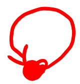
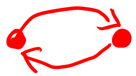
Max-Flow

CSE 6331

Flow Network:

Graph $G = (V, E)$

* No self-loops 
* No antiparallel edges 

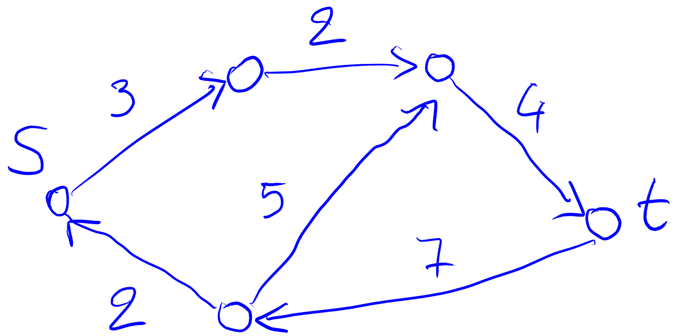
Capacity:

$$c(u,v) \geq 0 \quad , \forall u,v \in V$$

$$c(u,v) = 0 \quad , \text{if } (u,v) \notin E$$

Source vertex $s \in V$

Sink vertex $t \in V$



Flow

Function $f: V \times V \to \mathbb{R}$

* Capacity constraint:

$$\forall u, v \in V, \quad 0 \leq f(u,v) \leq c(u,v)$$

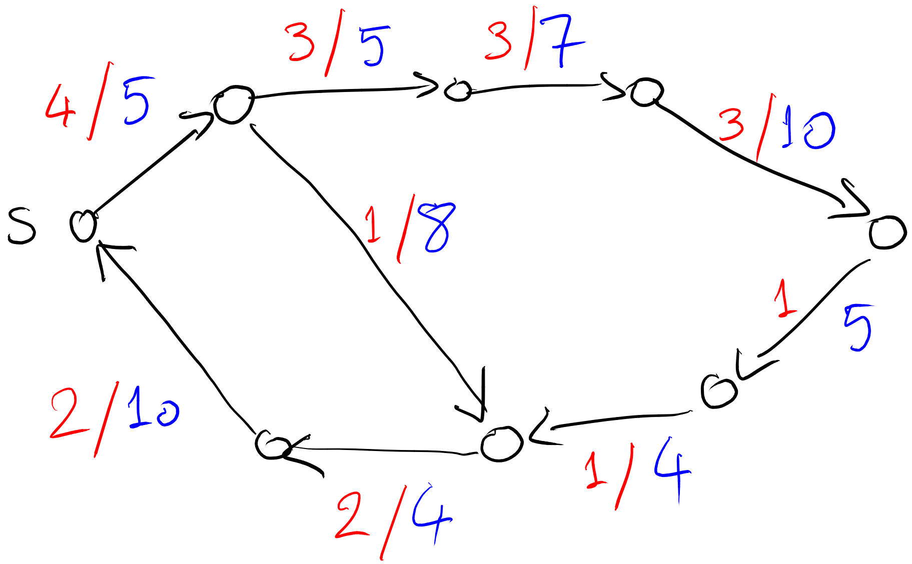
* Flow conservation:

$$\forall u - \{s,t\}, \quad \sum_{v \in V} f(v,u) = \sum_{v \in V} f(u,v)$$

Value of a flow:

$$|f| = \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s)$$

Example :

