1 Problem Statement

A Pythagorean triplet is a set of three natural numbers, $a < b < c$, for which,

$$a^2 + b^2 = c^2$$

For example, $32 + 42 = 9 + 16 = 25 = 52$.

There exists exactly one Pythagorean triplet for which $a + b + c = 1000$. Find the product abc.

2 Code

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# Project Euler #9 | Sidharth Baskaran | 01/21/2022

import time
import math

def solve(k):
    for c in range(math.ceil(k/3), math.ceil(k/2)):
        for b in range(math.ceil((k-c)/2), math.ceil(k/2)):
            a = k - (b + c)
            if a**2 + b**2 == c**2:
                return a*b*c

if __name__ == "__main__":
    s = time.time()
    print(solve(1000))
    e = time.time()

    print('%.3fms' % ((e-s)*1000))
```

3 Explanation

We have three variables and two equations–thus we need to iteratively solve for at least one of the variables. We begin by noticing that given $a < b < c$ and $a + b + c = k$ (where $k = 1000$), the minimum value of c is ceil($\frac{k}{3}$). We set c as a free variable and thus require complexity of the order $O(n)$ to find it.

Note that $a + b = k - c$ and $a^2 + b^2 = c^2$ allow us to use Lagrange multipliers to find the maximum value of $a^2 + b^2$ under our constraint. Let $f(a, b) = a^2 + b^2$ and $g(x, y) = a + b = k - c$.

$$\nabla f(x, y) = \nabla g(x, y) \implies \langle 2a, 2b \rangle = \lambda \langle 1, 1 \rangle$$

Thus, $a = b = \frac{\lambda}{2}$. Plugging this back into the constraint g, we get $\lambda = k - c$. Thus, the values of $a = b = \frac{k-c}{2}$ maximize $a^2 + b^2$ under the constraint.

We found a lower bound for c at ceil($\frac{k}{3}$), but not an upper bound. k itself would clearly not make a good bound, so let us attempt $\frac{k}{2}$. It helps to verify that this agrees with max($a^2 + b^2$) found earlier:

$$\left(\frac{k-c}{2}\right)^2 + \left(\frac{k-c}{2}\right)^2 = \frac{(k-c)^2}{2}$$

Plugging in our new upper bound, we get

$$\frac{(k-\frac{k}{2})^2}{2} = \frac{k^2}{8} < c = \left(\frac{k}{2}\right)^2,$$

so this upper bound can save time complexity. Now, we cannot find either a or b explicitly, but finding one will allow us to find the other. Let us arbitrarily make b a free variable, thus increasing complexity to $O(n^2)$. We already have a lower bound of $\frac{k-c}{2}$, and we know the higher bound must be less than $\frac{k}{2}$. It is thus reasonable to keep the same upper bound for b and c.

After searching through possible values for b and c, we can solve for a using $a = k-(b+c)$ and check the three values using the Pythagorean constraint, returning an answer if satisfied.