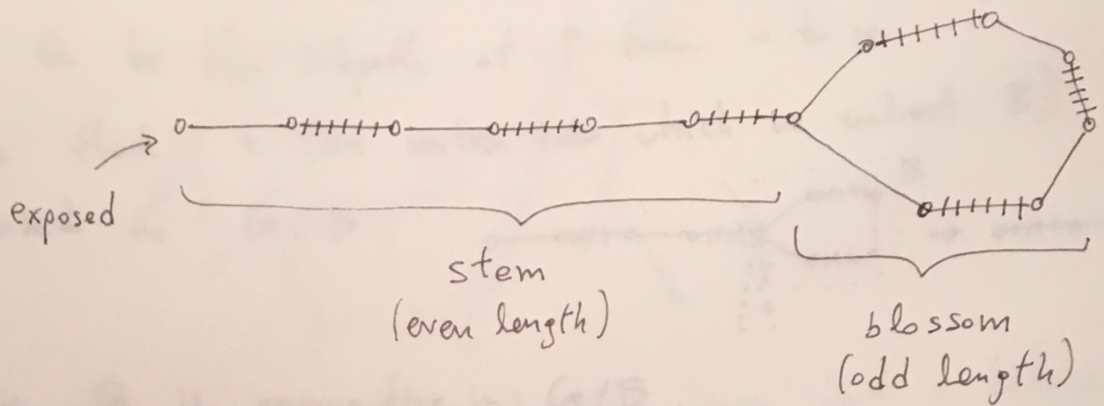
Maximum Matching (Non-bipartite)

Definition: Flower (w.r.t. matching M)



Algorithm:

$M = \emptyset$

Repeat until there are no augmenting paths and no flowers:

if $\exists$ an augmenting path P, let $M = M \triangle P$

if $\exists$ flower F then:

Let Q be the stem of F, and B be the blossom of F.

Let $M = M \triangle Q$

Let $G = G/B$ (contract B into single vertex)

end

end

Lemma: M is a maximum matching in G iff M/B is a maximum matching in G/B.

Proof: ($\Rightarrow$) Let N be a matching in G/B bigger than M/B.

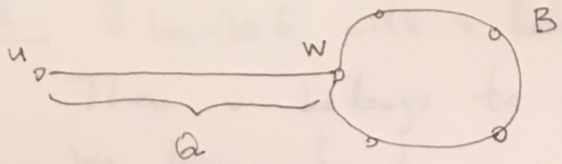
N is incident to at most one vertex of B in G (after pulling back).

Extend N to a matching N' in G by adding $\frac{1}{2}(|B|-1)$ edges in B.

Thus $|N'| - |M| = |N| - |M/B|$.

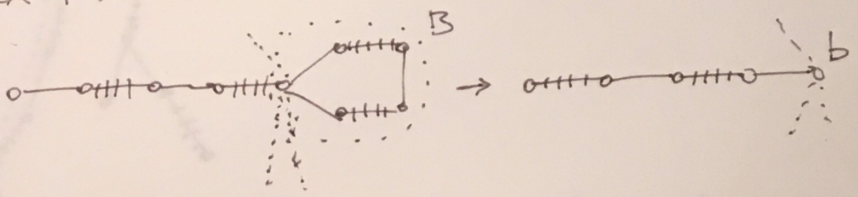
Thus $|N'| > |M|$.

($\Leftarrow$) If M is not maximum then $\exists$ augmenting path P (w.r.t. M) in G with endpoints u and v. B has only one exposed vertex, thus we may assume wlog that $u \notin B$. If $P \cap B = \emptyset$ then P is also augmenting in G/B. Otherwise let w be the first vertex of P in B. $\longrightarrow$



Let Q be the subpath of P from u to w.

Note that b (the vertex into which we contract B) is exposed in G/B:

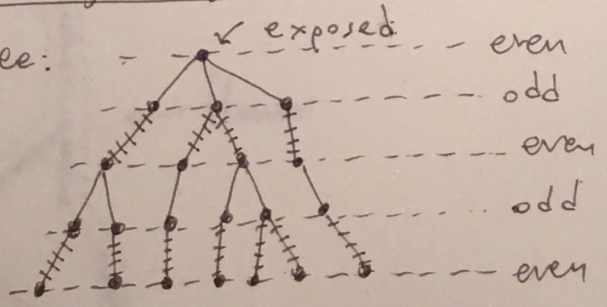


Thus Q is augmenting in G/B. ///

Corollary: The algorithm is correct.

How to find an augmenting path or a flower:

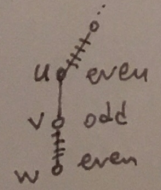
Alternating tree:



Start "growing" an alternating tree rooted at each exposed node.
All exposed nodes are marked "even".

Processing some even vertex u:

Case 1: $\exists (u,v) \in E$ with v unlabelled.
Label v "odd". v is not exposed (o/w it would have been labeled even).
Label its "match" w "even":

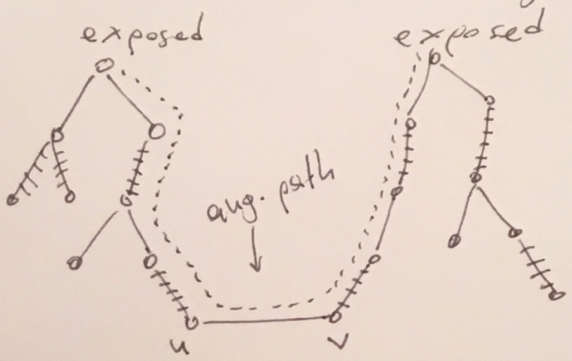


→

Case 2: $\exists (u,v) \in E$ with v labelled "even".

Then v belongs to another alternating tree.

We have found an augmenting path:



Case 3: $\exists (u,v)$ with v labelled "even" and v belongs to the same alternating tree as u.

Then we have found a flower: