The geometry of topological graphs, and its algorithmic applications

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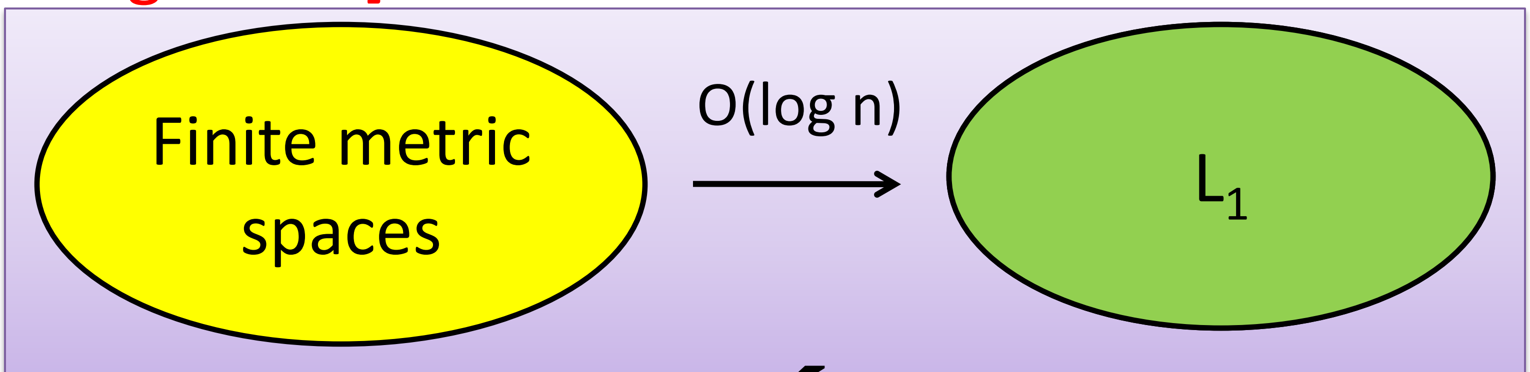
Metric embeddings

[Bourgain ’85]



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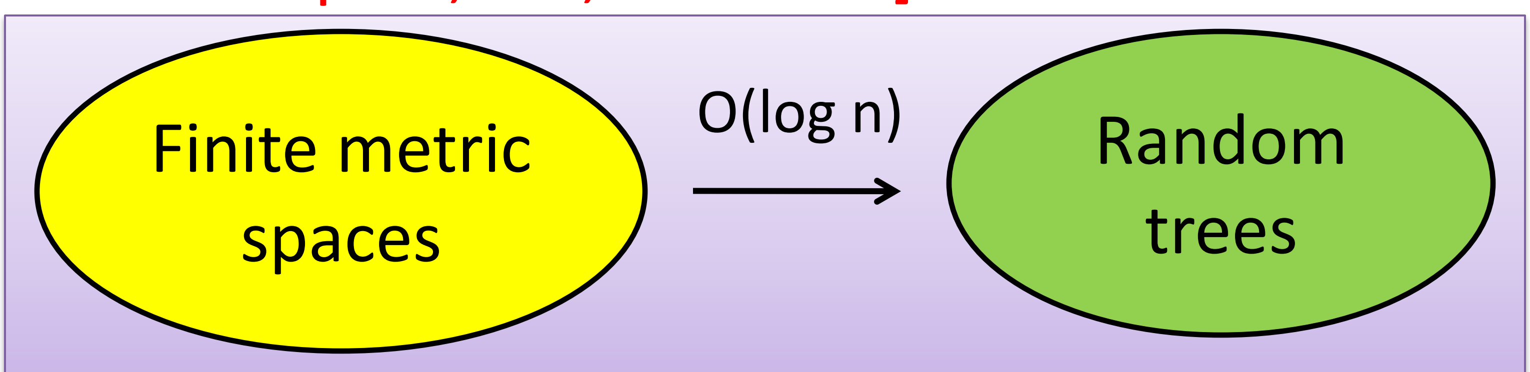
[Alon,Karp,Peleg,West’91], [Bartal’96], [Bartal’98], [Fakcharoenphol,Rao,Talwar’03]

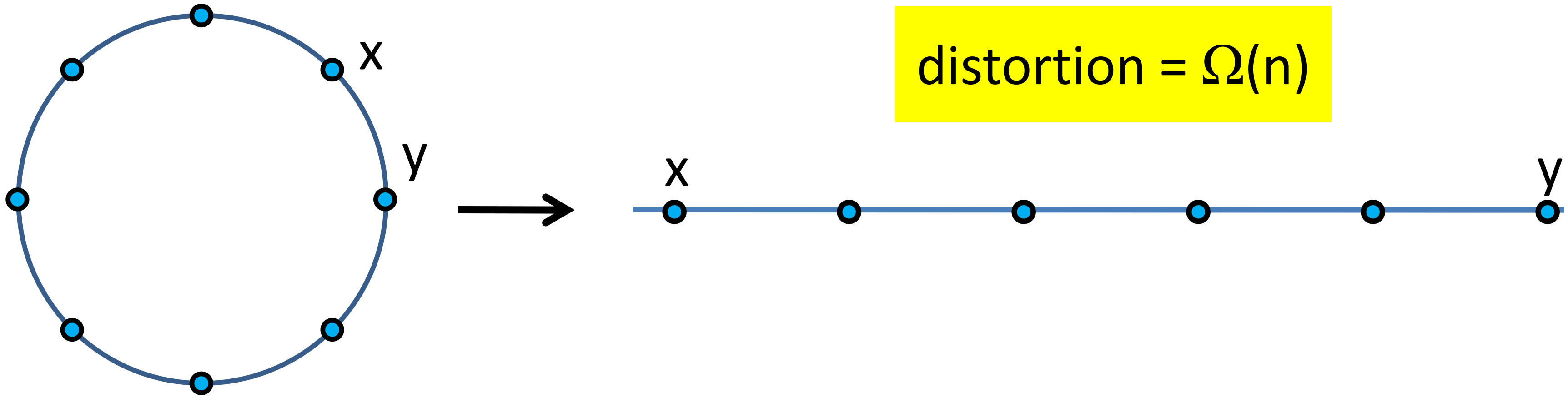


Topological simplification

- Topological simplification of a metric space M=(X,D)
- Low distortion embeddings
  - Mapping $f : X \rightarrow Y$
  - Preserve distances up to small distortion
- Relaxation: Stochastic embeddings
  - Random mapping $f : X \rightarrow Y$
  - Preserve distances in expectation

Stochastic embeddings: example

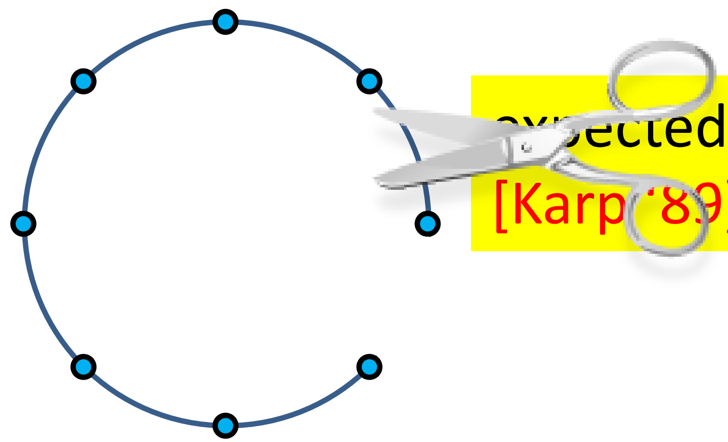
- Deterministic embedding of the cycle into R^1



distortion = $\Omega(n)$

- Randomization: Cut an edge at random!

•Pr[edge is cut] = 1/n
•If {x,y} is cut, then
D'(x,y)=n-1



expected distortion = O(1)
[Karp '89]

Stochastic embeddings

- Finite metric space $M=(X,D)$
- Distribution $\Phi=\{(M_1,f_1),\ldots,(M_k,f_k)\}$
  - $M_i=(X_i,D_i)$
  - $f_i : X \rightarrow X_i$

  such that for all u, v in X
  - for all M_i in F, $D_i(u,v) \geq D(u,v)$
  - $\mathbf{E}_N\,[D_N(f(u), f(v))] \leq \boldsymbol{\alpha} \cdot D(u,v)$

α : distortion

Stochastic embeddings

- $n \times n$ grid $\longrightarrow$ tree: $\Omega(\log n)$ [Alon,Karp,Peleg,West'91]
- planar $\longrightarrow$ $O(1)$-treewidth: $\Omega(\log n)$ [Carroll,Goel'04]
- genus-g $\longrightarrow$ planar:
  - $2^{O(g)}$ [Indyk, S '07]
  - $g^{O(1)}$ [Borradaile, Lee, S '09]
  - $O(\log g)$ [S '10]
  - $\Omega(\log g)$ [Borradaile, Lee, S '09]

Implications: Approximations algorithms

Let A be a minimization problem, s.t. the objective depends linearly on the distances of the input metric.

(e.g. Distance Oracles, MST, TSP, k-Median, Clustering, Metric Labeling, etc.)

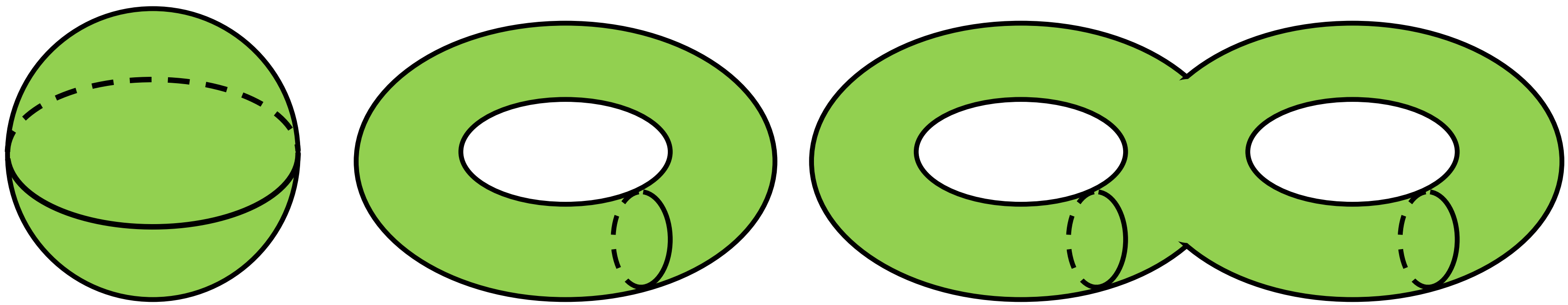
Theorem [S '10]

If there exists an **α**-approx. for A on planar graphs, then there exists an **O(α log g)**-approx. on genus-**g** graphs.

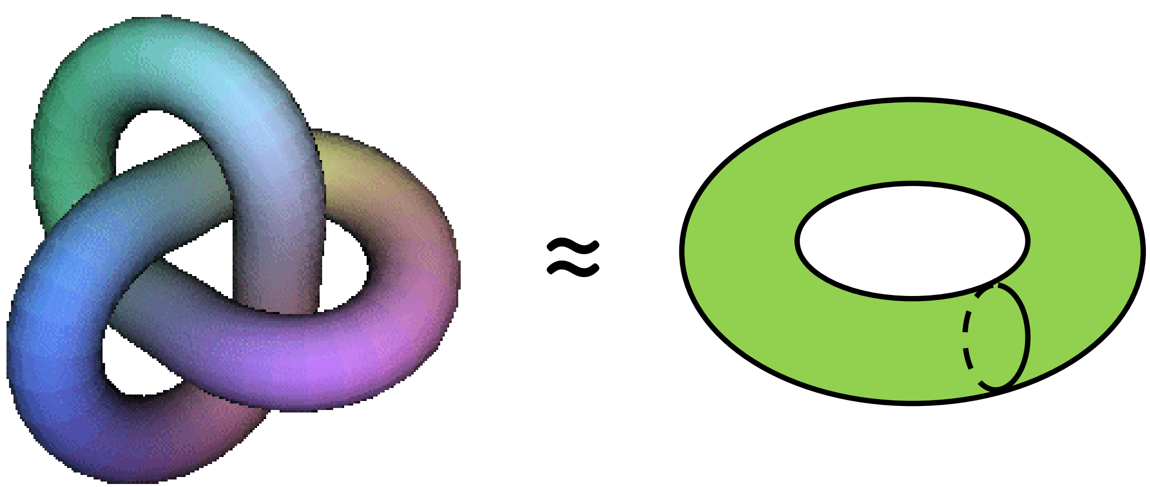
Implications

- O(log g)-approx for uniform Sparsest-cut [KPR'93]
- O(((log g)·(log n))$^{1/2}$)-approx for Non-uniform Sparsest-cut [Krauthgamer,Lee,Mendel,Naor'04]
- O(log g)-approx for treewidth [Feige,Hajiaghayi,Lee'05]
- k-th Laplacian eigenvalue: O(kg/n)·(log g)^2 [Kelner,Lee,Price,Teng'09]
- O(log g)-approx for 0-extension [Lee,Naor'04],[Calinescu,Karloff,Rabani'01]
- Similar improvements for Lipschitz extensions [Lee,Naor'04]
- Vertex Sparsifiers [Leighton, Moitra '10]
- Similar improvements for Minimum Crossing Number [Even,Guha,Schieber'02]

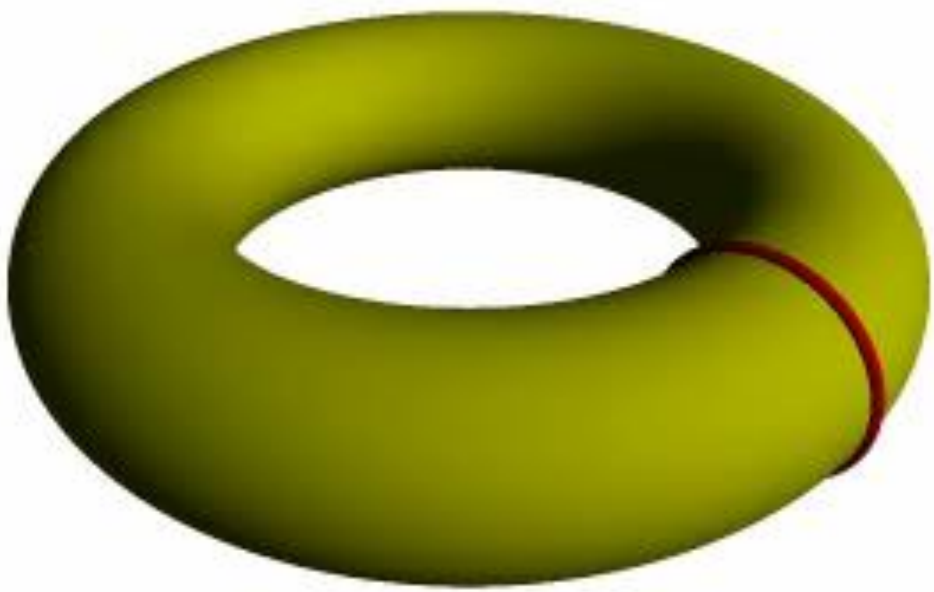
Orientable surfaces



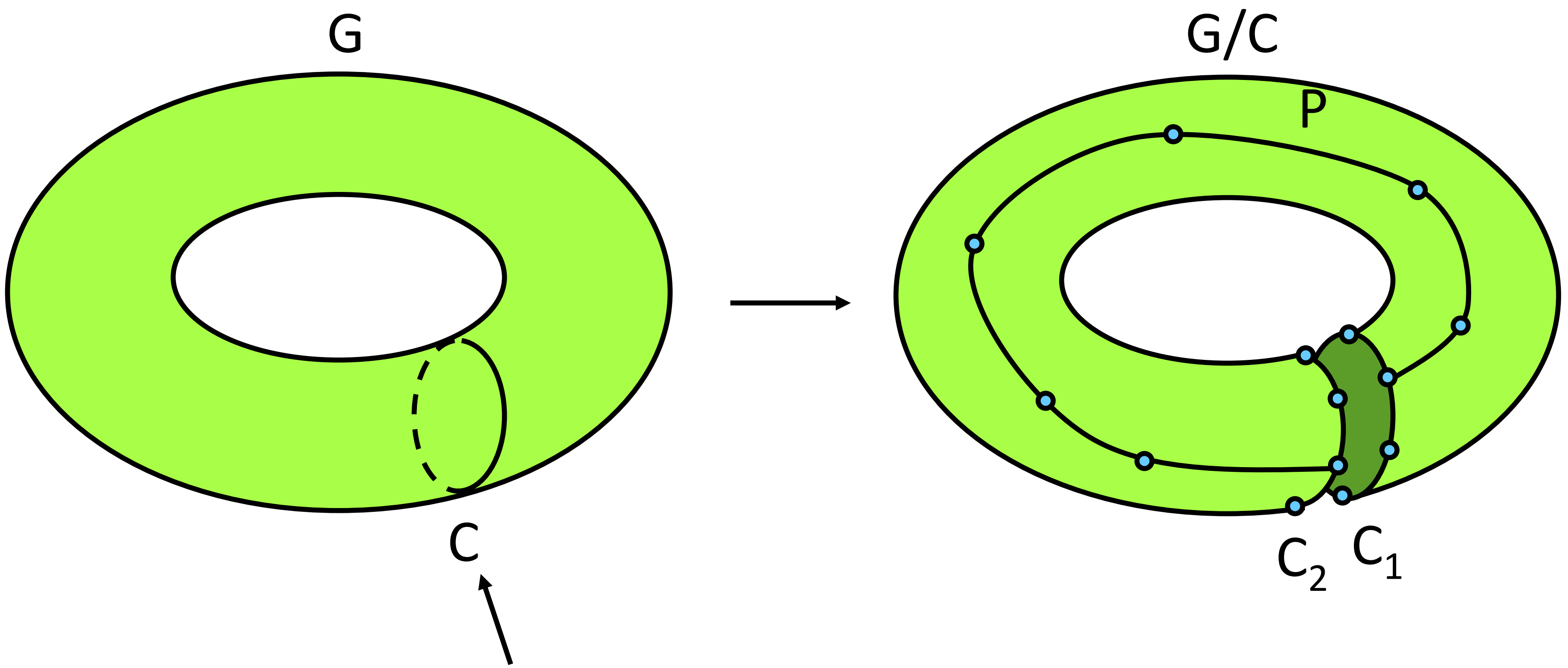
genus 0 genus 1 genus 2



Random cuts



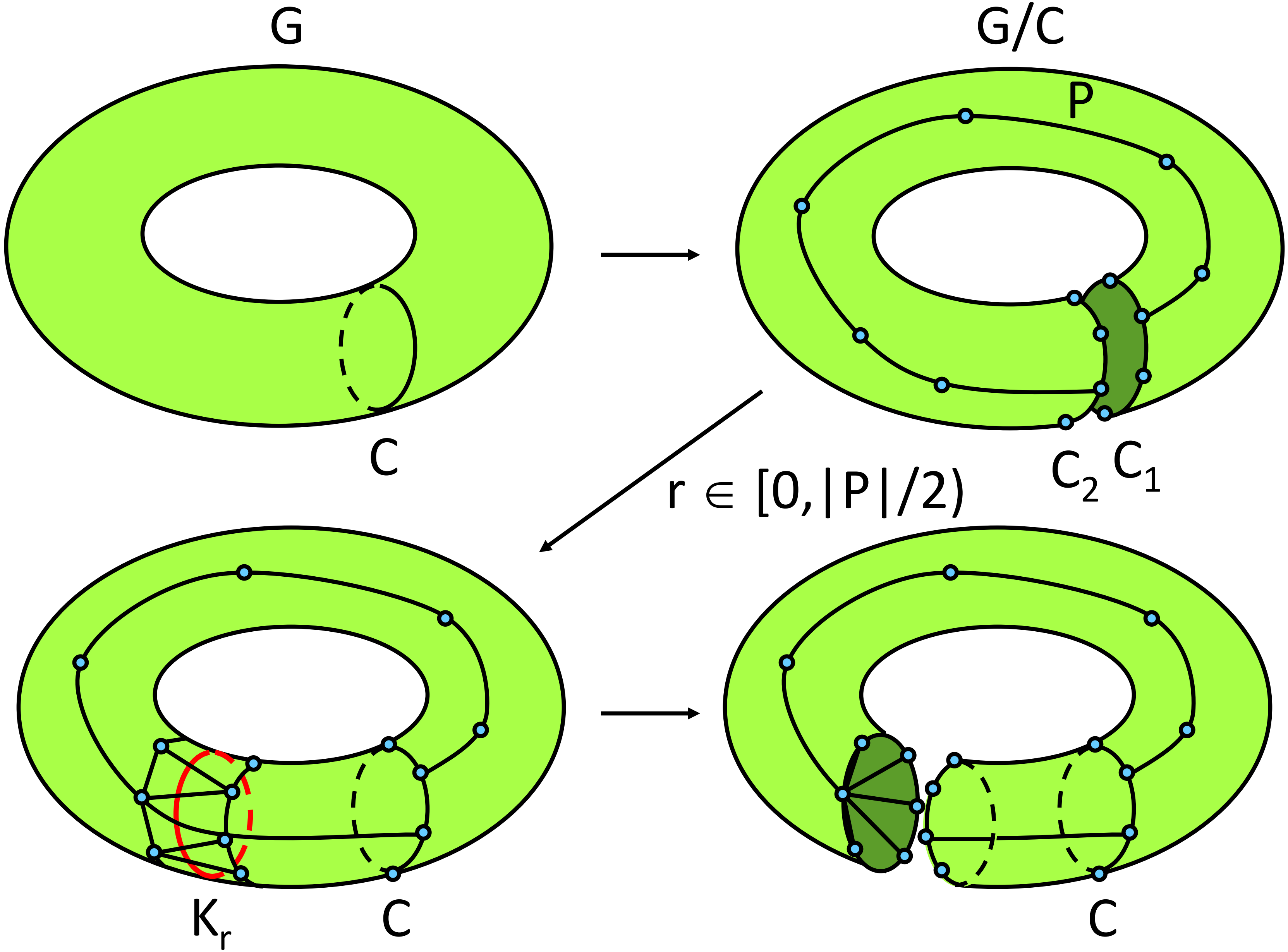
Random cuts



Shortest non-separating cycle
Can be computed e.g. by [Cabello,Chambers'07]

Claim: $|P| \geq |C|/2$

Random cuts



Random cuts: analysis

Consider edge e={u,v}

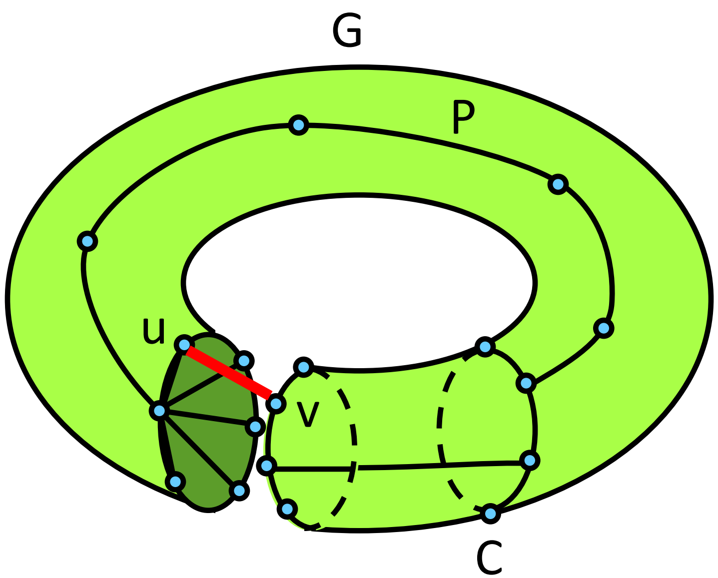
- Pr[e is cut] $\leq$ 2D(u,v) / |P|
- If e is cut, then

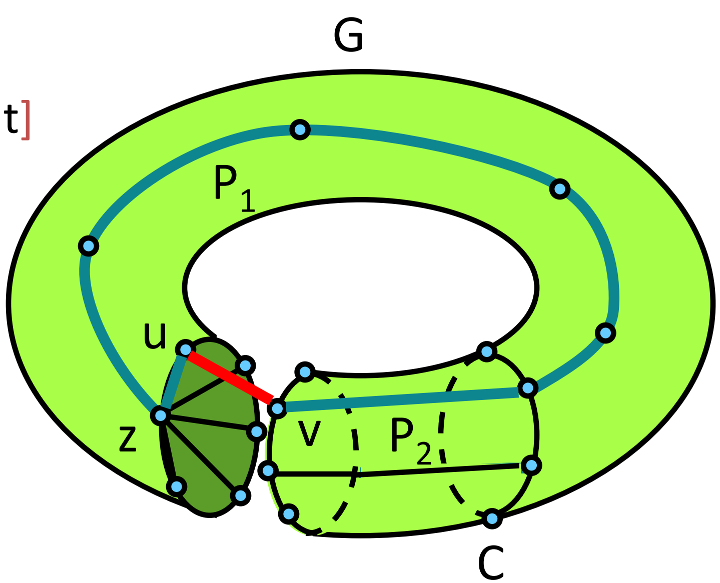
$$D'(u,v) \leq D'(u,z) + |P_1| + |P_2|$$
$$\leq D(u,z) + |P| + |P|$$
$$\leq 2|P| + |P|/2 + |P|/2$$
$$= O(|P|)$$

- Thus,

$$E[D'(u,v)] = D(u,v)\,\Pr[e \text{ not cut}] + O(|P|)\,\Pr[e \text{ is cut}]$$
$$= O(D(u,v))$$

For arbitrary paths, apply linearity of expectation.





