, that of geodesic convexity. This is a natural concept in the analysis on metric spaces: is a certain functional convex *along geodesics* of the metric space? in the case of the Wasserstein spaces, and in particular of W_2, this notion, introduced by McCann in [87] is called *displacement convexity* and turns out to be quite different, and very useful in many cases, than the usual convexity.

7.1 Semicontinuity

Since the main goal is to use these functionals in variational models, in order to minimize them we give first some semicontinuity criteria. We will consider semicontinuity or continuity of these functionals with respect to weak convergence. We start from the simplest functionals.

7.1.1 Potential and interaction energies, Wasserstein distances and dual norms

The easiest case is the one of the potential energy.

Proposition 7.1.1. *If $V \in C_b(\Omega)$ then $\mathcal{V}$ is continuous for the weak convergence of probability measures. If V is l.s.c. and bounded from below than $\mathcal{V}$ is semicontinuous.*

Moreover, semicontinuity of V (respectively, continuity) is necessary for the semicontinuity (continuity) of $\mathcal{V}$.

Proof. The continuity of $\mu \mapsto \int V d\mu$ for $V \in C_b$ is straightforward by definition of weak convergence. If V is l.s.c. and bounded from below, we know that there is a sequence of Lipschitz and bounded functions V_k increasingly converging to V. Then, by monotone convergence, we infer that $\mathcal{V}(\mu) = \lim_k \int V_k d\mu = \sup_k \int V_k d\mu$. This allows to see $\mathcal{V}$ as a supremum of continuous functionals, and hence it is l.s.c..

The necessity part is straightforward if one considers sequences of points $x_k \to x$ and the associated Dirac masses. Since $\delta_{x_k} \rightharpoonup \delta_x$ then continuity of $\mathcal{V}$ implies $V(x_k) \to V(x)$ and semicontinuity implies $\liminf_k V(x_k) \geq V(x)$. $\square$

We pass now to the case of the interaction functional $\mathcal{W}$.

Proposition 7.1.2. *If $W \in C_b(\Omega)$ then $\mathcal{W}$ is continuous for the weak convergence of probability measures. If W is l.s.c. and bounded from below than $\mathcal{W}$ is semicontinuous.*

This result is an easy consequence of the following lemma.

Lemma 7.1.3. *If $\mu_k \rightharpoonup \mu$ then $\mu_k \otimes \mu_k \rightharpoonup \mu \otimes \mu$.*

Proof. We want to prove that for any function $\xi \in C_b(\Omega \times \Omega)$ we have $\int \xi d\mu_k \otimes \mu_k \to \int \xi d\mu \otimes \mu$. First consider the case $\xi(x, y) = \phi(x)\psi(y)$. In this case

$$\int \xi d\mu_k \otimes \mu_k = \int \phi d\mu_k \cdot \int \psi d\mu_k \to \int \phi d\mu \cdot \int \psi d\mu = \int \xi d\mu \otimes \mu.$$

This proves that the desired convergence is true for all functions $\xi \in \mathcal{A}(\Omega)$, where the class $\mathcal{A}(\Omega)$

$$\mathcal{A}(\Omega) = \{\xi(x, y) = \sum_{i=1}^{N} \phi_i(x)\psi_i(y), \quad \phi_i, \psi_i \in C_b(\Omega)\}.$$

The class $\mathcal{A}$ is an algebra of functions which contains the constants and separates the points of $\Omega \times \Omega$ and hence, by Stone-Weierstrass theorem, if $\Omega \times \Omega$ is compact, it is dense in $C(\Omega \times \Omega)$. It is a general fact that weak-* convergence in a dual space may be tested on a dense set, provided that the sequence we are considering is bounded. Here weak convergence corresponds in the compact case to the weak-* convergence in the duality with $C(\Omega \times \Omega)$ and we are considering a sequence of probability measures, which is bounded. This proves $\mu_k \otimes \mu_k \rightharpoonup \mu \otimes \mu$ in the compact case.

If Ω is non compact, one can use Stone-Weierstrass theorem to prove that the convergence holds against any compactly supported function. The conclusion follows if one notices that a sequence of probability measures weakly converges (in the duality with C_b) to a given probability measure (in this case $\mu_k \otimes \mu_k$ to $\mu \otimes \mu$) if and only if weakly-* converges in the duality with C_c. This is the object of following lemma. □

Lemma 7.1.4. *Consider a sequence of positive measures ν_k and a measure ν on a same space E. Suppose $\nu_k(E) \to \nu(E)$ and $\int \xi \, d\nu_k \to \int \xi \, d\nu$ for all $\xi \in C_c(E)$. Then $\nu_k \rightharpoonup \nu$, i.e. $\int \xi \, d\nu_k \to \int \xi \, d\nu$ for all $\xi \in C_b(E)$.*

Proof. Take a positive function $\xi \in C_b(E)$. By considering a sequence of cutoff functions it is easy to express ξ as an increasing limit of compactly supported functions ξ_h. This gives

$$\int \xi_h \, d\nu = \lim_k \int \xi_h \, d\nu_k \leq \liminf_k \int \xi \, d\nu_k,$$

and, passing to the sup in h, $\int \xi \, d\nu \leq \liminf_k \int \xi \, d\nu_k$. It is not difficult to check that for negative ξ the opposite inequality holds $\int \xi \, d\nu \geq \limsup_k \int \xi \, d\nu_k$.

If ξ is bounded, then it may be written as $\xi = (\xi + C) - C$ with $\xi + C \geq 0$. This gives

$$\begin{aligned} \liminf_k \int \xi \, d\nu_k &= \liminf_k \int (\xi + C) d\nu_k - \lim_k C\nu_k(E) \\ &\geq \int (\xi + C) d\nu - C\nu(E) = \int \xi \, d\nu. \end{aligned}$$

By writing $\xi = (\xi - C) + C$ with $\xi - C \leq 0$ it is also possible to get

$$\limsup_k \int \xi \, d\nu_k = \int \xi \, d\nu$$

and hence, $\lim_k \int \xi \, d\nu_k = \int \xi \, d\nu$. □

It is worthwhile to notice that besides these two classes of functionals one could consider higher order interaction energies defined through

$$\mu \mapsto \int \int \dots \int W(x_1, x_2, \dots, x_n) d\mu(x_1) d\mu(x_2) \dots d\mu(x_n).$$

In particular the set of all these functional could be considered as "polynomials" on the space $\mathcal{P}(\Omega)$ and some analysis of the space of functions over measures are based on this very class of functionals (see for instance the lectures at Collège de France by P.-L. Lions).

We pass now to some different functionals.

Proposition 7.1.5. *For any $p < +\infty$, the Wasserstein distance $W_p(\cdot, \nu)$ to any fixed measure $\nu \in \mathcal{P}(\Omega)$ is continuous w.r.t. weak convergence provided Ω is compact. If Ω is not compact and $\nu \in \mathcal{P}_p(\Omega)$, then $W_p(\cdot, \nu)$ is well-defined over $\mathcal{P}_p(\Omega)$ and it is only l.s.c..*

Proof. Continuity in the compact case is straightforward since the convergence for the distance W_p exactly metrizes the weak convergence and every distance is continuous w.r.t. itself (as a consequence of triangle inequality).

For the non-compact case, consider a sequence $\mu_k \rightharpoonup \mu$ and a sequence of optimal transport plans $\gamma_k \in \Pi(\mu_k, \nu)$. Since μ_k is tight, γ_k is also tight. First extract a subsequence such that $\lim_h W_p(\mu_{k_h}, \nu) = \liminf_k W_p(\mu_k, \nu)$ and then extract once more so as to guarantee $\gamma_{k_h} \rightharpoonup \gamma$. We know that images through continuous functions pass to the limit, so that $\gamma \in \Pi(\mu, \nu)$.

Now we have

$$W_p^p(\mu, \nu) \le \int |x-y|^p d\gamma \le \liminf_h \int |x-y|^p d\gamma_{k_h} = \lim_h W_p(\mu_{k_h}, \nu) = \liminf_k W_p(\mu_k, \nu),$$

where the first inequality comes from the fact that γ is admissible but maybe not optimal, and the second because $|x-y|^p$ is positive and continuous but not bounded: thanks to Proposition 7.1.1 we get semicontinuity. $\square$

Finally, here is another class of functional with some interesting examples.

Proposition 7.1.6. *Let X be a Banach space of functions over Ω such that $X \cap C_b(\Omega)$ is dense in X. Then*

$$\mu \mapsto ||\mu||_{X'} = \sup_{\phi \in X, ||\phi|| \le 1} \int \phi \, d\mu = \sup_{\phi \in X \setminus \{0\}} \frac{\int \phi \, d\mu}{||\phi||}$$

is l.s.c. for the weak convergence.

Proof. This fact is straightforward if one notices that we can write

$$||\mu||_{X'} = \sup_{\phi \in C_b(\Omega) \cap X, ||\phi|| \le 1} \int \phi \, d\mu,$$

which expresses $||\mu||_{X'}$ as a supremum of linear functionals, continuous for the weak convergence. $\square$

It is interesting to see two examples.

Proposition 7.1.7. *Suppose $X = H_0^1(\Omega)$ (endowed with the L^2 norm of the gradient) and let, for every $\mu \in H^{-1}(\Omega)$, ϕ_μ be the solution of*

$$\begin{cases} -\Delta\phi = \mu & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega \end{cases}.$$

Then $||\mu||^2_{H^{-1}} = \int \phi_\mu \, d\mu = \int\int G(x,y) d\mu(x) d\mu(y)$, where $G(x,\cdot)$ is defined as the solution of

$$\begin{cases} -\Delta\phi = \delta_x & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega \end{cases}.$$

In particular, if $\Omega = (-1,1) \subset \mathbb{R}$, then $G(x,y) = \frac{1}{2}((1+x) \wedge (1+y))((1-x) \wedge (1-y))$.

Suppose on the contrary $X = H^1(\Omega)$ (endowed with the full H^1 norm) and let, for every $\mu \in (H^1(\Omega))' = X'$, ϕ_μ be the solution of

$$\begin{cases} -\Delta\phi + \phi = \mu & \text{in } \Omega, \\ \frac{\partial\phi}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}.$$

Then $||\mu||^2_{X'} = \int \phi_\mu \, d\mu = \int\int G(x,y) d\mu(x) d\mu(y)$, where $G(x,\cdot)$ is defined as the solution of

$$\begin{cases} -\Delta\phi + \phi = \delta_x & \text{in } \Omega, \\ \frac{\partial\phi}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}.$$

In particular, if $\Omega = (-1,1) \subset \mathbb{R}$, then $G(x,y) = \cosh((1+x) \wedge (1+y)) \cosh((1-x) \wedge (1-y)) / \sinh(2)$.

Proof. We just prove that $||\mu||^2_{X'} = \int \phi_\mu \, d\mu$ in the two cases.

First case : H_0^1. Notice that, for every function $\psi \in H_0^1$, one has

$$\int \psi \, d\mu = \int \psi(-\Delta\phi_\mu) = \int \nabla\psi \cdot \nabla\phi_\mu \leq ||\psi||_X ||\phi_\mu||_X$$

and equality holds for $\psi = \phi_\mu$ (we use the homogeneous norm $||\psi||_X := ||\nabla\psi||_{L^2}$). This shows that the supremum is realized by a multiple of ϕ_μ and that $\int \phi_\mu \, d\mu = ||\phi_\mu||^2_X$. As a consequence, we have

$$||\mu||^2_{X'} = \left(\frac{\int \phi_\mu d\mu}{||\phi_\mu||_X} \right)^2 = \int \phi_\mu d\mu.$$

The case of H^1 is really similar, with the only exception of the first computation

$$\int \psi \, d\mu = \int \psi(-\Delta\phi_\mu + \phi_\mu) = \int \nabla\psi \cdot \nabla\phi_\mu + \psi\phi_\mu \leq ||\psi||_X ||\phi_\mu||_X$$

and, again, equality holds for $\psi = \phi_\mu$.

Finally, expressing $\int \phi_\mu d\mu$ as a double integral $\int\int G(x,y)d\mu(x)d\mu(y)$ is only a matter of expressing $\phi_\mu(x)$ as $\int G(x,y)d\mu(y)$. This is possible by using the theory of Green functions and, for the one dimensional case $\Omega = (-1,1)$, it is enough to compute that $\int G(x,y)d\mu(y)$ is a solution of the desired equation. □

In this way we have seen that, thanks to Green functions, we have expressed these dual norms functionals as interaction functionals.

7.1.2 Local Functionals

Local functionals over measures are defined as those functionals $F : \mathcal{M}(\Omega) \to \mathbb{R}$ such that $F(\mu+\nu) = F(\mu)+F(\nu)$ whenever μ and ν are mutually singular (i.e. there exists $A, B \subset \Omega$ with $A \cup B = \Omega$, $\mu(A) = 0$ and $\nu(B) = 0$). This obviously does not make much sense for probability measures, since if μ and ν are probabilities then $\mu + \nu$ is not a probability. Yet, we will see in this section some functionals that can be defined over probability measures but could be also seen as the restriction to probabilities of functionals defined on more general measures. All our proofs could be easily generalized to finite positive measures; on the contrary, for simplicity we will not consider at all the case of signed or vector measures. We refer to [30] for the general theory of these functionals.

First, let us consider functionals of the form

$$\mu = \rho \cdot \lambda \mapsto \int f(\rho(x))\lambda(dx),$$

where λ is a given positive measure over Ω (for instance the Lebesgue measure).

Let us first see which are the natural conditions on f so as to ensure lower semicontinuity. Then, we will give a precise result.

As a first point, consider a sequence ρ_n of density, taking values a and b on two sets A_n and B_n chosen in the following way: we fix a partition of Ω into small cubes of diameter $\varepsilon_n \to 0$ and then we select set A_n and B_n such that, for every cube Q of this partition, we have $\lambda(A_n \cap Q) = (1-t)\lambda(Q)$ and $\lambda(B_n \cap Q) = t\lambda(Q)$. It is clear in this case that ρ_n converges weakly to a uniform density $\rho = (1-t)a + tb$. Semicontinuity of the functional would imply

$$f((1-t)a + tb) \le (1-t)f(a) + tf(b),$$

i.e. convexity of f. Hence it is clear that one needs to require f to be convex.

Another requirement concerns the growth of f. For simplicity it is always possible to assume $f(0) = 0$ (up to adding a constant to f). Suppose that f satisfies $f(t) \leq Ct$ for all $t \geq 0$. Then, if the functional is defined as

$$\mathcal{F}(\mu) : \begin{cases} \int f(\rho(x))d\lambda(x) & \text{if } \mu = \rho \cdot \lambda, \\ +\infty & \text{otherwise .} \end{cases}$$

and we take a sequence μ_n of absolutely continuous probability measures weakly converging to a singular measure μ, then we get $\mathcal{F}(\mu_n) \leq C$ while $\mathcal{F}(\mu) = +\infty$, thus violating the semicontinuity. This suggests that one should have $C = +\infty$, i.e. f superlinear. The following theorem gives a general result which also includes compensation for non-superlinearity.

Proposition 7.1.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be convex function, with $f(0) = 0$, and set $L := \lim_{t\to\infty} f(t)/t = \sup_{t>0} f(t)/t \in \mathbb{R} \cup \{+\infty\}$. Let λ be a fixed finite positive measure on Ω. For every measure μ write $\mu = \rho \cdot \lambda + \mu_s$, where $\rho \cdot \lambda$ is the absolutely continuous part of μ and μ_s be the singular one (w.r.t. λ). Then, the functional defined through*

$$\mathcal{F}(\mu) = \int f(\rho(x))d\lambda(x) + L\mu_s(\Omega)$$

is l.s.c.

Proof. We will try to use the equality

$$f(t) = \sup_{a\in\mathbb{R}} \quad at - f^*(a),$$

which is due to the fact that, f being convex (and lower semicontinuous, since it is real valued), $f^{**} = f$. Moreover, it is easy to see that for $a > L$ we have $f^*(a) = +\infty$: this is easy to see since $f^*(a) = \sup_t at - f(t) \geq \lim_{t\to+\infty} t(a - f(t)/t)$. Hence, we can also write

$$f(t) = \sup_{a\leq L} \quad at - f^*(a).$$

Let us consider the following functional

$$\tilde{\mathcal{F}}(\mu) := \sup_{a\in C_b(\Omega),\, a\leq L} \int a(x)d\mu(x) - \int f^*(a(x))d\lambda(x).$$

$\tilde{\mathcal{F}}$ is obviously l.s.c. since it is the supremum of a family of affine functionals, continuous w.r.t. the weak convergence. We want to prove $\tilde{\mathcal{F}} = \mathcal{F}$.

