Problem Set VIII

Introduction to Graph Theory, MATH 3545
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8 Optimal Colorings

Problem 1. Let $c : V(G) \to \{1, 2, \ldots, \chi(G)\}$ be a proper coloring of G with the smallest number of colors. Show that there must exist a vertex of color 1 that is adjacent to vertices of every other possible color.

Solution. *Proof.* We will prove this statement by contradiction. Assume that in our minimal proper coloring $c : V(G) \to \{1, 2, \ldots, \chi(G)\}$, **no vertex of color** 1 is adjacent to vertices of **every other color** (i.e., colors 2 through $\chi(G)$).

Assumption (for contradiction): For every vertex v with $c(v) = 1$, there exists at least one color $k_v \in \{2, 3, \ldots, \chi(G)\}$ such that v is *not* adjacent to any vertex with that color k_v. k_v can be different colors for each vertex of color 1.

Constructing a New Coloring:

1. **Define a Function:** For each vertex v with $c(v) = 1$, select a color $k_v \in \{2, 3, \ldots, \chi(G)\}$ such that v is not adjacent to any vertex of color k_v.
2. **Recolor Vertices of Color 1:** Recolor each vertex v (originally of color 1) with the color k_v for that vertex. Since v is not adjacent to any vertex of color k_v, this recoloring will still be a proper coloring.
3. **Resulting Coloring Uses Fewer Colors:** After recoloring, all vertices originally colored 1 now have colors in $\{2, 3, \ldots, \chi(G)\}$. Therefore, the new coloring will use only $\chi(G) - 1$ colors.

Contradiction:

- The new coloring is a proper coloring of G using fewer than $\chi(G)$ colors.
- This contradicts the definition of $\chi(G)$ as the minimal number of colors needed to properly color G.

Conclusion: Therefore, for the original coloring to be $\chi(G)$, our orginal assumption must be false. Therefore, there must exist at least one vertex v with $c(v) = 1$ such that v is adjacent to vertices of *every other color* in $\{2, 3, \ldots, \chi(G)\}$.

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Remark 1. A visual representation can help illustrate why recoloring the vertices reduces the total number of colors used. Consider the following diagram:

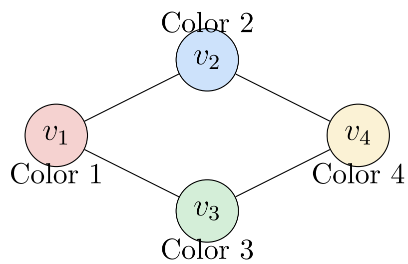

Figure 1: Example of a graph with a proper but not optimal coloring

In this simplified example, if vertex v_1 (color 1) is not adjacent to a vertex of color 4, we can recolor v_1 with color 4, reducing the total number of colors used.

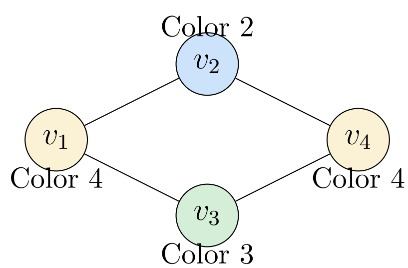

Figure 2: Example of a graph with a proper and optimal coloring

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Problem 2. Show that for any graph G there is an ordering of its vertices such that the greedy coloring algorithm uses exactly $\chi(G)$ colors, where $\chi(G)$ is the chromatic number of G.

Solution. We will prove this by constructing an ordering of the vertices that allows the greedy coloring algorithm to use exactly $\chi(G)$ colors.

Proof. 1. **Obtain an Optimal Coloring:**
Begin by finding a proper coloring of G that uses exactly $\chi(G)$ colors. Let the color classes (independent sets) be denoted as $S_1, S_2, \ldots, S_{\chi(G)}$, where each S_i consists of vertices colored with color i.

2. **Order the Vertices:**
   Construct an ordering of the vertices by listing all the vertices in S_1 first, followed by all the vertices in S_2, and so on, up to $S_{\chi(G)}$. That is, the ordering is:

$$v_1, v_2, \ldots, v_{|S_1|}, v_{|S_1|+1}, \ldots, v_{|S_1|+|S_2|}, \ldots, v_n$$

   where n is the total number of vertices in G.

3. **Apply the Greedy Coloring Algorithm:**
   Run the greedy coloring algorithm on G using the constructed ordering:

   - **Vertices in S_1:** Since S_1 is an independent set, none of its vertices are adjacent. When the algorithm processes each vertex in S_1, it assigns color 1 (the smallest available color).
   - **Vertices in S_2:** Each vertex in S_2 may be adjacent to vertices in S_1 (all colored 1) but not to other vertices in S_2. The algorithm assigns color 2 to each vertex in S_2. However, some vertices in S_2 may not be adjacent to any vertices in S_1 and can also be colored 1. This will not affect the number of colors used but sets a bound that the color assigned to a set can be $\leq$ the number of the independent set.
   - **Continue for $S_3, \ldots, S_{\chi(G)}$:** Similarly, for each S_i, the algorithm assigns color i to all its vertices, as they are only adjacent to vertices in earlier color classes (which have colors $< i$).

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Problem 3. For every $k \geq 0$ construct a tree T_k with $\Delta(T_k) = k$ and an ordering of its nodes such that the greedy coloring uses exactly $k + 1$ colors.

Solution. For a tree to to have a $\Delta(T_k) = k$, the node with degree of k must be adjacent to nodes with colors $1, 2, \ldots, k$. This would then make the node with degree k have color $k + 1$.

Proof. Therefore we can start with a graph like this where the color size is:

$$\{1, 2, 3, 4, 5\}$$

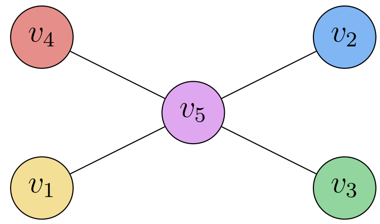

Figure 3: Tree T_4 with $\Delta(T_4) = 4$ and greedy coloring where the center node must be colored 5

From here, we can work backwards and construct a edges that force the adjacent edge to be that next smallest color. For example, starting at v_2, we can make it connected to node that is colored 1. And for v_3, we can make it connected to a node that is colored 2 and 1 and so on. Ultimately, as we branch out we will have a graph that uses exactly $k + 1$ colors.

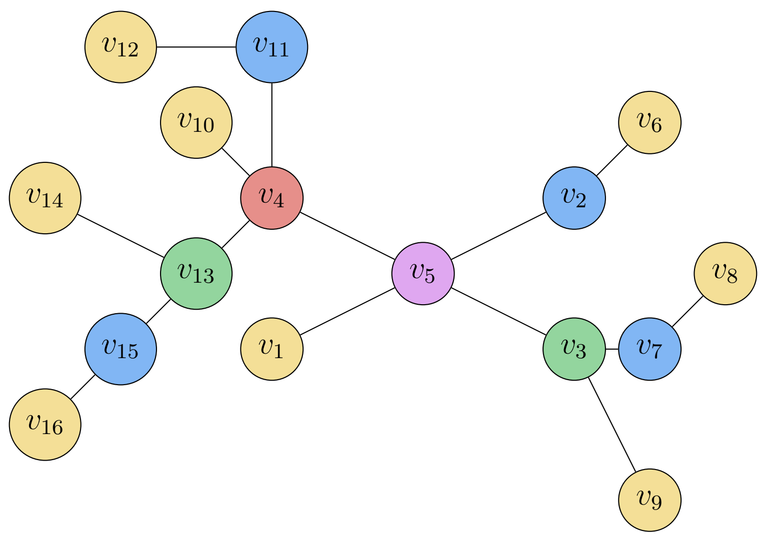

Figure 4: Tree T_k with $\Delta(T_k) = k$ and greedy coloring using $k + 1$ colors

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Remark 2. As the graph depicts, this construction requires $k + 1$ colors to properly color the graph. When you break down the graph you can see that certain components are repeated. For example the component of v_3 is the same as the compoenent of v_{13} connected to v_4. This indicates that a contructions of graphs using $k + 1$ colors involve repeating components of graphs that are required to construct graphs of smaller k values. The process of constructing a graph with $k + 1$ colors involves bulding components of $\{T_0, T_1, \ldots, T_{k-1}\}$ and the T_k is added in the center and conects to all these compoenents.

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Problem 4. Suppose we color every point in the plane red, green, or blue. Show that there will be 2 points of the same color whose distance is exactly 1 inch.

Solution. We will prove this by constructing a specific geometric configuration.

