Differential Equations

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1 Definitions, Families of Curves

1.1 Definitions

Definition 1.1 (Order)**.** Order of a DE is the highest-ordered derivative appearing in it. So

$$\frac{d^2y}{dx^2} + 2b(\frac{dy}{dx})^3 + y = 0 \tag{1}$$

is a 2nd order DE. In general,

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0. \tag{2}$$

is an n-th order DE. Under restrictions on F, can find a solution in terms of the other $n+1$ variables

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}). \tag{3}$$

Definition 1.2 (Solution)**.** A function ϕ on interval $x \in (a, b)$ is a solution to the DE (3) if the n derivatives exist on $x \in (a, b)$ and $\phi^{(n)}(x) = f(x, \phi(x), \ldots, \phi^{(n-1)}(x))$.

Definition 1.3 (First order DE)**.** A first order DE is of the form

$$\frac{dy}{dx} = f(x, y) \tag{4}$$

with solution of the form $y = f(x)$. Can be rewritten for convenience in the form

$$M(x, y)dx + N(x, y)dy = 0 \tag{5}$$

Definition 1.4 (Linear ODE)**.** An ODE of order n is linear if it can be written in the form

$$b_0(x)\frac{d^ny}{dx^n} + b_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + b_{n-1}(x)\frac{dy}{dx} + b_n(x)y = R(x) \tag{6}$$

Definition 1.5 (Partial DE)**.** Is of the form, for example

$$b_0(x, y)\frac{\partial w}{\partial x} + b_x(x, y)\frac{\partial w}{\partial y} = R(x, y) \tag{7}$$

1.2 Families of Solutions

Solutions to the DE

$$\frac{dy}{dx} = f(x, y) \Leftrightarrow y = \int f(x)dx + c \tag{8}$$

exist as one-parameter families with parameter c.

1.3 Isoclines

Let there be the DE

$$\frac{dy}{dx} = y \tag{9}$$

Isoclines are lines $f(x, y) = y = c$. Example:

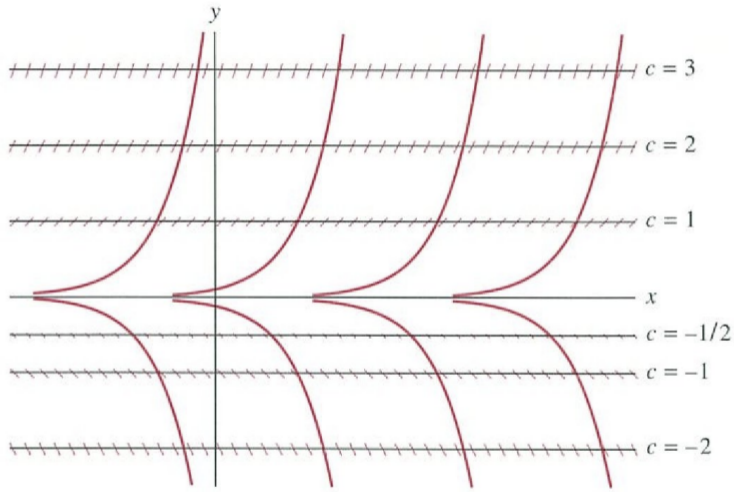

Figure 1: Isoclines of $\frac{dy}{dx} = y$

1.4 Existence Theorem

Consider equation

$$\frac{dy}{dx} = f(x, y) \tag{10}$$

Further, let T denote the rectangle defined by

$$|x - x_0| \leq a \tag{11}$$
$$|y - y_0| \leq b \tag{12}$$

with the point (x_0, y_0) as the center. Also let $f, \frac{\partial f}{\partial y}$ be continuous functions of x, y in T.

With these conditions an interval exists for x_0 where $|x - x_0| \leq h$, and function $y(x)$ which has properties

1. $y = y(x)$ is a sol'n of the DE on interval $|x - x_0| \leq h$
2. On this interval, $|y(x) - y_0| \leq b$
3. $y = y(x_0) = y_0$ at $x = x_0$
4. $y(x)$ is unique on interval $|x - x_0| \leq h$ where it is the only function with above 3 properties

2 Equations of Order One

2.1 Separation of Variables

We begin with equations of the form

$$M dx + N dy = 0 \tag{13}$$

where M and N can be multivariate of x, y.

It is separable iff

$$A(x)dx + B(y)dy = 0. \tag{14}$$

Then find a function F with total differential being the LHS of above, so $F = c$.

2.2 Homogeneity

Definition 2.1 (Homogeneity of polynomials)**.** Polynomials where all terms are of the same degree are homogeneous.

Homoegeneity of functions is analogous to assigning physical dimensions (e.g. length) to all of the variables. If the function has the length dimension to the kth power, then it is homogeneous of degree k.

Example 2.2. If x, y are lengths, then the following is homogeneous of degree 3.

$$f(x, y) = 2y^3 \exp(\frac{y}{z}) - \frac{x^4}{x + 3y} \tag{15}$$

Alternate definition also suffices for generality.