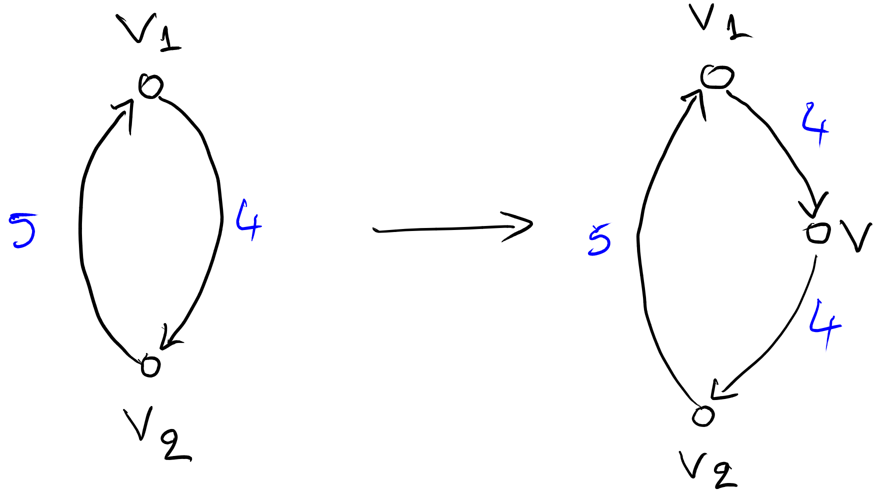


Capacity

Flow

$$\boxed{|f| = 2}$$

Modeling antiparallel edges

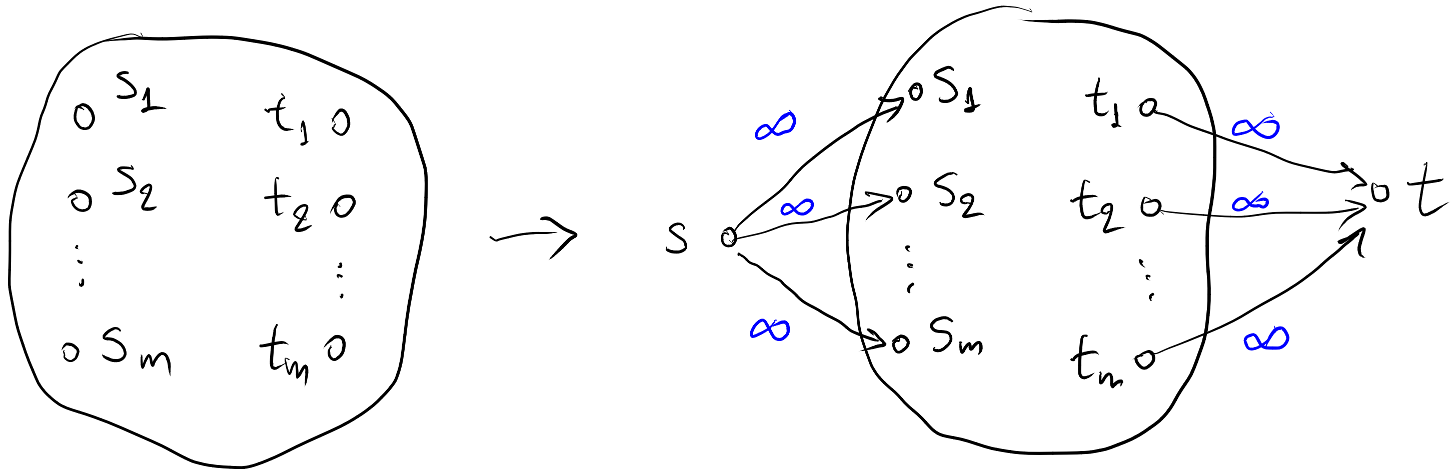


We can remove antiparallel pairs by introducing a new vertex v.

$$c(v_1, v) = c(v, v_2) = c(v_1, v_2).$$

Modelling multiple sources and sinks.

Suppose we have many sources $s_1, \ldots, s_m$, and sinks $t_1, \ldots, t_n$.



We add a "super-source" s, and "super-sink" t.

The Max-Flow problem

Given flow network G, c, s, t

Compute a flow f with maximum value $|f|$.

Fundamental problem in Computer Science.

Ford-Fulkerson-Method(G, s, t)

Initialize flow f to 0.

while there exists "augmenting path" p
in the "residual network" G_f,

augment flow f along p

return f.

Residual networks

Let G: flow network, f: flow in G.

"Residual capacity":

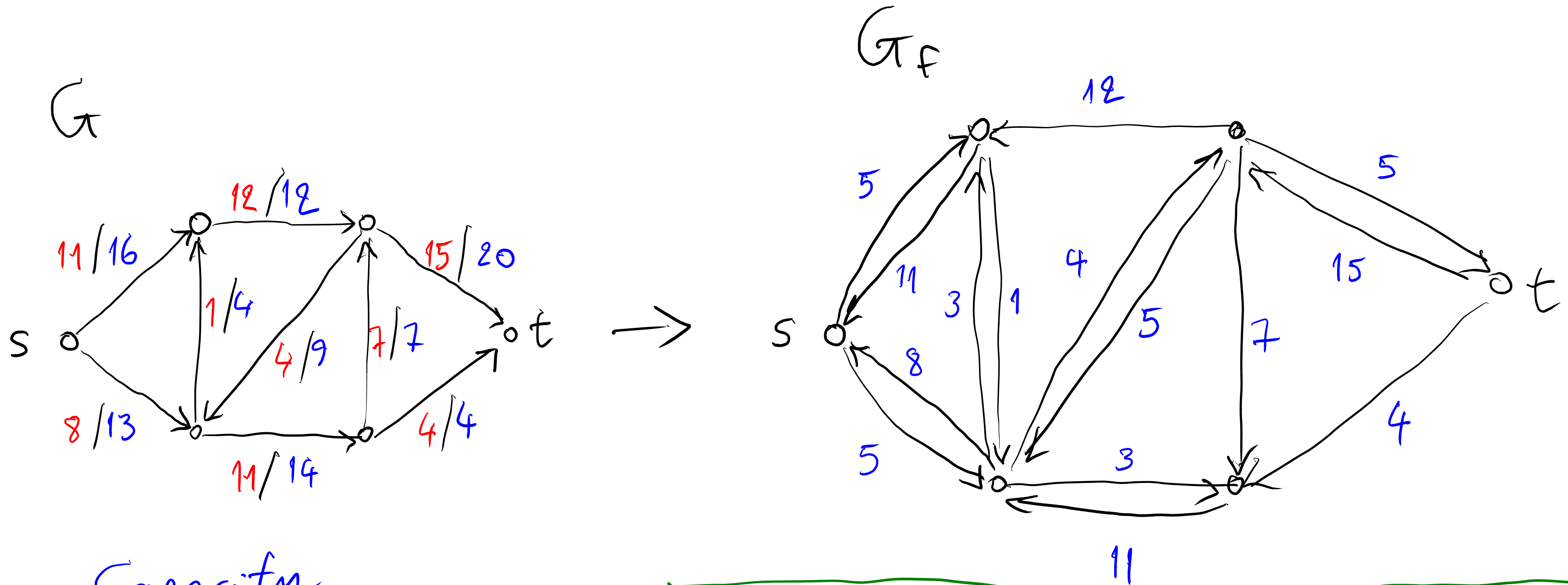
$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & , \text{ if } (u,v) \in E \\ f(v,u) & , \text{ if } (v,u) \in E \\ 0 & , \text{ otherwise} \end{cases}$$

"Residual network"

$$G_f = (V, E_f)$$

$$E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$$

Residual network example:



Capacity

Flow

Note: G_f can have antiparallel edges.

Augmentation:

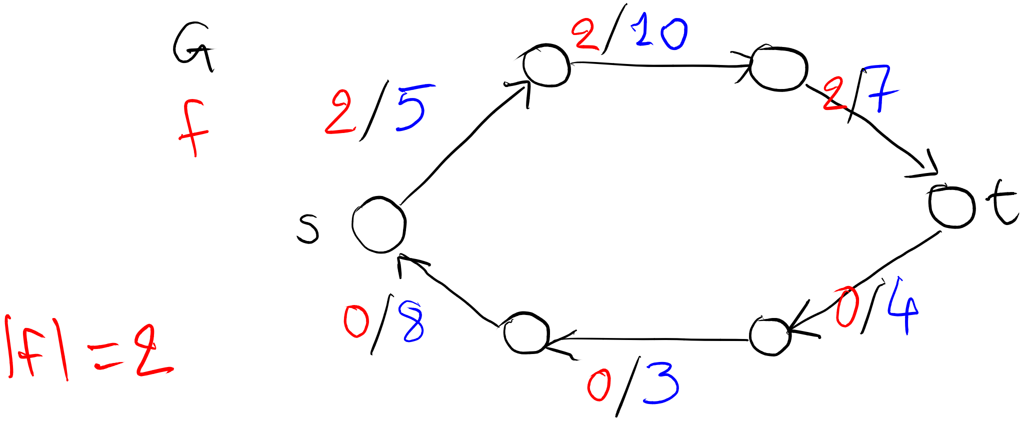
Let f be a flow in G.