In order to do so, first notice that, thanks to Lusin's theorem (applied to the measure $\lambda + \mu$), it is not difficult to prove that

$$\tilde{\mathcal{F}}(\mu) := \sup_{a\in L^\infty,\, a\le L} \int a(x)d\mu(x) - \int f^*(a(x))d\lambda(x).$$

Then take a set A such that $\mu_s(\Omega \setminus A) = \lambda(A) = 0$: this allows to write

$$\tilde{\mathcal{F}}(\mu) := \sup_{a\in L^\infty,\, a\le L} \int_{\Omega\setminus A} [a(x)\rho(x) - f^*(a(x))]\, d\lambda(x) + \int_A a(x)d\mu_s(x).$$

The values of $a(x)$ may be chosen independently on A and $\Omega \setminus A$ and it is not difficult to check that

$$\sup_{a\in L^\infty,\, a\le L} \int_{\Omega\setminus A} [a(x)\rho(x) - f^*(a(x))]\, d\lambda(x) = \int f(\rho(x))d\lambda(x),$$
$$\sup_{a\in L^\infty,\, a\le L} \int_A a(x)d\mu_s(x) = L\mu_s(A) = L\mu_s(\Omega),$$

which allows to conclude $\tilde{\mathcal{F}} = \mathcal{F}$. □

Remark 24. The assumption that λ is a finite measure is necessary to avoid integrability issues for the term $\int f^*(a(x))d\lambda$. A typical case where this term could give troubles is the entropy $f(t) = t\log t$, where $f^*(s) = e^{s-1}$. Suppose that λ is the Lebesgue measure on the whole $\mathbb{R}^d$; it is easy to see that, for any L^∞ function a, the integral $\int_{\mathbb{R}^d} e^{a(x)-1}dx$ does not converge, and this provides $\tilde{\mathcal{F}}(\mu) = -\infty$ for every μ. However, if λ is σ-finite (which is the case of the Lebesgue measure on $\mathbb{R}^d$) and $f \ge 0$, it is possible to prove the semicontinuity by approximating λ from below with finite measures.

If we come back to the interpretation of $\mathcal{F}$, it is not difficult to check that $\mathcal{F}$ "favors" dispersed measure: first it is only finite for absolutely continuous measures, second, due to the Jensen inequality, the value of $\mathcal{F}$ is minimal for the constant density.

If we look for functionals having an opposite behavior and favoring the concentrated part of the measure, there are at least two different choices. We can look at an interaction functional such as $\mu \mapsto \int\int |x-y|^2\mu(dx)\mu(dy)$ (where the square of the distance could be replaced by any increasing function of it). This is a global and spatially dependent functional, and has a different flavour than $\mathcal{F}$. Indeed, we can find in the same class of local functionals some functionals which favor concentration, by looking in particular at the atomic part of μ. It is the case of the functional

$$\mathcal{G}(\mu) : \begin{cases} \sum_i g(a_i) & \text{if } \mu = \sum_i a_i \delta_{x_i},\ x_i \neq x_j \text{ for } i \neq j, \\ +\infty & \text{if } \mu \text{ is not purely atomic.} \end{cases}$$

As before, let us first understand which are the basic properties of g so as to guarantee semicontinuity.

We also assume $g(0) = 0$ (which is by the way necessary if we want to avoid ambiguities due to zero-mass atoms). Suppose that g satisfies $g(t) \leq Ct$ for all $t > 0$ and take, similar to what we didi before, a sequence $\mu_n \rightharpoonup \mu$ where μ_n is purely atomic but μ is not (take for instance n points with mass $1/n\dots$). Then we have $\mathcal{G}(\mu_n) \leq C$ but $\mathcal{G}(\mu) = +\infty$. This is a contradiction with semicontinuity and suggests us to consider functions g such that $\lim_{t\to 0} g(t)/t = +\infty$.

Second consideration, take $\mu_n = a\delta x_n + b\delta_{y_n} + \nu$ (with ν finitely atomic, for instance)and suppose $x_n \to x$ and $y_n \to y$. Then $\mu_n \rightharpoonup \mu = (a+b)\delta_x + \nu$. In this case semicontinuity would imply

$$g(a+b) \leq g(a) + g(b),$$

i.e. we need g to be subadditive.

It is useful to establish the following.

Lemma 7.1.9. *Suppose that $g(0) = 0$ and that g is subadditive. Suppose that $\mu_n = \sum_{i=1}^N a_{i,n}\delta_{x_{i,n}}$ is a sequence of atomic measures with bounded number of atoms. Then, up to a subsequence, $\mu_n \rightharpoonup \mu$ where μ is also atomic with at most N atoms and $\mathcal{G}(\mu) \leq \liminf \mathcal{G}(\mu_n)$.*

Proof. Let us precise that we impose to the points $x_{i,n}$ to be distinct. If a measure μ_n has less than N atoms then we chose points $x_{i,n}$ at random to complete the list of atoms and set $a_{i,n} = 0$ for those extra indices.

Now extract a subsequence (not relabeled) such that for each $i = 1, \dots, N$ one has $a_{i,n} \to a_i$ and $x_{i,n} \to x_i$. For this subsequence one has $\mu_n \rightharpoonup \mu := \sum_{i=1}^N a_i\delta_{x_i}$. It is possible that the points x_i are not distinct. If they are distinct we have $\mathcal{G}(\mu) = \sum_{i=1}^n g(a_i)$, otherwise we have (thanks to subadditivity) $\mathcal{G}(\mu) \leq \sum_{i=1}^n g(a_i)$. Anyway we have

$$\mathcal{G}(\mu) \leq \sum_{i=1}^n g(a_i) = \lim_n \sum_{i=1}^n g(a_{i,n}) = \lim_n \mathcal{G}(\mu_n),$$

which proves the desired result (one can choose the subsequence so that it realizes the liminf of the whole sequence). □

It is now possible to prove the following

Lemma 7.1.10. *Suppose that $g(0) = 0$, that $g(t) \geq 0$, that g is subadditive and l.s.c. and that $\lim_{t\to 0} g(t)/t = +\infty$. Then $\mathcal{G}$ is l.s.c.*

Proof. Fix a number $M > 0$ and use $\lim_{t\to 0} g(t)/t = +\infty$ to say that there exists ε_0 such that for all $t < \varepsilon_0$ we have $g(t) > Mt$. Consider a sequence $\mu_n \rightharpoonup \mu$, assume $\mathcal{G}(\mu_n) \leq C < +\infty$ and decompose it into $\mu_n = \mu_n^s + \mu_n^b$, where $\mu_n^b = \sum_{i=1}^N a_{i_n}\delta_{x_{i,n}}$ is the sum of the atoms of μ_n with mass at least ε_0. In particular, these atoms are no more than $N := \varepsilon_0^{-1}$. The other part μ_n^s (the "small" part, μ_n^b being the "big" one) is just defined as the remainding atoms (every μ_n is purely atomic since $\mathcal{G}(\mu_n) < +\infty$).

If we write

$$C \geq \mathcal{G}(\mu_n) \geq \mathcal{G}(\mu_n^s) = \sum_i g(a_{i,n}) \geq M \sum_i a_{i,n} = M\mu_n^s(\Omega)$$

we get an estimate on the mass of the "small" part. Hence it is possible to get, up to subsequences,

$$\mu_n^b \rightharpoonup \mu^b \quad \text{and} \quad \mu_n^s \rightharpoonup \mu^s,\ \mu^s(\Omega) \leq \frac{C}{M}.$$

We can now apply lemma 7.1.9 to prove that μ^b is purely atomic and that $\mathcal{G}(\mu^b) \leq \liminf_n \mathcal{G}(\mu_n^b) \leq \liminf_n \mathcal{G}(\mu_n)$.

This proves that μ must be purely atomic, since the possible non-atomic part of μ must be contained in μ^s, but $\mu^s(\Omega) \leq C/M$. This implies that the mass of the non-atomic part is smaller than C/M, which is arbitrarily small, and hence the mass of the non-atomic part must be zero.

We have now proven that μ is purely atomic and we have an estimate of $\mathcal{G}(\mu^b)$, where μ^b is a part of μ depending on M. If we write $(a_i)_i$ for the masses of μ and $(a_i^M)_i$ for those of μ^b we have

$$\sum_i g(a_i^M) \leq \liminf_n \mathcal{G}(\mu_n) := \ell.$$

We want to prove $\sum_i g(a_i) \leq \ell$ and, to this aim, it is enough to let $M \to \infty$. Actually, $\mu^s(\Omega) = \sum_i (a_i - a_i^M) \leq C/M \to 0$ implies that for each i we have $a_i - a_i^M \to 0$ and thus $a_i^M \to a_i$. Using the semicontinuity of g we have $g(a_i) \leq \liminf_{M\to\infty} g(a_i^M)$. If we fiw an arbitrary number N we get

$$\sum_{i=1}^N g(a_i) \leq \liminf_{M\to\infty} \sum_{i=1}^N g(a_i^M) \leq \ell.$$

By passing to the supremum over N we finally get

$$\sum_{i=1}^{\infty} g(a_i) \leq \ell,$$

which is the thesis. $\square$

As a particular example for the functionals of type $\mathcal{F}$ we can consider the L^p norms to the power p, i.e.

$$\mathcal{F}(\mu) = ||\mu||_{L^p}^p = \begin{cases} \int \rho(x)^p dx & \text{if } \mu = \rho \cdot dx, \\ +\infty & \text{otherwise .} \end{cases}$$

As a particular example for the functionals of type $\mathcal{G}$, we can consider the cardinality of the support, obtained for $g(t) = 1$ if $t > 0$ and $g(0) = 0$:

$$\mathcal{G}(\mu) = \#(\mathrm{spt}(\mu)).$$

For the sake of generality – *Characterization of local l.s.c. functionals on* $\mathcal{M}^d(\Omega)$

It is also useful to notice that functionals of the form $\mathcal{F}$ and $\mathcal{G}$ could be mixed, obtaining local functionals accepting both absolutely continuous and atomic parts. There is a general lower semicontinuity result that comes from the general theory developed in [30, 31, 32], which also characterizes the semicontinuity. It also covers the case of vector-valued measures and can be expressed as follows.

Theorem Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be convex and l.s.c. and $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be subadditive and l.s.c.. For every vector $v \in \mathbb{R}^d$ define

$$f^\infty(v) := \lim_{t\to+\infty} \frac{f(tv)}{t} \quad \text{and} \quad g^0(v) := \lim_{t\to 0} \frac{g(tv)}{t},$$

and suppose $f^\infty = g^0$. Given a finite positive measure λ, decompose every vector measure $\mu \in \mathcal{M}^d(\Omega)$ as $\mu = \rho \cdot \lambda + \mu^c + \mu^\#$, where $\mu^c + \mu^\#$ is the singular part of μ w.r.t. λ, which is also decomposed into a purely atomic part $\mu^\# = \sum_i a_i \delta_{x_i}$ and the remainder, usually refereed to as the Cantor part; $\rho : \Omega \to \mathbb{R}^d$ is a vector density and the $a_i = \mu(\{x_i\}) \in \mathbb{R}^d$ are also vectors; we also denote by $w : \Omega \to S^{d-1}$ the density of μ^c w.r.t. its own total variation measure $|\mu^c|$. Then the functional

$$\mu \mapsto \int f(\rho(x))\, d\lambda(x) + \int f^\infty(w(x))\, d|\mu^c|(x) + \sum_i g(a_i)$$

is local and lower semicontinuous for the weak convergence of measures.

Conversely, every local and lower semicontinuous functional on $\mathcal{M}^d(\Omega)$ can be written in the form above, for suitable choices of f, g and λ.

7.2 Convexity, first variations and subdifferentials

We pass in this section to another important notion about these functionals. If semicontinuity is crucial to establish existence results for minimization problems, convexity is crucial for uniqueness. Also, with convexity comes the notion of sub-differential, and hence one comes to another very natural question in calculus of variations: how to compute first variations of these functionals?

We start from the very first question: which among these functionals are convex?

Convexity and strict convexity

- $\mathcal{V}$ is linear and hence convex but not strictly convex.

- $\mathcal{W}$ is quadratic, but not always convex. Take for instance $W(x,y) = |x-y|^2$ and compute

$$\begin{aligned}
&\iint |x-y|^2\,d\mu(x)\,d\mu(y) \\
&= \iint |x|^2 d\mu(x)\,d\mu(y) + \iint |y|^2 d\mu(x)\,d\mu(y) - 2\iint x\cdot y\,d\mu(x)\,d\mu(y) \\
&\qquad\qquad = 2\int |x|^2 d\mu(x) - 2\left(\int x\,d\mu(x)\right)^2.
\end{aligned}$$

  This shows that $\mathcal{W}$ has in this case a linear part to which we subtract the square of another linear one; it is then concave rather than convex.

- The $p-$th power of the $p-$th Wasserstein distance can be expressed by duality formula as a supremum of linear functionals

$$W_p^p(\mu,\nu) = \sup_{\phi(x)+\psi(y)\le |x-y|^p} \int \phi\,d\mu + \int \psi\,d\nu$$

  and it is hence convex (but W_p is in general not). Strict convexity is discussed below (it is true for $p>1$ and $\nu \ll \mathcal{L}^d$).

- The norm in a dual functional space is always convex since it is a norm, but is never strictly convex because it is $1-$homogeneous. Notice that also the square of a norm could be non-strictly convex (as it is the case for the L^∞ or the L^1 norms).

- The functionals $\mathcal{F}$ that we considered above are actually convex due to the assumptions on f. Strict convexity is true if both f is convex and $L = +\infty$ (notice that if one takes $f(t) = \sqrt{1+t^2} - 1$, which is strictly convex, then $\mathcal{F}$ is not strictly convex because it is finite and linear on singular measures).

- On the contrary, the functionals $\mathcal{G}$ that we considered above are typically concave since the typical examples of sub-additive functions are concave.

First variations

We now pass to the computation of first variations of this functionals. Since many of them are only considered in the convex set $\mathcal{P}(\Omega)$, which is a proper (but convex) subset of the Banach space $\mathcal{M}(\Omega)$, we prefer to give an ad-hoc, and pragmatic, definition.

Given a functional $F : \mathcal{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ we call $\frac{\delta F}{\delta \rho}(\rho)$, if it exists, any measurable function such that

$$\frac{d}{d\varepsilon} F(\rho + \varepsilon\chi)_{|\varepsilon=0} = \int \frac{\delta F}{\delta \rho}(\rho) d\chi$$

for every perturbation $\chi \in \mathcal{M}(\Omega)$ such that, at least for $\varepsilon \in [0, \varepsilon_0]$, the measure $\rho + \varepsilon\chi$ belongs to $\{F < +\infty\} \subset \mathcal{P}(\Omega)$. From the fact that we necessarily have $\int d\chi = 0$, it is clear that $\frac{\delta F}{\delta \rho}(\rho)$ is defined up to additive constants. If the space of admissible perturbations χ is "large enough", then $\frac{\delta F}{\delta \rho}(\rho)$ is indeed unique up to additive constants.

- From $\mathcal{V}(\mu + \varepsilon\chi) = \mathcal{V}(\mu) + \varepsilon \int V d\chi$ we infer $\frac{\delta \mathcal{V}}{\delta \rho}(\rho) = V$.

- As $\mathcal{W}$ is quadratic, the computation is easy:

$$\begin{aligned}
&\mathcal{W}(\mu + \varepsilon\chi) - \mathcal{W}(\mu) = \\
&\qquad \varepsilon \int\int W(x,y) d\mu(x) d\chi(y) + \varepsilon \int\int W(x,y) d\mu(y) d\chi(x) \\
&\qquad\qquad + \varepsilon^2 \int\int W(x,y) d\chi(x) d\chi(y).
\end{aligned}$$

  This provides

$$\frac{\delta \mathcal{W}}{\delta \rho}(\rho)(y) = \int W(x,y) d\mu(x) + \int W(y,y') d\mu(y').$$

The formula becomes simpler when W is symmetric (since the two terms are equal) and even simpler when $W(x,y) = h(x-y)$ for an even function h, in which case it takes the form of a convolution $\frac{\delta \mathcal{W}}{\delta \rho}(\rho) = 2h * \mu$.

- For $\mu \mapsto W_p^p(\mu,\nu)$ and more generally for all transport costs, the first variation is given by the Kantorovich potential ϕ, provided it is unique, in the transport from μ to ν, but this will be discussed below.

