Problem Set VII

Introduction to Graph Theory, MATH 3545
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7 Blocks and Chromatic Numbers

Remark 1. A *proper coloring* of a graph G with k colors is a function $f : V(G) \to \{1, 2, \ldots, k\}$ such that $f(a) \neq f(b)$ for any edge $ab \in E(G)$.

The *chromatic number* $\chi(G)$ is the smallest k for which there is a proper coloring of G with k colors.

An *independent set* is a subset $S \subseteq V(G)$ such that there are no edges between nodes of S. The *independence number* $\alpha(G)$ is the size of the largest independent set. The significance of independent sets for coloring is that each color class (the set of nodes getting the same color) must form an independent set.

Problem 1. Find the chromatic numbers of the following graphs:

- K_n
- P_n
- C_n
- any tree T
- the complement of C_n

Solution. We will determine the chromatic number $\chi(G)$ for each graph G individually.

1. **Complete Graph** K_n

   Proof. In a complete graph K_n, every pair of distinct vertices is connected by an edge. Therefore, no two adjacent vertices can share the same color. Thus, we need a unique color for each vertex.

$$\chi(K_n) = n$$

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2. **Path Graph P_n**

Proof. A path graph P_n is a sequence of n vertices connected in a line. It is a bipartite graph, meaning its vertices can be divided into two disjoint sets such that no two vertices within the same set are adjacent.

For $n = 1$, only one vertex exists, requiring one color.

For $n \geq 2$, we can color the graph using two colors by alternating colors along the path.

$$\chi(P_n) = \begin{cases} 1, & \text{if } n = 1; \\ 2, & \text{if } n \geq 2. \end{cases}$$

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3. **Cycle Graph C_n**

Proof. A cycle graph C_n forms a closed loop with n vertices. The chromatic number depends on whether n is even or odd:

- **Even** n: The graph is bipartite; two colors suffice by alternating colors around the cycle.
- **Odd** n: The graph is not bipartite due to the presence of an odd-length cycle; at least three colors are needed.

$$\chi(C_n) = \begin{cases} 2, & \text{if } n \text{ is even}; \\ 3, & \text{if } n \text{ is odd}. \end{cases}$$

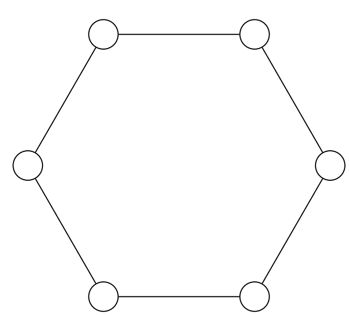

Figure 1: Cycle graph C_6 (even n)

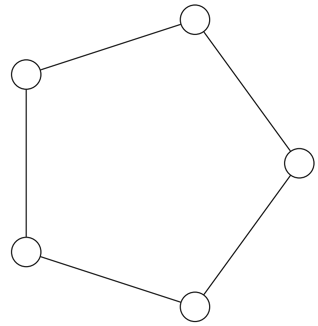

Figure 2: Cycle graph C_5 (odd n)

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4. **Any Tree T**

Proof. A tree is an acyclic connected graph. Trees are inherently bipartite because they contain no cycles.

For a tree with a single vertex:

$$\chi(T) = 1.$$

For a tree with $n \geq 2$ vertices, we can partition the vertices into two sets such that no two vertices within the same set are adjacent, allowing us to color the tree with two colors.

$$\chi(T) = \begin{cases} 1, & \text{if } T \text{ has one vertex;} \\ 2, & \text{if } T \text{ has more than one vertex.} \end{cases}$$

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5. **Complement of a Cycle Graph $\overline{C_n}$**

Definition: The **complement** of a graph G, denoted $\overline{G}$, is a graph on the same set of vertices where two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Chromatic Number:

$$\chi(\overline{C_n}) = \left\lceil \frac{n}{2} \right\rceil$$

Proof. To determine the chromatic number of the complement of a cycle graph $\overline{C_n}$, we analyze its structure based on whether the number of vertices n is even or odd.

1. Even n:

When n is even, $\overline{C_n}$ decomposes into a perfect matching. Specifically, each vertex is connected to exactly one other vertex such that no two edges share a common vertex. For example, consider $\overline{C_4}$:

- Vertices: 1, 2, 3, 4
- Edges in C_4: (1-2), (2-3), (3-4), (4-1)
- Edges in $\overline{C_4}$: (1-3), (2-4)

In this case, $\overline{C_4}$ consists of two disjoint edges. Each edge can be assigned the same color since they are independent of each other. Therefore, two colors suffice:

$$\chi(\overline{C_4}) = \left\lceil \frac{4}{2} \right\rceil = 2$$

2. Odd n:

When n is odd, $\overline{C_n}$ consists of $\left\lfloor \frac{n}{2} \right\rfloor$ disjoint edges plus an additional vertex that is connected to all other vertex color pairs. This additional vertex is odd with all its adjacent vertices, necessitating an extra color. For instance, consider $\overline{C_5}$:

- Vertices: 1, 2, 3, 4, 5
- Edges in C_5: (1-2), (2-3), (3-4), (4-5), (5-1)
- Edges in $\overline{C_5}$: (1-3), (1-4), (2-4), (2-5), (3-5)

Here, $\overline{C_5}$ includes two disjoint edges: (1-3) and (2-4), along with vertex 5 connected to both 1 and 3. Vertex 5 cannot share a color with either vertex 1 or 3, requiring a third color. Therefore:

$$\chi(\overline{C_5}) = \left\lceil \frac{5}{2} \right\rceil = 3$$

General Case:

For any n, whether even or odd, the chromatic number can be expressed as:

$$\chi(\overline{C_n}) = \left\lceil \frac{n}{2} \right\rceil$$

This is because:

- Even n: The graph decomposes into $\frac{n}{2}$ disjoint edges (perfect matching), requiring $\frac{n}{2}$ colors.
- Odd n: The graph has $\left\lfloor \frac{n}{2} \right\rfloor$ disjoint edges plus an additional vertex connected to multiple vertices, necessitating $\left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lceil \frac{n}{2} \right\rceil$ colors.

Therefore, combining both bounds, we conclude:

$$\chi(\overline{C_n}) = \left\lceil \frac{n}{2} \right\rceil$$

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Problem 2. Show that for any graph $\chi(G) \leq \Delta(G) + 1$.

Solution. *Proof.* We will prove the statement by constructing an algorithm that colors the graph using at most $\Delta(G) + 1$ colors.

Algorithm:

1. **Initialization:** Start with any edge uv in the graph G. Assign different colors to vertices u and v:

$$\text{Color}(u) = 1, \quad \text{Color}(v) = 2.$$

2. **Vertex Addition:** Add the remaining vertices one at a time in any order. For each new vertex w, proceed as follows:

1. Let S_w be the set of colors assigned to the neighbors of w that have already been colored.

2. Assign to w a color not in S_w that is perferably a color that already exists. If all colors up to $\Delta(G)$ are in S_w, assign a new color $\Delta(G) + 1$ to w.

Justification:

- Each vertex w has at most $\Delta(G)$ neighbors. - Therefore, $|S_w| \leq \Delta(G)$. - There are $\Delta(G) + 1$ available colors, so there is always at least one color not in S_w. - Assigning w a color not in S_w ensures that no two adjacent vertices share the same color.

Why a Vertex with Maximum Degree May Require a New Color:

- A vertex v_{max} with degree $\Delta(G)$ may be adjacent to vertices of all existing colors. - In this case, v_{max} must be assigned a new color, as all other colors are used by its neighbors. - Since the maximum number of neighbors is $\Delta(G)$, the total number of colors needed does not exceed $\Delta(G) + 1$. And if a new vertex x is added to the graph that requires a new color not used before, it must be adjacent to nodes that have all the colors used before, so it must then become the largest degree. We know this recursively as each neighbor of x can't be connected to more nodes than x or else it will be connected to a node that has the same color as itself.

Conclusion:

By following this algorithm, we can color all vertices of G using at most $\Delta(G) + 1$ colors, which shows that:

$$\chi(G) \leq \Delta(G) + 1.$$

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Problem 3. Find a connected graph where $\chi(G) > 2\frac{|E|}{|V|} + 1$, that is, the maximum degree cannot be replaced by the average degree in the previous problem.

Solution. Consider the connected graph G formed by taking a complete graph K_4 with 4 vertices and attaching a path of length 4 to one of its vertices.

Proof. **Number of vertices $|V|$:**

- K_4 has 4 vertices.
- The path of length 4 has 4 vertices.
- Since the path is attached to one vertex of K_4, the total number of vertices is: $|V| = 4 + 4 = 8$.

Number of edges $|E|$:

- K_4 has $\frac{4\times3}{2} = 6$ edges.
- The path of length 4 has 4 edges.
- The total number of edges is: $|E| = 6 + 4 = 10$.

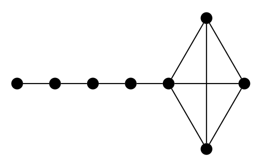

Figure 3: Graph G formed by K_4 and a path of length 4

Chromatic number $\chi(G)$:

- The complete graph K_4 requires 4 colors since every pair of vertices is connected.
- The path can be colored using 2 colors.
- Since the path is attached to a single vertex of K_4, the chromatic number of the entire graph remains: $\chi(G) = 4$.