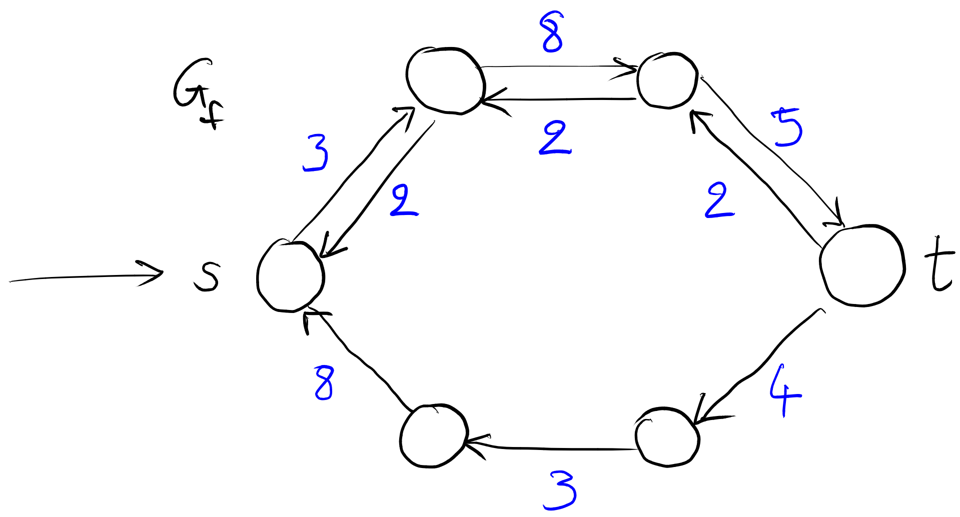
Let f' be a flow in G_f.

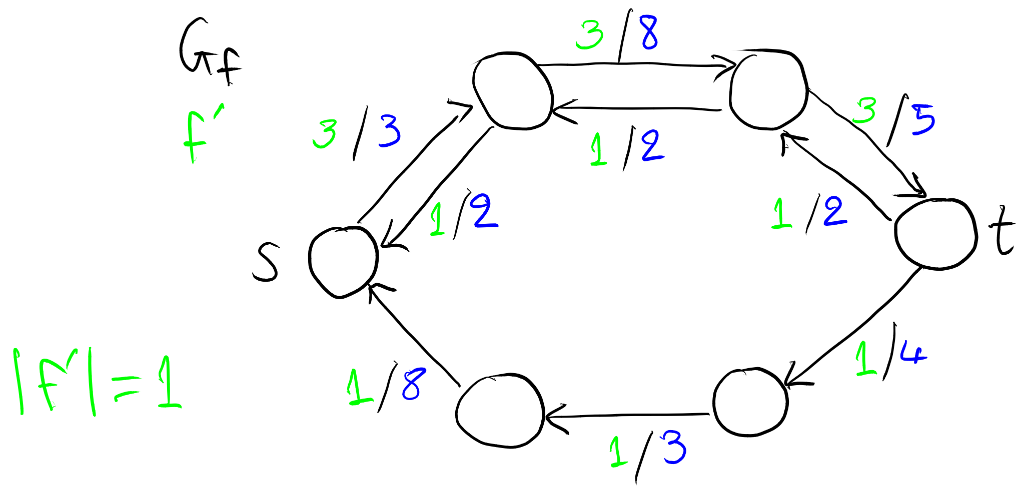
The "augmentation" of f by f' is a function $f \uparrow f' : V \times V \to \mathbb{R}$, where $\forall\, u,v \in V$,

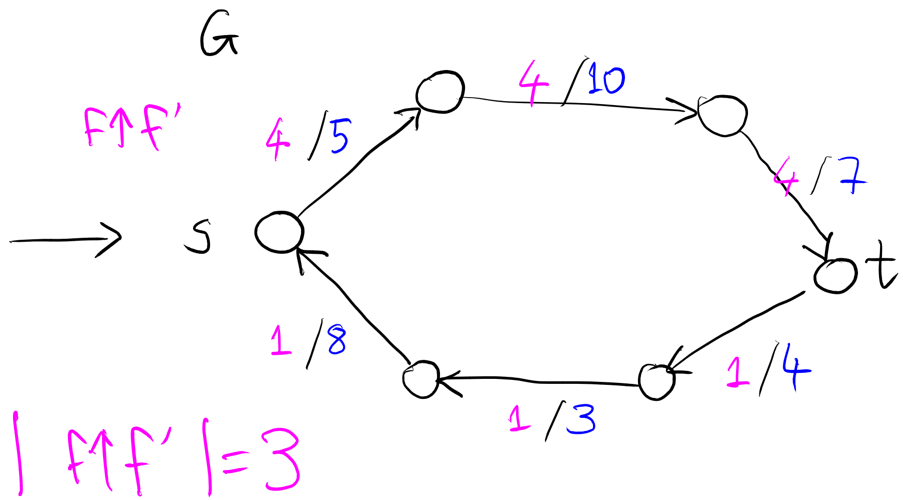
$$(f \uparrow f')(u,v) = \begin{cases} f(u,v) + f'(u,v) - f'(v,u) & \text{, if } (u,v) \in E \\ 0 & \text{, otherwise} \end{cases}$$

Example augmentation









Lemma

Let f be a flow in G.

Let f' be a flow in G_f.