- Analogously, the first variation of $\mu \mapsto ||\mu||_{X'}$ is given by the function ϕ realizing the maximum in $\max\{\int \phi \, d\mu \,:\, ||\phi||_X \leq 1\}$, provided it exists and is unique.

- If f has polynomial growth ($f(t) \leq Ct^p + C$, one can see that $\frac{\delta \mathcal{F}}{\delta \rho}(\rho) = f'(\rho)$. Indeed, writing $\chi = \theta \cdot \lambda$, we have $\mathcal{F}(\rho + \varepsilon\chi) = \int f(\rho(x) + \varepsilon\theta(x))d\lambda(x)$ and, differentiating under the integral sign, (which is the reason to require some growth conditions, and guarantee the assumptions we need to differentiate this way), we get

$$\frac{d}{d\varepsilon}\mathcal{F}(\rho + \varepsilon\chi)_{|\varepsilon=0} = \int f'(\rho(x))\theta(x)d\lambda = \int f'(\rho)\, d\chi.$$

- Notice that in general the functionals $\mathcal{G}$ are not differentiable along this kind of perturbations. Indeed, from $g'(0) = +\infty$ we can infer that $G(\mu + \varepsilon\chi)$ is usually differentiable only if χ is concentrated on the same atoms as μ, even if for χ finitely atomic one would have $\mathcal{G}(\mu + \varepsilon\chi) < +\infty$.

7.2.1 Dual norms

We develop here the non-trivial issues above, i.e. the case of the Wasserstein distances and of the dual norms. Both are characterized by the fact they are defined as suprema of linear functionals. Since they are convex functionals, in order to provide not only the first variation, but also the sub-differential, we start from this general fact in convex analysis.

Lemma 7.2.1. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be convex and l.s.c.. Set $f^*(y) = \sup < x,y > -f(x)$ for $y \in X'$, where $< x,y >$ denotes the duality between X and X'. Then $\partial f^*(y_0) = \operatorname{argmax}_x\{< x,y_0 > -f(x)\}$.*

Proof. We know $x_0 \in \partial f^*(y_0) \iff y_0 \in \partial f(x_0)$. This is equivalent to the fact that 0 belongs to the subdifferetial of $x \mapsto f(x) - < x, y_0 >$ at x_0, but this is also equivalent to the fact that x_0 minimizes the same expression. $\square$

Proposition 7.2.2. *Suppose that X is reflexive separable Banach space of functions on Ω such that $X \cap C_b(\Omega)$ is dense in X. Let $F(\mu) := ||\mu||_{X'} = \sup\{\int \phi \, d\mu \; : \; ||\phi||_X \le 1\}$. Suppose that, for a given μ, the function $\phi_\mu \in X$ realizing the maximum in the definition of $F(\mu)$ exists and is unique. Then we have $\frac{\delta F}{\delta \rho}(\mu) = \phi_\mu$. Moreover, the subdifferential $\partial F(\mu)$ is always equal to the set (not necessarily a singleton) of maximizers.*

Proof. The second part of the statement is an easy consequence of Lemma 7.2.1, using $f : X \to \mathbb{R} \cup \{+\infty\}$ defined as $f(\phi) = 0$ for $||\phi|| \le 1$ and $f(\phi) = +\infty$ for $||\phi|| > 1$ (the indicator function of the unit ball of X, in the sense of convex analysis).

For the first part, take a perturbation χ such that $\mu + \varepsilon\chi$ belongs to X' for small ε. This implies that both μ and χ belong to X'. From the definition of F, we can write

$$\int \phi_\mu d\chi \le \frac{F(\mu + \varepsilon\chi) - F(\mu)}{\varepsilon} \le \int \phi_{\mu+\varepsilon\chi} d\chi.$$

In order to pass to the limit, we just need to show that the right hand side tends to $\int \phi_\mu d\chi$ as $\varepsilon \to 0$. First, notice that the functions $\phi_{\mu+\varepsilon\chi}$ belong to the unit ball of a reflexive separable Banach space. This means that, up to subsequences, we have $\phi_{\mu+\varepsilon\chi} \rightharpoonup \phi$, for a function ϕ with $||\phi|| \le 1$. We just need to prove $\phi = \phi_\mu$. From the uniqueness of the optimizer ϕ_μ, we just need to show that ϕ is also an optimizer for μ, i.e. $\int \phi d\mu = F(\mu)$.

To do this, notice that $\mu + \varepsilon\chi$ converges strongly in X' to μ, and hence we have

$$\lim_{\varepsilon \to 0} F(\mu + \varepsilon\chi) = \lim_{\varepsilon \to 0} \int \phi_{\mu+\varepsilon\chi} d(\mu + \varepsilon\chi) \to \int \phi \, d\mu \le F(\mu) = \lim_{\varepsilon \to 0} F(\mu + \varepsilon\chi),$$

where the last equality comes from the continuity of F for strong convergence in X'. Hence the inequality $\int \phi \, d\mu \le F(\mu)$ is indeed an equality and $\phi = \phi_\mu$. $\square$

7.2.2 Transport costs

We consider here the case of Wasserstein distances to a fixed measure, and in particular functional of the form $\mu \mapsto W_p^p(\mu, \nu)$. More generally, one could consider transport costs $\mathcal{T}_c(\mu, \nu) := \min\{\int c(x, y) \, d\gamma, \; , \gamma \in \Pi(\mu, \nu)\}$, i.e.

minimal transport costs between measures and a fixed one, without caring whether this cost is of the form $(x,y) = |x-y|^p$ or not, and whether it is linked to a distance on $\mathcal{P}(\Omega)$ or not. Notice that for continuity issues it was useful to take advantage of the distance structure of W_p, but for convexity and first variation issued this is not especially the case.

For the sake of simplicity, the following result will only be given in the case of a compact domain $\Omega \subset \mathbb{R}^d$.

Proposition 7.2.3. *Let $\Omega \subset \mathbb{R}^d$ be compact and $c : \Omega \times \Omega \to \mathbb{R}$ be continuous. Then the functional $\mu \mapsto \mathcal{T}_c(\mu,\nu)$ is convex, and its subdifferential at μ_0 coincides with the set of Kantorovich potentials $\{\phi \in C^0(\Omega) : \int \phi\, d\mu_0 + \int \phi^c\, d\nu = \mathcal{T}_c(\mu,\nu)\}$. Moreover, if the Kantorovich potential from μ_0 to ν is unique up to additive constants, i.e. if the above set is made of a same function ϕ_μ plus arbitrary additive constants, then we also have $\frac{\delta \mathcal{T}_c(\cdot,\nu)}{\delta \rho}(\mu) = \phi_\mu$.*

Proof. Convexity comes from the expression

$$\mathcal{T}_c(\mu,\nu) = \sup\left\{ \int \phi\, d\mu + \int \phi^c\, d\nu \ : \ \phi \in C^0(\Omega) \right\},$$

which allows to see $\mathcal{T}_c(\cdot,\nu)$ as a supremum of affine functionals of μ. Notice that the very same supremum gives $+\infty$ as a result if we take $\mu \in \mathcal{M}(\Omega$ with $\mu(\Omega) \neq \nu(\Omega) = 1$. Indeed, we can always add a constant λ to any ϕ, and $(\phi+\lambda)^c = \phi^c - \lambda$. Taking for instance $\phi = 0$ we get

$$\sup\left\{ \int \phi\, d\mu + \int \phi^c\, d\nu \ : \ \phi \in C^0(\Omega) \right\} \geq \int 0^c(y)\, d\nu(y) + \lambda\big(\mu(\Omega) - \nu(\Omega)\big),$$

which can be made as large as we want if $\mu(\Omega) - \nu(\Omega) \neq 0$ (here 0^c is a bounded function whose precise expression depends on c).

In order to identify the subdifferential, we apply Lemma 7.2.1, with $X = C^0(\Omega)$ (endowed with the L^∞ norm) and $f : X \to \mathbb{R}$ given by $f(\phi) := -\int \phi^c\, d\nu$. We just need to see that this functional is convex and semicontinuous (indeed, it is continuous). Notice that if we take $\phi_0, \phi_1 \in C^0(\Omega)$ we have $\phi_1^c(y) = \inf_x c(x,y) - \phi_1(x) \leq \inf_x c(x,y) - \phi_0(x) + ||\phi_1 - \phi_0|| = \phi_0^c(y) + ||\phi_1 - \phi_0||$. By interchanging the roles of ϕ_0 and ϕ_1 and using the arbitrariness of y we get $||\phi_1^c - \phi_0^c|| \leq ||\phi_1 - \phi_0||$, which implies $\left|\int \phi_1^c\, d\nu - \int \phi_0^c\, d\nu\right| \leq ||\phi_1 - \phi_0||$ and hence the continuity of the functional. As far as convexity is concerned, set $\phi_t = (1-t)\phi_0 + t\phi_1$; we have $\phi_t^c(y) = \inf_x c(x,y) - (1-t)\phi_0(x) - t\phi_1(x) \geq (1-t)\phi_0^c(y) + t\phi_1^c(y)$. This implies the concavity $\int \phi^c_t \, d\nu \geq (1-t) \int \phi^c_0 \, d\nu + t \int \phi^c_1 \, d\nu$ and the convexity we looked for.

In order to get the first variation, we take $\mu_\varepsilon = \mu + \varepsilon\chi$ and we estimate the ratio $(\mathcal{T}_c(\mu_\varepsilon, \nu) - \mathcal{T}_c(\mu, \nu))/\varepsilon$. First, by using that (ϕ_μ, ϕ^c_μ) is optimal in the dual formulation for μ, but not necessarily for μ_ε, we have

$$\frac{\mathcal{T}_c(\mu_\varepsilon, \nu) - \mathcal{T}_c(\mu, \nu)}{\varepsilon} \geq \frac{\int \phi_\mu d\mu_\varepsilon + \int \phi^c_\mu d\nu - \int \phi_\mu d\mu - \int \phi^c_\mu d\nu}{\varepsilon} = \int \phi_\mu d\chi,$$

which gives the lower boundlim $\inf_{\varepsilon\to 0}(\mathcal{T}_c(\mu_v e, \nu) - \mathcal{T}_c(\mu, \nu))/\varepsilon \geq \int \phi_\mu d\chi$.

To look at the lim sup, first fix a sequence of values of ε_k such that $\lim_k(\mathcal{T}_c(\mu_{\varepsilon_k}, \nu) - \mathcal{T}_c(\mu, \nu))/\varepsilon_k = \limsup_{\varepsilon\to 0}(\mathcal{T}_c(\mu_\varepsilon, \nu) - \mathcal{T}_c(\mu, \nu))/\varepsilon$. Then we can estimate the same ratio using the optimality of a pair (ϕ_k, ϕ^c_k), Kantorovitch potentials in the transport from μ_{ε_k} to ν (we can assume them to be $c-$concave functions and also assume $\phi_k(x_0) = 0$ for a fixed point $x_0 \in \Omega$).

$$\frac{\mathcal{T}_c(\mu_{\varepsilon_k}, \nu) - \mathcal{T}_c(\mu, \nu)}{\varepsilon_k} \leq \frac{\int \phi_k d\mu_{\varepsilon_k} + \int \phi^c_k d\nu - \int \phi_k d\mu - \int \phi^c_k d\nu}{\varepsilon_k} = \int \phi_k d\chi. \tag{2.1}$$

As in Proposition 7.2.2, we need to pass to the limit in k. To do this, first notice that the families of functions $(\phi_k)_k$ and $(\phi^c_k)_k$ are both equicontinuous (all $c-$concave functions have the same modulus of continuity of c). Moreover, the condition $\phi_k(x_0) = 0$ gives a bound on ϕ_k which implies a bound on ϕ^c_k as well. Hence, Ascoli's Theorem allows to pass to the limit up to a subsequence (not relabeled). This gives $(\phi_k, \phi^c_k) \to (\phi, \phi^c)$ uniformly (we already noticed above the continuity of the $c-$transform for the uniform convergence). As we did in Proposition 7.2.2, it is easy to check that (ϕ, ϕ^c) must be optimal in the duality formula for the transport between μ and ν (one of the key ingredients was the continuity of the cost, which is justified here by theorem 1.6.11) Actually, from

$$\mathcal{T}_c(\mu_{\varepsilon_k}, \nu) = \int \phi_k d\mu_{\varepsilon_k} + \int \phi^c_k d\nu$$

it is easy to pass to the limit and get

$$\mathcal{T}_c(\mu, \nu) = \int \phi d\mu + \int \phi^c d\nu,$$

which implies $\phi = \phi_\mu$ by uniqueness. Finally, passing to the limit in (2.1) we get also $\limsup_{\varepsilon\to 0}(\mathcal{T}_c(\mu_\varepsilon, \nu) - \mathcal{T}_c(\mu, \nu))/\varepsilon \leq \int \phi_\mu d\chi$. $\square$

As Proposition 7.2.3 requires uniqueness of the Kantorovich potential in order to efficiently compute the first variation, we also give here a sufficient condition

Proposition 7.2.4. *If Ω is the closure of a bounded connected ope set, c is C^1, and at least one of the measures μ or ν is supported on the whole Ω, then the Kantorovitch potential in the transport from μ to ν is unique up to additive constants.*

Proof. Suppose that $\mathrm{spt}(\mu) = \Omega$. First notice that $c \in C^1$ implies that c is Lipschitz on $\Omega \times \Omega$, and hence all Kantorovich potentials, which are $c-$concave, are Lipschitz as well, and are differentiable a.e. Consider two different Kantorovich potentials ϕ_0 and ϕ_1. We use Proposition 1.3.1, which guarantees that their gradients must agree a.e. on Ω. Since Ω is the closure of a connected open set, this means that the difference $\phi_0 - \phi_1$ is constant and provides the desired result. If the measure with full support is ν, just apply the same procedure to the transport from ν to μ and get the uniqueness of the potential ψ. Then, from $\phi = \psi^c$ one also recovers the uniqueness of ϕ. □

Notice that in order to apply the above result to Proposition 7.2.3, the uniqueness $\mu-$a.e. would not be enough (since one integrates it also against other measures μ_ε, and, anyway, without connectedness assumptions on $\mathrm{spt}(\mu)$ it would be impossible to deduce the uniqueness of ϕ.

We finish this section with a remark on strict convexity. We come back for simplicity to the case $c(x, y) = |x - y|^p$, i.e. to the functional $W_p^p(\cdot, \nu)$, but the reader will see that everything works the same under a suitable twist condition (or if $c(x, y) = h(x - y)$ with h strictly convex). Also, the assumption $\nu \ll \mathcal{L}^d$ is not sharp, at least for $p = 2$ (see Section 1.3.1

Proposition 7.2.5. *If $\nu \ll \mathcal{L}^d$ the functional $\mu \mapsto W_p^p(\mu, \nu)$ is strictly convex.*

Proof. Suppose by contradiction that $\mu_0 \neq \mu_1$ and $t \in]0, 1[$ are such that $W_p^p(\mu_t, \nu) = (1 - t)W_p^p(\mu_0, \nu) + tW_p^p(\mu_1, \nu)$, where $\mu_t = (1 - t)\mu_0 + t\mu_1$. Let γ_0 be the optimal transport plan in the transport from ν to μ_0 (pay attention to the direction: it is a transport map if we see it backward: *from ν to μ_0*, since $\nu \ll \mathcal{L}^d$; we write $\gamma_0 = (T_0, id)_\#\nu)$ and, analogously, take $\gamma_1 = (T_1, id)_\#\nu$ optimal from ν to μ_1. Set $\gamma_t := (1 - t)\gamma_0 + t\gamma_1 \in \Pi(\mu_t, \nu)$.

We have

$$(1-t)W_p^p(\mu_0,\nu)+tW_p^p(\mu_1,\nu)=W_p^p(\mu_t,\nu)\leq\int|x-y|^p\,d\gamma_t$$
$$=(1-t)W_p^p(\mu_0,\nu)+tW_p^p(\mu_1,\nu),$$

which implies that γ_t is actually optimal in the transport from ν to μ_t. Yet γ_t is not induced from a transport map, unless $T_0=T_1$. This is a contradiction with $\mu_0\neq\mu_1$ and proves strict convexity. □

7.3 Displacement convexity

In all the considerations of the previous section about convexity, we always considered the standard convex interpolations $[0,1]\ni t\mapsto(1-t)\mu_0+t\mu_1$. Yet, another notion of convexity, more linked to the metric structure of the space $\mathcal{P}(\Omega)$ endowed with the distance W_p, may be useful.