Proof. Consider the following configuration:

1. **Initial Setup:**
   - Let point P be located at the center of the plane and colored red.
   - Draw a circle C_1 with a radius of 1 inch centered at P.
   - Place six points $A_1, A_2, \ldots, A_6$ equally spaced around C_1 at every $60°$.

2. **Distances Between Points:**
   - The distance between P and any A_i is exactly 1 inch.
   - The distance between adjacent points A_i and A_{i+1} is:

$$|A_iA_{i+1}| = 2 \times 1 \times \sin\left(\frac{60°}{2}\right) = 1 \text{ inch.}$$

3. **Coloring the Points on C_1:**
   - None of the points A_i can be colored red; otherwise, they would be 1 inch from P, resulting in two red points 1 inch apart.
   - Therefore, the points A_i are colored either green or blue.
   - To avoid adjacent same-color points (since $|A_iA_{i+1}| = 1$ inch), we must color them alternately: green, blue, green, blue, green, blue.

4. **Constructing Outer Points:**
   - For each pair of adjacent points A_i (green) and A_{i+1} (blue), there exists a point B_i such that $|A_iB_i| = |A_{i+1}B_i| = 1$ inch.
   - Each B_i forms an equilateral triangle with A_i and A_{i+1}.
   - The points B_i lie on a larger circle C_2 centered at P with radius $R = \sqrt{3}$ inches.

5. **Coloring the Points on C_2:**
   - Since B_i is 1 inch from both A_i and A_{i+1} (which are colored green and blue), to avoid creating a monochromatic pair at a distance of 1 inch, B_i must be colored red.

6. **Analyzing Distances Between Red Points:**
   - All B_i and P are colored red.

- Therefore there exists infinite points colored red on C_2.
- And there are infinite red points on C_2 that are exactly 1 inch apart.

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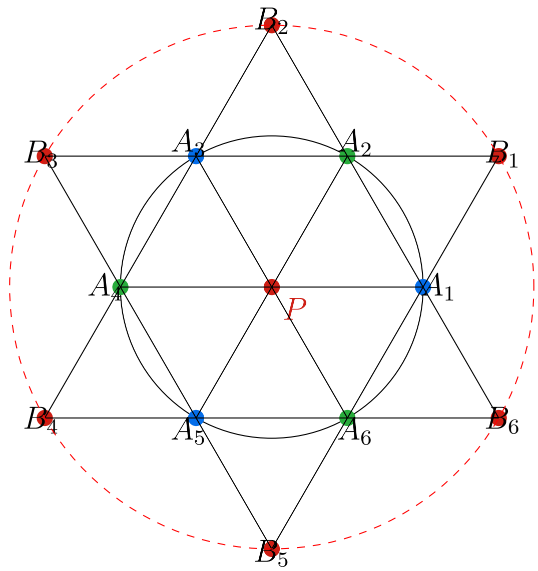

Figure 5: Geometric configuration with points colored red, blue, and green

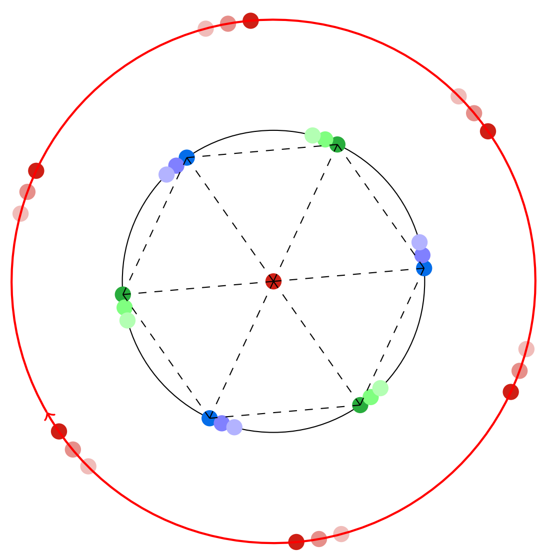

Figure 6: Geometric configuration shifted around the center point P illustrating the infinite number of red points on circle C_2.

Remark 3. This proof illustrates that in any 3-coloring of the plane, there must exist two points of the same color that are exactly 1 inch apart. The graph above illustrates this configuration where an infinite number of red points creates a circle of radius $\sqrt{3}$ inches, with infinite pairs of points exactly 1 inch apart.

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[1]With Désirée DeGennaro, Allison Kennedy, and Na'Ama Nevo