Then, $f \uparrow f'$ is a flow in G, with

$$|f \uparrow f'| = |f| + |f'|$$

Proof:

$\forall (u,v) \in E$, we have

$$(f \uparrow f')(u,v) = f(u,v) + f'(u,v) - f'(v,u) \quad \text{(by definition of } f \uparrow f')$$

$$\geq f(u,v) + f'(u,v) - f(u,v) \quad \text{(since } c_f(v,u) = f(u,v))$$

$$= f'(u,v)$$

$$\geq 0$$

Proof (cont.):

Also, $\forall (u,v) \in E$,

$$(f \uparrow f')(u,v) = f(u,v) + f'(u,v) - f'(v,u) \quad \text{(by definition of } f \uparrow f')$$

$$\leq f(u,v) + f'(u,v) \quad (\text{since } f' \geq 0)$$

$$\leq f(u,v) + c_f(u,v) \quad \text{(by capacity constraint)}$$

$$= f(u,v) + c(u,v) - f(u,v) \quad (\text{definition of } c_f)$$

$$= c(u,v)$$

Proof (cont.):

Therefore, $\forall (u,v) \in E$,

$$0 \le (f \uparrow f')(u,v) \le c(u,v).$$

I.e., the capacity constraint is satisfied.

Proof (cont.):

$\forall u \in V - \{s,t\}$,

$$\sum_{v \in V} (f \uparrow f')(u,v) = \sum_{v \in V} \left(f(u,v) + f'(u,v) - f'(v,u) \right)$$

$$= \sum_{v \in V} f(u,v) + \sum_{v \in V} f'(u,v) - \sum_{v \in V} f'(v,u)$$

$$= \sum_{v \in V} f(v,u) + \sum_{v \in V} f'(v,u) - \sum_{v \in V} f'(u,v) \quad \text{(by flow conservation)}$$

$$= \sum_{v \in V} \left(f(v,u) + f'(v,u) - f'(u,v) \right)$$

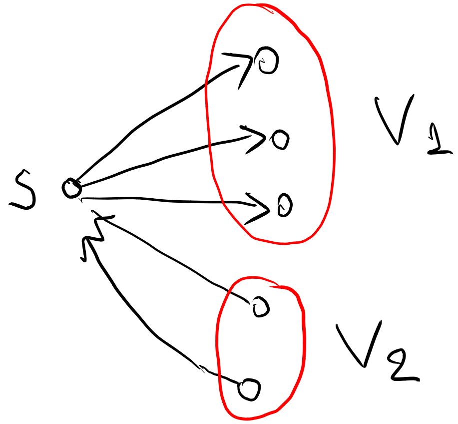
$$= \sum_{v \in V} (f \uparrow f')(v,u)$$

Thus, $f \uparrow f'$ satisfies flow conservation

Proof (cont.):

Let $V_1 = \{ v : (s,v) \in E \}$

$V_2 = \{ v : (v,s) \in E \}$



$$|f \uparrow f'| = \sum_{v \in V_1} (f \uparrow f')(s,v) - \sum_{v \in V_2} (f \uparrow f')(v,s)$$

$$= \sum_{v \in V_1} \left(f(s,v) + f'(s,v) - f'(v,s) \right) - \sum_{v \in V_2} \left(f(v,s) + f'(v,s) - f'(v,s) \right)$$

$$= \sum_{v \in V_1} f(s,v) - \sum_{v \in V_2} f(v,s) + \sum_{v \in V_1 \cup V_2} f'(s,v) - \sum_{v \in V_1 \cup V_2} f'(v,s)$$

$$= \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) + \sum_{v \in V} f'(s,v) - \sum_{v \in V} f'(v,s) = |f| + |f'| \quad \blacksquare$$

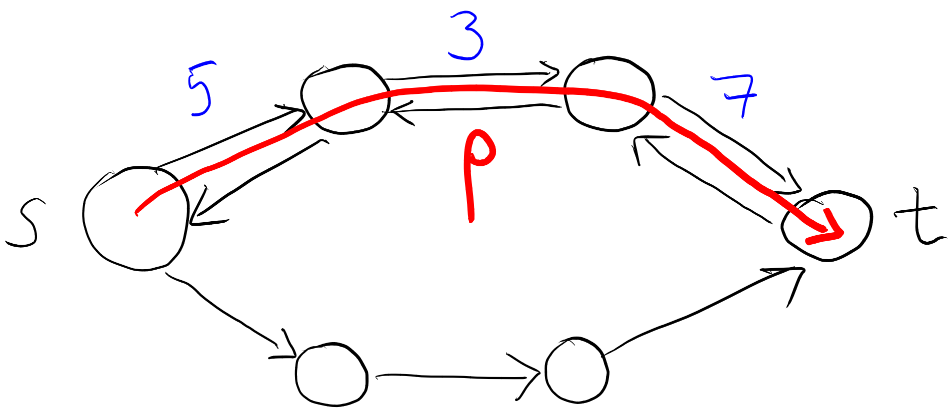
Augmenting paths

* An "augmenting path" is a simple $s \leadsto t$ path p in G_f.

* "Residual capacity" of p

$$c_f(p) = \min\{c_f(u,v) : (u,v) \text{ is on } p\}$$

Example:



$c_f(p) = 3$

Lemma:

Let f be a flow in G.

Let p be an augmenting path in G_f.

Define $f_p : V \times V \longrightarrow \mathbb{R}$:

$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \text{ is on } p \\ 0 & \text{otherwise} \end{cases}$$

Then, f_p is a flow in G_f, with $|f_p| = c_f(p) > 0$.

Corollary:

Suppose we augment f by f_p.

Then, $f \uparrow f_p$ is a flow in G

with value $|f \uparrow f_p| = |f| + |f_p| > |f|$.

* The Ford-Fulkerson method augments a flow along augmenting paths, until there are no augmenting paths left.

* Question: Is the final flow a maximum flow?

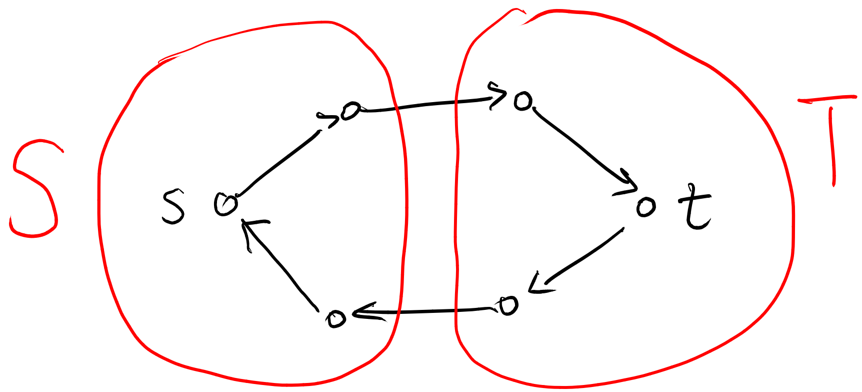
Cuts of flow networks

* A "cut" of flow network $G=(V,E)$ is a bipartition (S,T) of V.