Let us recall that, in a general metric space X (actually, it is better if we are in a geodesic space), we can define $F:X\to\mathbb{R}\cup\{+\infty\}$ to be *geodesically convex* if for every two points $x_0,x_1\in X$ there exists a constant speed geodesic ω connecting $\omega(0)=x_0$ to $\omega(1)=x_1$ such that $[0,1]\ni t\mapsto F(\omega(t))$ is convex. This obviously reduces to usual convexity in $\mathbb{R}^n$ or in any other normed vector space where the segments are the unique geodesics (notice that in any normed vector space segments are geodesics but they may not be the unique ones, as it is the case in L^1 or L^∞ or any non-strictly convex space). By the way, in spaces where there is not uniqueness of the geodesics the definition that we gave could be discussed, since one could also choose to define geodesically convex those functionals that satisfy the geodesic inequality on every geodesic ω... Yet, this is the definition that is usually chosen as it satisfies some extra stability results that would not be true with the more restrictive one requiring convexity for every geodesic.

The notion of geodesic convexity in the space $\mathcal{P}(\Omega)$ for the distance W_p has been introduced by McCann in [87] and is particularly interesting since we know how to characterize the geodesics in such a space. This notion of convexity is usually referred to as *displacement convexity*. Notice that it could a priori depend on the exponent p.

7.3.1 Displacement convexity of $\mathcal{V}$ and $\mathcal{W}$

We recall that the geodesics for the W_p distance are of the form $\mu_t=(\pi_t)_\#\gamma$ where γ is an optimal transport and $\pi_t(x,y)=(1-t)x+ty$.

Proposition 7.3.1. *The functional $\mathcal{V}$ is displacement convex if and only if V is convex.*

The functional $\mathcal{W}$ is displacement convex if W is convex.

Proof. First consider $\mathcal{V}$ and suppose that V is convex. Let us evaluate $\mathcal{V}(\mu_t)$:

$$\mathcal{V}(\mu_t) = \int V\, d(\pi_t)_\# \gamma = \int V((1-t)x + ty) d\gamma.$$

It is clear from this formula that $t \mapsto \mathcal{V}(\mu_t)$ is convex if V is convex.

On the other hand, the convexity of V is a necessary condition for $\mathcal{V}$ being convex as one can easily check by considering geodesics $\mu_t = \delta_{(1-t)x+ty}$ since $\mathcal{V}(\mu_t) = V((1-t)x + ty)$.

The proof for the convexity of $\mathcal{W}$ is similar:

$$\begin{aligned}\mathcal{W}(\mu_t) &= \int W(x, x')\, d(\pi_t)_\#\gamma(x) d(\pi_t)_\#\gamma(x') \\ &= \int W((1-t)x + ty, (1-t)x' + ty')\, d\gamma(x, y)\, d\gamma(x', y'). \quad \square\end{aligned}$$

Notice that we did not state that convexity of W is a necessary condition for $\mathcal{W}$, since it is not true in general. Consider for instance the one-dimensional case.

Proposition 7.3.2. *Let $\Omega = (a, b)$, $T^+ = \{(x, y) \in \Omega \times \Omega,\ y \geq x\}$ and $T^- = \{(x, y) \in \Omega \times \Omega,\ y \leq x\}$. Then it is sufficient that W is convex when restricted to T^+ and T^-, in order to have displacement convexity of $\mathcal{W}$.*

Proof. It is sufficient to consider the proof of Proposition 7.3.1 and check that all the segments $t \mapsto ((1-t)x+ty, (1-t)x'+ty')$ for $(x, y), (x', y') \in \operatorname{spt}\gamma$ are contained either in T^+ or in T^-. This is true thanks to the monotonicity properties of $\operatorname{spt}\gamma$ that we showed in Section 2.2. Actually, we showed that, for every strictly convex cost, the (unique) optimal transport plan γ satisfies

$$(x, y), (x', y') \in \operatorname{spt}\gamma, \quad x < x' \quad \Rightarrow \quad y \leq y'.$$

This means that if $x < x'$ then we have $(x, x') \in T^+$ and $(y, y') \in T^+$ and, by convexity, $((1-t)x + ty, (1-t)x' + ty') \in T^+$. Analogously, for $x > x'$ we get that the segment is contained in T^-. If $x = x'$ it is enough to look at the transport plan from the point of view of y, thus getting the implication $y < y' \quad \Rightarrow \quad x \leq x'$ and concluding in the same way. The only case that stays apart is $x = x'$ and $y = y'$ but in this case the segment reduces to a point.

This proves convexity of $\mathcal{W}$ along the geodesics in W_p for $p > 1$. It also works for W_1 if we choose the same γ as an optimal transport plan (in this case it will not be the unique one, but this will be enough to build a geodesic where there is convexity). $\square$

An interesting consequence of this criterion is the fact that the squared dual norm $\mu \mapsto ||\mu||^2_{X'}$ for $X = H^1(-1, 1)$ is actually displacement convex, as a corollary of the characterization of Proposition 7.1.7 (but it does not work for $= H^1_0$).

On the other hand, no characterization for the multidimensional case is available.

7.3.2 Displacement convexity of $\mathcal{F}$

The most interesting displacement convexity result is the one for functionals depending on the density.

To consider these functionals, we need some technical facts.

The first is just a consequence of standard change-of-variable techniques and we already saw it in the *Change-of-variable and image measures* Memo of Section 4.1.2. It states that if μ has density ρ and $T : \Omega \to \Omega$ is injective and countably Lipschitz with $\det DT(x) \neq 0$ a.e., then $T_\# \mu$ is absolutely continuous and its density ρ^T is given by

$$\rho^T(y) = \frac{\rho(T^{-1}(y))}{(\det DT)(T^{-1}(y))}.$$

The second is just a computation.

Lemma 7.3.3. *Let A be a $d \times d$ matrix with real eigenvalues $\lambda_i \geq -1$ (for instance this is the case when A is symmetric and $I + A \geq 0$). Then $[0, 1] \ni t \mapsto g(t) := \det(I + tA)^{1/d}$ is concave.*

Proof. We can write A in a suitable basis on $\mathbb{C}$ so that it is triangular, and we get

$$g(t)^d = \prod_{i=1}^d (1 + t\lambda_i),$$

and, differentiating,

$$dg(t)^{d-1} g'(t) = \sum_{j=1}^d \lambda_j \prod_{i=1,\, i\neq j}^d (1 + t\lambda_i) = g(t)^d \sum_{j=1}^d \frac{\lambda_j}{1 + t\lambda_j},$$

i.e. $dg'(t) = g(t)\sum_{j=1}^{d} \frac{\lambda_j}{1+t\lambda_j}$. If we differentiate once more we get

$$dg''(t) = g'(t)\sum_{j=1}^{d} \frac{\lambda_j}{1+t\lambda_j} - g(t)\sum_{j=1}^{d} \frac{\lambda_j^2}{(1+t\lambda_j)^2}.$$

Here we use the quadratic-arithmetic mean inequality which gives

$$\sum_{j=1}^{d} \frac{\lambda_j^2}{(1+t\lambda_j)^2} \geq \frac{1}{d}\left(\sum_{j=1}^{d} \frac{\lambda_j}{1+t\lambda_j}\right)^2 = d\left(\frac{g'(t)}{g(t)}\right)^2$$

and hence

$$dg''(t) \leq d\,\frac{g'(t)^2}{g(t)} - d\,\frac{g'(t)^2}{g(t)} = 0,$$

which proves the concavity of g. □

We can now state the main theorem.

Theorem 7.3.4. *Suppose that f is convex and superlinear, $f(0) = 0$ and that $s \mapsto s^{-d}f(s^d)$ is convex and decreasing. Then $\mathcal{F}$ is geodesically convex for the distance W_p.*

Proof. Let us consider two measures μ_0, μ_1 and the constant speed geodetic μ_t between them. The goal being to prove $\mathcal{F}(\mu_t) \leq (1-t)\mathcal{F}(\mu_0) + \mathcal{F}(\mu_1)$, we can assume $\mathcal{F}(\mu_0), \mathcal{F}(\mu_1) < +\infty$, and in particular μ_0 and μ_1 may be taken absolutely continuous. This implies in particular the uniqueness of the optimal transport between them and the fact the geodesic has the form $\mu_t = (T_t)_\#\mu_0$, where $T_t = id + t(T - id)$. Notice that T_t is injective because of Lemma 4.3.4.

We first look at the case $p = 2$, which is easier. In this case we have $T_t(x) = x - t\nabla\phi(x)$, where ϕ is such that $\frac{x^2}{2} - \phi$ is convex. This implies, by Theorem 3.1.9, that $\nabla\phi$ is countably Lipschitz, and so is T_t. Moreover, $D^2\phi$ (which exists a.e.) is symmetric and $D^2\phi \leq I$. Let us define $A(x) = -D^2\phi(x)$. From the formula for the density of the image measure, we know that μ_t is absolutely continuous and we can write its density ρ_t as $\rho_t(y) = \rho(T_t^{-1}(y))/\det(I + tA(T_t^{-1}(y))$ and hence

$$\mathcal{F}(\mu_t) = \int f\left(\frac{\rho(T_t^{-1}(y))}{\det(I + tA(T_t^{-1}(y))}\right)dy = \int f\left(\frac{\rho(x)}{\det(I + tA(x)}\right)\det(I+tA(x))dx,$$

where we used the change of variable $x = T_t^{-1}(y)$, which gives $y = T_t(x)$ and $dy = \det DT_t(x)dx = \det(I + tA(x))dx$.

From Lemma 7.3.3 we know that $\det(I+tA(x)) = g(t,x)^d$ for a function $g : [0,1]\times\Omega$ which is concave in t. It is a general fact that the composition of a convex and decreasing function with a concave one gives a convex function. This implies that

$$t \mapsto f\left(\frac{\rho(x)}{g(t,x)^d}\right) g(t,x)^d$$

is convex (if $\rho(x) \neq 0$ this uses the assumption on f and the fact that $t \mapsto g(t,x)/\rho(x)^{1/d}$ is concave; if $\rho(x) = 0$ then this function is simply zero).

Finally, we have proven convexity of $t \mapsto \mathcal{F}(\mu_t)$.

We now have to adapt to the case $p \neq 2$. In this case, setting $h(z) = \frac{1}{p}|z|^p$, we have $T_t(x) = x - t\nabla h^*(\nabla\phi(x))$. Notice that both h and h^* are $C^2(\mathbb{R}^d \setminus \{0\})$. First, we notice that we can decompose, up to negligible sets, Ω into two measurable parts: the set Ω' where $\nabla\phi = 0$ and the set Ω'' where $\nabla\phi \neq 0$. The analysis that we do is very much similar to what we did in Section 3.3.2. Ω' is the set where T is the identity, and so is T_t. Since T_t is injective, the density is preserved on this set, and we can apply Lemma 7.3.3 with $A = 0$. Ω'', on the contrary, can be decomposed into a countable union of sets Ω_{ij} where $x \in B_i$, $T(x) \in B_j$, where $(B_i)_i$ is a countable family of balls generating the topology of $\mathbb{R}^d$, and only take pairs such that $B_i \cap B_j = \emptyset$. On this sets $\phi = \psi^c$ also coincides with ϕ_{ij} defined as the restriction to B_i of the function $x \mapsto \inf_{y\in B_j} h(x-y) - \psi(y)$, which is $\lambda-$concave for $\lambda = \sup\{||D^2h(z)|| : z \in B_j - B_i\}$. This proves that ϕ_{ij} has the same regularity of concave functions, and that $\nabla\phi$ and $\nabla h^*(\nabla\phi)$ are countably Lipschitz on Ω''. We can also compute $D^2\phi$ a.e.

If we fix a point $x_0 \in \Omega''$ where $\nabla\phi(x_0)$ and $D^2\phi(x_0)$ exist, then we can write $\phi(x) \leq h(x - T(x_0)) - \psi(T(x_0))$, an inequality which is true for every x, with equality at $x = x_0$. In particular we get $D^2\phi(x_0) \leq D^2h(x_0 - T(x_0)) = D^2h(\nabla h^*(\nabla\phi(x_0)))$. From general properties of Legendre transforms, we have $D^2h(\nabla h^*(z)) = [D^2h^*(z)]^{-1}$ (just differentiate the relation $\nabla h(\nabla h^*(z)) = z$). Hence, we can apply Lemma 7.3.3 with $A(x) = -D^2h^*(\nabla\phi(x))D^2\phi(x)$, which is diagonalizable and has eigenvalues larger than -1 (see below). □

Memo – *Diagonalizing products of symmetric matrices*

it is well known that Symmetric matrices can be diagonalized on $\mathbb{R}$. A trickier result is the following

Theorem: If A, B are symmetric $d \times d$ matrices, and A is positive definite, then AB is diagonalisable.

We do not prove it, but just prove the following weaker result.

Proposition: If A, B are symmetric $d \times d$ matrices, and A is positive definite, then AB has real eigenvalues.

Proof: suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of AB, i.e. $ABv = \lambda v$ for a vector $v \in \mathbb{C}^n \setminus \{0\}$. Take the Hermitian product with $A^{-1}v$, i.e. $\bar{v}^t Bv = \lambda \bar{v}^t A^{-1} v$, where the symbol ^{t} denotes transposition in the sense of matrices. Notice that, by diagonalizing independently B and A^{-1}, both the terms $\bar{v}^t Bv$ and $\bar{v}^t A^{-1}v$ are real, and $\bar{v}^t A^{-1} v > 0$. Hence, $\lambda \in \mathbb{R}$.

Another important point that we need is the following.

Proposition: If $B \leq A^{-1}$ (in the sense of symmetric matrices), then all the eigenvalues λ_i of AB satisfy $\lambda_i \leq 1$

Proof: again, write $ABv = \lambda v$ and deduce $\bar{v}^t Bv = \lambda \bar{v}^t A^{-1} v$. This implies $\bar{v}^t A^{-1} v \geq \bar{v}^t Bv = \bar{v}^t A^{-1} v$, which implies $\lambda \leq 1$.

Let us see some easy example of convex functions satisfying the assumptions of Theorem 7.3.4. For instance

- for any $p > 1$, the function $f(t) = t^p$ satisfies these assumptions, since $s^d f(s^{-d}) = s^{-d(p-1)}$ is convex and decreasing;
- the function $f(t) = t \ln t$ also satisfies the assumptions since $s^d f(s^{-d}) = -d \ln s$ is convex and decreasing;
- if $1 - \frac{1}{d} \leq m < 1$ the function $f(t) = -t^m$ is convex, and if we compute $s^d f(s^{-d}) = -t^{m(1-d)}$ we get a convex and decreasing function since $m(1-d) < 1$; yet these functions lack the superlinearity assumption, but this does not prevent us from applying the same proof of Theorem 7.3.4 to the case where μ_0 and μ_1 are supposed to be a priori absolutely continuous.

Let us see some consequences of Theorem 7.3.4 in the case $f(t) = t^p$.

Proposition 7.3.5. *Consider an exponent $1 < p \leq +\infty$ and two probability measures $\mu_0, \mu_1 \in L^p(\Omega)$ (in the sense that they are absolutely continuous and their densities are L^p). Then the measures μ_t are L^p aussi and $||\mu_t||_{L^p} \leq \max\{||\mu_0||_{L^p}, ||\mu_1||_{L^p}\}$.*

Proof. The case $p < +\infty$ is an easy consequence of Theorem 7.3.4. Actually, if we use $f(t) = t^p$ we have $\mathcal{F}(\mu) = ||\mu||^p_{L^p}$. Hence

$$||\mu_t||^p_{L^p} \leq (1-t)||\mu_0||^p_{L^p} + t||\mu_1||^p_{L^p} \leq (\max\{||\mu_0||_{L^p}, ||\mu_1||_{L^p}\})^p .$$

This allows to get the desired L^p estimate.

