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Assignment 1

Q1. Given,

$$P(\text{error} \mid x) = \min\{P(\omega_1 \mid x), P(\omega_2, x)\}$$

Let the min ω_i be represented by ω_m

a) $P(\text{error} \mid x) = P(\omega_m \mid x)$

for this, ~~cos~~ let ~~us con~~ us consider $\omega_m = \omega_2$

i.e $m = 2$ $\therefore P(\text{Error} \mid x) = P(\omega_2 \mid x)$

$$P(\omega_2 \mid x) = 1 - P(\omega_1 \mid x) \rightarrow (1)$$

$$P(\omega_1 \mid x) > P(\omega_2 \mid x) \rightarrow (2)$$

from (1) and (2)

$$P(\omega_1 \mid x) > \frac{1}{2} \quad \text{i.e } 2P(\omega_1 \mid x) > 1$$

now multiplying by $P(\omega_2 \mid x)$

$$2P(\omega_1 \mid x) P(\omega_2 \mid x) > P(\omega_2 \mid x)$$

$$2P(\omega_1 \mid x) P(\omega_2 \mid x) > P(\text{error} \mid x)$$

by integrating this of every x

$$\int 2P(\omega_1 \mid x) P(\omega_2 \mid x)\, dx > \int P(\text{error} \mid x)\, dx$$

$\downarrow$

upperbound on full error.

* we get same answer if we consider
$P(\text{error} \mid x) = P(\omega_1 \mid x)$

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1(b) We know that $P(\omega_1|x) \geq \frac{1}{2}$.

Consider $\alpha = \cancel{1.5}\ 1.2$

$P(\omega_1|x) = \cancel{0.8}\ 0.7$

$\therefore P(\omega_2|x) = \cancel{0.2}\ 0.3$

$\therefore P(error|x) = \cancel{0.2}\ 0.3$

$\alpha P(\omega_1|x) P(\omega_2|x) = \cancel{1.5 \times 0.8 \times 0.2}$

$= 1.2 \times 0.7 \times 0.3$

$= 0.252$

$0.252 < P(error|x)$

$\therefore$ This cannot possibly give upper bound ~~gua~~ guarantee

1(c) Since $P(error|x) = P(\omega_2|x)$

we have

$P(\omega_1|x) < P(\omega_2|x)$ $\quad P(error|x)$

$P(\omega_1|x) P(\omega_2|x) < P(\omega_2|x) P(\omega_2|x)$

$\int P(\omega_1|x) P(\omega_2|x) dx < \int P(\omega_2|x) P(error|x) dx$

we know

$\int P(\omega_2|x) P(error|x) dx < \int P(error|x) dx$

as $P(\omega_2|x) < 1 \ \forall x$

$\therefore \int P(error|x) dx > \int P(\omega_1|x) P(\omega_2|x) dx$

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Hence we have lower bound.

1(d) let $\beta = 1.2$ $p(\omega_1|x) = 0.7$

Same as 1(b)

2\. Given,

$$p(x|\omega_i) \propto e^{-|x-a_i|/b_i} \rightarrow (1)$$

from (1) we can get

$$p(x|\omega_i) = Ke^{-|x-a_i|/b_i}$$

2a) To normalize, we integrate over full range and $=1$

$$\int_{-\infty}^{\infty} p(x|\omega_i)\,dx = 1$$

$x - a_i < 0$ if $x < a_i$

$x - a_i \geq 0$ if $x \geq a_i$

$$\therefore \int_{-\infty}^{\infty} Ke^{-|x-a_i|/b_i} = K\int_{-\infty}^{a_i} e^{(x-a_i)/b_i}\,dx$$

$$+ K\int_{a_i}^{\infty} e^{-(x-a_i)/b_i}\,dx$$

$$= \left[K\,b_i\,e^{(x-a_i)/b_i}\right]_{-\infty}^{a_i}$$

$$+ \left[-K.b_i\,e^{-(x-a_i)/b_i}\right]_{a_i}^{\infty}$$

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$$= b_i k_0 - 0 + 0 - (-b_i k)$$