  $T = V - S$

  $s \in S$

  $t \in T$

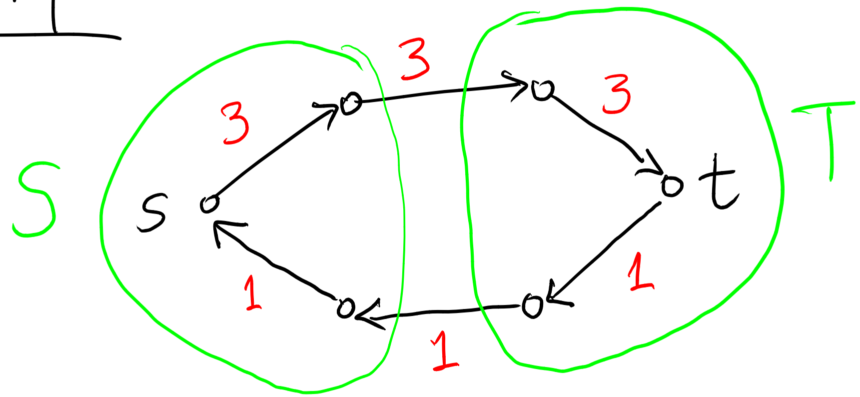


Let f be a flow.

The "net flow" $f(S,T)$ across the cut (S,T) is defined to be:

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$

Example:

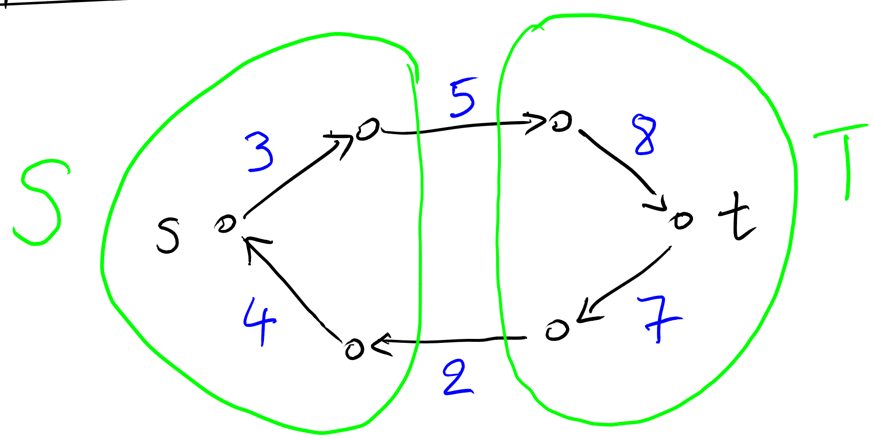


$f(S,T) = 3 - 1 = 2$

"Capacity" of a cut:

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$

Example:



$c(S,T) = 5$

Lemma:

Let f be a flow in flow network G.

Let (S,T) by any cut of G.

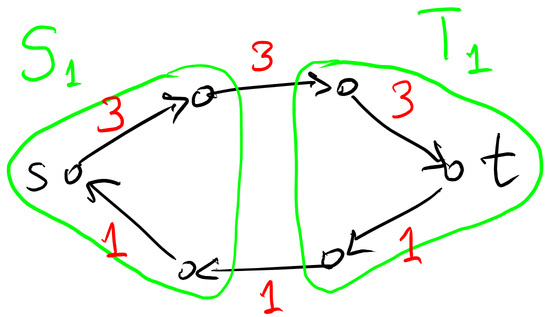
Then, $f(S,T) = |f|$.

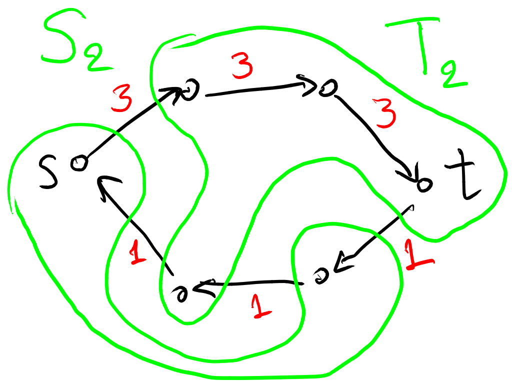
Example:

$f(S_1,T_1) = 3 - 1 = 2 = |f|$

$f(S_2,T_2) = 3 + 1 - 1 - 1$

$= 2 = |f|$





Proof:

By flow-conservation, $\forall\ u \in V - \{s, t\}$, we have

$$\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u) = 0$$

Thus,

$$\sum_{u \in S - \{s\}} \left(\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u) \right) = 0$$

Proof (cont.)

Thus,

$$|f| = \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s)$$

$$= \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) + \sum_{u \in S - \{s\}} \left(\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u) \right)$$

$$= \sum_{v \in V} \sum_{u \in S} f(u,v) - \sum_{v \in V} \sum_{u \in S} f(v,u)$$

Proof (cont.)

Thus,

$$|f| = \sum_{v \in T} \sum_{u \in S} f(u,v) + \overbrace{\sum_{v \in S} \sum_{u \in S} f(u,v)}^{(*)} - \sum_{v \in T} \sum_{u \in S} f(v,u) - \overbrace{\sum_{v \in S} \sum_{u \in S} f(v,u)}^{(*)}$$

$$= \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$

$$= f(S,T)$$

$\blacksquare$

Corollary:

$\forall$ cut (S,T), we have $|f| \le c(S,T)$.

Proof:

$$|f| = f(S,T) = \sum_{u\in S}\sum_{v\in T} f(u,v) - \sum_{u\in S}\sum_{v\in T} f(v,u)$$

$$\le \sum_{u\in S}\sum_{v\in T} f(u,v)$$

$$\le \sum_{u\in S}\sum_{v\in T} c(u,v)$$

$$= c(S,T)$$

$\blacksquare$

Theorem (Max-flow min-cut theorem):

The following conditions are equivalent:

1. f is a maximum flow in G.
2. G_f contains no augmenting paths.
3. $|f| = c(S,T)$, for some cut (S,T) of G.