The case $p = +\infty$ is just obtained by passing to the limit $p \to +\infty$. □

7.4 Discussion

7.4.1 A case study: $\min F(\rho) + W_2^2(\rho, \nu)$

7.4.2 Brunn-Minkowski inequality

Another interesting consequence of the displacement convexity criterion is a transport-based proof of a well-known geometric inequality called Brunn-Minkowski inequality. This inequality states that for every two sets $X, Y \subset \mathbb{R}^d$ then

$$\mathcal{L}^d(X+Y)^{1/d} \geq \mathcal{L}^d(X)^{1/d}+\mathcal{L}^d(Y)^{1/d}, \quad \text{where } X+Y = \{x+y \,:\, x \in X,\, y \in Y\}.$$

This inequality may be proven in the following way: consider the measure μ with constant density $1/\mathcal{L}^d(X)$ on X and ν with constant density $1/\mathcal{L}^d(Y)$ on Y. Consider the measure $\mu_{1/2}$ obtained at time $t = 1/2$ on the constant speed geodesic between them. Since $\mu_{1/2} = (\frac{1}{2}id + \frac{1}{2}T)_{\#}\mu_0$ we know that $\mu_{1/2}$ is concentrated on $\frac{1}{2}X+\frac{1}{2}Y$. Moreover, it is clear that $\mu_{1/2}$ is absolutely continuous and we call ρ its density (by the way, we know $\rho \in L^\infty$ because $\mu, \nu \in L^\infty$). Notice that, for convex f, we have

$$\frac{1}{\mathcal{L}^d(\frac{1}{2}X + \frac{1}{2}Y)} \int_{\frac{1}{2}X+\frac{1}{2}Y} f(\rho(x))dx \geq f\left(\frac{\int_{\frac{1}{2}X+\frac{1}{2}Y} \rho(x)dx}{\mathcal{L}^d(\frac{1}{2}X + \frac{1}{2}Y)}\right) = f\left(\mathcal{L}^d\left(\frac{1}{2}X + \frac{1}{2}Y\right)^{-1}\right).$$

Geodesic convexity of $\mathcal{F}$ for $f(t) = -t^{1-1/d}$ implies

$$\begin{aligned} &-\frac{1}{2}\mathcal{L}^d(X)^{1/d} - \frac{1}{2}\mathcal{L}^d(Y)^{1/d} = \frac{1}{2}\mathcal{F}(\mu) + \frac{1}{2}\mathcal{F}(\nu) \geq \mathcal{F}(\mu_{1/2}) \\ &= -\int_{\frac{1}{2}X+\frac{1}{2}Y} f(\rho(x))dx \geq \mathcal{L}^d(\frac{1}{2}X+\frac{1}{2}Y) f\left(\mathcal{L}^d\left(\frac{1}{2}X + \frac{1}{2}Y\right)^{-1}\right) = -\mathcal{L}^d\left(\frac{1}{2}X + \frac{1}{2}Y\right)^{1/d}. \end{aligned}$$

If we multiply this inequality by -2 on each side, using scaling properties of the $\mathcal{L}^d$ measure, we get exactly Brunn-Minkowski inequality.

7.4.3 Displacement convexity, game theory and spatial economics: urban equilibria

Chapter 8

Gradient Flows

8.1 Gradient flows in $\mathbb{R}^n$ and in metric spaces

First of all, let us present what a gradient flow is in the simplest situation. Suppose you have a function $F : \mathbb{R}^n \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}^n$. A gradient flow is an evolution stemming from x_0 and always moving in the direction where F decreases the most, thus "minimizing" F gradually, starting from x_0. More precisely, it is just the solution of the Cauchy problem

$$\begin{cases} x'(t) = -\nabla F(x(t)) & \text{for } t > 0, \\ x(0) = x_0. \end{cases}$$

This is a standard Cauchy problem which has a unique solution if ∇F is Lipschitz continuous, i.e. if $F \in C^{1,1}$. We will see that existence and uniqueness can also hold without this strong assumption, thanks to the variational structure of the equation.

A first interesting property is the following, concerning uniqueness and estimates.

Proposition 8.1.1. *Suppose that F is convex and let x_1 and x_2 be two solutions of $x'(t) = -\nabla F(x(t))$ (if F is not differentiable we can consider $x'(t) \in \partial F(x(t))$). Then we have $|x_1(t) - x_2(t)| \le |x_1(0) - x_2(0)|e^{-\lambda t}$. In particular this gives uniqueness of the solution of the Cauchy problem.*

Proof. Let us consider $g(t) = \frac{1}{2}|x_1(t) - x_2(t)|^2$ and differentiate it. We have

$$g'(t) = (x_1(t)-x_2(t))\cdot(x_1'(t)-x_2'(t)) = -(x_1(t)-x_2(t))\cdot(\nabla F(x_1(t))-\nabla F(x_2(t))).$$

Here we use the basic property of gradient of convex functions, i.e. that for every x, y we have

$$(x_1 - x_2) \cdot (\nabla F(x_1) - \nabla F(x_2)) \ge 0.$$

More generally, it is also true that for every x_1, x_2 and every $p_1 \in \partial F(x_1)$, $p_2 \in \partial F(x_2)$, we have

$$(x_1 - x_2) \cdot (p_1 - p_2) \geq 0.$$

From these considerations, we obtain $g'(t) \leq 0$ and $g(t) \leq g(0)$. This gives the first part of the thesis.

Then, if we take two different solutions of the same Cauchy problem, we have $x_1(0) = x_2(0)$, and this implies $x_1(t) = x_2(t)$ for any $t > 0$, which gives uniqueness. □

Remark 25. From the same proof, one can also deduce uniqueness and stability estimates in the case where F is only $\lambda-$convex (i.e. $x \mapsto F(x) - \frac{\lambda}{2}|x|^2$ is convex). Indeed, in this case we obtain $|x_1(t) - x_2(t)| \leq |x_1(0) - x_2(0)|e^{-\lambda t}$, which also proves, if $\lambda > 0$, exponential convergence to the unique minimizer of F. The key point is that, if F is $\lambda-$convex it is easy, by applying the monotonicity inequalities above to $x \mapsto F(x) - \frac{\lambda}{2}|x|^2$, to get

$$(x_1 - x_2) \cdot (\nabla F(x_1) - \nabla F(x_2)) \geq \lambda |x_1(t) - x_2(t)|^2.$$

This implies $g'(t) \leq -2\lambda g(t)$ and allows to conclude, by Gronwall's lemma, $g(t) \leq g(0)e^{-2\lambda t}$. For the exponential convergence, if $\lambda > 0$ then F is coercive and admits a minimizer, which is unique by strict convexity. Let us call it $\bar{x}$. Take a solution $x(t)$ and compare it to the constant curve $\bar{x}$, which is a solution since $0 \in \partial F(\bar{x})$. Then we get $|x_1(t) - \bar{x}| \leq e^{-\lambda t}|x_1(0) - \bar{x}|$.

Another interesting feature of those particular Cauchy problems which are gradient flows is their discretization in time. Actually, one can fix a small time step parameter $\tau > 0$ and look for a sequence of points $(x_k^\tau)_k$ defined through

$$x_{k+1}^\tau \in \operatorname{argmin}_x F(x) + \frac{|x - x_k^\tau|^2}{2\tau}.$$

Under very mild assumptions on F (l.s.c. and some lower bounds, for instance $F(x) \geq C_1 - C_2|x|^2$) these problems admit a solution (notice that one can consider τ arbitrarily small, which gives coercivity for the sum of F plus the quadratic part). By the way, if F is $\lambda-$convex this also gives uniqueness, even if λ is negative, since for τ small the sum will be strictly convex.

We can interpret this sequence of points as the values of the curve $x(t)$ at times $t = 0, \tau, 2\tau, \dots, k\tau, \dots$. It happens that the optimality conditions of the recursive minimization exactly give a connection between these minimization problems and the equation, since we have

$$x_{k+1}^\tau \in \operatorname{argmin} F(x) + \frac{|x - x_k^\tau|^2}{2\tau} \Rightarrow \nabla F(x_{k+1}^\tau) + \frac{x_{k+1}^\tau - x_k^\tau}{\tau} = 0,$$

i.e.

$$\frac{x_{k+1}^\tau - x_k^\tau}{\tau} = -\nabla F(x_{k+1}^\tau).$$

This expression is exactly the discrete-time implicit Euler scheme for solving $x' = -\nabla F(x)$! It would be possible to prove that, for $\tau \to 0$, the sequence we found, suitably interpolated, converges to the solution of the problem. It even suggests how to define solutions for functions F which are only l.s.c., with no gradient at all!

But a huge advantage of this discretized formulation is also that it can easily be adapted to metric spaces. Actually, if one has a metric space (X, d) and a l.s.c. function $F : X \to \mathbb{R} \cup \{+\infty\}$, one can define

$$x_{k+1}^\tau \in \operatorname{argmin}_x F(x) + \frac{d(x, x_k^\tau)^2}{2\tau} \tag{1.1}$$

and study the limit as $\tau \to 0$. Obviously the assumption on F have to be adapted, since we need existence of the minimizers and hence a little bit of compactness. But if X is compact than everything works for F only l.s.c..

We can consider two different interpolations of the points x_k^τ

$$x^\tau(t) = x_k^\tau, \quad \tilde{x}^\tau(t) = \omega_{x_{k-1}^\tau, x_k^\tau}\left(\frac{t-(k-1)\tau}{\tau}\right) \quad \text{for } t \in](k-1), \tau, k\tau],$$

where $\omega_{x,y}(s)$ denotes any constant speed geodesic connecting a point x to a point y, parametrized on the unit interval $s \in [0, 1]$. The interpolation $\tilde{x}^\tau$ only makes sense in spaces where geodesic exist, obviously. It is in this case a continuous (locally Lipschitz) curve, which coincides with the piecewise constant interpolation x^τ at times $t = k\tau$.

De Giorgi, in [59], defined what he called *Generalized Minimizing Movements*:

Definition 19. A curve $x : [0, T] \to X$ is said to be a Generalized Minimizing Movement if there exists a sequence of time steps $\tau_j \to 0$ such that the the piecewise constant interpolations x^{τ_j}, built from a sequence of solutions of the iterated minimization scheme (1.1), uniformly converge to x on $[0, T]$.

Compactness results guaranteeing the existence of Generalized Minimizing Movements are derived from a simple property giving a Hölder behaviour for the curves x^τ: for every τ and every k, the optimality of x_{k+1}^τ provides

$$F(x_{k+1}^\tau) + \frac{d(x_{k+1}^\tau, x_k^\tau)^2}{2\tau} \le F(x_k^\tau), \tag{1.2}$$

which implies
$$d(x^\tau_{k+1},x^\tau_k)^2 \le 2\tau\left(F(x^\tau_k)-F(x^\tau_{k+1})\right).$$
If $F(x_0)$ is finite and F is bounded from below, taking the sum over k, we have
$$\sum_{k=0}^{l} d(x^\tau_{k+1},x^\tau_k)^2 \le 2\tau\left(F(x^\tau_0)-F(x^\tau_{l+1})\right)\le C\tau.$$
Cauchy-Schwartz inequality implies, for $t<s$, $t\in](k-1)\tau,k\tau]$ and $s\in](l-1)\tau,l\tau]$ (hence $|l-k|\le \frac{|t-s|}{\tau}+1$)

$$d(x^\tau(t),x^\tau(s)) \le \sum_{k=0}^{l} d(x^\tau_{k+1},x^\tau_k) \le \left(\sum_{k=0}^{l} d(x^\tau_{k+1},x^\tau_k)^2\right)^{1/2}\left(\frac{|t-s|}{\tau}+1\right)^{1/2}$$
$$\le C\left(|t-s|^{1/2}+\sqrt{\tau}\right).$$

This means that the curves x^τ - if we forget for a while that they are actually discontinuous - are morally equi-Hölder continuous of exponent $1/2$. The situation is even clearer for $\tilde{x}^\tau$. Indeed, we have

$$\tau\left(\frac{d(x^\tau_{k-1},x^\tau_k)}{\tau}\right)^2 = \int_{(k-1)\tau}^{k\tau} |(\tilde{x}^\tau)'|^2(t)dt,$$

which implies, by summing up

$$\int_0^T |(\tilde{x}^\tau)'|^2(t)dt \le C.$$

This means that the curves $\tilde{x}^\tau$ are bounded in $H^1([0,T];X)$, which also implies a $C^{0,\frac{1}{2}}$ bound by usual Sobolev embedding.

If the space X, the distance d, and the functional F sont connus explicitement, dans certains cas il est déjà possible de passer à la limite dans les conditions d'optimalité de chaque problème d'optimisation en temps discret, et de caractériser les courbes (ou la courbe) limite $x(t)$. Il sera possible de faire ainsi dans le cadre des mesures de probabilité dont il est question dans la section 2, mais pas dans d'autres cas. De fait, sans un petit peu de structure (différentielle) sur l'espace X, cela est pratiquement impossible. Si l'on souhaite développer une théorie générale pour les flots de gradient dans les espaces métriques, il faut utiliser des instruments plus fins, qui permettent vraiment de caractériser, à l'aide seulement de quantités métriques, le fait qu'une courbe continue $x(t)$ soit un flot de gradient.

Le livre d'Ambrosio-Gigli-Savaré [5], et en particulier sa première partie (la deuxième étant dédiée aux espaces de mesures de probabilité), se donne exactement cet objectif.

Nous présentons ici deux inégalités qui sont satisfaites par les flots de gradient dans le cas euclidien régulier, et qui peuvent être utilisées comme définition de flot de gradient dans un cadre métrique, toutes les quantités qui y apparaissent ayant une contrepartie métrique.

La première observation est la suivante: pour toute courbe $x(t)$ on a

$$\begin{aligned} F(x(s)) - F(x(t)) = \int_s^t -\nabla F(x(r)) \cdot x'(r)\, dr &\le \int_s^t |\nabla F(x(r))||x'(r)|\, dr \\ &\le \int_s^t \left(\frac{1}{2}|x'(r)|^2 + \frac{1}{2}|\nabla F(x(r))|^2\right) dr. \end{aligned}$$

Ci-dessus, la première inégalité est une égalité si et seulement si $x'(r)$ et $\nabla F(x(r))$ sont des vecteurs de directions opposées pour presque tout r, et la deuxième est une égalité si et seulement si leurs modules sont égaux. Ainsi, la condition, appelée EDE (*Energy Dissipation Equality*)

$$F(x(s)) - F(x(t)) = \int_s^t \left(\frac{1}{2}|x'(r)|^2 + \frac{1}{2}|\nabla F(x(r))|^2\right) dr, \quad \text{pour tous } s < t$$

(ou même la simple inégalité $F(x(s)) - F(x(t)) \ge \int_s^t \left(\frac{1}{2}|x'(r)|^2 + \frac{1}{2}|\nabla F(x(r))|^2\right) dr$) est équivalente à $x' = -\nabla F(x)$ p.p., et pourrait être prise comme définition de flot de gradient. This is what is done in a series of works by Ambrosio, Gigli and Savaré, and in particular in their book [5]. Developing this (huge) theory is not among the scopes of this book. The reader who is curious about this abstract approach but wants to find a synthesis of their work from the point of view of the author can have a look at [**?**]. The role of the different parts of the theory with respect to the possible applications is clarified as far as possible (we also discuss in short some of these issues in the Discussion Section). Unfortunately, some knowledge of French is required (even if not forbidden, English is unusual in the Bourbaki seminar, since "Nicolas Bourbaki a une préférence pour le français").

8.2 Gradient flows for W_2, derivation of the PDE

In this section we give a short and sketchy presentation of what can be done when we consider the gradient flow of a functional $F : \mathcal{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$. The functional will be supposed l.s.c. for the weak convergence of probabilities and Ω compact. In particular, we will give an heuristics on how to derive a PDE from the optimality conditions at each time step. This is obviously something that we can only do in this particular metric space and does not work in the general case of an abstract metric space.

Indeed, we exploit the fact that we know, from Chapter 5, that all absolutely continuous curves in the space $(\mathcal{P}(\Omega), W_2)$ are solution of the continuity equation $\partial\rho + \nabla\cdot(\rho v) = 0$, for a suitable vector field v. The goal is to identify the vector field v_t, which will depend on the measure ρ_t, in a way which is ruled by the functional F.

We consider the iterated minimization scheme

$$\rho^\tau_{k+1} \in \operatorname{argmin} F(\rho) + \frac{W_2^2(\rho, \rho^\tau_k)}{2\tau}.$$

Each of these problems has a solution, by compactness of $\mathcal{P}(\Omega)$ and semicontinuity of the criterion. Exactly as we did for the Euclidean case (F defined on $\mathbb{R}^n$), we want to study the optimality conditions of these problems so as to guess the limit equation.

We recall the notation $\frac{\delta G}{\delta\rho}(\rho)$ for the first variation of a functional $G : \mathcal{P}(\Omega) \to \mathbb{R}\cup\{+\infty\}$.

We will suppose that $\frac{\delta F}{\delta\rho}$ is known and try to write the limit equation in terms of this operator.

Now, take an optimal measure $\bar\rho$ for the minimization problem at step k and compute variations with respect to perturbations of the form $\rho_\varepsilon := (1-\varepsilon)\bar\rho+\varepsilon\tilde\rho$, where $\tilde\rho$ is any other probability measure. This means choosing a perturbation $\chi = \tilde\rho - \bar\rho$, which guarantees that, for $\varepsilon > 0$, the measure ρ_ε is actually a probability over Ω.

We now compute the first variation and, due to optimality, we have

$$0 \le \frac{d}{d\varepsilon}\left(F(\bar\rho+\varepsilon\chi) + \frac{1}{\tau}\frac{W_2^2(\bar\rho+\varepsilon\chi, \rho^\tau(k))}{2}\right)_{|\varepsilon=0} = \int\left(\frac{\delta F}{\delta\rho}(\rho) + \frac{\phi}{\tau}\right)d\chi.$$

From the arbitrariness of χ , which is a measure with zero mass (actually, it is not an arbitrary zero-mass measure, since we need to impose positivity of ρ_ε, but we will fix this point when doing the rigorous proof), we deduce from the condition above that $\frac{\delta F}{\delta\rho}(\rho) + \frac{\phi}{\tau}$ must be constant.

If we combine the fact that the above sum is constant, and that we have $T(x) = x - \nabla\phi(x)$ for the optimal T, we get

$$\frac{T(x)-x}{\tau} = -\frac{\nabla\phi(x)}{\tau} = \nabla\big(\frac{\delta F}{\delta\rho}(\rho)\big)(x). \tag{2.3}$$

We will denote by $-v$ the ratio $\frac{T(x)-x}{\tau}$. Why? because, as a ratio between a displacement and a time step, it has the meaning of a velocity, but since it is the displacement associated to the transport from ρ^τ_{k+1} to ρ^τ_k, it is better to view it rather as a backward velocity (which justifies the minus sign).

Since here we have $v = -\nabla\big(\frac{\delta F}{\delta\rho}(\rho)\big)$, this suggests (and we will analyze it in the next section) that at the limit $\tau \to 0$ we will find a solution of

$$\partial_t\rho - \nabla\cdot\big(\rho\,\nabla\big(\frac{\delta F}{\delta\rho}(\rho)\big)\big) = 0.$$

Before entering into the details making the above approach rigorous (next section), we want to present some examples of this kind of equations. We will consider two functionals that we already analyzed in Chapter 7, and more precisely

$$\mathcal{F}(\rho) = \int f(\rho(x))dx, \quad \text{and} \quad \mathcal{V}(\rho) = \int V(x)d\rho.$$

In this case we already saw that we have

$$\frac{\delta\mathcal{F}}{\delta\rho}(\rho) = f'(\rho), \quad \frac{\delta\mathcal{V}}{\delta\rho}(\rho) = V.$$

An interesting example is the case $f(t) = t\ln t$. In such a case we have $f'(t) = \ln t + 1$ and $\nabla(f'(\rho)) = \frac{\nabla\rho}{\rho}$: this means that the gradient flow equation associated to the functional $\mathcal{F}$ would be the *Heat Equation* $\partial_t\rho - \Delta\rho = 0$, and that for $\mathcal{F}+\mathcal{V}$ we would have the *Fokker-Planck Equation* $\partial_t\rho - \Delta\rho - \nabla\cdot(\rho\nabla V) = 0$. We will see a list of other Gradient Flow equations in the Discussion session, including the so-called *Porous media* equation, obtained for other choices of f.

Notice that all these PDEs come accompanied by Neumann boundary conditions $\rho\frac{\partial}{\partial_\nu}(\frac{\delta F}{\delta\rho}(\rho)) = 0$ on $\partial\Omega$, as a consequence of the Neumann boundary conditions for the continuity equation of Chapter 5. We will see in the Discussion Session a case of extension to Dirichlet boundary conditions.

We finish this section with some philosophical thoughts. Why to study some PDEs considering them as gradient flows for the distance W_2? there are at least three reasons. The first is that it allows to give an existence result (of a weak solution of such a PDE), with the technique that we will see in the next session. This is obviously useless when the equation is already well-known, as it is the case for the Heat equation, or its Fokker-Planck variant. But the same strategy could be applied to variants of the same equations (or to similar equations in stranger spaces, as it is nowadays done for the Heat flow in metric measure spaces). It is also very useful when the equation is of new type (we will see the case of the crowd motion models in the Discussion Section). The second goal could be to derive properties of the flow and of the solution, once we know that it is a gradient flow. A simple one is the fact that $t \mapsto F(\rho_t)$ must be decreasing in time. This can be often deduced from the PDE in different ways and does not require in general to have a gradient flow. The third goal concerns numerics: the discretized scheme to get a gradient flow is itself a sort of algorithm to approximate the solution. Even if we do not claim that it can be efficiently applied as it is in order to do numerics, it can be a source of inspiration for more specific algorithms for these equations.

Finally, all the procedure we presented is related to the study and the existence of weak solutions. What about uniqueness? we stress that in PDE applications the important point is to prove uniqueness for weak solutions of the continuity equation with $v = -\nabla\frac{\delta F}{\delta\rho}$ (i.e., we do not care at the metric structure, and at the definitions of EVI and EDE). In some cases, this uniqueness could be studied independently of the gradient-flow structure (this is the case for the Heat equation, for instance). anyway, should we use PDE approaches for weak solutions, or abstract approaches in metric spaces, it turns out that usually uniqueness is linked to some kind of convexity, or $\lambda-$convexity, and in particular to displacement convexity. This is why a large part of the theory has been developed for $\lambda-$geodesically convex functionals.

8.3 Analysis of the Fokker-Planck case

We consider here the case study of the Fokker-Planck equation, which is the gradient flow of the functional

$$J(\rho) = \int_\Omega \rho \ln \rho + \int_\Omega V \, d\rho,$$

where V is a Lipschitz function on the compact domain Ω, starting from a given measure $\rho_0 \in \mathcal{P}(\Omega)$ such that $F(\rho_0) < +\infty$.

We stress that the first term of the functional is defined as

$$\mathcal{F}(\rho) := \begin{cases} \int_\Omega \rho(x) \ln \rho(x) dx & \text{if } \rho \ll \mathcal{L}^d, \\ +\infty & \text{otherwise,} \end{cases}$$

where we identify the measure ρ with its density, when it is absolutely continuous. This functional is l.s.c. thanks to Proposition 7.1.8.

Semicontinuity allows to establish the following

Proposition 8.3.1. *The functional J has a unique minimum over $\mathcal{P}(\Omega)$. In particular J is bounded from below. Moreover, for each $\tau > 0$ the following sequences of optimization problems recursively defined is well-posed*

$$\rho^\tau(k+1) \in \operatorname{argmin}_\rho J(\rho) + \frac{W_2^2(\rho, \rho^\tau(k))}{2\tau}, \tag{3.4}$$

which means that there is a minimizer at every step, and this minimizer is unique.

Proof. Just apply the direct method, noticing that $\mathcal{P}(\Omega)$ is compact for the weak convergence, which is the same as the convergence for the W_2 distance. And for this convergence $\mathcal{F}$ is l.s.c. and the other terms are continuous. This gives at the same time the existence of a minimizer for J and of a solution to each of the above minimization problems (3.4). Uniqueness comes from the fact that all the functionals are convex (in the usual sense) and $\mathcal{F}$ is strictly convex. □

Optimality conditions at each time step We will use Propositions 7.2.3 and 7.2.4, and to do so we first need to prove the following result.

Lemma 8.3.2. *Any minimizer $\bar\rho$ in (3.4) must satisfy $\bar\rho > 0$ a.e.*

Proof. Consider the measure $\tilde\rho$ with constant positive density c in Ω (i.e. the density equals $|\Omega|^{-1}$). Let us define ρ_ε as $(1-\varepsilon)\bar\rho + \varepsilon\tilde\rho$ an compare $\bar\rho$ to ρ_ε.

We write

$$\mathcal{F}(\bar\rho) - \mathcal{F}(\rho_\varepsilon) \le \int V d\rho_\varepsilon - \int V d\bar\rho + \frac{W_2^2(\rho_\varepsilon, \rho_k)}{2\tau} - \frac{W_2^2(\bar\rho, \rho(k))}{2\tau}.$$

The Wasserstein term in the right hand side may be estimated by convexity, which gives

$$\frac{W_2^2(\rho_\varepsilon, \rho_k)}{2\tau} \le (1-\varepsilon)\frac{W_2^2(\bar\rho, \rho_k)}{2\tau} + \varepsilon\frac{W_2^2(\bar\rho, \rho_k)}{2\tau}.$$

This shows that the right hand side is estimated by $C\varepsilon$ and we get

$$\int f(\bar\rho) - f(\rho_\varepsilon) \le C\varepsilon$$

where we use $f(t) = t\ln t$. Since f is convex we write, on the set $A = \{x \in \Omega \,:\, \bar\rho(x) > 0\}$, $f(\bar\rho(x)) - f(\rho_\varepsilon(x)) \ge (\bar\rho(x) - \rho_\varepsilon(x))f'(\rho_\varepsilon(x)) = \varepsilon(\bar\rho(x) -$

$\tilde{\rho}(x))(1 + \ln \rho_\varepsilon(x))$. On the set $B = \{x \in \Omega : \bar{\rho}(x) = 0\}$ we simply write $f(\bar{\rho}(x)) - f(\rho_\varepsilon(x)) = -\varepsilon c \ln(\varepsilon c)$. This allows to write

$$-\varepsilon c \ln(\varepsilon c)|B| + \varepsilon \int_A (\bar{\rho}(x) - c)(1 + \ln \rho_\varepsilon(x))dx \le C\varepsilon.$$

Notice that the function $(\bar{\rho}(x) - c)(1 + \ln \rho_\varepsilon(x))$ is always larger than $(\bar{\rho}(x)-c)(1+\ln c)$ (just distinguish between the case $\bar{\rho}(x) \ge c$ and $\bar{\rho}(x) \le c$). Thus, after dividing by ε, we have

$$-c \ln(\varepsilon c)|B| + \int_A (\bar{\rho}(x) - c)(1 + \ln c)dx \le C.$$

Letting $\varepsilon \to 0$ provides a contradiction, unless $|B| = 0$. □

We can now compute the first variation and give optimality conditions on the optimal $\rho^\tau(k+1)$.

Proposition 8.3.3. *The optimal measure $\rho^\tau(k+1)$ in (3.4) satisfies*

$$\ln(\rho^\tau(k+1)) + V + \frac{\bar{\phi}}{\tau} = \textit{constant a.e.}$$

where $\bar{\phi}$ is the (unique) Kantorovitch potential from $\rho^\tau(k+1)$ to $\rho^\tau(k)$. In particular, it T_k^τ is the optimal transport from $\rho^\tau(k+1)$ to $\rho^\tau(k)$, then it satisfies

$$v^\tau(k) := \frac{id - T_k^\tau}{\tau} = -\nabla\left(\ln(\rho(k+1)) + V\right) \text{ a.e.} \tag{3.5}$$

Proof. Take the optimal measure $\bar{\rho} := \rho^\tau(k+1)$ and compute variations with respect to perturbations of the form $\rho_\varepsilon := (1-\varepsilon)\bar{\rho} + \varepsilon\tilde{\rho}$, where $\tilde{\rho}$ is any other probability measure, with L^∞ density (so as to ensure every integrability condition). This means choosing a perturbation $\chi = \tilde{\rho} - \bar{\rho}$, which guarantees that, for $\varepsilon > 0$, the measure ρ_ε is actually a probability over Ω.

We now compute the first variation and, due to optimality, we have

$$0 \le \frac{d}{d\varepsilon}\left(J(\bar{\rho} + \varepsilon\chi) + \frac{1}{\tau}\frac{W_2^2(\bar{\rho} + \varepsilon\chi, \rho^\tau(k))}{2}\right)_{|\varepsilon=0} = \int \left(\frac{\delta J}{\delta\rho}(\bar{\rho}) + \frac{\bar{\phi}}{\tau}\right) d\chi.$$

If we set for a while $\psi = \frac{\delta J}{\delta \rho}(\bar{\rho}) + \frac{\bar{\phi}}{\tau}$ we would have

$$\int \psi \, d\chi \ge 0 \quad \text{i.e.} \int \psi \, d\tilde{\rho} \ge \int \psi \, d\bar{\rho} \quad \text{for all } \tilde{\rho} \in L^\infty(\Omega). \tag{3.6}$$

Set $l = ess\inf\psi$: on the one hand, the right hand side in (3.6) is larger than l, on the other hand, choosing $\tilde{\rho}$ concentrated on a set $\{\psi < l+\varepsilon\}$ (which has positive measure), we can get the left hand side smaller than $l+\varepsilon$. Hence, we finally get

$$l = \int \psi \, d\bar{\rho} \text{ and } \psi \geq l \quad \bar{\rho} - a.e.$$

This gives $\psi = l$ a.e. w.r.t. $\bar{\rho}$, but since we know $\bar{\rho} > 0$ a.e., we get $\psi = l$ a.e.

This gives the first part of the thesis if we replace $\frac{\delta F}{\delta \rho}$ with $f'(\rho) + V = \ln(\rho) + 1 + V$. In particular it implies that $\rho(k+1)$ is Lipschitz continuous, since it holds

$$\rho^\tau(k+1)(x) = \exp\left(C - V(x) - \frac{\bar{\phi}(x)}{\tau}\right).$$

Then, one differentiates and gets the equality

$$\nabla\bar{\phi}(x) = \frac{id - T_k^\tau}{\tau} = -\nabla\left(\ln(\rho^\tau(k+1)) + V\right) \text{ a.e.}$$

and this allows to conclude. □

Interpolation between time steps and uniform estimates. Let us collect some other tools

Proposition 8.3.4. *For any $\tau > 0$, the sequence of minimizers satisfies*

$$\sum_k \frac{W_2^2(\rho^\tau(k+1), \rho^\tau(k))}{\tau} \leq C = 2(F(\rho_0) - \inf F),$$

where the constant C is finite and independent of ρ.

Proof. This is obtained by comparing the optimizer $\rho^\tau(k+1)$ to the previous measure ρ_k^τ. We get

$$F(\rho^\tau(k+1)) + \frac{W_2^2(\rho^\tau(k+1), \rho^\tau(k))}{2\tau} \leq F(\rho_k^\tau),$$

which implies

$$\sum_k \frac{W_2^2(\rho^\tau(k+1), \rho^\tau(k))}{\tau} \leq \sum_k 2(F(\rho_k^\tau) - F(\rho^\tau(k+1))),$$

and this last sum is telescopic and gives the thesis. □

Let us define two interpolations between the measures ρ^τ_k.

With this time-discretized method, we have obtained, for each $\tau > 0$, a sequence $(\rho^\tau(k))_k$. We can use it to build at least two interesting curves in the space of measures:

- first we can define some piecewise constant curves, i.e. $\rho^\tau_t := \rho^\tau(k+1)$ for $t \in]k\tau, (k+1)\tau]$; associated to this curve we also define the velocities $v^\tau_t = v^\tau(k+1)$ for $t \in]k\tau, (k+1)\tau]$, where $v^\tau(k)$ is defined as in (3.5): $v^\tau(k) = (id - T^\tau_k)/\tau$, taking as T^τ_k the optimal transport from $\rho^\tau(k+1)$ to $\rho^\tau(k)$; we also define the momentum variable $E^\tau = \rho^\tau v^\tau$;

- then, we can also consider the densities $\tilde{\rho}^\tau_t$ that interpolate the discrete values $(\rho^\tau(k))_k$ along geodesics:

$$\tilde{\rho}^\tau_t = \left(\frac{k\tau - t}{\tau} v^\tau(k) + id\right)_{\#} \rho^\tau(k), \quad \text{for } t \in](k-1)\tau, k\tau[; \tag{3.7}$$

  the velocities $\tilde{v}^\tau_t$ are defined so that $(\tilde{\rho}^\tau, \tilde{v}^\tau)$ satisfy the continuity equation and $||\tilde{v}^\tau_t||_{L^2(\tilde{\rho}^\tau_t)} = |(\tilde{\rho}^\tau)'|(t)$, taking

$$\tilde{v}^\tau_t = v^\tau_t \circ \left((k\tau - t)v^\tau(k) + id\right)^{-1};$$

  as before, we define: $\tilde{E}_\tau = \tilde{\rho}^\tau \tilde{v}^\tau$.

After these definitions we consider some a priori bounds on the curves and the velocities that we defined. We start from some estimates which are standard in the framework of Minimizing Movements (this is the name of the discrete procedure which minimizes the funtional plus a quadratic penalization on the distance, see [1, 59]).

