FE 621: HW3

Due date: March 27th at 11:59 pm

- Each of the four problems is worth 30 points but 100 points is considered a full score for this assignment.

Problem 1 (Delta hedging)

In this problem we use simulation to measure the hedging error resulting from discrete rebalancing of a hedge. We sell a 3-month European call option at the Black-Scholes price and hedge it by holding "delta" shares of the underlying stock. We can borrow money from the bank at constant interest rate, and any money left in our account earns the same constant rate of interest.

- At initiation of the contract, we get the option premium from the client and buy delta shares of the stock. We may need to borrow extra money to set up the hedging portfolio.
- At each time step, the stock price has evolved from the previous step and the delta must be adjusted. Depending on how it has changed, we need either to buy or sell shares. We also pay or earn interest on any money borrowed or deposited over the previous period.
- At maturity, we close our position. This means selling all shares we own, reimbursing the bank for the money we owe or get what is left in our account, and paying our client the amount $(S_T - K)^+$. The cash left after that is our profit or loss (PnL).

Assume the underlying asset to follow a geometric Brownian motion with the parameters specified below. Black-Scholes theory says that as the rebalancing frequency goes to infinity (with T fixed), the hedging error (PnL) goes to zero. Since continuous trading is impossible in practice, the hedge is imperfect and we want to study this imperfection.

Initial price: $S_0 = 50$,	Rate of return: $\mu = 10\%$
Volatility: $\sigma = 30\%$,	Interest rate: $r = 5\%$
Strike: $K = 50$,	Expiration: $T = 0.25$

(a) Simulate paths of the stock price and find the mean and standard deviation of the hedging error with daily and weekly rebalancing. Make a histogram of the distribution of hedging errors for both cases. Specify the number of simulations (n) you use (it should be at least $n = 100,000$). Comment on your findings.

Note: For each simulated path, there will be one hedging error (PnL). Hence, your histogram will be based on n simulated values.

(b) Take different values for μ (e.g., μ in the range 0 to 1). Does this impact the mean and standard deviation of the hedging error with daily and weekly rebalancing? Explain your findings.

(c) Let Δt denote the rebalancing interval. We know that as $\Delta t \to 0$ the hedging error goes to 0. What does your simulation suggest about the rate of convergence? Does the hedging error appear to be of order $(\Delta t)^\alpha$ for some $\alpha > 0$, and if so, what α?

Hint: You may want to draw a log-log plot and note that

$$error \sim (\Delta t)^\alpha \quad \Longrightarrow \quad \log(error) \sim \alpha \log(\Delta t).$$

For the hedging error you can for example consider the root-mean-squared-error: $RMSE = \sqrt{Bias^2 + Var}$.

Problem 2 (Stop-loss start-gain hedging strategy)

In this problem we test another hedging scheme for a call option. This strategy consists first in charging the client $(S_0 - Ke^{-rT})^+$ for the option, and then in hedging the option by holding one share of the stock when $S_t > Ke^{-r(T-t)}$ and zero shares of the stock when $S_t \leq Ke^{-r(T-t)}$. This scheme is based on the simple observation that the seller of the option will need one share at expiry if $S_t > K$ and none if $S_T \leq K$.

Modify your code in Problem 1 and find the mean and standard deviation of the hedging error with daily and weekly rebalancing.

Make a histogram of the distribution of hedging errors for both cases. How does the hedge perform? Does higher rebalancing frequency help? Explain your findings.

Note 1: The only thing you need to change from Problem 1 is how delta is computed. The stock price simulation and the computation of hedging error is exactly the same.

Note 2: Black-Scholes theory says that we need to charge $e^{-rT}\mathbb{E}[(S_T - K)^+]$ for the option in order to set up a self-financing replicating portfolio with zero hedging error as the rebalancing frequency increases.

This problem indicates that we can charge $(S_0 - Ke^{-rT})^+$ in order to set up a self-financing replicating portfolio. If that is the case, then from the inequality

$$(S_0 - Ke^{-rT})^+ \leq e^{-rT}\mathbb{E}[(S_T - K)^+],$$

we can conclude that the Black-Scholes price is too high! This paradox was eventually solved in a paper by Peter Carr and Robert Jarrow where it is explained that the hedging strategy in this problem is not self-financing and consistently loses money.[1]

[1] P. Carr and R. Jarrow. The Stop-Loss Start-Gain Paradox and Option Valuation: A New Decomposition into Intrinsic and Time Value. *The Review of Financial Studies*, 3(1), 469–492, 1990.

Problem 3 (Effect of non-constant volatility)

In this problem we consider what happens to the Black-Scholes hedging strategy if volatility is not constant. Let us suppose that the stock follows a CEV (constant elasticity of variance) process:

$$\frac{dS_t}{S_t} = \mu dt + \alpha S_t^\beta dW_t.$$

Take $\beta = -0.8$ and α such that $\alpha S_0^\beta = 0.3$ so the volatility at initiation of the contract is 30% like in Problem 1. Use the same values for S_0, μ, r, K and T as in Problem 1.

To simulate the CEV process on a partition $(t_1, \ldots, t_m)$ with $t_i = iT/m = i\Delta t$, use the Euler discretization scheme:

$$\hat{S}_{i+1} = \hat{S}_i\big(1 + \mu\Delta t + \alpha\hat{S}_i^\beta\sqrt{\Delta t}Z_{i+1}\big),$$

where the Z_i's are i.i.d. standard Gaussian random variables.

Modify your simulation in Problem 1 but keep hedging the option with the Black-Scholes delta at constant volatility 30%. Repeat Problem 1(a) and (c) and compare your results with the constant volatility case.

Note: The only thing that changes from Problem 1 is how the stock price paths are simulated. Computation of delta and the hedging error is exactly the same.

Problem 4 (basket options)

Consider d assets $S = (S^1, \dots, S^d)$ following a d-dimensional geometric Brownian motion,

$$\frac{dS_t^i}{S_t^i} = rS_t^i dt + \sigma^i S_t^i dW_t^i,$$

where the d-dimensional Brownian motion $W = (W^1, \dots, W^d)$ has covariance matrix

$$\Sigma = \begin{pmatrix} 1 & \rho & \rho & \dots & \rho \\ \rho & 1 & \rho & \dots & \rho \\ \rho & \rho & 1 & \dots & \rho \\ \vdots & & & \ddots & \vdots \\ \rho & \rho & \rho & \dots & 1 \end{pmatrix}.$$

(a) Write a function that uses Cholesky decomposition to find a lower triangular matrix A such that $\Sigma = AA^T$. Demonstrate that your code works for $d = 3$ and $\rho = 0.1$.

Note: An increment of the Brownian motion between times t and $t + \Delta$ can then be simulated as $W_{t+\Delta} - W_t = \sqrt{\Delta}AZ$ where Z is a standard multivariate Gaussian random variable.

(b) Use Monte Carlo simulation to price a European basket option with payoff

$$f(S_T) = \big(\max_{1 \le i \le d} S_T^i - K\big)^+.$$

Use parameters $S_0^i = 1$, $r = 0$, $\sigma^i = 0.3$, $T = 1$ and $K = 1$. Do this for values of ρ between $-\frac{1}{d-1}$ and 1. Report your results graphically for $d = 3$ and $d = 4$. Comment on your findings. In particular, what is the effect of ρ and d on the option price?

Note: You simulation procedure should use your function from part (a).

(c) Repeat part (b) for a European basket option with payoff

$$f(S_T) = \big(\frac{1}{d}\sum_{i=1}^{d} S_T^i - K\big)^+.$$

Comment on your findings. In particular, what is the effect of ρ and d on the option price?

(d) *Extra credit (hard!):* Show that Σ is a valid covariance matrix if and only if

$$-\frac{1}{d-1} \le \rho \le 1.$$

For example, if $d = 2$ we need $-1 \le \rho \le 1$, for $d = 3$ we need $-0.5 \le \rho \le 1$, and for $d = 11$ we need $-0.1 \le \rho \le 1$. Can you see intuitively why a larger d prevents us from choosing negative values of ρ?

Hint: Use that the symmetric matrix Σ is a covariance matrix if and only if it is positive semidefinite, which is equivalent to all of its eigenvalues being nonnegative.