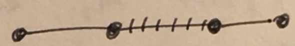
Maximum Bipartite Matching

* A "matching" in a graph $G=(V,E)$ is some $M \subseteq E$, s.t. every vertex in V is incident to at most one edge in M.
* A vertex v is "exposed" (or "unmatched") if no edge in M is incident to v.
* A matching is "perfect" if no vertex is exposed.

Maximum Bipartite Matching Problem:

Input: Bipartite graph $G=(V,E)$

Goal: Find matching M in G, maximizing $|M|$.

Note: "Greedy" fails. E.g. 

Definition: An "alternating path with respect to some matching M" is a path that alternates between edges in M and edges in $E \setminus M$.

Definition: An "augmenting path with respect to some matching M" is an alternating path w.r.t. M in which the first and last vertices are exposed.

Definition: Let M be a matching, and let P be an augmenting path w.r.t. M.
"Augmenting M along P" means replacing M by the matching $M' = M \Delta P := (M \setminus P) \cup (P \setminus M)$

symmetric difference.

Note: M' is indeed a matching. (Why?)

Claim: $|M'| = |M|+1$ (Why?)

Theorem: A matching M is maximum iff there are no augmenting paths w.r.t. M.

Proof:

($\Rightarrow$) If $\exists$ augmenting path P, then $M' = M \Delta P$ is a matching with $|M'| > |M|$. Thus, M is not a maximum matching.

($\Leftarrow$) If M is not maximum, then let M^* be a maximum matching. Thus $|M^*| > |M|$.

Let $Q = M \Delta M^*$.

We have:

(*) Q has more edges from M^* than from M (since $|M^*| > |M| \Rightarrow |M^* \setminus M| > |M \setminus M^*|$).

(*) Each vertex in V is incident to at most one edge in $M \cap Q$ and at most one edge in $M^* \cap Q$.

(*) Thus Q is a union of paths and cycles that alternate between M and M^*.

(*) All cycles in Q have even length (by alternating property).

(*) Thus there must exist a path in Q with more edges in M^* than in M.

This path is augmenting w.r.t. M, a contradiction.

$\blacksquare$

Note: The above theorem holds for arbitrary graphs (i.e. not just bipartite).

Algorithm:

Start with arbitrary matching M (possibly empty).

While there exists augmenting path P w.r.t. M,

augment M along P.

end.

How can we find an augmenting path w.r.t. M if one exists?

* Direct edges in G as follows:

Let A & B be the two partitions of G.

If $e \notin M$ then orient e from A to B.

If $e \in M$ then orient e from B to A.

Let D be the resulting directed graph.

Lemma: $\exists$ augmenting path in G w.r.t. M iff $\exists$ a directed path in D from an exposed vertex in A to an exposed vertex in B.

Proof: ...