Random cuts: summary

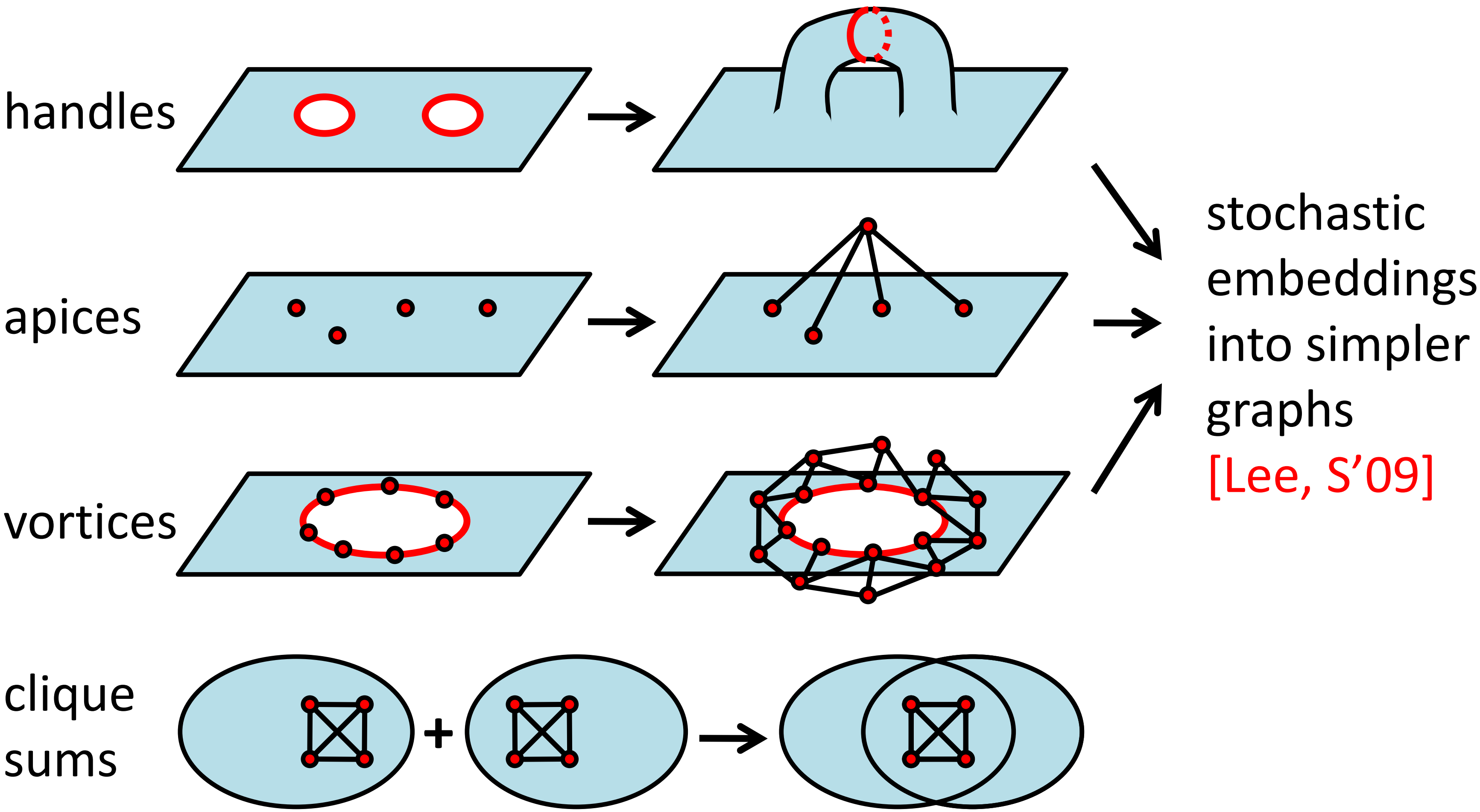
•Pr[edge is cut] = 1/L
•If {x,y} is cut, then D'(x,y)=O(L)

→ expected distortion = O(1)

- Repeating g times gives a planar graph [Indyk,S'07]
- Distortion $2^{O(g)}$
- Can we cut all the handles at once?

The Graph Minor Theorem

How to construct any non-trivial minor-closed graph family [Robertson, Seymour ’99]



Multi-flows in minor-free graphs

Conjecture:

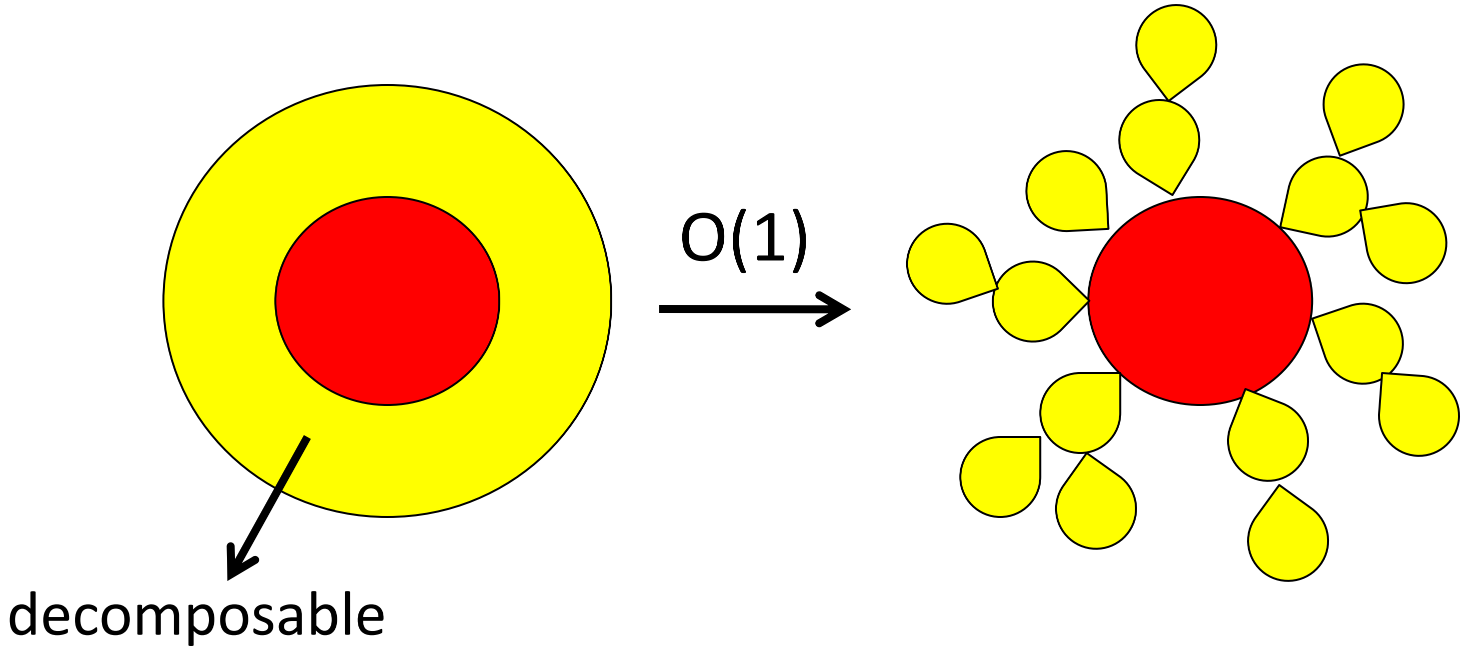
[Gupta,Newman,Rabinovich,Sinclair'99] For every nontrivial minor-closed graph family, the multi-flow/cut gap is O(1).

Theorem: [Lee, S '08]

The **only** hard instances for the GNRS conjecture are *essentially* planar graphs, and O(1)-treewidth graphs.