$$= 2 b_i k$$

$$2 b_i k = 1$$

$$k = \frac{1}{2 b_i}$$

$$\therefore p(x | w_i) = \frac{1}{2 b_i} e^{-|x - a_i| / b_i}$$

2 b) Likelihood ratio is given by

$$\frac{p(x|w_2)}{p(x|w_1)} = \frac{\frac{1}{2b_2} e^{-|x-a_2|/b_2}}{\frac{1}{2b_1} e^{-|x-a_1|/b_1}}$$

$$= \frac{b_1}{b_2} e^{-\left[|x-a_2|/b_2 - |x-a_1|/b_1\right]}$$

2 c) $$\frac{p(x|w_1)}{p(x|w_2)} = \frac{b_2}{b_1} e^{-\frac{|x-a_1|}{b_1} + \frac{|x-a_2|}{b_2}}$$

$a_1 = 0 \quad a_2 = 1 \quad b_1 = 1 \quad b_2 = 2$

when $x < 0$

$$x - \frac{x}{2} + \frac{1}{2} = \frac{1}{2}(x+1)$$

when $x \le 1, x > 0$

$$-x + \frac{(-(x-1))}{2}$$

$$-x - \frac{x}{2} + \frac{1}{2} = \frac{1}{2}(\cancel{3x}\,1 - 3x)$$

when $x > 1$

$$-x + \frac{x}{2} - \frac{1}{2} = -\frac{1}{2}(x+1)$$

$$\frac{p(x \mid \omega_1)}{p(x \mid \omega_2)} = \begin{cases} 2e^{(x+1)/2} & x \le 0 \\ 2e^{(1-3x)/2} & 0 < x \le 1 \\ 2e^{-(1+x)/2} & x > 1 \end{cases}$$

8. $$p(x \mid \omega_i) = \frac{1}{\pi b} \cdot \frac{1}{1 + \left(\frac{x-a_i}{b}\right)^2} \qquad i = 1, 2$$

a) $$\int_{-\infty}^{\infty} p(x \mid \omega_i)\,dx = \int_{-\infty}^{\infty} \frac{1}{\pi b} \frac{1}{1 + \left(\frac{x-a_i}{b}\right)^2}\,dx$$

$$\cancel{= \frac{1}{\pi b} \int_{-\infty}^{\infty}}$$

substituting $\frac{x-a_i}{b} = y$

$dy = \frac{1}{b}\,dx \quad dx = b\,dy$

$$= \frac{1}{\pi \cancel{b}} \int_{-\infty}^{\infty} \frac{1}{1+y^2}\,\cancel{b}\,dy$$

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$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+y^2} \, dy$$

$$\frac{1}{\sqrt{1+y^2}} = \sin\theta \qquad \frac{1}{\sin^2\theta} = 1+y^2$$

$$y^2 = 1 - \frac{1}{\sin^2\theta}$$

$$d.y = \frac{1}{\sin^2\theta} \, d\theta$$

$$K = \frac{1}{\pi} \int_{\theta=-\pi}^{\theta=0} \frac{\sin^2\theta}{\sin^2\theta} \, d\theta$$

$$= \frac{1}{\pi} \left[0 - (-\pi) \right] = 1$$

Hence it is normalized.

8 b) Given,

$$P(\omega_1) = P(\omega_2)$$

TPT $P(\omega_1 | x) = P(\omega_2 | x)$ if $x = (a_1 + a_2)/2$

We have to find the point of decision boundary

$$\frac{p(x|\omega_1) P(\omega_1)}{Z} = \frac{p(x|\omega_2) P(\omega_2)}{Z}$$

$$Z = p(x|\omega_1) P(\omega_1) + p(x|\omega_2) P(\omega_2)$$

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$$\frac{1}{1+\left(\frac{x-a_1}{b}\right)^2} = \frac{1}{1+\left(\frac{x-a_2}{b}\right)^2}$$

$$x-a_1 = \pm(x-a_2)$$

$$a_1 \neq a_2 \qquad x = \frac{a_1+a_2}{2}$$

which is the midpoint.