Compute $\frac{2|E|}{|V|} + 1$:

$$\frac{2|E|}{|V|} + 1 = \frac{2 \times 10}{8} + 1 = \frac{20}{8} + 1 = 2.5 + 1 = 3.5.$$

Comparison:

$$\chi(G) = 4 > 3.5 = \frac{2|E|}{|V|} + 1.$$

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Remark 2. This example demonstrates that the chromatic number $\chi(G)$ can indeed be greater than $\frac{2|E|}{|V|}+1$. The complete graph K_4 ensures a high chromatic number, while the attached path increases the total number of vertices and edges without decreasing/increasing the chromatic number. This effectively lowers the average degree term, showing that the maximum degree cannot be replaced by the average degree in the inequality.

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Problem 4. Let $B_1, \ldots, B_k$ be the blocks of a connected graph G. Show that $\chi(G) = \max_i \chi(B_i)$.

Solution. *Proof.* We aim to prove that the chromatic number of G is equal to the maximum chromatic number among its blocks:

$\chi(G) = \max_i \chi(B_i)$

Step 1: Lower Bound

Since each block B_i is a subgraph of G, it follows that

$$\chi(G) \geq \chi(B_i) \quad \text{for all } i = 1, 2, \ldots, k.$$

Therefore,

$$\chi(G) \geq \max_i \chi(B_i).$$

Step 2: Upper Bound

We will construct a proper coloring of G using $\max_i \chi(B_i)$ colors.

Block-Cut Tree Structure

Consider the block-cut tree T of G, where:

- Each vertex represents either a block B_i or a cut-vertex.
- Edges connect blocks to their corresponding cut-vertices.

Since T is a tree, it is acyclic and can be traversed systematically.

Coloring Procedure

1. Choose an arbitrary block B_0 as the root.
2. Perform a depth-first traversal of T.
3. For each block B_i, proceed as follows:
   (a) Identify cut-vertices that connect B_i to previously colored blocks. These vertices have predetermined colors.
   (b) Since $\chi(B_i) \leq k$, where $k = \max_i \chi(B_i)$, we can extend the coloring of these cut-vertices to a proper k-coloring of B_i.

Justification

The pre-colored cut-vertices act as constraints, but they do not increase the chromatic number beyond k. This is because:

- The cut-vertices are shared between blocks and have already been assigned colors.
- Each block B_i can be properly colored with $\chi(B_i) \leq k$ colors, even with some vertices pre-colored. This is because since you are going in a DFS order, you will never end up at a block that is not colored but has > 1 nodes' colors predetermined as the path to get to the block is unique and can't be reached from another set of blocks.

Conclusion

By coloring each block in this manner, we ensure that:

- Adjacent vertices have different colors.
- No more than k colors are used throughout the graph.

Thus,

$$\chi(G) \leq \max_i \chi(B_i).$$

Combining this with the lower bound, we have:

$$\chi(G) = \max_i \chi(B_i).$$

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Remark 3. The key insight is that blocks are connected via cut-vertices, and the block-cut tree structure allows us to color each block independently while maintaining proper coloring across G.

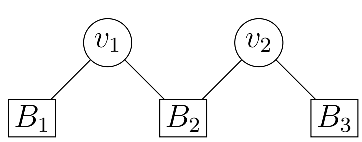
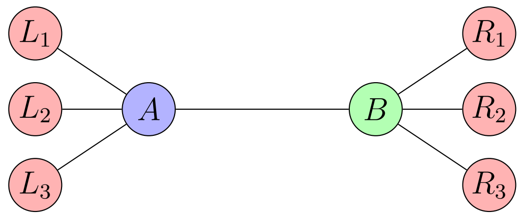


Figure: Block-Cut Tree of G

Problem 5. Find a graph where the largest independent set cannot be colored with a single color in any optimal coloring.

Solution. Consider the following graph:



Proof. In this graph:

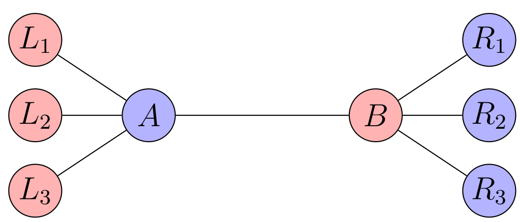
- **Left Cluster**: Node A connected to leaf nodes L_1, L_2, L_3.
- **Right Cluster**: Node B connected to leaf nodes R_1, R_2, R_3.
- **Connecting Path**: Node A connects B A–B.

An **independent set** is a set of vertices with no edges connecting any pair of them. The largest independent set in this graph includes:

$$\{L_1, L_2, L_3, B, R_1, R_2, R_3\}$$

With the largest independent set colored the same, the coloring is suboptimal because the connecting path A–B requires different colors for A and B. However, if you alternate the colors of red and blue, the graph can be optimally colored.

Consider the following graph:



In this graph, the largest independent set is $\{L_1, L_2, L_3, B, R_1, R_2, R_3\}$, which cannot be colored with a single color in any optimal coloring. However, by alternating the colors of the independent set, the graph can be optimally colored.

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[1]With Désirée DeGennaro, Allison Kennedy, and Na'Ama Nevo