The peeling lemma

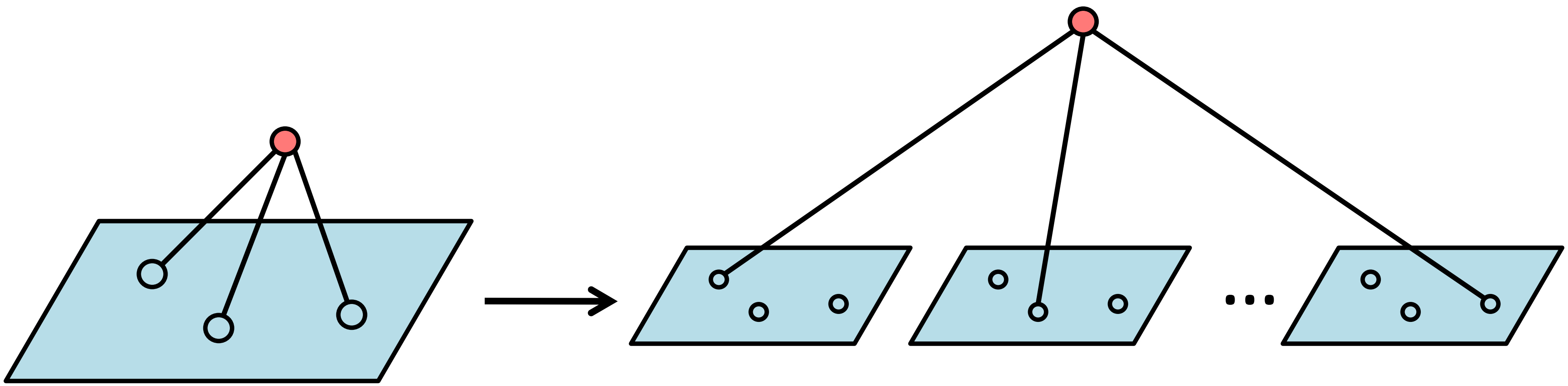
The peeling lemma



Peeling Lemma [Lee, S ’09]

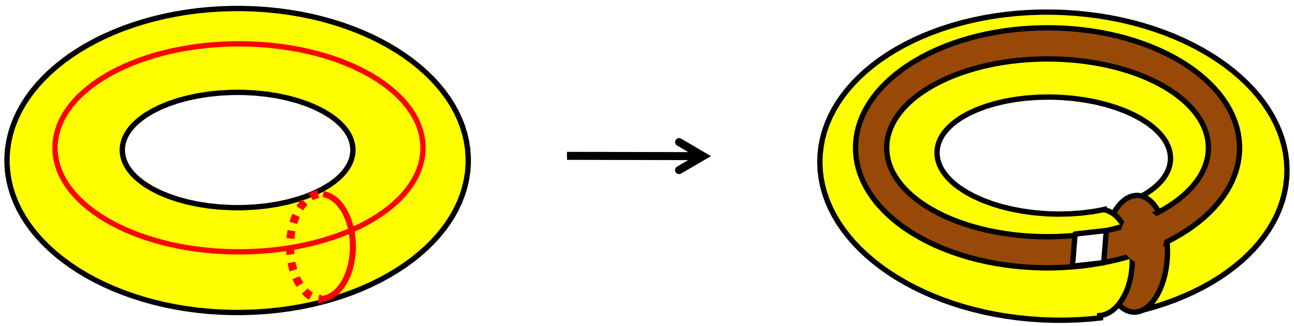
A ∪ B stochastically O(1)-embeds into 1-sums of A with B

The peeling lemma: removing apices



Homotopy generators

- Greedy system of loops [Erickson,Whittlesey'05]
  - Set H of cycles s.t. G\H is planar

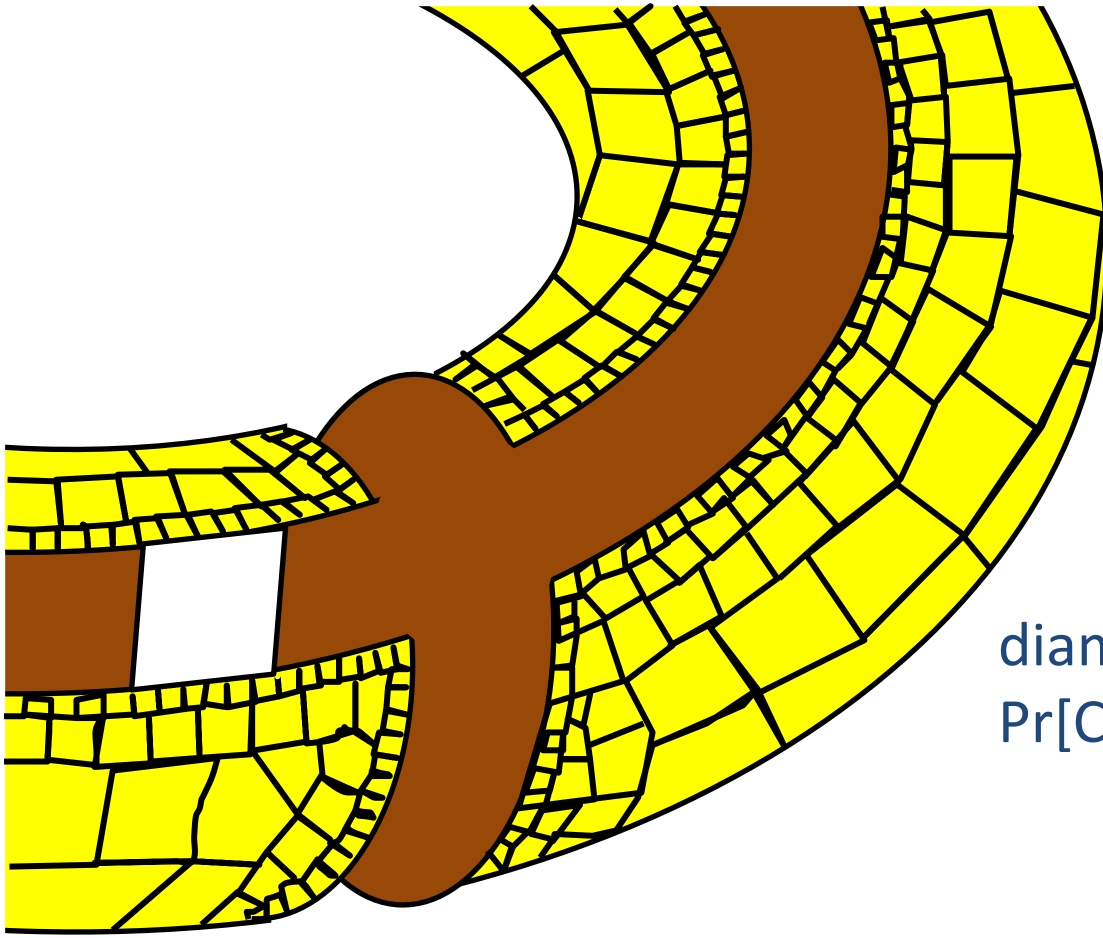


Lemma: [Borradaile, Lee, S '09]

The graph H has dilation O(g).

