The Simplex Algorithm

CSE 6331

Standard Form

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to:} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad , \quad \forall\, i \in \{1, \dots, m\}$$

$$x_j \geq 0 \quad , \forall\, j \in \{1, \dots, n\}$$

Slack Form: (N, B, A, b, c, v)

maximize $\quad v + \sum_{j=1}^{n} c_j x_j$

subject to $\quad x_{n+i} = b_i + \sum_{j=1}^{n} a_{ij} x_j \quad , \forall i \in \{1, \dots, m\}$

$$x_i \geq 0 \quad , \quad \forall i \in \{1, \dots, n+m\}$$

"Basic" variables: $B = \{x_{n+1}, \dots, x_{n+m}\}$

"Non-basic" variables: $N = \{x_1, \dots, x_n\}$

Simplex(A, b, c):

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(N, B, A, b, c, v) = Init-Simplex(A, b, c)
while ∃ j ∈ N, with c_j > 0
    choose e ∈ N, with c_e > 0
    for each i ∈ B
        if a_ie > 0
            Δ_i = b_i / a_ie
        else Δ_i = ∞
    choose l ∈ B, minimizing Δ_l
    if Δ_l = ∞ return "unbounded"
    else (N, B, A, b, c, v) = Pivot(N, B, A, b, c, v, l, e)
for i = 1 to n
    if i ∈ B then x̄_i = b_i, else x̄_i = 0
return (x̄_1, ..., x̄_n)
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Basic solution

Set non-basic variables to 0.
Solve for the basic variables.

Example:

$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution: $(x_1, \dots, x_6) = (0, 0, 0, 30, 24, 36)$

Objective value of the basic solution

$$v + \sum_{j=1}^{n} c_j \underbrace{x_j}_{\text{non-basic variables}} = V$$

Pivot (N, B, A, b, c, v, l, e)

Solve for x_e: express x_e as a function of x_l and x_i, $i \in N - \{l\}$.

Add a constrain for x_e

Remove constrain for x_l

Substitute x_e in the right-hand-side of remaining constraints.

Substitute x_e in the objective function.

$\hat{N} = N - \{e\} \cup \{l\}$

$\hat{B} = B - \{l\} \cup \{e\}$

return $(\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{v})$

Pivoting example

$$z = 3x_1 + x_2 + 2x_3$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

$\longrightarrow$

$$z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}$$
$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$
$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}$$
$$x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}$$

$l = 6$, $e = 1$

Lemma:

The current slack form is uniquely determined by the current set of basic variables B.

Corollary:

The Simplex algorithm either terminates in $\binom{n+m}{n}$ iterations, or it enters on infinite loop.

Avoiding infinite loops

* Solution 1: Choose e and ℓ to be smallest possible.

* Solution 2: Perturb input slightly so that no two basic solutions have the same value.

Lemma:

Suppose Init-Simplex returns a slack form with a feasible basic solution.

Then, if Simplex returns a solution, then that solution is feasible.

<u>Proof :</u>

We will prove by induction that the following loop invariants are maintained:

1. The current slack form is equivalent to the slack form returned by Init-Simplex.
2. $\forall i \in B$, we have $b_i \geq 0$
3. The basic solution of the current slack form is feasible.

$\longrightarrow$

Proof (cont.):

Initialization of loop invariants:

1 & 3 are immediate.

Let $i \in B$.

All non-basic variables are set to 0.

Thus: $$x_i = b_i + \sum_{j \in N} a_{ij} x_j = b_i$$

By feasibility, we have

$$x_i \geq 0 \implies b_i \geq 0$$

Thus, 2 holds.

$\longrightarrow$

Proof (cont.):

Maintenance of 1:

The current slack form is obtained by a call to Pivot on the previous one.

The Pivot procedure just rewrites the set of equations by moving x_e to the LHS, and x_ℓ to the RHS.

Therefore, the new slack form is equivalent to the previous one.

Proof (cont.):

Maintenance of 2

Let b_i, a_{ij} be the coefficients before the call to Pivot, and $\hat{b}_i$ the coefficients after the call.

We have $a_{\ell e} > 0$, and $b_\ell \geq 0$ (by loop invariant).

Thus, $\hat{b}_e = b_\ell / a_{\ell e} \geq 0$.

Proof (cont.):

$\forall\ i \in B \sim \{l\}$, we have

$$\hat{b}_i = b_i - a_{ie}\hat{b}_e = b_i - a_{ie} \cdot \frac{b_l}{a_{le}}$$

* If $a_{ie} > 0$, then

we have $\frac{b_l}{a_{le}} \le \frac{b_i}{a_{ie}}$ (by choice of l)

Thus: $\hat{b}_i \ge b_i - a_{ie}\frac{b_i}{a_{ie}} = b_i - b_i = 0$

* If $a_{ie} \le 0$, then since $a_{le}, b_i, b_l \ge 0$, we have

$$\hat{b}_i = b_i - a_{ie}\frac{b_l}{a_{le}} \ge 0$$

$\longrightarrow$

Proof (cont.):

Maintenance of 3:

Clearly, all equality constraints are satisfied. (Why?)

$\forall\, i \in N$, we have $x_i = 0$.

$\forall\, i \in B$, we have $x_i = b_i - \sum_{j \in N} a_{ij} x_j = b_i \geq 0$

Thus, all variables are non-negative, and the basic solution is feasible.

$\blacksquare$

Lemma:

If Simplex returns "unbounded" then the linear program is unbounded.

Proof: $\forall\, i \in B$, we have $a_{ie} \leq 0$.

Let
$$\bar{x}_i = \begin{cases} \infty & , \text{if } i = e \\ 0 & , \text{if } i \in N - \{e\} \\ b_i - \sum_{j \in N} a_{ij} \bar{x}_j & , \text{if } i \in B \end{cases}$$

Note that $\forall\, i \in N - \{e\}$, $\bar{x}_i = 0$, and $\bar{x}_e = \infty$

Also, $\forall\, i \in B$, $\bar{x}_i = b_i - a_{ie} \bar{x}_e \geq 0$ (since $a_{ie} \leq 0$).

Thus, $\forall\, i$, $\bar{x}_i \geq 0$, and $(\bar{x}_1, \dots, \bar{x}_n)$ is a feasible solution.

Proof (cont.):

The objective value is:

$$z = v + \sum_{j \in N} c_j \overline{x}_j$$

$$= v + c_e \overline{x}_e$$

$$= \infty \quad (\text{since } c_e > 0, \text{ and } \overline{x}_e = \infty)$$

Thus, the linear program is unbounded.

$\blacksquare$

<u>Corollary</u> :

The simplex algorithm either correctly reports that a linear program in unbounded, or it terminates with a feasible solution in at most $\binom{n+m}{m}$ iterations.

Duality

"Primal" Linear Program:

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i \,, \forall i \in \{1, \dots, m\}$$

$$x_j \ge 0 \,, \forall \; j \in \{1, \dots, n\}$$

"Dual" Linear Program:

$$\text{minimize} \quad \sum_{i=1}^{m} b_i y_i$$

$$\text{s.t.} \quad \sum_{i=1}^{m} a_{ij} y_i \ge c_j \,, \forall j \{1, \dots, n\}$$

$$y_i \ge 0 \,, \forall i \in \{1, \dots, m\}$$

Example:

Primal:

maximize $3x_1 + x_2 + 2x_3$

s.t. $x_1 + x_2 + 3x_3 \leq 30$

$2x_1 + 2x_2 + 5x_3 \leq 24$

$4x_1 + x_2 + 2x_3 \leq 36$

$x_1, x_2, x_3 \geq 0$

Dual:

minimize $30y_1 + 24y_2 + 36y_3$

s.t. $y_1 + 2y_2 + 4y_3 \geq 3$

$y_1 + 2y_2 + y_3 \geq 1$

$3y_1 + 5y_2 + 2y_3 \geq 2$

$y_1, y_2, y_3 \geq 0$

Lemma: (Weak linear-programming duality)

Let $\bar{x}$ be any feasible solution to the primal LP, and let $\bar{y}$ be any feasible solution to the dual LP.

Then,
$$\sum_{j=1}^{n} c_j \bar{x}_j \leq \sum_{i=1}^{m} b_i \bar{y}_i$$

Proof:

$$\sum_{j=1}^{n} c_j \bar{x}_j \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \bar{y}_i \right) \bar{x}_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \bar{x}_j \right) \bar{y}_i \leq \sum_{i=1}^{m} b_i \bar{y}_i$$

by dual constraints

by primal constraints

Corollary:

If $\sum_{j=1}^{n} c_j \bar{x}_j = \sum_{i=1}^{m} b_i \bar{y}_i$,

then $\bar{x}$ is an optimal solution to the primal LP,

and $\bar{y}$ is an optimal solution to the dual LP.

