FE 621: HW1

Due date: February 14th at 11:59 pm

- Please make sure that your solutions are presented in an organized and readable manner.

Problem 1

(a) Download price and implied volatility data for S&P 500 index options (SPX options). You also need the value of the S&P 500 index. The data can be obtained from, e.g., Yahoo Finance, Google Finance, or Bloomberg.

(b) Download interest rate data from `http://www.federalreserve.gov/releases/H15/Current/`.[1] Make sure to use data for the same date as your options data.

(c) Implement the Black-Scholes option pricing formula as a function of current stock price S_0, volatility σ, time to maturity $\tau = T - t$ (in years), strike price K, and short-term interest rate r (annual). You need to implement the function yourself - no financial toolbox function is allowed.

(d) Implement a function that uses either the bisection method or Newton's method to compute the implied volatility of options. Note: Newton's method requires computing the derivative of the Black-Scholes price with respect to the volatility σ. This derivative is known as *vega* and it has a closed-form formula in the Black-Scholes model.

(e) Use your function in (d) to compute the implied volatility of the options in your data set. For the price of an option, use the average of the bid and ask prices.

Remark: For simplicity you *may* skip options with the shortest maturities. For such options there may only be a liquid market for a small set of options that are *near-the-money.* Similarly, for a given maturity, be mindful of the fact that there may not be a liquid market for deep in-the-money or out-of-the-money options.

(f) For at least three different maturities ("short", "medium", and "long"), visualize the implied volatilities for both types of options by plotting it as a function of strike K. Highlight the at-the-money (ATM) implied volatility in your plots. How do the computed volatilities compare to the downloaded volatilities?

Remark: (i) There is not a well-defined boundary between ATM/OTM/ITM. Moneyness of an option is defined as the ratio S_0/K of the current stock price and the strike price. In the literature it is common to use moneyness between 0.9 and 1.1 to define ATM, but smaller and larger intervals are also used. (ii) In practice it is common to plot the implied volatilities as a function of moneyness S_0/K. You may experiment with this.

(g) Comment on what happens to implied volatility when time-to-maturity increases. For a fixed maturity, comment on what happens when to implied volatility when options become ITM and OTM.

(h) For the same maturities as used in (f), use the put-call parity to calculate for each call (put) option the price of the corresponding put (call). Visualize the resulting prices together with the downloaded prices of the corresponding options (if they exist). Does the put-call parity seem to hold?

(i) For the same maturities as used in (f), plot the Black-Scholes delta and gamma as a function of strike K. Comment on the patterns you observe. Note that closed-form formulas can be used to compute the delta and gamma of call and put options in the Black-Scholes model.

Optional: Plot delta and gamma computed using finite-difference approximations. For example, if $C(K,T)$ is the price of a call option with strike K and maturity T we have

$$\Delta(K,T) = \frac{\partial C(K,T)}{\partial S} \approx \frac{C(K+\delta K,T) - C(K-\delta K,T)}{2\delta K},$$

$$\Gamma(K,T) = \frac{\partial^2 C(K,T)}{\partial S^2} \approx \frac{C(K+\delta K,T) - 2C(K,T) + C(K-\delta K,T)}{(\delta K)^2},$$

where δK is "small".

[1] Note that multiple types of interest rates are listed. There is not a universal rule when it comes to which one to use, but the *effective federal funds rate* is commonly used in option pricing applications. Also note the data on this website is reported in percent per annum.

Problem 2

(a) Write pseudocode (i.e., outline the algorithm) for using the bisection method and Newton's method to compute the square root of a positive number a. For the bisection method, remember to state how you choose upper and lower bounds for the initial search interval.

(b) Implement your procedures in part (a) with tolerance $\epsilon = 10^{-6}$. Which one seems to require a larger number of iterations to converge?

(c) *Optional:* Show that Newton's method has quadratic order of convergence. Specifically, let x^* be a solution of $f(x) = 0$ for some function f, and show that

$$|x_{n+1} - x^*| \leq K|x_n - x^*|^2,$$

for some constant K, where x_n and x_{n+1} are the n-th and $n+1$-st iteration of the algorithm. You can make any assumptions needed about the function f and the initial guess x_0.