Definition 2.3 (Homogeneous function)**.** $f(x, y)$ is homogeneous of degree k iff $f(\lambda x, \lambda y) = \lambda^k f(x, y)$.

Definition 2.4 (Alternate definition of homogeneity)**.** If $f(x, y)$ can be rewritten as $f(\frac{y}{x})$ or $f(\frac{x}{y})$ then it is homogeneous.

2.3 Homogeneous Differential Equations

Corollary 2.5 (Homogeneous DEs)**.** If $M(x, y)$ and $N(x, y)$ are homogeneous and of same degree, then $M(x, y)dx + N(x, y)dy = 0$ is a homogeneous DE.

Corollary 2.6 (Homogeneous DEs)**.** $M(x, y)/N(x, y)$ is homogeneous of degree 0.

Corollary 2.7 (Homogeneous DEs)**.** If $f(x, y)$ is homogeneous of degree 0 in x, y, then $f(x, y)$ is a function of y/x alone.

The ratio M/N is a function of y/x, so the above can be rewritten as

$$\frac{dy}{dx} + g(\frac{y}{x}) = 0 \tag{16}$$

$$\frac{d}{dx}(vx) + g(v) = \frac{dv}{dx} + v + g(v) = 0 \tag{17}$$

Can thus transform into SOV problem by substituting $y = vx$ or $x = vy$, where v is a function of y or x. Then, substitute back $v = \frac{y}{x}$ to obtain a general solution.

2.4 Exact Equations

If there exists an equation of the form $A(x)dx + B(y)dy = 0$, the solution is a function with differential $A(x)dx + B(y)dy$. Idea works for equations of form

$$dF = Mdx + Ndy. \tag{18}$$

So, $F(x, y) = c \implies dF = 0$ and

$$Mdx + Ndy = 0. \tag{19}$$

If there's a function F such that $Mdx + Ndy$ is the **total differential** of F, then Eq. 5 is an *exact equation* by definition. Can rewrite the total differential from the chain rule:

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy. \tag{20}$$

So $M = \frac{\partial F}{\partial x}, N = \frac{\partial F}{\partial y}$. We can take 2nd derivative to show these are equal because the partials are continuous (Clairaut's theorem).

$$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} \tag{21}$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial y \partial x}. \tag{22}$$

Definition 2.8 (Exactness)**.**

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{23}$$

Proof. Let $\phi(x, y)$ be a function where $\frac{\partial \phi}{\partial x} = M$. ϕ is the function you get from integrating Mdx wrt x and holding y. Then

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{24}$$

Integrating both sides wrt x:

$$\frac{\partial \phi}{\partial x} = N + B'(y) \tag{25}$$

where $B'(y)$ is the integration constant. Let

$$F = \phi(x, y) - B(y) \tag{26}$$

such that

$$\begin{aligned} dF &= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy - B'(y) dy && (27) \\ &= M dx + \left[N + B'(y)\right] dy - B'(y) dy && (28) \\ &= M dx + N dy && (29) \end{aligned}$$

$\square$

Example 2.9. We have the DE

$$3x(xy - 2)dx + (x^3 + 2y)dy = 0. \tag{30}$$

Then,

$$\frac{\partial M}{\partial y} = 3x^2, \frac{\partial N}{\partial y} = 3x^2 \tag{31}$$

The DE is exact, and $F = c$ is the solution.

$$\begin{aligned} \frac{\partial F}{\partial x} &= M = 3x^2 y - 6x && (32) \\ \frac{\partial F}{\partial y} &= N = x^3 + 2y && (33) \end{aligned}$$

Try to find F from 18, integrate both sides wrt x with an integration constant $T(y)$.

$$F = x^3 y - 3x^2 + T(y) \tag{34}$$

Using Eq. 19, can can find $\frac{\partial F}{\partial y}$ from Eq. 20 and equate:

$$x^3 + T'(y) = x^3 + 2y \implies T'(y) = 2y \tag{35}$$

Because $F = c$ is the I.C., can conclude

$$T(y) = y^2 \tag{36}$$

Thus,

$$F = x^3y - 3x^2 + y^2 \Leftrightarrow x^3y - 3x^2 + y^2 = c \tag{37}$$

2.5 Linear Equations of Order 1

If an equation is not exact, can attempt to do so by multiplying DE by an integrating factor.

Definition 2.10 (Linear DE of order 1)**.**

$$A(x)\frac{dy}{dx} + B(x)y = C(x) \tag{38}$$

Divide each side by $A(x)$ to obtain

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{39}$$

Suppose there exists for Eq. 25 a I.F. $v(x) > 0$. Then,

$$v(x)\left[\frac{dy}{dx} + P(x)y\right] = v(x)Q(x) \tag{40}$$

becomes exact, or of form $Mdx + Ndy = 0$. Here,

$$M = vPy - vQ \tag{41}$$
$$N = v \tag{42}$$

Because the requirement is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,

$$vP = \frac{dv}{dx} \tag{43}$$
$$Pdx = \frac{dv}{v} \tag{44}$$
$$\ln v = \int Pdx \tag{45}$$
$$v = \exp(\int Pdx) \tag{46}$$

