

AP Calculus BC Reference

Sidharth Baskaran

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1 Limits and Continuity

1.1 Evaluating Limits

1. To simply evaluate $\lim_{x \rightarrow c} f(x)$, plug in c such that $f(c) = L$, the value of the limit.
2. If $f(c) = \frac{0}{0}$, factor numerator and denominator, then cancel terms.

$$\lim_{x \rightarrow 0} \frac{x^4 + x^2}{x^3 + 3x^2} = \lim_{x \rightarrow 0} \frac{x^2 + 1}{x + 3} = \frac{1}{3}$$

3. If $f(c) = \frac{0}{0}$ and radicals are involved, then rationalize using conjugate and resubstitute.

$$\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9}$$

4. For limits that approach $\pm\infty$, cancel everything but greatest degree terms from numerator and denominator, then re-evaluate.

Given the problem to find the value of constant c as below, observe that when the limit is indeterminate, it is in the form $\frac{0}{0}$. This is because a limit with just denominator as 0 does not exist, so the indeterminate form is required.

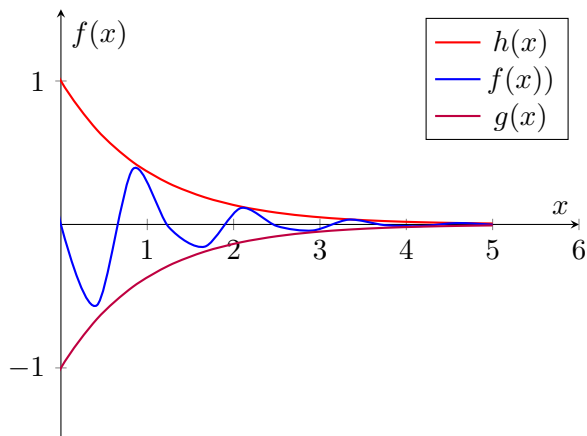
$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 + cx + c - 10}{x^2 - 3x + 2} \\ x^2 - 3x + 2|_{x=2} = 0 \Rightarrow \lim_{x \rightarrow 2} \frac{x^2 + cx + c - 10}{x^2 - 3x + 2} = \frac{0}{0} \\ x^2 + cx + c - 10|_2 = 3c - 6 = 0 \\ c = 2 \end{aligned}$$

1.2 Horizontal Asymptotes

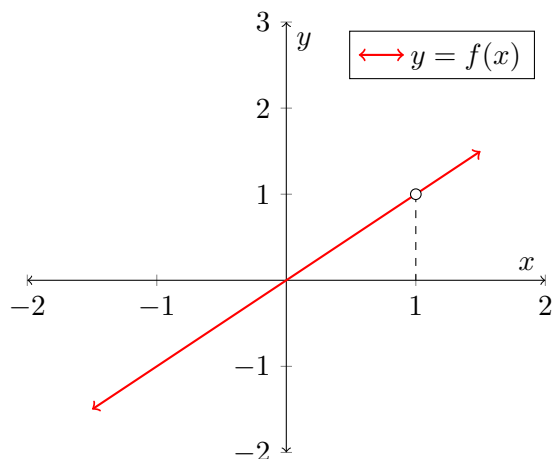
Take limit to $\pm\infty$ (end behavior).

1.3 Squeeze Theorem

Near $x = c$, let $g(x) \leq f(x) \leq h(x) \forall x$, if $\lim_{x \rightarrow c} g(x) = L$ and $\lim_{x \rightarrow c} h(x) = L$, then $\lim_{x \rightarrow c} f(x) = L$ must be true.



1.4 Graphical Limits



Notice that the limit of $f(x)$ exists at $x = 1$ though it is undefined at $(1, 1)$. Formally, $\lim_{x \rightarrow 1} f(x) = 1$ where $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x)$.

1.5 Continuity

A function is continuous at an interior point if $\lim_{x \rightarrow c} f(x) = f(c)$. It is continuous at a left endpoint if $\lim_{x \rightarrow a^+} f(x) = f(a)$ and at a right endpoint if $\lim_{x \rightarrow b^-} f(x) = f(b)$.