The peeling lemma

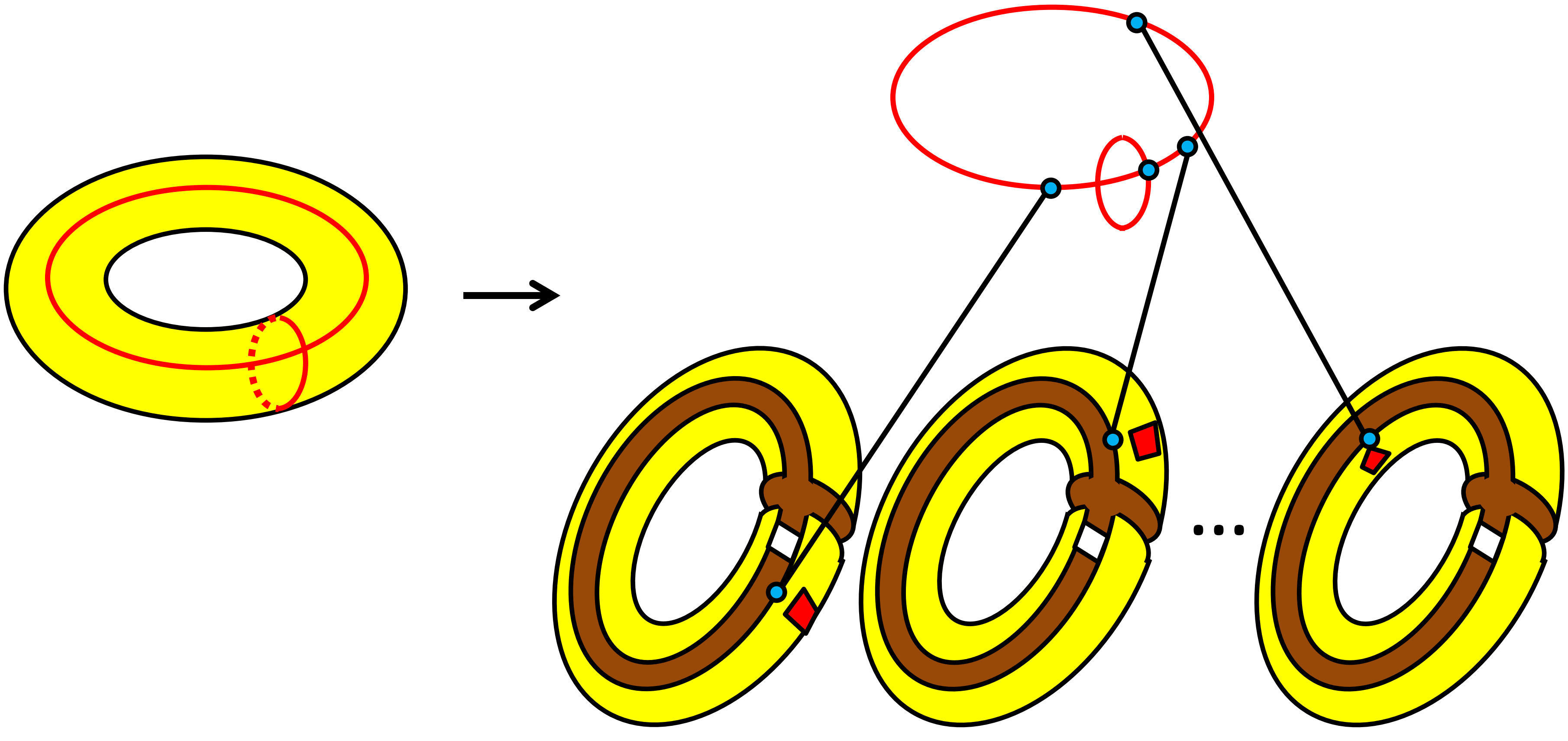
Random retractions [Lee,Naor]



$\mathrm{diam}(C) \leq d(C,H)$

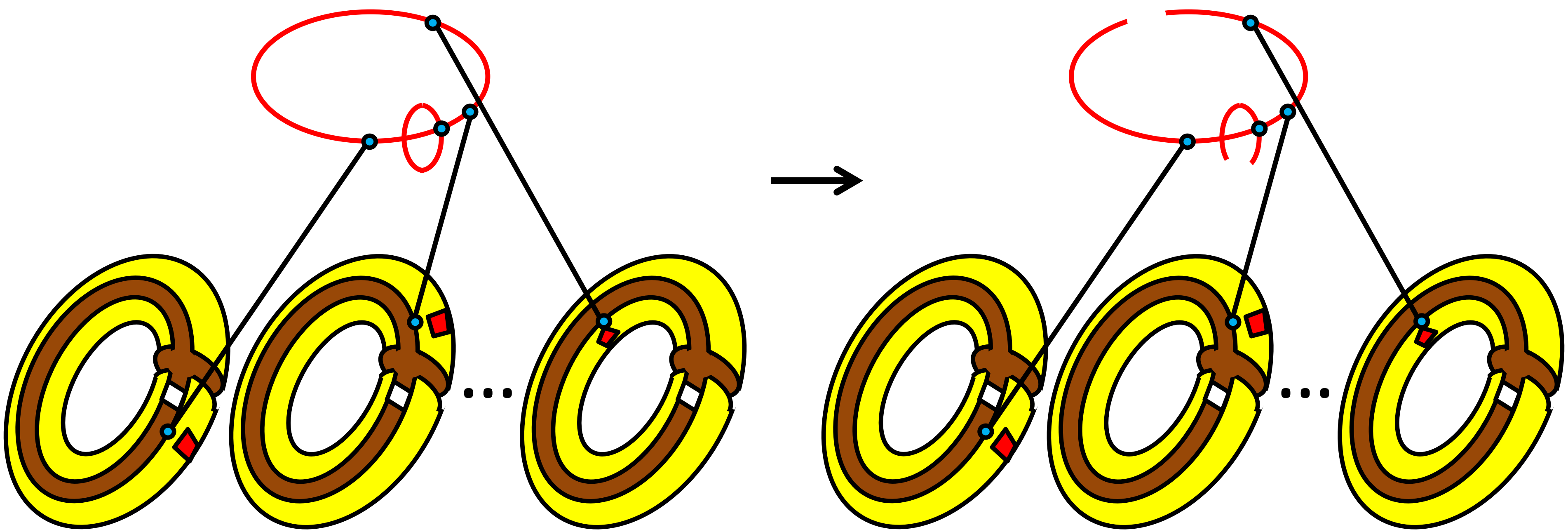
$\Pr[C(x) \neq C(y)] \approx d(x,y)/d(xy,H)$

The peeling lemma



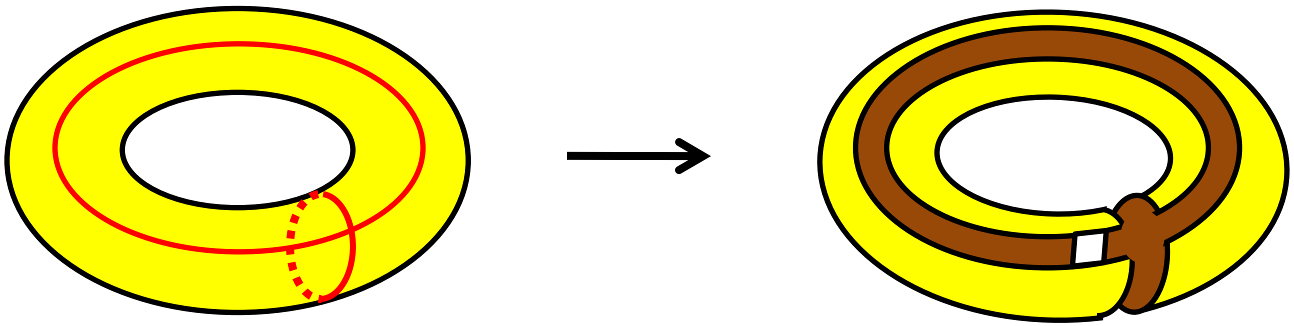
The final embedding

Resulting distortion $g^{O(1)}$



Homotopy generators revisited

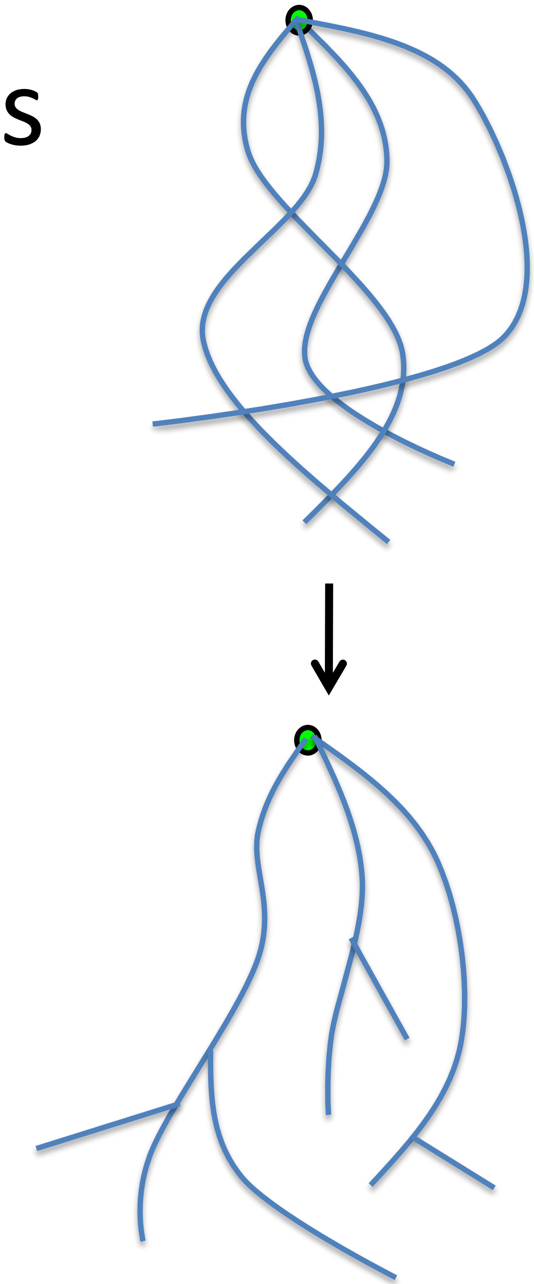
- Greedy system of loops [Erickson,Whittlesey'05]
  - Set H of cycles s.t. G\H is planar



Fact: H consists of O(g) shortest paths with a common end-point.