We can then multiply both sides of the DE by this I.F. One side of this eqn will be of the product rule form, the derivative of $y \exp(\int P dx)$:

$$\exp(\int P dx)\frac{dy}{dx} + P\exp(\int P dx)y = Q\exp(\int P dx) \tag{47}$$

2.6 General Solution of a Linear Equation

Given the original form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{48}$$

suppose P and Q are continuous on $x \in (a, b)$ and $x = x_0$ is such a number. $y = y_0$ satisfies the initial condition. This sol'n satisfies Eq. 34 for all x in the interval. Multiplying Eq. 34 by integrating factor $\exp(\int P dx)$ gives

$$yv = \int vQ\, dx + c \tag{49}$$

Because $v \neq 0$,

$$y = v^{-1}\int vQ\, dx + cv^{-1} \tag{50}$$

Given any x_0, y_0 in the interval, can find c s.t. the DE is satisfied. Every eqn of above form will have P, Q with common interval of continuity and a unique set of solutions with one I.C. obtained by using the integrating factor. These solutions are unique, so any other method yields a solution that aligns with the general solution–all possible solutions satisfying the DE on $x \in (a, b)$.

2.7 Application of Mixing Problem

Strategy is to determine the differential equation describing rate of change of a certain quantity, then finding the particular solution with some trivial IC.

Example 2.11. 100 liter tank contains 10 kg salt mixed with 60 liter water. Sol'n with concentration 0.1 $\frac{\text{kg}}{\text{liter}}$ flows in at rate 5 liters/min. Solution is well stirred (asume equal distribution), outflow rate of 3 liters/min. Need to find salt in tank when it is full.

Note that the tank will become full, as in $-$ out > 0. Let x be kg of salt. Then, inflow rate is $0.1\frac{\text{kg}}{\text{liter}} \cdot 5\frac{\text{liter}}{\text{min}} = 0.5\frac{\text{kg}}{\text{min}}$. Out is $x\frac{\text{kg}}{60\text{ liter}} \cdot 3\frac{\text{liter}}{\text{min}} = \frac{x}{20}\frac{\text{kg}}{\text{min}}$. We then express the DE as

$$\frac{dx}{dt} = 0.5 - \frac{x}{20} \tag{51}$$

Then just express in linear form, solve with I.F. method.

Example 2.12. Initially 50 gallons of brine, 10 lb dissolved salt. Inflow of 2 lb salt/gal at 5 gal/min, outflow of 3 gal/min, but **mixture kept uniform**.

Inflow is thus 10 lb/min, outflow is $\frac{3x}{50+2t}$. Key here is that mixture concentration on outflow does not change, so the volume dynamically adapts for changing weight. DE is thus

$$\frac{dx}{dt} = 10 - \frac{3x}{50+2t} \tag{52}$$

2.8 Integrating Factor by Inspection

By recognizing differentials in a problem, can find the integrating factor by inspection.

Example 2.13. Given

$$ydx + (x + x^3y^2)dy = 0 \tag{53}$$

the terms can be grouped by like degree so

$$(ydx + xdy) + x^3y^2dy = 0. \tag{54}$$

Can be rewritten as

$$d(xy) + x^3y^2dy = 0 \tag{55}$$

then divide by $(xy)^3$ for it does not affect integrability of $d(xy)$ term but keeps function of y with dy term,

$$\frac{d(xy)}{(xy)^3} + \frac{dy}{y} = 0. \tag{56}$$

Integrating:

$$\int (xy)^{-3}d(xy) + \int \frac{dy}{y} = 0 \tag{57}$$

$$\frac{(xy)^{-2}}{-2} + \ln|y| = C \tag{58}$$

$$Cy = \frac{(xy)^{-2}}{2} \tag{59}$$

$$Cy(xy)^2 = 1 \tag{60}$$

2.9 Determining Complex Integrating Factors

Let there be the DE

$$Mdx + Ndy = 0. \tag{61}$$

Suppose $\exists u$, possibly of both x, y that is an integrating factor such that

$$uMdx + uNdy = 0 \tag{62}$$

and for it to be exact,

$$\frac{\partial}{\partial y}(uM) = \frac{\partial}{\partial x}(uN) \tag{63}$$

so u satisfies

$$u\frac{\partial M}{\partial y} + M\frac{\partial u}{\partial y} = u\frac{\partial N}{\partial x} + N\frac{\partial u}{\partial x} \tag{64}$$

$$u(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y}. \tag{65}$$

This does not lead anywhere, so let u be a function of x. Thus, $\partial u/\partial y = 0, \partial u/\partial x = du/dx$. So the above reduces to

$$u(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = N\frac{du}{dx} \Leftrightarrow \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})dx = \frac{du}{u} \tag{66}$$

and integrating factor is the following, assuming LHS of above is a function of x or y alone

$$u = \exp\left[\int f(x)dx\right] \text{ (for } x\text{)} \tag{67}$$

$$u = \exp\left[\int -g(y)dy\right] \text{ (for } y\text{)} \tag{68}$$