Proof :

① ⇒ ②

If there exists an augmenting path p in G_f, then augmenting f along p results in a flow of bigger value, contradicting the maximality of f.

Proof (cont.):

(2) $\Rightarrow$ (3) Suppose G_f contains no augmenting path.

Let $S = \{v \in V : \exists \text{ a path } s \rightsquigarrow v \text{ in } G_f\}$

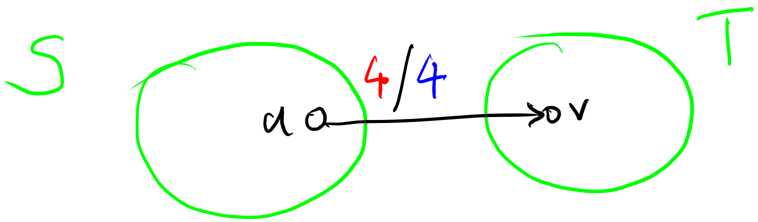
Let $T = V - S$.

(S, T) is a cut because $s \not\rightsquigarrow t$.

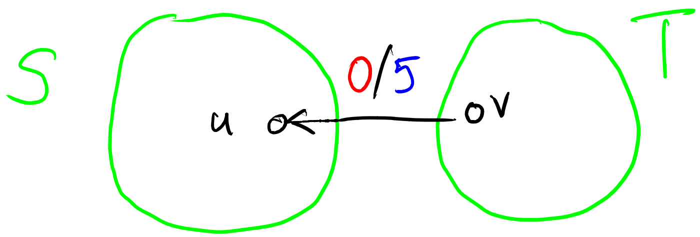
Proof (cont.):

Let $u \in S$, $v \in T$.

If $(u,v) \in E$, then $f(u,v) = c(u,v)$, since otherwise $(u,v) \in E_f$, and $v \in S$.



If $(v,u) \in E$, then $f(v,u) = 0$, since otherwise $c_f(u,v) = f(v,u) > 0 \Rightarrow (u,v) \in E \Rightarrow v \in S$.



Proof (cont.):

Thus,

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{v \in T} \sum_{u \in S} f(v,u)$$

$$= \sum_{u \in S} \sum_{v \in T} c(u,v) - 0$$

$$= c(S,T)$$

Thus, $|f| = f(S,T) = c(S,T)$.

Proof (cont.)

③ ⟹ ①

We have $|f| \leq c(S,T)$ for all cuts (S,T).

Since $|f| = c(S,T)$, it follows that f is a maximum flow. ■

Ford-Fulkerson(G, s, t):

for each $(u,v) \in E$

$\quad (u,v).f = 0$

while $\exists\, p : s \rightsquigarrow t$ in G_f

$\quad c_f(p) = \min\{c_f(u,v) : (u,v) \text{ in } p\}$

$\quad$ for each (u,v) in p

$\quad\quad$ if $(u,v) \in E$

$\quad\quad\quad (u,v).f = (u,v).f + c_f(p)$

$\quad\quad$ else

$\quad\quad\quad (v,u).f = (v,u).f - c_f(p)$

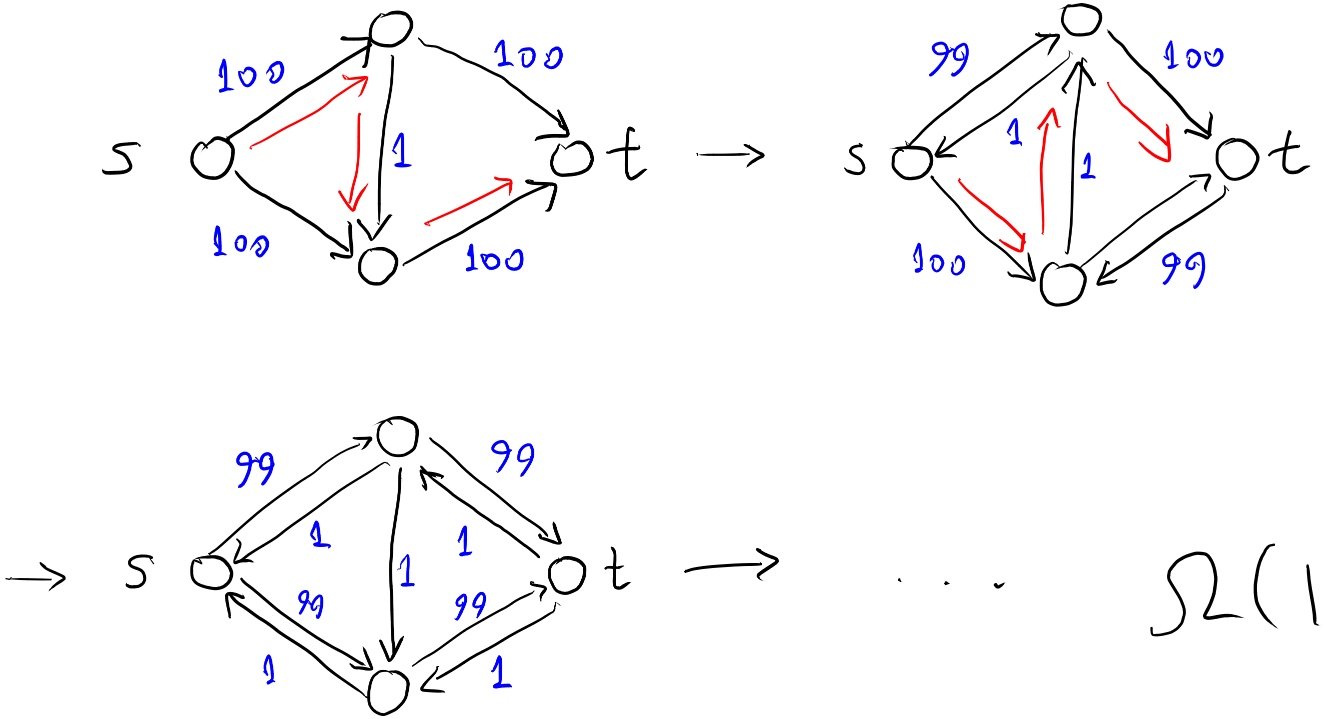
Running time :

* $O(|V|+|E|)$ time to find augmenting path.