Untangling paths

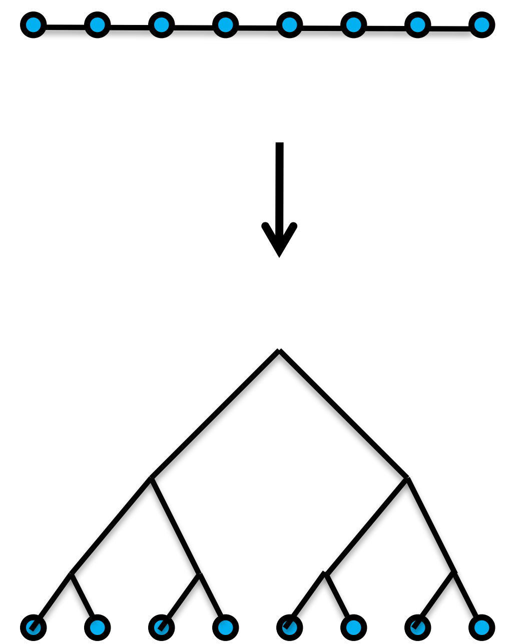
Fact: The cut graph consists of O(g) shortest paths with a common end-point.



Theorem: [S '10] Let **X** be the union of **g** shortest paths in a graph G, with a common end-point.
Then, **(X,d)** embeds into a random tree with distortion **O(log g)**.

The ultrametric barrier

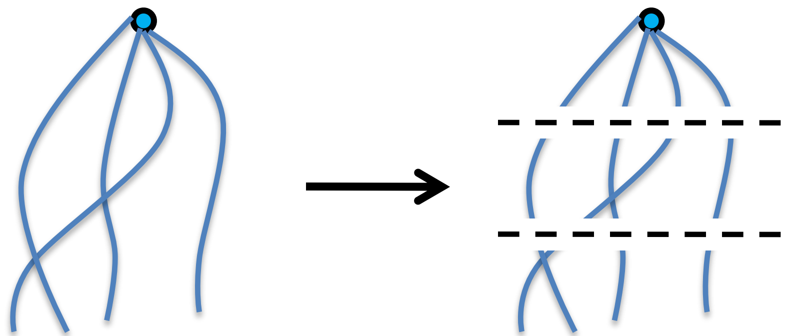
- Essentially all known tree embeddings:
  - Compute a partition for every scale $1,2,4,\ldots,2^i,\ldots$
  - Merge partitions into a tree.
  - The resulting tree is an **ultrametric**.



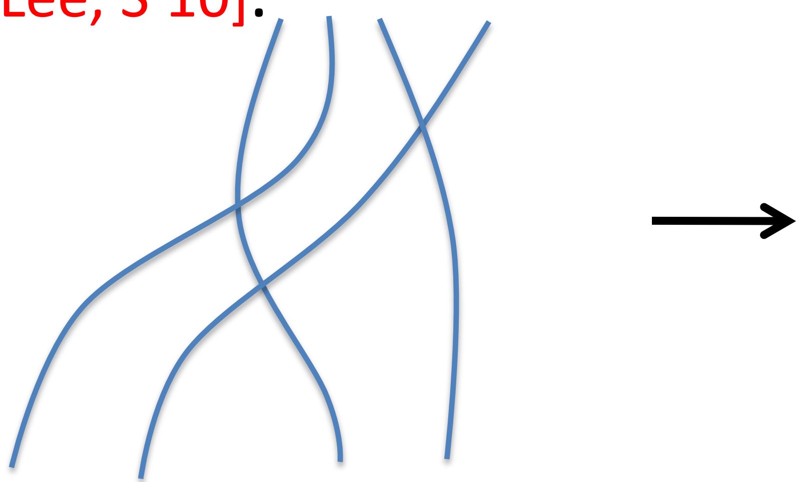
- Any embedding of the **n**-path into a random ultrametric has distortion $\mathbf{\Omega(\log n)}$.

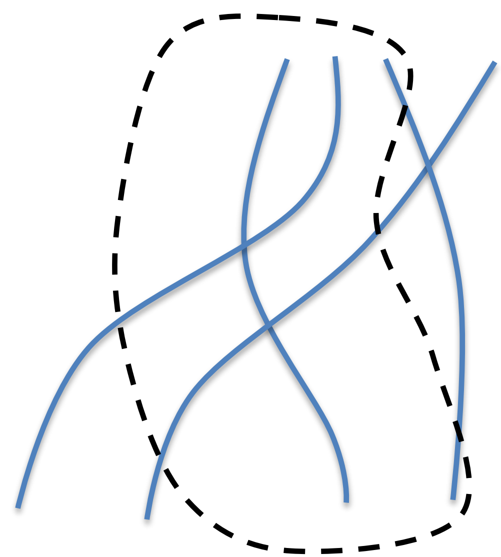
Key idea: Alternating partitions

- Combine two partitions at every scale:
  - **Vertical** partition, similar to [Klein,Plotkin,Rao'93].



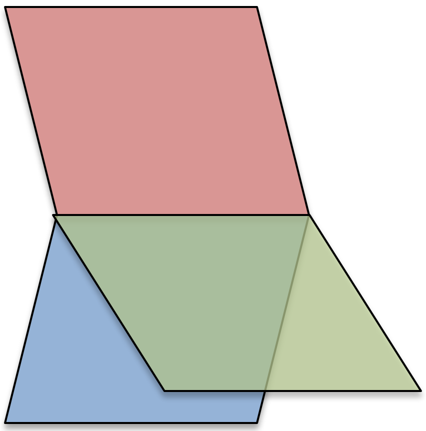
  - **Horizontal** partition, similar to [Calinescu,Karloff,Rabani'01], [Lee, S'10].



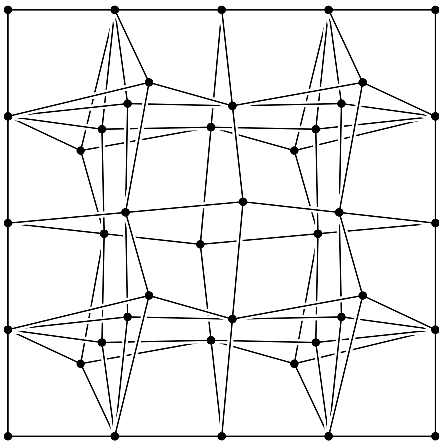


Beyond surfaces

- Main message:
  - Let **X** be a space obtained by glueing **simple** subspaces $\mathbf{X_1}$, $\mathbf{X_2}$, …, $\mathbf{X_k}$, along geodesics.
  - Then, **X** can be stochastically simplified.

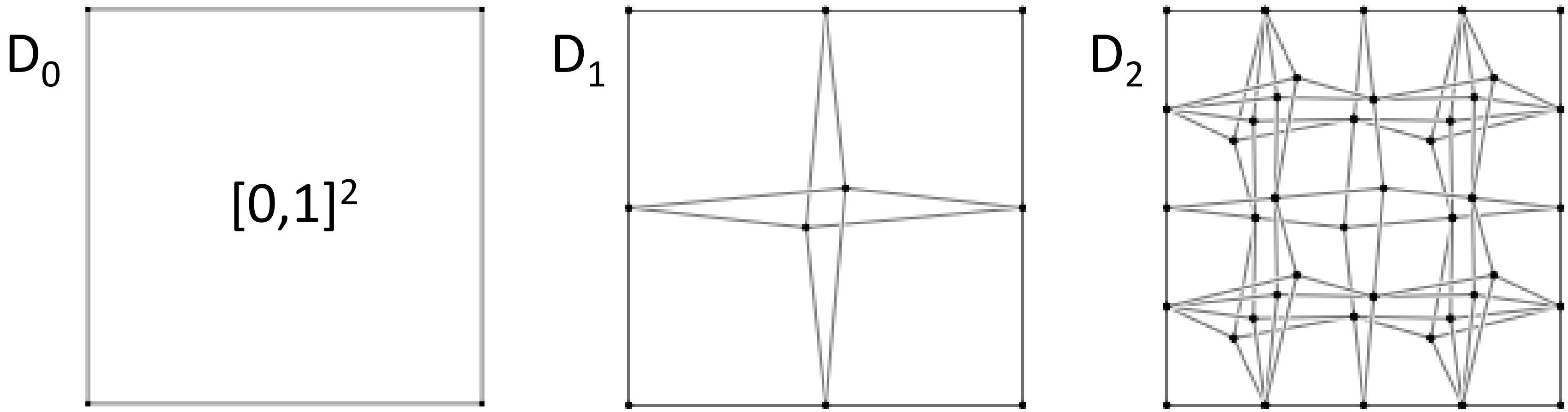


- Reverse direction:
  - [Lee, S '11] By glueing grids along shortest paths, we can construct a "complicated" space.