2 Derivatives

2.1 Derivative Rules

1. Power Rule: $\frac{d}{dx} x^n = nx^{n-1}$
2. Product Rule: $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + g'(x)f(x)$
3. Quotient Rule: $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

4. Chain Rule: $\frac{d}{dx} f(g(x)) = g'(x)f'(g(x))$

5. Logarithmic Derivatives

(a) $\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}$

(b) $\frac{d}{dx} \log_a(f(x)) = \frac{f'(x)}{\ln a f(x)}$

6. Trigonometric Derivatives

(a) $\frac{d}{dx} \sin x = \cos x$

(b) $\frac{d}{dx} \cos x = -\sin x$

(c) $\frac{d}{dx} \tan x = \sec^2 x$

(d) $\frac{d}{dx} \cot x = -\csc^2 x$

(e) $\frac{d}{dx} \sec x = \sec x \tan x$

(f) $\frac{d}{dx} \csc x = -\csc x \cot x$

7. Inverse Trigonometric Derivatives

(a) $\frac{d}{dx} \arcsin(f(x)) = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$

(b) $\frac{d}{dx} \arccos(f(x)) = \frac{-f'(x)}{\sqrt{1-(f(x))^2}}$

(c) $\frac{d}{dx} \arctan(f(x)) = \frac{f'(x)}{1+(f(x))^2}$

(d) $\frac{d}{dx} \operatorname{arccot}(f(x)) = \frac{-f'(x)}{1+(f(x))^2}$

(e) $\frac{d}{dx} \operatorname{arcsec}(f(x)) = \frac{f'(x)}{|f(x)|\sqrt{(f(x))^2-1}}$

(f) $\frac{d}{dx} \operatorname{arccsc}(f(x)) = \frac{-f'(x)}{|f(x)|\sqrt{(f(x))^2-1}}$

2.1.1 Example of logarithmic differentiation

Take \ln of both sides, differentiate, then get in terms of $f'(x)$ and simplify.

$$\begin{aligned}f(x) &= 2^x \\ \ln(f(x)) &= x \ln(2) \\ \frac{f'(x)}{f(x)} &= \ln 2 \\ f'(x) &= \ln 2 \cdot f(x) \\ f'(x) &= \ln 2 \cdot 2^x\end{aligned}$$

3 Applications of Derivatives

3.1 Implicit Differentiation

Explained in the following example.

$$\frac{dy}{dx}(x^2 + y^2 + y = 25)$$

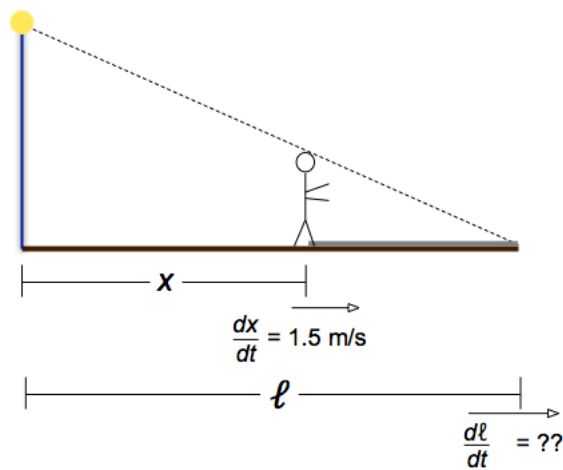
$$2x + 2yy' + y' = 0$$

$$y'(2y + 1) = -2x$$

$$\frac{dy}{dx} = y' = \frac{-2x}{2y + 1}$$

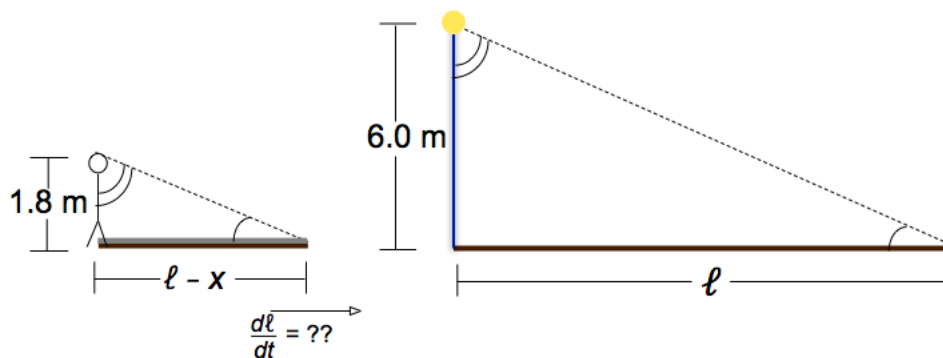
3.2 Related Rates

3.2.1 Shadow Problem



A 1.8-meter tall man walks away from a 6.0-meter lamp post at the rate of 1.5 m/s. The light at the top of the post casts a shadow in front of the man. How fast is the “head” of his shadow moving along the ground?