Theorem (Linear-Programming duality)

Suppose Simplex returns $(\bar{x}_1, \ldots, \bar{x}_n)$.

Let N, B be the non-basic, and basic vars in the final slack form.

Let c' be the final coefficients.

Let $(\bar{y}_1, \ldots, \bar{y}_m)$ where

$$\bar{y}_i = \begin{cases} -c'_{n+i} & \text{, if } (n+i) \in N \\ 0 & \text{, otherwise} \end{cases}$$

Then, $\sum_{j=1}^{n} c_j \bar{x}_j = \sum_{i=1}^{m} b_i \bar{y}_i$.

In particular, $\bar{x}$ is an optimal solution.

Proof:

The objective function of the final slack form is

$$z = v' + \sum_{j \in N} c'_j x_j .$$

Since Simplex terminates, we have

$$c'_j \leq 0 \quad , \forall i \in N .$$

Define $c'_j = 0 \quad , \forall i \in B$,

We have $z = v' + \sum_{j \in N} c'_j x_j = v' + \sum_{j=1}^{n+m} c'_j x_j$

Since the final slack form is equivalent with the first one:

$$\sum_{j=1}^{n} c_j \bar{x}_j = v' + \sum_{j=1}^{n+m} c'_j \bar{x}_j = v'$$

$\longrightarrow$

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Proof (cont.):

$\forall (x_1, \dots, x_n)$, we have:

$$\sum_{j=1}^{n} c_j x_j = v' + \sum_{j=1}^{n+m} c'_j x_j = v' + \sum_{j=1}^{n} c'_j x_j + \sum_{i=1}^{m} (-y_i)\, x_{n+i}$$

$$= v' + \sum_{j=1}^{n} c'_j x_j + \sum_{i=1}^{m} (-y_i) \cdot \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right)$$

$$= \left(v' - \sum_{i=1}^{m} b_i y_i \right) + \sum_{j=1}^{n} \left(c'_j + \sum_{i=1}^{m} a_{ij} y_i \right) x_j$$

$\longrightarrow$

Proof (cont):

This implies:

$$v' - \sum_{i=1}^{m} b_i \bar{y}_i = 0$$

$$c'_j + \sum_{i=1}^{m} a_{ij} \bar{y}_i = c_j \quad , \quad \forall\, j \in \{1, \dots, n\}$$

$\Rightarrow (\bar{y}_1, \dots, \bar{y}_m)$ is a feasible dual solution of value v'.

$\blacksquare$

Finding an initial feasible solution

Lemma:

L: maximize $\sum_{j=1}^{n} c_j x_j$

s.t. $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$, $\forall i \in \{1, \dots, m\}$

$x_j \geq 0$, $\forall j \in \{1, \dots, n\}$

L_{aux}: maximize $-x_0$

s.t. $\sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i$, $\forall i \in \{1, \dots, m\}$

$x_j \geq 0$, $\forall j \in \{0, \dots, n\}$

L is feasible if and only if the optimal objective value of L_{aux} is 0.

Initialize-Simplex(A, b, c):

Let k be the index of the minimum b_i

if $b_k \geq 0$

return $(\{1, \dots, n\}, \{n+1, \dots, n+m\}, A, b, c, 0)$

Let (N, B, A, b, c, v) be the slack form of L_{aux}

$l = n + k$

$(N, B, A, b, c, v) = \text{Pivot}(N, B, A, b, c, v, l, 0)$

This pivot returns a slack form with feasible basic solution.

Find an optimal solution $(\bar{x}_0, \dots, \bar{x}_n)$ to L_{aux} using "Simplex".

If $\bar{x}_0 = 0$

if x_0 is basic

perform a Pivot to make x_0 non-basic

remove x_0 from the final slack form

return the resulting slack form

else return infeasible

Theorem (Fundamental theorem of linear programming)

Any linear program L, given in standard form, either

1. has an optimal solution with a finite objective value,
2. is infeasible, or
3. is unbounded.

If L is infeasible, Simplex returns "infeasible".

If L is unbounded, Simplex returns "unbounded".

Otherwise, Simplex returns an optimal solution.