* Assuming integer capacities :
  * Let f^* be a maximum flow
  * $|f^*|$ iterations

* $O(|f^*| \cdot |E|)$ total time.

A bad input:



$\ldots \quad \Omega(|f^*|)$ iterations

The Edmonds-Karp Algorithm

At every iteration, augment along a <u>shortest</u> $s \rightsquigarrow t$ path in G_f.

Lemma:

During the execution of the Edmonds-Karp algorithm, $\forall v \in V - \{s,t\}$, the shortest-path distance $\delta_f(s,v)$ in G_f increases monotonically.

<u>Proof:</u>

Suppose there exists augmentation $f \to f'$, that causes some distance to decrease.

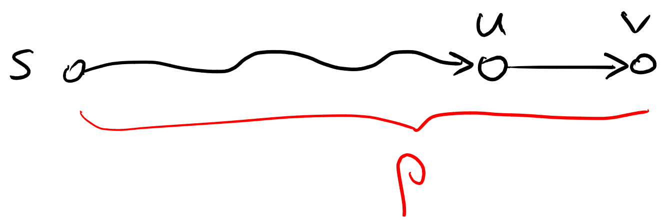
Let $v \in V - \{s, t\}$, s.t.

$$\delta_{f'}(s,v) < \delta_f(s,v).$$

Pick such v, minimizing $\delta_{f'}(s,v)$.

Proof (cont.):

Let $p = s \rightsquigarrow u \rightarrow v$ be shortest path in $G_{f'}$.



We have: $\delta_{f'}(s,u) = \delta_{f'}(s,v) - 1$.

$$\delta_{f'}(s,u) \geq \delta_f(s,u) \quad \text{(by choice of } v\text{)}$$

If $(u,v) \in E_f$, then

$$\delta_f(s,v) \leq \delta_f(s,u) + 1 \leq \delta_{f'}(s,u) + 1 = \delta_{f'}(s,v)$$

a contradiction, since $\delta_{f'}(s,v) < \delta_f(s,v)$.

$\Longrightarrow$ $(u,v) \in E_{f'}$, and $(u,v) \notin E_f$

Proof (cont.):

* We have $(u,v) \notin E_f$ and $(u,v) \in E_{f'}$.

$\Rightarrow$ The augmentation increased the flow on (v,u).

* Since we augment along shortest paths, the shortest path $s \leadsto u$ in G_f contains (v,u).

$$\Rightarrow \delta_f(s,v) = \delta_f(s,u) - 1$$
$$\leq \delta_{f'}(s,u) - 1$$
$$= \delta_{f'}(s,v) - 2,$$

contradicting the assumption that $\delta_{f'}(s,v) < \delta_f(s,v)$.

$\blacksquare$

Theorem:

The Edmonds-Karp performs $O(|V|\cdot|E|)$ augmentations.

Proof:

We say that edge (u,v) is "critical" on an augmenting path p, if $c_f(u,v) = c_f(p)$.

We will show that each edge can become critical at most $O(|V|)$ times.

Proof (cont.):

let $(u,v) \in E$.

* The first time (u,v) becomes critical, we have

$$\delta_f(s,v) = \delta_f(s,u) + 1$$

* After the augmentation, (u,v) disappears from the residual network.
* (u,v) can reappear only after flow on (u,v) is decreased.
* So, (v,u) must first appear in an augmenting path.
* when this occurs:

$$\delta_{f'}(s,u) = \delta_{f'}(s,v) + 1$$

Proof (cont.):

We have:

$$
\begin{aligned}
\delta_{f'}(s,u) &= \delta_{f'}(s,v)+1 \\
&\geq \delta_f(s,v)+1 \\
&= \delta_f(s,u)+2
\end{aligned}
$$

Since $\delta_f(s,u) \in \{1, \dots, |V|\}$ (while u reachable from s), and every time (u,v) becomes critical, $\delta_f(s,u)$ increases by at least 2, it follows that (u,v) can become critical at most $O(|V|)$ times.

Proof (cont.):

* Every edge can become critical $O(|V|)$ times.
* There are at most $2 \cdot |E|$ edges that can become critical.
* Every augmenting path has at least one critical edge.

$\Rightarrow$ There are $O(|V| \cdot |E|)$ augmentations.

$\blacksquare$

Running time of Edmonds-Karp:

Each augmentation can be performed in $O(|V|)$ time.

Total time: $O(|V|^2 \cdot |E|)$.

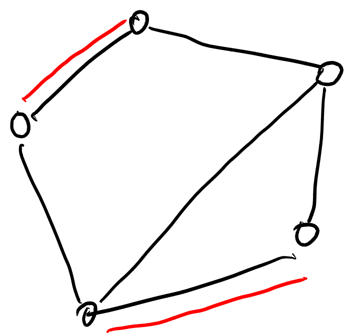
Maximum bipartite matching

Graph $G=(V,E)$.

$M \subseteq E$ is a "matching" if $\forall v \in V$,

at most one edge incident to v is in M.

Example:



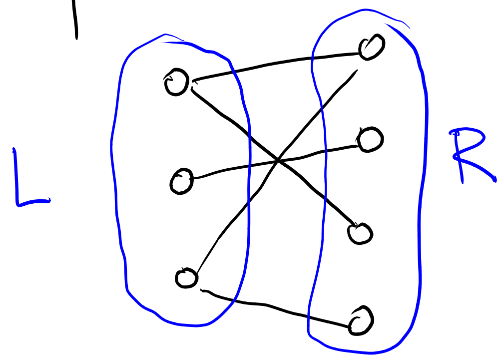
Maximum matching problem:

Given G, find matching M,

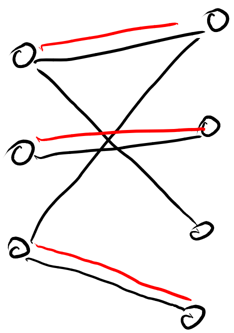
of maximum cardinality $|M|$.