8d) ~~$\lim_{x\to\infty} p(x|\omega_i) = \frac{1}{\pi b}\frac{1}{1+\left(\frac{x-a_i}{b}\right)^2}$~~

8d) $$p(\omega_i|x) = \frac{p(x|\omega_i)P(\omega_i)}{Z}$$

$$Z = \sum_i p(x|\omega_i)\,p(\omega_i)$$

$$\therefore \lim_{x\to\infty} p(\omega_i|x) = \lim_{x\to\infty} \frac{\left[\frac{1}{\pi b}\cdot\frac{1}{1+\left(\frac{x-a_i}{b}\right)^2}\right)}{Z}$$

Z (similar to num but i = 1, 2.)

$$= \lim_{x\to\infty} \frac{b^2+(x-a_i)^2}{b^2+(x-a_1)^2+b^2+(x-a_2)^2}$$

$$= \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

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9 9a) Consider $a_2 > a_1$ → we can do this without loss of generality

from 8 we know that the decision boundary is at $(a_1 + a_2)/2$.

$$P(\text{error}) = \int_{-\infty}^{(a_1+a_2)/2} p(\omega_2|x)\,dx + \int_{a_1+a_2/2}^{\infty} P(\omega_1|x)\,dx$$

$$= \frac{1}{\pi b}\int_{-\infty}^{\frac{a_1+a_2}{2}} \frac{1}{1+\left(\frac{x-a_2}{b}\right)^2}\,dx + \frac{1}{\pi b}\int \frac{1}{1+\left(\frac{x-a_1}{b}\right)^2}$$

$y = x - a_2/b$

$$\therefore \frac{1}{\pi}\int_{-\infty}^{a_1-a_2/2} \frac{1}{1+y^2}\,dy$$

$$\int \frac{1}{1+y^2}\,dy = \tan^{-1} y$$

$$\therefore P(\text{error}) = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{a_1-a_2}{2b}\right) - \tan^{-1}[-\infty]\right]$$

$$= \frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left|\frac{a_2-a_1}{2b}\right|$$

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9c) ~~Bi~~ max $P(\text{error}) = \frac{1}{2}$

occurs for $\left|\frac{a_2 - a_1}{2b}\right| = 0$

ie if $a_1 = a_2$ ~~or~~ → when both dist or same

or

$b = \infty$ → when it is flat

~~b~~

12 Given,

$\omega_{max}(x)$

$P(\omega_{max}(x)|x) \geq P(\omega_i|x)$

12a) TST $P(\omega_{max}|x) \geq \frac{1}{c}$

if $P(\omega_i|x) = P(\omega_j|x)\ \forall i, j$

then $P(\omega_i|x) = \frac{1}{c}$

$\therefore P(\omega_{max}|x) = \frac{1}{c}$

If $\exists i$ st $P(\omega_i|x) < \frac{1}{c}$

then $\exists j, P(\omega_j|x) > \frac{1}{c}$

~~that is~~ which is not possible here

$P(\omega_{max}|x) = \frac{1}{c}$

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~~12(b) $P(\text{error}) = 1 - \int p(\omega_{max} | x$~~

12 b) ~~too~~ $p(\text{error}) = 1 - p(\text{correct})$

$\therefore$ ~~$P(\text{error}) =$~~

$$p(\text{correct}) = \int \max_i p(\omega_i | x)\, dx$$

$$= \int p(\omega_{max} | x)\, dx$$

$$\therefore p(\text{error}) = 1 - \int P(\omega_{max} | x)\, p(x)\, dx$$

12 c) consider $\alpha \geq 1/c$

$$p(\text{error}) \leq 1 - \int \alpha\, p(x)\, dx$$

$$\leq 1 - \alpha \int \underbrace{p(x)\, dx}_{=1} = 1 - \alpha$$

$$\therefore p(\text{error}) \leq 1 - \frac{1}{c} = \frac{(c-1)}{c}$$

12 d) This is possible only if all classes have same priors and identical distributions