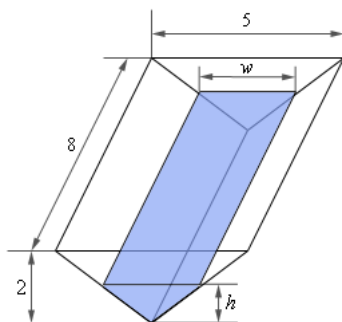
Must find $\frac{dl}{dt}$. The strategy is to use similar triangles to relate x and l .



$$\begin{aligned}\frac{\ell - x}{\ell} &= \frac{1.8}{6.0} \\ \ell - x &= 0.3\ell \\ x - \ell &= 0.30\ell \\ x &= 0.7\ell\end{aligned}$$

$$\begin{aligned}0.7 \frac{d\ell}{dt} &= \frac{dx}{dt} \\ \frac{d\ell}{dt} &= \frac{1.5}{0.7} \\ \frac{d\ell}{dt} &= 2.1 \frac{m}{s}\end{aligned}$$

3.2.2 Trough Problem



A trough of water is 8 meters in length and its ends are in the shape of isosceles triangles whose width is 5 meters and height is 2 meters. If water is being pumped in at a constant rate of $6 \text{ m}^3/\text{sec}$, at what rate is the height of the water changing when the water has a height of 120 cm? At what rate is the width of the water changing when the water has a height of 120cm?

It is known that $V' = 6m^3/sec$. Need h' when $h = 1.2$.

The volume of the water in the tank is given by:

$$\begin{aligned} V &= \frac{1}{2} base \times height \times depth \\ &= \frac{1}{2} hw(8) \\ &= 4hw \end{aligned}$$

Need to eliminate w as target is h' . Using similar triangles.

$$\begin{aligned} \frac{w}{5} &= \frac{h}{2} \Rightarrow w = \frac{5h}{2} \Rightarrow V = 10h^2 \\ V' &= 20hh' \Rightarrow 6 = 20(1.2)h' \Rightarrow h' = 0.25m/sec \end{aligned}$$

R.O.C of width can be found by manipulating similar triangles to get V in terms of w only.

$$h = \frac{2w}{5} \Rightarrow V = \frac{8w^2}{5}$$

Differentiate and substitute as before.

3.3 L'Hopital's Rule

3.3.1 Indeterminate Forms: $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$$

Suppose that $f(x) = 0$ and $g(x) = 0$ and that $c \in \mathbb{R}$.

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

Process can be repeated.

3.3.2 Indeterminate Forms: $\infty \cdot 0$ or $\infty - \infty$

Requires that the limit be rewritten in fractional form for differentiation.

$$\begin{aligned} \lim_{x \rightarrow 0^+} x \csc x &= \frac{x}{\sin x} \\ \lim_{x \rightarrow 0} \frac{x}{\sin x} &= \frac{1}{\cos x} \Big|_0 = \frac{1}{1} = 1 \end{aligned}$$

3.3.3 Indeterminate Forms: 0^0 , 1^∞ , or ∞^0

Incorporates changes using logarithms in order to get limit in to the form as shown before.

$$\begin{aligned}L &= \lim_{x \rightarrow 0^+} x^x \\ \ln L &= x \ln x \Rightarrow -\infty \cdot 0 \\ \ln L &= \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{\frac{1}{x}}{\frac{-1}{x^2}} \\ \ln L &= \lim_{x \rightarrow 0^+} -x = 0 \\ e^{\ln L} &= e^0 \\ L &= 1\end{aligned}$$

Special example in the following form:

$$\lim_{x \rightarrow c} \left(d + \frac{a}{x}\right)^{bx} = e^{ab}$$

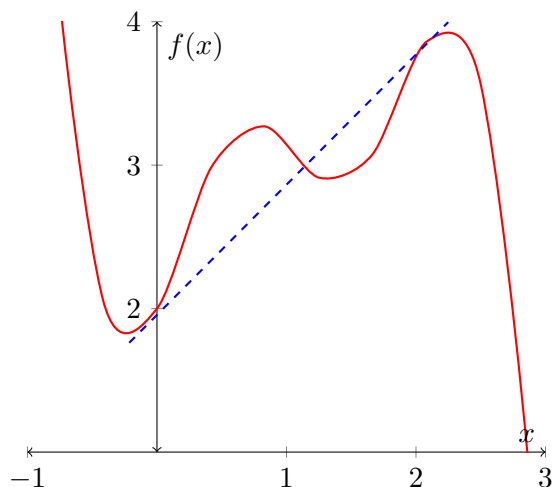
3.3.4 Limit Definition of Derivative

In the form $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$, where the derivative is taken with respect to h . By recognizing this form, the answer would be $\frac{d}{dx}f(x)$.

