Problem Set II

Introduction to Graph Theory, MATH 3545
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1 Cycles, Trees, and Bipartite Structures

Problem 1. Let G be a simple graph with $n >= 2$ nodes. Prove that G must have two nodes whose degrees are the same.

Solution. *A simple graph with n nodes can have node degrees ranging from 0 to $n-1$. This is because a node can be connected to at most n-1 other nodes (every other node once except for itself).*

In general the set of all possible degrees a node can have in a simple graph can be represented by

$$\{d(v) \mid d(v) \in \mathbb{Z}, 0 \leq d(v) \leq n-1\}$$

Since in this case, the size of this set is equivalent to n, for each node to have a unique count of degrees, one of the nodes must have a degree of $n-1$. However, if this is the case that means that this node is connected to every single other node in the simple graph. This would indicate that there cannot be a node with degree 0. The set will therefore be represented as

$$\{d(v) \mid d(v) \in \mathbb{Z}, 1 \leq d(v) \leq n-1\}$$

The count of this set is $n-1$ which is less that the number of nodes n. As a result, due to the Pigeonhole Principle, there must be at least two nodes with the same degree. As

$$\{\left|\{d(v) \mid d(v) \in \mathbb{Z}, 1 \leq d(v) \leq n-1\}\right| < n\}$$

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Problem 2. Let G be a bipartite graph with parts A, B. We happen to know that every node in A has degree 7, and every node in B has degree 9. We also know that A consists of 72 nodes. What are the possible values of the size of B?

Solution. *A bipartite graph is a graph where the vertices can be divided into two disjoint sets such that all edges connect a vertex in one set to a vertex in another set. As a result, the total number of edges incident to the nodes in A must be equal to the total number of edges incident to the nodes in B, since each edge connects a node in A to a node in B.*

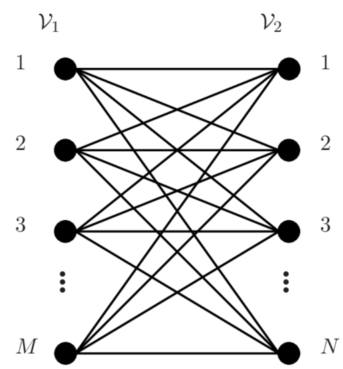

Figure 1: Example of a Bipartite Graph where v_1 and v_2 are A and B

Let $|A|$ and $|B|$ represent the sizes of each part. With the given information, we know that

$$|A| = 72$$
$$d(v) = 7 \quad \forall v \in A$$
$$d(v) = 9 \quad \forall v \in B$$

The total number of edges incident to A is

$$7|A| = 7 \times 72 = 504$$

Since the number of edges incident to $A = B$, we know the total number of edges incident to B and can solve for the unknown value of $|B|$

$$504 = 9|B|$$
$$|B| = \frac{504}{9} = 56$$

Therefore, the possible value(s) of the size of B ($|B|$) is/are 56. ■

Problem 3. What is the largest n for which there is an n-vertex graph G such that both G and G^c have no cycles at all?

Solution. *For this problem we are looking for a graph G where G and its complement G^c are acyclic. Acyclic graphs are trees or forests (collection of trees) are graphs that contain no cycles. Furthermore, since G is a complement of G^c, the number of edges in G is equal to the number of edges in G^c. The maximum number of edges that a graph with n vertices can have is $n(n-1)/2$.*

This problem can be solved by figuring out that number of edges both G and G^c can have must be less than the number of vertices in $V(G) + V(G_c)$.

$$\begin{aligned}
\text{Total Possible Edges in } G + G^c &= |E_1| + |E_2| \\
|E_1| + |E_2| &= \frac{n(n-1)}{2} \\
\text{Number of Vertices in } G + G^c &= 2|V| \\
2|V| &= 2n \\
\frac{n(n-1)}{2} &< 2n \\
n(n-1) &< 4n \\
(n-1) &< 4 \\
n &< 5
\end{aligned}$$

Therefore the maximum possible number (integer) of nodes a graph can be before the number of possible edges in a complete graph exceeds the number of edges in $G + G^c$ is 4. ■

Problem 4. Let G be an n-vertex simple graph with the following property: for any v in $V(G)$, deleting v from G results in a tree. Which graphs can G be?

Solution. *A tree is a connected graph with no cycles. A tree has $n-1$ edges where n is the number of nodes.*

Lets call the graph of v deleted from $V(G)$ as G_v. Since G_v is a tree, it must have $n-1$ edges relative to its nodes n. Lets denote the nodes of G and G_v as n and n_v.

Therefore

$$
\begin{aligned}
V(G) &= n \\
n-1 &= n_v \\
V(G_v) &= n_v \\
E(G_v) &= n_v - 1 \\
E(G_v) &= n - 2
\end{aligned}
$$

Since G_v must have one less vertex then G when the vertex is delted from G, the number of edges in G_v must be $n_v - 1$ and $n - 2$. Thus G must be a graph where for every v_i in V, the degree is $deg(v_i) = 2$. The only condition where that occurs is when G is a cycle graph.

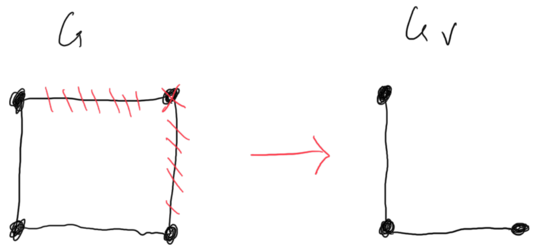

Figure 2: G is a cycle graph and removing a node v requires the removal of two edges resulting in a tree where $E(G_v) = n - 2$ and $E(G_v) = n_v - 1$ for the new graph G_v.

Additionally, the case where the graph is a collection of disconnected cycle graphs will still be a graph where every node has a degree of 2. However, this graph is invalid as trees cannot be disconnected and removing a vertex will not connect the cycle graphs.

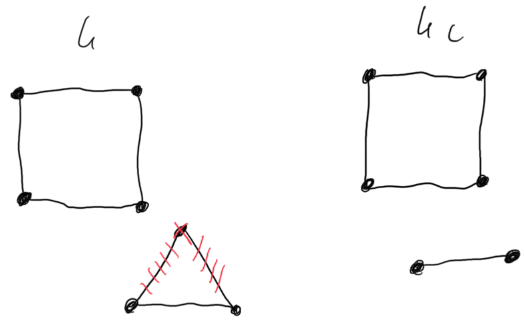

Figure 3: G is a graph of disconnected cycle graphs, while G_c is a graph with one vertex removed from one cycle. The graph is still disconnected and there still exists one cycle as well.

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