Baker's technique

Independent Set :

Input : $G = (V, E)$

Goal : Find $X \subseteq V$

s.t. $\forall u, v \in X$
with $u \neq v$,
we have $\{u, v\} \notin E$
maximizing $|X|$

Baker's Algorithm : (on planar graphs)

Pick arbitrary $r \in V$

$k = 1/\varepsilon$

compute BFS tree T in G rooted at r

for $i = 0$ to $k-1$
  Let V_i be the set of vertices at level i mod k in T
  Let $G_1^i, G_2^i, \ldots$ be the components of $G \setminus V_i$
  For each j, compute a min. I.S. S_i^j for G_i^j
  Let $S_i = \bigcup_j S_i^j$
end

$i^* = \arg\max_i \{|S_i|\}$

return S_{i^*}

<u>Definition</u>: An embedding of a graph G is "1-outerplanar", if it is planar, and all vertices lie on the outer face.
For $k \geq 2$, an embedding is "k-outerplanar", if it is planar, and when all vertices of the outer face are deleted, then a $(k-1)$-outerplanar embedding is obtained.
A graph is "k-outerplanar" if it admits a k-outerplanar embedding.

<u>Lemma</u>: $\forall i, \forall j$, G_i^j is $O(k)$-outerplanar.

Defn:

Let $G=(V,E)$, let $T=(V,F)$ be a maximal spanning forest of G (i.e. T contains a spanning tree of each component of G).

* For each $e=\{v,w\} \in E \setminus F$, the "fundamental cycle" of e is the unique cycle consisting of e and the v-w path in T.

* "Vertex remember number":

$$vr(G,T) = \max_{v \in V} \{\#\text{ of fundamental cycles that use } v\}$$

* "Edge remember number":

$$er(G,T) = \max_{e \in F} \{\#\text{ of fundamental cycles that use } e\}$$

Theorem: Let $G=(V,E)$ and $T=(V,F)$ be a maximal spanning forest of G.
Then $tw(G) \le \max\{vr(G,T),\ er(G,T)+1\}$.

Proof: Let T' be the tree with $T'=(V \cup F, F')$, where

$$F' = \{(v,e) : v \in V,\ e \in F,\ \exists w \in V : e=\{v,w\}\}$$

For each $\alpha \in V \cup F$ we shall construct bubble X_α:

For each $v \in V$, add v to X_v.

For each $\{v,w\} \in F$, add v and w to X_v.

For each $\{v,w\} \in E \setminus F$, choose an arbitrary endpoint, say v, and add v to each X_x, for all $x \in V$, $x \neq w$ that are on the fundamental cycle of (v,w).

Do not add v to X_w.

Add v to X_e, for all edges $e \in F$ on the fundamental cycle of $\{v,w\}$.

The result is a valid tree decomposition of width at most $\max\{vr(G,T),\ er(G,T)+1\}$. $\boxtimes$

Lemma: Let $G=(V,E)$ be a planar graph with some fixed planar embedding.
Let $H=(V,E')$ be the graph obtained from G by removing all edges in the exterior face.
Let $T'=(V,F')$ be a maximal spanning forest of H.
Then $\exists$ maximal spanning forest $T=(V,F)$ of G,
s.t. $er(G,T) \le er(H,T')+2$
and $vr(G,T) \le vr(H,T') + degree(G)$.

Proof: Let $K=(V,(E\setminus E')\cup F')$.
Let T be a maximal spanning forest of K, s.t. $T' \subseteq T$.
Every fundamental cycle in K (relative to T) is an interior face of K.
Each edge is adjacent to at most 2 faces.
Thus $er(K,T) \le 2$
Each vertex is contained in at most $degree(G)$ faces.
Thus $vr(K,T) \le degree(G)$.
Each fundamental cycle in G (w.r.t. T) is either a fundamental cycle in H or in K.
Thus $er(G,T) \le er(K,T) + er(H,T') \le er(H,T')+2$
and $vr(G,T) \le vr(K,T) + vr(H,T') \le vr(H,T') + degree(G)$.

$\square$

Lemma: Let $G=(V,E)$ be an outerplanar graph with $\text{degree}(G) \leq 3$. Then $\exists$ maximal spanning forest T of G with $er(G,T) \leq 2$ and $vr(G,T) \leq 2$.

Proof: Removing all edges in the exterior face we obtain forest $T'=(V,F')$.

Clearly, $er(T',T') = vr(T',T') = 0$.

The result follows as in previous Lemma by observing that each vertex is incident to at most 2 interior faces. $\boxtimes$

Lemma: Let $G=(V,E)$ be k-outerplanar with $\text{degree}(G) \leq 3$. Then $\exists$ maximal spanning forest $T=(V,F)$ with $er(G,T) \leq 2k$ and $vr(G,T) \leq 3k-1$.

Proof: Follows via induction from the previous two Lemmas. $\boxtimes$

Lemma: $\forall$ k-outerplanar $G=(V,E)$,
$\exists$ k-outerplanar $H=(V',E')$, s.t.
G is a minor of H and $\text{degree}(H) \leq 3$.

Proof: Replace each vertex of degree $d \geq 4$ by a path of $d-2$ vertices of degree 3, s.t. the new graph remains k-outerplanar.

. . .

$\square$

Lemma: The treewidth of a k-outerplanar graph G is at most $3k-1$.

Proof: $\exists$ k-outerplanar H s.t. $G \preceq H$ and $\text{degree}(H) \leq 3$.

$\exists$ maximal spanning forest T of H s.t.
$er(H,T) \leq 2k$ and $vr(H,T) \leq 3k-1$.

Thus $tw(H) \leq \max\{3k-1, 2k+1\} = 3k-1$.

$\Rightarrow tw(G) \leq tw(H) \leq 3k-1$.

$\blacksquare$

Other problems:

* Vertex Cover
* Dominating set
* Max 3-SAT