Notice that the velocity (i.e., metric derivative) of $\tilde{\rho}^\tau$ is constant on each interval $]k\tau, (k+1)\tau[$ and equal to $W_2(\rho^\tau(k+1), \rho^\tau(k))/\tau$. This distance W_2 also equals $(\int |id - T^\tau_k|^2 d\rho^\tau(k+1))^{1/2} = \tau ||v^\tau_{k+1}||_{L^2(\rho^\tau(k+1))}$, which gives

$$||\tilde{v}^\tau_t||_{L^2(\tilde{\rho}^\tau_t)} = |(\tilde{\rho}^\tau)'|(t) = \frac{W_2(\rho^\tau(k+1), \rho^\tau(k))}{\tau} = ||v^\tau_t||_{L^2(\rho^\tau_t)},$$

where we used the fact that the velocity field $\tilde{v}^\tau$ has been choses so that its L^2 norm equals the metric derivative of the curve $\tilde{\rho}^\tau$.

In particular we can obtain

$$\begin{aligned} |E^\tau|([0,T] \times \Omega) &= \int_0^T dt \int_\Omega |v^\tau_t| d\rho^\tau_t = \int_0^T ||v^\tau_t||_{L^1(\rho^\tau_t)} \le \int_0^T ||v^\tau_t||_{L^2(\rho^\tau_t)} \\ &\le T^{1/2} \int_0^T ||v^\tau_t||^2_{L^2(\rho^\tau_t)} = T^{1/2} \sum_k \tau \left(\frac{W_2(\rho^\tau(k+1), \rho^\tau(k))}{\tau}\right)^2 \le C. \end{aligned}$$

The estimate on $\tilde{E}^\tau$ is completely analogous

$$|\tilde{E}^\tau|([0,T]\times\Omega) = \int_0^T dt \int_\Omega |\tilde{v}_t^\tau| d\tilde{\rho}_t^\tau \leq T^{1/2} \int_0^T ||\tilde{v}_t^\tau||^2_{L^2(\tilde{\rho}_t^\tau)} = T^{1/2} \sum_k \tau \left(\frac{W_2(\rho^\tau(k+1), \rho^\tau(k))}{\tau} \right)^2 \leq C.$$

This gives compactness of E^τ and $\tilde{E}^\tau$ in the space of vector measures on space-time, or the weak convergence. As far as $\tilde{\rho}^\tau$ is concerned, we can obtain more than that. Consider the following estimate, for $t < t$

$$W_2(\tilde{\rho}_t^\tau, \tilde{\rho}_s^\tau) \leq \int_s^t |(\tilde{\rho}^\tau)'|(r) dr \leq (t-s)^{1/2} \left(\int_s^t |(\tilde{\rho}^\tau)'|(r)^2 dr \right)^{1/2}.$$

From the previous computations, we have again

$$\int_0^T |(\tilde{\rho}^\tau)'|(r)^2 dr = \sum_k \tau \left(\frac{W_2(\rho^\tau(k+1), \rho^\tau(k))}{\tau} \right)^2 \leq C,$$

and this implies

$$W_2(\tilde{\rho}_t^\tau, \tilde{\rho}_s^\tau) \leq C(t-s)^{1/2}, \tag{3.8}$$

which means that the curves $\tilde{\rho}^\tau$ are uniformly Hölder continuous. Since they are defined on $[0,T]$ and valued in $\mathcal{P}(\Omega)$ which is compact, when endowed with the Wasserstein distance, we can apply Ascoli's Theorem. This means that, up to subsequences, we have

$$E^\tau \rightharpoonup E \text{ in } \mathcal{M}([0,T]\times\Omega;\mathbb{R}^d),\ \tilde{E}^\tau \rightharpoonup \tilde{E} \text{ in } \mathcal{M}([0,T]\times\Omega;\mathbb{R}^d);\ \tilde{\rho}^\tau \to \rho \text{ uniformly for the } W_2 \text{ distance.}$$

As far as the curves ρ^τ are concerned, they also converge uniformly to the same curve ρ, since $W_2(\rho_t^\tau, \tilde{\rho}_t^\tau) \leq C\sqrt{\tau}$ (a consequence of (3.8), of the fact that $\tilde{\rho}^\tau = \rho^\tau$ on the points of the form $k\tau$ and of the fact that ρ^τ is constant on each interval $]k\tau, (k+1)\tau]$.

Let us now prove that $\tilde{E} = E$.

Lemma 8.3.5. *Suppose that we have two families of vector measures E^τ and $\tilde{E}^\tau$ such that*

- $\tilde{E}^\tau = \tilde{\rho}^\tau \tilde{v}^\tau$; $E^\tau = \rho^\tau v\tau$;
- $\tilde{v}_t^\tau = v_t^\tau \circ ((k\tau - t)v^\tau(k) + id)^{-1}$; $\tilde{\rho}^\tau = ((k\tau - t)v^\tau(k) + id)_\# \rho^\tau$;
- $\int\int |v^\tau|^2 d\rho^\tau \leq C$ *(with C independent of τ);*
- $E^\tau \rightharpoonup E$ *and* $\tilde{E}^\tau \rightharpoonup \tilde{E}$ *as* $\tau \to 0$

Then $\tilde{E} = E$*.*

Proof. It is sufficient to fix a Lipschitz function $f : [0, T] \times \Omega \to \mathbb{R}^d$ and to prove $\int f \cdot dE = \int f \cdot d\tilde{E}$. To do that, we write

$$\int f \cdot d\tilde{E}^\tau = \int_0^T dt \int_\Omega f \cdot \tilde{v}_t^\tau d\tilde{\rho}^\tau = \int_0^T dt \int_\Omega f \circ ((k\tau - t)v^\tau + id) \cdot v_t^\tau d\rho^\tau,$$

which implies

$$\left| \int f \cdot d\tilde{E}^\tau - \int f \cdot dE^\tau \right| \leq \int_0^T dt \int_\Omega |f \circ ((k\tau - t)v^\tau + id) - f| \, |v_t^\tau| d\rho^\tau \leq \mathrm{Lip}(f)\tau \int_0^T \int_\Omega |v_t^\tau|^2 d\rho^\tau \leq$$

This estimate proves that the limit of $\int f \cdot d\tilde{E}^\tau$ and $\int f \cdot dE^\tau$ is the same, i.e. $E = \tilde{E}$. □

Relation between ρ and E. We can obtain the following

Proposition 8.3.6. *The pair* (ρ, E) *satisfies, in distributional sense*

$$\partial_t \rho + \nabla \cdot E = 0, \quad E = -\nabla \rho - \rho \nabla V.$$

In particular we have found a solution to

$$\begin{cases} \partial_t \rho + \Delta \rho + \nabla \cdot (\rho \nabla V), \\ \rho(0) = \rho_0 & \textit{given.} \end{cases}$$

Proof. First, consider the weak convergence $(\tilde{\rho}^\tau, \tilde{E}^\tau) \rightharpoonup (\rho, E)$ (which is a consequence of $\tilde{E} = E$). Weak convergences let easily any linear condition pass to the limit and the continuity equation $\partial_t \tilde{\rho}^\tau + \nabla \cdot \tilde{E}^\tau = 0$ satisfied in the sense of distributions stays true at the limit (it is enough to test the equations against any C^1 function on $[0, T] \times \Omega$).

Then, use the convergence $(\rho^\tau, E^\tau) \rightharpoonup (\rho, E)$. Actually, using the optimality conditions of Proposition 8.3.3 and the defintiion of $E^\tau = v^\tau \rho^\tau$, we have, for each $\tau > 0$, $E^\tau = -\nabla \rho^\tau - \rho^\tau \nabla V$. It is not difficult to pass this condition to the limit neither. Take $f \in C_c^1([0, T] \times \Omega; \mathbb{R}^d)$ and test:

$$\int f \cdot dE^\tau = -\int f \cdot \nabla \rho^\tau - \int f \cdot \nabla V \rho^\tau = \int \nabla \cdot f d\rho^\tau - \int f \cdot \nabla V \rho^\tau.$$

These terms pass to the limit as $\rho^\tau \rightharpoonup \rho$, at least if $V \in C^1$, since all the test functions above are continuous. This would give $\int f \cdot dE = \int \nabla \cdot f d\rho - \int f \cdot \nabla V \rho$, which implies $E = -\nabla \rho - \rho \nabla V$.

To handle the case where V is only Lipschitz continuous, let us notice that for every τ, t we have $J(\rho_t^\tau) \le J(\rho_0)$. This gives a uniform bound on $\mathcal{F}(\rho_t^\tau)$ and Theorem 8.3.7 turns the weak convergence $\rho_t^\tau \rightharpoonup \rho_t$ as measures into a weak convergence in L^1. Once we have weak convergence in L^1, multiplying times a fixed L^∞ function, i.e. ∇V, preserves the limit. □

Theorem 8.3.7 (Dunford-Pettis). *Suppose that ρ_n is a sequence of probability densities weakly converging as measures to $\rho \in \mathcal{P}(\Omega)$. Suppose that $\int f(\rho_n(x))dx \le C$ for a finite constant C and a convex and superlinear function f. Then ρ is also absolutely continuous and the weak converence also holds in L^1 (i.e. in duality with L^∞ functions, and not only continuous ones).*

Proof. First notice that the absolute continuity of ρ is just a consequence of the lower semcontinuity of $\mathcal{F}$ defined as

$$\mathcal{F}(\rho) := \begin{cases} \int_\Omega f(\rho(x))dx & \text{if } \rho \ll \mathcal{L}^d, \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, the bound on $\int f(\rho_n(x))dx$ implies equiintegrability for the densities ρ_n. Equiintegrability means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that on every set A with $|A| \le \delta$ we have $\int_A \rho_n(x)dx \le \varepsilon$ for any n. This can be seen in this way: fix $\varepsilon > 0$ and take M such that for all t with $t > M$ we have $f(t)/t > 2C/\varepsilon$ (this is possible thanks to the superlinearity of f). Then we estimate

$$\int_A \rho_n(x)dx = \int_{A\cap\{\rho_n \le M\}} \rho_n(x)dx + \int_{A\cap\{\rho_n > M\}} \frac{\rho_n(x)}{f(\rho_n(x))} f(\rho_n(x))dx \le M|A| + \frac{\varepsilon}{2C} C.$$

It is now enough to take $\delta = \frac{\varepsilon}{2M}$ and we get $\int_A \rho_n(x)dx < \varepsilon$.

Once we have equiintegrability, we can fix a test function $\phi \in L^\infty$ and use the fact that for every $\varepsilon > 0$ there is a $\delta > 0$ with $|A| \le \delta \Rightarrow \int_A \rho_n(x)dx \le \varepsilon$ (equiintegrability) and that, by Lusin Theorem, for this $\delta > 0$ there exists a continuous function $\tilde\phi$ with $||\tilde\phi||_{L^\infty} = ||\phi||_{L^\infty}$ and $|\{\phi \ne \tilde\phi\}|$ is contained in an open set A with $|A| < \delta$. We want to prove $\int \phi(x)\rho_n(x)dx \to \int \phi(x)\rho(x)dx$. We have

$$\left| \int \phi(x)\rho_n(x)dx - \int \phi(x)\rho(x)dx \right| \le \left| \int \tilde\phi(x)\rho_n(x)dx - \int \tilde\phi(x)\rho(x)dx \right| + 2||\phi||_{L^\infty} \int_A (\rho_n(x)+\rho(x))dx.$$

Since A is open and $\rho_n \rightharpoonup \rho$ as measures we have $\int_A \rho(x)dx \le \liminf \int_A \rho_n(x)dx \le \varepsilon$ and finally we get

$$\limsup \left| \int \phi(x)\rho_n(x)dx - \int \phi(x)\rho(x)dx \right| \le 0 + 2||\phi||_{L^\infty}\varepsilon,$$

which implies, ε being arbitrary, $\int \phi(x)\rho_n(x)dx \to \int \phi(x)\rho(x)dx$.

$\square$

Notice that we were quite sloppy about the boundary conditions for the PDE that we got, which are actually Neumann (a consequence of the fact that we can test against any C^1 function, with no need to vanish on the boundary).

Last remark: this proof is not the main proof used in [5, 8] or [79], and the main different point is the use of "vertical" perturbations, i.e. $\rho_\varepsilon := (1-\varepsilon)\bar\rho + \varepsilon\tilde\rho$ rather than $\rho_\varepsilon := (id + \varepsilon\xi)_\#\bar\rho$.

8.4 Discussion

8.4.1 EVI, uniqueness and geodesic convexity

Pour les résultats d'unicité, une autre caractérisation est proposée. Elle se base sur l'observation suivante : si $F : \mathbb{R}^n \to \mathbb{R}$ est convexe, alors l'inégalité

$$F(y) \geq F(x) + p \cdot (y - x) \quad \text{pour tout } y \in \mathbb{R}^n$$

caractérise (par définition) les vecteurs $p \in \partial F(x)$ et, si $F \in C^1$, elle est vérifiée uniquement par $p = \nabla F(x)$. De même, si F est λ-convexe, l'inégalité qui caractérise le gradient est

$$F(y) \geq F(x) + \frac{\lambda}{2}|x-y|^2 + p \cdot (y - x) \quad \text{pour tout } y \in \mathbb{R}^n.$$

Ainsi, on peut prendre une courbe $x(t)$ et un point y et calculer

$$\frac{d}{dt}\frac{1}{2}|x(t)-y|^2 = (y - x(t)) \cdot (-x'(t)).$$

Par conséquent, imposer que, pour tout t et tout y, on ait

$$\frac{d}{dt}\frac{1}{2}|x(t)-y|^2 \leq F(y) - F(x(t)) - \frac{\lambda}{2}|x(t)-y|^2,$$

sera donc équivalent à l'égalité $-x'(t) = \nabla F(x(t))$ pour tout t. Cela donnera une deuxième caractérisation (appelée EVI, *Evolution Variational Inequality*) des flots de gradient dans un environnement métrique. Même si on oubliera souvent la dépendance en λ, il faut remarquer que la condition EVI devrait être indiquée comme EVI_λ, puisqu'elle fait intervenir un paramètre λ, a priori arbitraire. D'ailleurs, remarquons aussi que la λ-convexité de

F n'est pas nécessaire pour définir la propriété EVI$_\lambda$, mais le sera pour l'existence de courbes la satisfaisant, ce qui nous amènera à définir ce qu'est une fonction λ-convexe dans un espace métrique (ce sera d'ailleurs une propriété nécessaire pour l'existence de ces courbes).

8.4.2 Other gradient-flow PDEs

8.4.3 Dirichlet boundary conditions

8.4.4 Optimal transport and PDE: not only gradient flows

The End

Bibliography

[1] L. Ambrosio, Movimenti minimizzanti, *Rend. Accad. Naz. Sci. XL Mem. Mat. Sci. Fis. Natur.* 113 (1995) 191–246.

[2] L. Ambrosio, Lecture Notes on Optimal Transport Problems, *Mathematical Aspects of Evolving Interfaces,* Springer Verlag, Berlin, Lecture Notes in Mathematics (1812), 1–52, 2003.

[3] L. Ambrosio and G. Crippa, Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields, *Transport Equations and Multi-D Hyperbolic Conservation Laws, Lecture Notes of the Unione Matematica Italiana*, Vol. 5.

[4] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford mathematical monographs. Oxford University Press, Oxford, 2000.

[5] L. Ambrosio, N. Gigli and G. Savaré, *Gradient flows in metric spaces and in the spaces of probability measures.* Lectures in Mathematics, ETH Zurich, Birkhäuser, 2005.

[6] L. Ambrosio and G. Savaré, *Gradient flows in spaces of probability measures.* Handbook of Differential Equations. Evolutionary equations III, North Holland 2007.

[7] Ambrosio, L., Kirchheim, B., Pratelli, A.: Existence of optimal transport maps for crystalline norms, Duke Math. J., 125, 207–241 (2004).

[8] L. Ambrosio, G. Savaré, Gradient flows of probability measures, *Handbook of differential equations, Evolutionary equations* **3**, ed. by C.M. Dafermos and E. Feireisl (Elsevier, 2007).

[9] L. Ambrosio and A. Pratelli. Existence and stability results in the L^1 theory of optimal transportation, in *Optimal transportation and appli-*

cations, Lecture Notes in Mathematics (CIME Series, Martina Franca, 2001) 1813, L.A. Caffarelli and S. Salsa Eds., 123-160, 2003.

[10] L. Ambrosio and P. Tilli, *Topics on analysis in metric spaces.* Oxford Lecture Series in Mathematics and its Applications (25). Oxford University Press, Oxford, 2004.

[11] M. Beckmann, A continuous model of transportation, *Econometrica* (20), 643–660, 1952.

[12] M. Beckmann, C. McGuire and C. Winsten, C., *Studies in Economics of Transportation.* Yale University Press, New Haven, 1956.

[13] M. Beckmann and T. Puu, *Spatial Economics: Density, Potential and Flow*, North-Holland, Amsterdam, 1985.

[14] M. Beiglboeck and N. Juillet On a problem of optimal transport under marginal martingale constraints, preprint, 2012

[15] J.-D. Benamou and Y. Brenier, A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem, *Numer. Math.*, 84, pp. 375–393, 2000.

[16] Benamou, J.D., Brenier, Y., A numerical method for the optimal time-continuous mass transport problem and related problems, *Contemp. Math.*, 226, pp. 1–11, 1999.

[17] F. Benmansour, G. Carlier, G. Peyré and F. Santambrogio, Numerical Approximation of Continuous Traffic Congestion Equilibria, *Net. Het. Media*, 4 (3), 2009, 605–623.

[18] F. Benmansour, G. Carlier, G. Peyré and F. Santambrogio, Fast Marching Derivatives with Respect to Metrics and Applications, *Numerische Mathematik*, 116, no 3, 357–381, 2010

[19] P. Bernard and B. Buffoni, The Monge Problem for supercritical Mañé Potentials on compact Manifolds, *Advances in Mathematics*, 207 n. 2, 691-706, 2006.

[20] M. Bernot, V. Caselles and J.-M. Morel, Traffic plans, *Publicacions Matemàtiques* (49), no. 2, 417–451, 2005.

[21] M. Bernot, V. Caselles and J.-M. Morel, The structure of branched transportation networks, *Calc. Var. Par. Diff. Eq.*, 32 (2008), no. 3, 279–317.

[22] M. Bernot, Optimal transport and irrigation, *PhD Thesis*, ENS Cachan, 2005, available at `http://perso.crans.org/~bernot`.

[23] M. Bernot, V. Caselles and J.-M. Morel, Traffic plans, *Publicacions Matemàtiques* (49), no. 2, 417–451, 2005.

[24] M. Bernot, V. Caselles and J.-M. Morel, *Optimal transportation networks, Models and Theory*, Lecture Notes in Mathematics, Springer, Vol. 1955 (2008).

[25] M. Bernot, A. Figalli and F. Santambrogio, Generalized solutions for the Euler equations in one and two dimensions, *J. Math. Pures et Appl.* 91, no. 2, 2009, 137–155

[26] J. Bertrand, M. Puel, The optimal mass transport problem for relativistic costs, Calculus of Variations and Partial Differential Equations, 2013, 46 (1-2), pp. 353-374

[27] S. Bianchini and M. Gloyer, On the Euler-Lagrange equation for a variational problem: the general case II, preprint available online at the page `https://digitallibrary.sissa.it/handle/1963/2551`, 2008.

[28] V. I. Bogachev and A. V. Kolesnikov, The Monge–Kantorovich problem: achievements, connections, and perspectives, *Russian Math. Surveys* 67:5 785–890, Dedicated to the centenary of the birth of Leonid Vital'evich Kantorovich

[29] G. Bouchitté, G. Buttazzo and P. Seppecher, Shape optimization solutions via Monge-Kantorovich equation. *C. R. Acad. Sci. Paris Sér. I Math.* 324, n. 10, 1185–1191, 1997.

[30] G. Bouchitté and G. Buttazzo, New lower semicontinuity results for nonconvex functionals defined on measures, *Nonlinear Anal.*, 15 (1990), pp. 679–692.

[31] G. Bouchitté and G. Buttazzo, Integral representation of nonconvex functionals defined on measures, *Annales de l'institut Henri Poincaré (C) Analyse non linéaire* 9 (1), 101-117, 1992

[32] G. Bouchitté and G. Buttazzo, Relaxation for a class of nonconvex functionals defined on measures, *Annales de l'institut Henri Poincaré (C) Analyse non linéaire* 10 (3), 345-361, 1993

[33] Braides, A. Γ-*convergence for beginners*, Oxford University Press, Oxford, 2002.

[34] L. Brasco, G. Carlier and F. Santambrogio, Congested traffic dynamics, weak flows and very degenerate elliptic equations, *J. Math. Pures et Appl.*, 93, No 6, 2010, 652–671. Notice that this is the correct version, since a first version containing a substantial mistake appeared before

[35] L. Brasco, G. Buttazzo and F. Santambrogio, A Benamou-Brenier approach to branched transportation, *SIAM J. Math. An.*, Vol. 43, Nr. 2 (2011) , p. 1023–1040.

[36] Y. Brenier, Décomposition polaire et réarrangement monotone des champs de vecteurs. (French) *C. R. Acad. Sci. Paris Sér. I Math.* (305), no. 19, 805–808, 1987.

[37] Brenier, Y., Polar factorization and monotone rearrangement of vector-valued functions, *Communications on Pure and Applied Mathematics* 44, 375-417, 1991.

[38] Y. Brenier: The least action principle and the related concept of generalized flows for incompressible perfect fluids. *J. Amer. Mat. Soc.*, **2** (1989), 225–255.

[39] Y.Brenier: Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations. *Comm. Pure Appl. Math.*, **52** (1999), 411–452.

[40] H. Brezis, *Analyse fonctionnelle, Théorie et applications*, Masson, Paris (1983)

[41] G. Buttazzo, L. De Pascale, P. Gori-Giorgi Optimal transport meets electronic density functional theory. *Phys. Rev. A*, 85 (2012), (to appear),

[42] G. Buttazzo and F. Santambrogio, A model for the optimal planning of an urban area. *SIAM J. Math. Anal.* (37), no. 2, 514–530, 2005.

[43] L. Caravenna *A proof of Sudakov theorem with strictly convex norms*, Math. Z. (2011) 268:371–407

[44] L. Caravenna and S. Daneri, The disintegration of the Lebesgue measure on the faces of a convex function, preprint, 2009, available at `http://cvgmt.sns.it`

[45] G. Carlier, L. De Pascale and F. Santambrogio, A strategy for non-strictly convex transport costs and the example of $||x-y||^p$ in $\mathbb{R}^2$, *Comm. Math. Sci.*, vol 8, no 4, 931–941, 2010.

[46] G. Carlier, A. Galichon and F. Santambrogio, From Knothe's transport to Brenier's map and a continuation method for optimal transport, *SIAM J. Math. An.* vol 41, no 6, 2554–2576, 2010.

[47] G. Carlier, C. Jimenez and F. Santambrogio, Optimal transportation with traffic congestion and Wardrop equilibria, *SIAM J. Control Optim.* (47), 2008, 1330-1350.

[48] G. Carlier and F. Santambrogio, A Variational Model for Urban Planning with Traffic Congestion, ESAIM Control Optimization and Calculus of Variations, Vol. 11, No. 4, 2005, 595–613.

[49] G. Carlier and F. Santambrogio, A continuous theory of traffic congestion and Wardrop equilibria, proceedings of the conference *Optimization and stochastic methods for spatially distributed information*, St Petersburg, 2010.

[50] T. Champion, L. De Pascale, P. Juutinen, The $\infty-$Wasserstein distance: local solutions and existence of optimal transport maps, *SIAM J. Math. An.* (40), no. 1,1–20, 2008.

[51] T. Champion, L. De Pascale, The Monge problem for strictly convex norms in $\mathbb{R}^d$.*J. Eur. Math. Soc.* 12(6), 1355–1369 (2010).

[52] T. Champion, L. De Pascale, The Monge problem in $\mathbb{R}^d$. *Duke Math. J.* Vol 157, No 3 (2011), 551-572.

[53] P. Chen, F. Jiang, X.-P. Yang, Two dimensional optimal transportation for a distance cost with a convex constraint, preprint.

[54] P. Chen, F. Jiang, X.-P. Yang, Optimal transportation in $\mathbb{R}^n$ for a distance cost with convex constraint, *Zeitschrift fuer Angewandte Mathematik und Physik*, to appear

[55] M. Colombo, A. Figalli Regularity results for very degenerate elliptic equations *J. Math. Pures Appl.*, to appear

[56] B. Dacorogna and J. Moser, On a partial differential equation involving the Jacobian determinant. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 7 (1990), no. 1, 1–26.

[57] G. Dal Maso: *An Introduction to* $\Gamma-$*convergence*. Birkhauser, Basel, 1992.

[58] E. De Giorgi and T. Franzoni, Su un tipo di convergenza variazionale, *Atti Acc. Naz. Lincei Rend.*, (58) no. 8, 1975, pp. 842-850.

[59] E. De Giorgi, New problems on minimizing movements, *Boundary Value Problems for PDE and Applications, C. Baiocchi and J. L. Lions eds.* (Masson, 1993) 81–98.

[60] L. De Pascale and A. Pratelli, Regularity properties for Monge Transport Density and for Solutions of some Shape Optimization Problem, *Calc. Var. Par. Diff. Eq.* 14, n. 3, pp. 249–274, 2002.

[61] L. De Pascale, L. C. Evans and A. Pratelli, Integral estimates for transport densities, *Bull. of the London Math. Soc.*. 36, n. 3, pp. 383–385, 2004.

[62] L. De Pascale and A. Pratelli, Sharp summability for Monge Transport density via Interpolation, *ESAIM Control Optim. Calc. Var.* 10, n. 4, pp. 549–552, 2004.

[63] C. Dellacherie, P.-A. Meyer, Probabilities and potential, North-Holland Mathematics Studies, **29**, North-Holland Publishing Co., Amsterdam-New York, 1978.

[64] R. J. DiPerna, P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. **98** (1989), 511-547.

[65] L. C. Evans and W. Gangbo, Differential equations methods for the Monge-Kantorovich mass transfer problem, Mem. Amer. Math. Soc. **137**, 653 (1999)

[66] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[67] H. Federer, *Geometric Measure Theory*, Classics in Mathematics, Springer, 1996 (reprint of the 1st ed. Berlin, Heidelberg, New York 1969 edition)

[68] M. Feldman and R. McCann, Uniqueness and transport density in Monge's mass transportation problem, *Calc. Var. Par. Diff. Eq.* 15, n. 1, pp. 81–113, 2002.

[69] A. FIGALLI AND N. JUILLET, Absolute continuity of Wasserstein geodesics in the Heisenberg group, *J. Funct. Anal.*, 255, n. 1, pp. 133-141, 2008.

[70] A. FIGALLI AND N. GIGLI, A new transportation distance between non-negative measures, with applications to gradients flows with Dirichlet boundary conditions, preprint.

[71] A. FIGALLI, F. MAGGI et A. PRATELLI – *A mass transportation approach to quantitative isoperimetric inequalities*, Inventiones Mathematicae, **182**, no 1, (2010), 167–211.

[72] Gangbo, W., An elementary proof of the polar factorization of vector-valued functions, *Arch. Rational Mech. Anal.*, 128, pp. 381–399, 1994.

[73] W. GANGBO, R. MCCANN, *The geometry of optimal transportation*, Acta Math., **177** (1996), 113–161.

[74] M. GHISI, M. GOBBINO The monopolist's problem: existence, relaxation and approximation, *Calc. Var. Partial Differential Equations*, Vol 24, no 1, 2005, pp 111–129

[75] E. N. GILBERT, Minimum cost communication networks, *Bell System Tech. J.* (46), 2209–2227, 1967.

[76] E. N. Gilbert and H. O. Pollak, Steiner minimal trees. *SIAM J. Appl. Math.* (16), 1–29, 1968.

[77] C. Jimenez, Optimisation de Problèmes de Transport, PhD thesis of Université du Sud-Toulon-Var, 2005.

[78] C. JIMENEZ AND F. SANTAMBROGIO, Optimal transportation in the quadratic case with a convex constraint, *J. Math. Pures Appl.*, Volume 98, no 1, 2012, 103–113

[79] R. Jordan, D. Kinderlehrer, F. Otto, The variational formulation of the Fokker-Planck equation, *SIAM J. Math. Anal.* **29**(1) (1998) 1–17.

[80] L. Kantorovich, On the transfer of masses. *Dokl. Acad. Nauk. USSR*, (37), 7–8, 1942.

[81] Knothe, H., Contributions to the theory of convex bodies, *Michigan Mathematical Journal* 4, pp. 39–52, 1957.

[82] Jean-Michel Lasry and Pierre-Louis Lions, Mean-Field Games, *Japan. J. Math.* 2, 229–260(2007)

[83] Q. R. Li, F. Santambrogio, X. J. Wang, Regularity in Monge's mass transfer problem, preprint (2013)

[84] F. Maddalena and S. Solimini, Transport distances and irrigation models, *J. Conv. An.*, 2009.

[85] F. Maddalena, S. Solimini and J.-M. Morel, A variational model of irrigation patterns, *Interfaces and Free Boundaries*, **5** (2003), 391–416.

[86] B. Maury, A. Roudneff-Chupin and F. Santambrogio A macroscopic crowd motion model of gradient flow type, *Math. Models and Methods in Appl. Sciences* Vol. 20, No. 10 (2010), 1787–1821

[87] R. J. McCann, A convexity principle for interacting gases. *Adv. Math.* (128), no. 1, 153–159, 1997.

[88] McCann, R.J., Existence and uniqueness of monotone measure preserving maps, *Duke Math. J.*, 80, pp. 309–323, 1995.

[89] G. Monge, Mémoire sur la théorie des déblais et des remblais, *Histoire de l'Académie Royale des Sciences de Paris*, 666–704, 1781.

[90] J.-M. Morel and F. Santambrogio, Comparison of distances between measures, *Appl. Math. Lett.*, (20), no. 4, 2007, 427–432.

[91] P. Pegon, D. Piazzoli, F. Santambrogio, Full characterization of optimal transport plans for concave costs, preprint (2013), available at `http://cvgmt.sns.it/paper/2280/`.

[92] L. Ruschendorf, On $c-$optimal random variables, *Statistics and Probability Letters*, Vol. 27, 1996, 267–270

[93] Rachev, S.T., Rüschendorf, L., *Mass Transportation Problems. Vol. I: Theory; Vol. II : Applications*, Springer-Verlag 1998.

[94] R. Tyrell Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

[95] Rosenblatt, M., Remarks on a multivariate transformation, *Annals of Mathematical Statistics* 23, pp. 470–472, 1952.

[96] JEAN-CHARLES ROCHET; PHILIPPE CHONÉ Ironing, Sweeping, and Multidimensional Screening. . Econometrica, Vol. 66, No. 4. (Jul., 1998), pp. 783-826.

[97] F. SANTAMBROGIO, Optimal Channel Networks, Landscape Function and Branched Transport, *Interfaces and Free Boundaries*, **9** (2007), 149–169.

[98] F. SANTAMBROGIO, Absolute continuity and summability of transport densities: simpler proofs and new estimates, *Calc. Var. Par. Diff. Eq.* 36, no. 3, 2009, 343-354.

[99] F. Santambrogio, Models and applications of Optimal Transport Theory, lecture notes for the Summer School *Optimal transportation : Theory and applications*, Grenoble, 2009.

[100] F. SANTAMBROGIO, Inégalités Isopérimétriques quantitatives via le transport optimal, d'après A. Figalli, F. Maggi et A. Pratelli, proceedings of the *Bourbaki Seminar*, Janvier 2011 (in French).

[101] F. SANTAMBROGIO, Flots de gradient dans les espaces métriques et leurs applications (d'après Ambrosio-Gigli-Savaré), proceedings of the *Bourbaki Seminar*, Janvier 2013 (in French).

[102] F. SANTAMBROGIO AND V. VESPRI, Continuity for a very degenerate elliptic equation in two dimensions, accepted by *Nonlinear Analysis: Theory, Methods & Applications* 73 (2010), 3832–3841.

[103] SUDAKOV, V. N., Geometric problems in the theory of infinite-dimensional probability distributions. Cover to cover translation of Trudy Mat. Inst. Steklov 141 (1976). Proc. Steklov Inst. Math. 1979, no. 2, i–v, 1–178.

[104] C. Villani. *Topics in Optimal Transportation.* Graduate Studies in Mathematics, AMS, 2003.

[105] C. Villani, *Optimal transport: Old and New*, Springer Verlag (Grundlehren der mathematischen Wissenschaften), 2008

[106] J. G. WARDROP, Some theoretical aspects of road traffic research, *Proc. Inst. Civ. Eng.* **2** (1952), 325-378.

[107] Q. XIA, Optimal Paths related to Transport Problems. *Comm. Cont. Math.* (5), no. 2, 251–279, 2003.