Maximum bipartite matching

Given bipartite G: $V = L \cup R$



Find maximum matching:



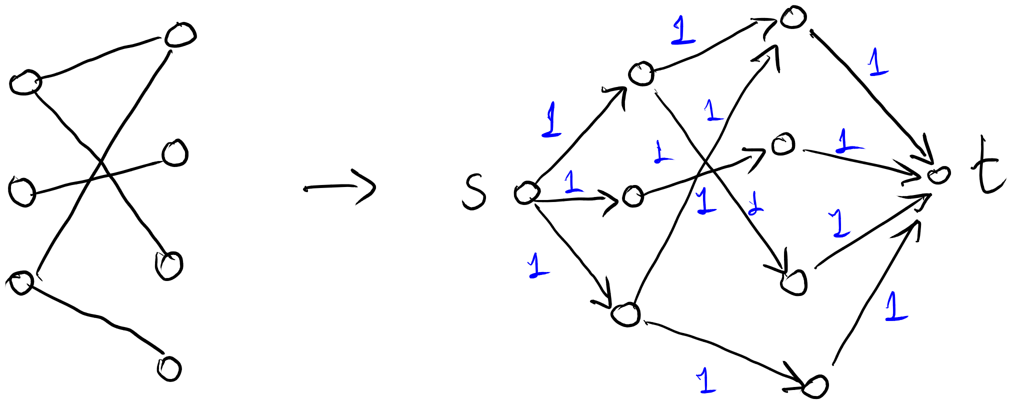
Finding a maximum bipartite matching

Given $G=(V,E)$, build flow network $G'=(V',E')$.

$V' = V \cup \{s,t\}$

$E' = \{(s,u) : u \in L\} \cup \{(u,v) : (u,v) \in E\} \cup \{(v,t) : v \in R\}$

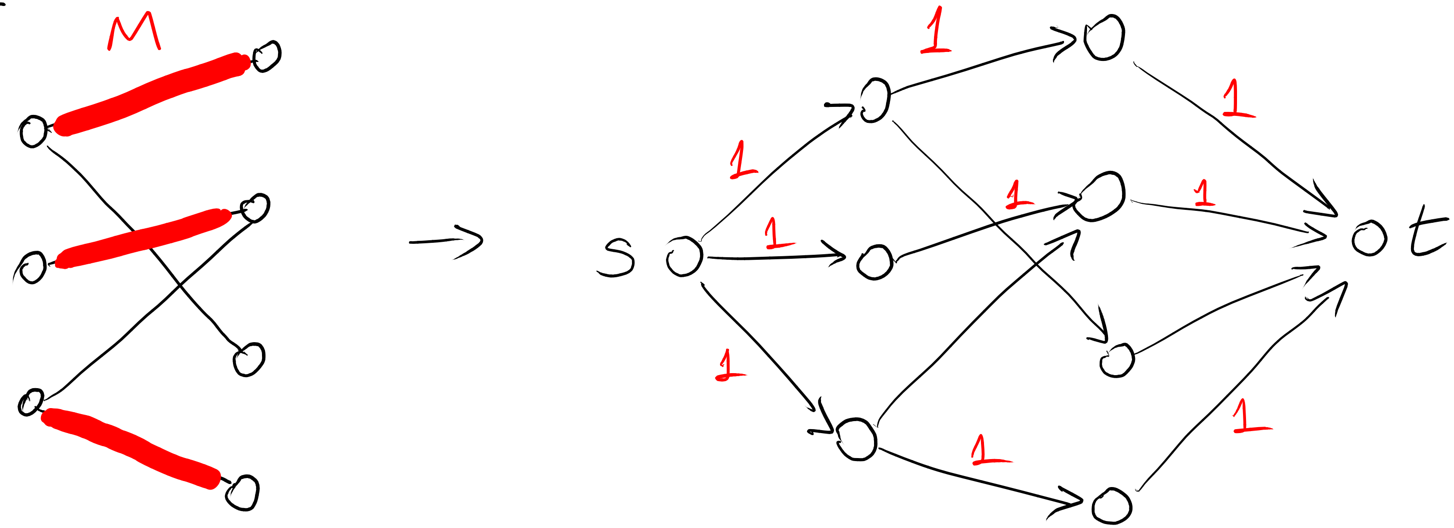
All capacities $= 1$.



Lemma:

If M is a matching in G, then $\exists$ integer-valued flow f in G', with $|f| = M$.

Proof:



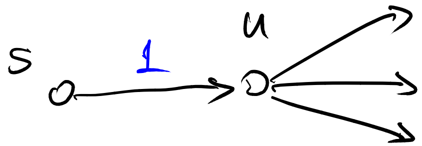
Lemma:

If f is an integer-valued flow in G', then $\exists$ matching M in G, with $|M| = |f|$.

Proof:

Let $M = \{(u,v) : u \in L, v \in R, \text{ and } f(u,v) > 0\}$.

Each $u \in L$ has at most 1 unit of flow entering.

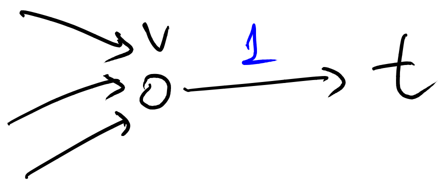


Thus, at most 1 unit of flow is leaving u.

By integrality, at most one edge incident to u is in M.

Proof (cont.)

Similarly, $\forall\, v \in R$, at most one unit of flow leaves v:



Thus, at most one edge incident to v has positive flow.

Thus, M is a matching with $|M| = |f|$.

$\blacksquare$

Theorem:

In any flow network, if the capacities are integers, then the maximum flow produced by the Ford-Fulkerson method satisfies:

* $|f|$ is an integer
* $\forall\, u,v \in V$, $f(u,v)$ is an integer.

Proof: Induction on # of augmentations. $\blacksquare$

Corollary:

The cardinality of a maximum matching in G equals the value of a maximum flow in G'.

Corollary:

We can compute a maximum bipartite matching in time $O(|V| \cdot |E'|) = O(|V| \cdot |E|)$.