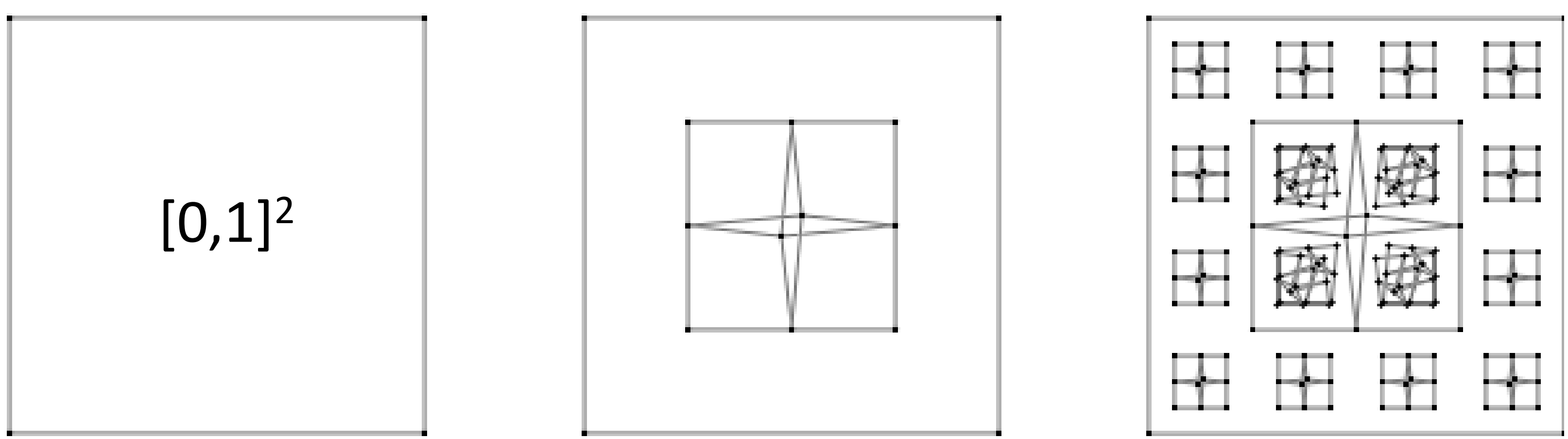


Constructing a "complicated" space

The diamond-fold

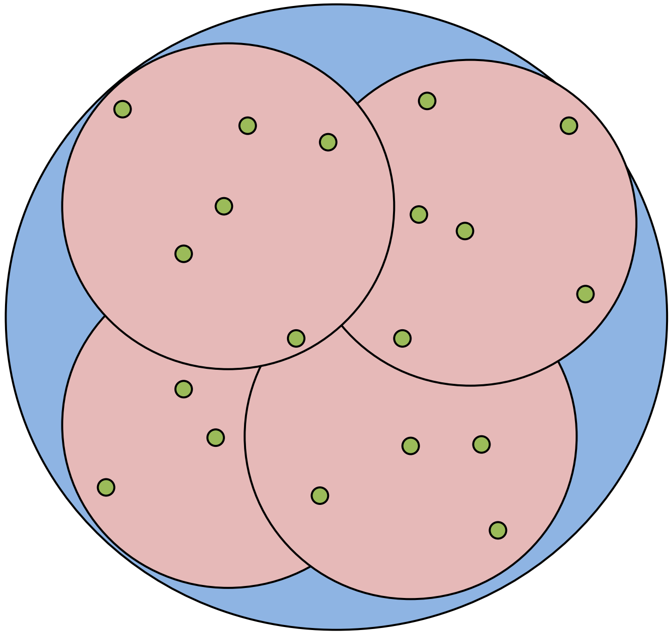


The Laakso-fold



Doubling spaces

- A metric (X, d) is **doubling** if every ball of radius r can be covered by O(1) balls of radius r/2.
- Metric notion of “bounded dimension”



Distortion of L_1 embeddings

- n-point metrics: $O(\log n)$ [Bourgain ’85]
- n-vertex expanders: $\Omega(\log n)$ [Linial,London,Rabinovich ’95]
- Doubling metrics : $O(\log n)^{1/2}$ [Gupta,Krauthgamer,Lee ’03]
- Doubling metrics : $\Omega(\log n)^{\delta}$, for some $\delta>0$ [Cheeger,Kleiner,Naor ’09]
- Doubling metrics : $\Omega(\log n)^{1/2}$ [Lee, S ’11]

Negative type

(X,d) is in **NEG** if $c_2(X,d^{1/2}) = 1$

(X,d) is in **soft-NEG** if $c_2(X,d^{1/2}) = O(1)$

Our result

Theorem [Lee,S]

There exists a doubling space that requires distortion $\Omega((\log n / \log\log n)^{1/2})$ to be embedded into L_1.

Theorem [Assoud'83]

Every doubling space is in soft-NEG.

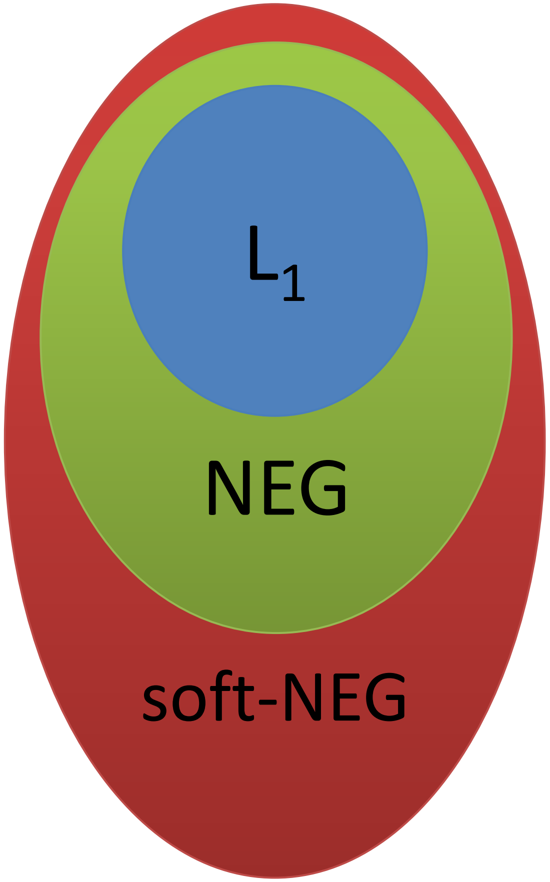
Corollary [Lee,S]

There exists a metric in soft-NEG that requires distortion $\Omega((\log n / \log\log n)^{1/2})$ to be embedded into L_1.

Soft negative-type

- All known algorithms for Sparsest-Cut require only optimization over soft-NEG
- This fact is essential for some fast algorithms [Sherman'09]

Corollary [Arora,Lee,Naor'05],[Lee,S'11] The integrality gap of the soft-SDP for Sparsest-Cut is $\widetilde{\boldsymbol{\Theta}}(\sqrt{\log \boldsymbol{n}})$.



Further directions

- Stochastic embeddings of genus-**g** graphs without knowing the drawing. [Makarychev, S ‘12]
- Approximate the genus of a graph.
  – Work in progress [Makarychev, Nayyeri, S]
- Optimal embeddings for graphs that exclude a minor **H**, in terms of **|H|**.
  – Only **Ω(log |H|)** lower bounds are known.
  – Almost all upper bounds are super-exponential in **|H|**.
  – Work in progress [Lee, S ‘12]