Multivariable Calculus Reference

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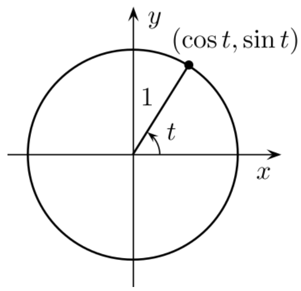
1 Paths whose image curve is a circle

1.1 Unit Circle

Unit circle is set of points in $\mathbb{R}^2$ defined as $C = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$. Ellipse is $C = \{(x,y) \in \mathbb{R}^2 | \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$ Has standard parameterization of $\vec{c}(t) = \big(\cos(t), \sin(t)\big)$.



Properties

- Image of $\vec{c}$ is a closed curve (has no endpoints, plane is divided into ≥ 2 disjoint regions)
- Image of $\vec{c}$ is a simple curve; no self-intersection
- $\vec{c}(t)$ is an **injective path**; path is considered injective if $\vec{c}(t_1) = \vec{c}(t_2)$, which implies that $t_1 = t_2$ where these are on the open interval (a,b) even if $a = b$
- Orientation of $\vec{c}$ is counter-clockwise in traversal

1.2 Observations

$$\boxed{\vec{p}(t) = (a\cos(\pm nt) + x_0, b\sin(\pm nt) + y_0)}$$

If $-t$ for t, orientation is CW, CCW is t. If $a = b$, then curve is a circle of radius a or b, else an ellipse with horizontal and vertical radii. x_0 and y_0 simply shift the center coordinate. $n > 0 \in \mathbb{R}$ determines how many times the circle is traversed given $t \in [0, 2\pi]$, for example.

2 Paths whose image is a line or line segment in the plane

2.1 Line Parametrics

A line is a 1D subspace of $\mathbb{R}^2$, so $L = \{t\vec{m} | t \in \mathbb{R}\}$ for $\vec{m} \in \mathbb{R}^2$. $\vec{m} = \begin{bmatrix} m_x \\ m_y \end{bmatrix}$ is the **slope vector**. Path given by image of L:

$$\vec{c}(t) = (m_x t, m_y t), t \in \mathbb{R}$$

Can represent $\vec{c}(t) = t\vec{m}$ as well.

Lines Main Ideas

- Image of a line is a curve (e.g. $y = x$ represents image curve of $\vec{c}(t) = (t, t)$)
- Lines can have nonzero intercepts, so $\vec{c}(t) = t\vec{m}$ represents $y = 2x+1$. Line that has intercept vector $P_0 = (x_0, y_0) \parallel \vec{m} = (m_x, m_y)$ can be expressed as:

$$\boxed{\vec{c}(t) = (x_0 + tm_x, y_0 + m_y t) = \vec{P_0} + t\vec{m}}$$

  Note endpoint of $\vec{c}(t)$ is on image line (curve).

2.2 General Forms

2 parametric lines **collide** if they intersect and the point of intersection corresponds to the same t in both curves. If you set the parameter vector coordinates equal to each other and solve for t, a solution indicates they collide. Intersection is found by **eliminating** the parameter (solve for t in terms of either x or y and plug into the other).

General form of parameterized curve can be expressed as the following:

$$\boxed{\vec{c}(t) = (\frac{m_x}{\Delta t}(t-a) + x_0, \frac{m_y}{\Delta t}(t-a) + y_0)}$$

where Δt is the domain interval over $[a, b]$ and (x_0, y_0) represents the desired **starting coordinate**. This is important as when going in reverse, other coordinate can be used and slope might be negative. a is used in $(t - a)$ because everything is conventionally done with respect to starting coordinate.

3 Paths whose image curve is a line in R3

3.1 R3 parameterization

If $\vec{m}$ is a nonzero vector along L through origin in $\mathbb{R}^3$, then $L = \{t\vec{m} | t \in \mathbb{R}\}$; follows that $\vec{m} = (m_x, m_y, m_z)$, the slope or direction vector of the line. The basic parameterization is:

$$\vec{c}(t) = (m_x t, m_y t, m_z t)$$

Basis vectors in $\mathbb{R}^3$ are $\vec{i}, \vec{j}, \vec{k}$. Rewriting parameterization:

$$\vec{c}(t) = (x_0 + m_x t)\vec{i} + (y_0 + m_y t)\vec{j} + (z_0 + m_z t)\vec{k}$$

2 lines $\vec{c}_1(t) = P_0 + \vec{m}_1 t$ and $\vec{c}_2(t) = Q_0 + \vec{m}_2 t$ are parallel if direction vectors are parallel $(\vec{m}_1 = k\vec{m}_2)$. Collisions still exist. If neither parallel nor intersecting, considered as skew.

To determine skew, parallel, or coincide, use parameters s, t for each line and solve SOE. If same slope, rule out skew clearly, then check if $s, t \in \mathbb{R}$: if not, then parallel, if so, then they coincide. If intersecting and want to check if collide, some t must satisfy all relations.