4 Graphing and Analytical Applications

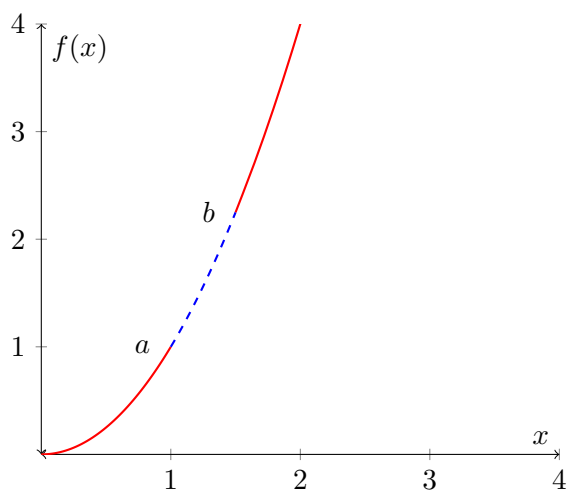
4.1 Mean Value Theorem (MVT)

Given that $f(x)$ is continuous $\forall x \in [a, b]$ and differentiable $\forall x \in (a, b)$, $\exists c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.



4.2 Intermediate Value Theorem (IVT)

Let $f(x)$ be continuous $\forall x \in [a, b]$. Let m between $f(a)$ and $f(b)$. $\exists c \in (a, b)$ such that $f(c) = m$.

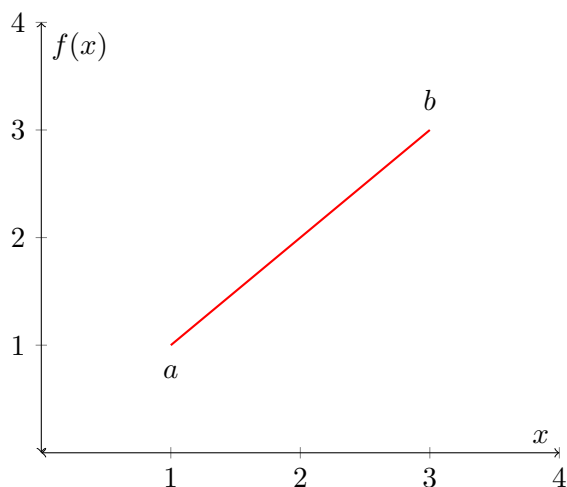


4.3 Rolles' Theorem

Given that $f(x)$ is continuous $\forall x \in [a, b]$ and differentiable $\forall x \in (a, b)$ and $f(a) = f(b)$, then $\exists c \in (a, b)$ such that $f'(c) = 0$.

4.4 Extreme Value Theorem (EVT)

If $f(x)$ is continuous $\forall x \in [a, b]$ then $f(x)$ has a max and min value on $[a, b]$.

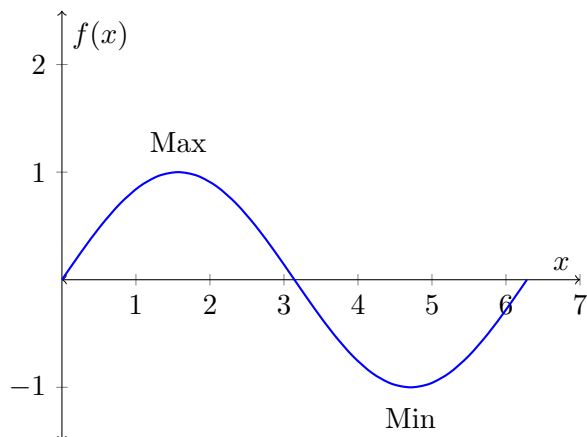


4.5 Fermat's theorem

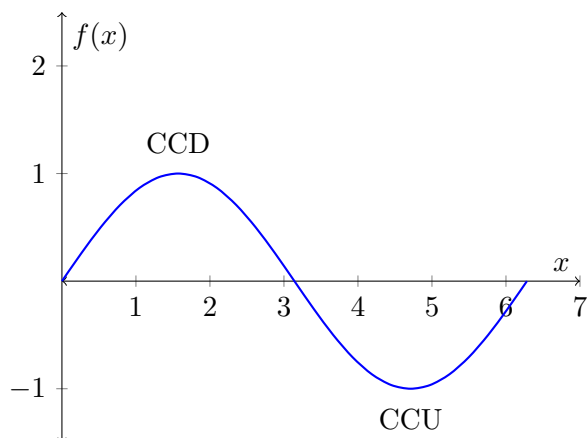
If $f(x)$ has a rel. max or min at $x = c$ and $f'(c)$ exists, then $f'(c) = 0$. However, there can be a rel. max/min when $f'(c)$ does not exist but implements a sign change.

4.6 First Derivative Test for Local Extrema and Second Derivative Test

Determines where a function increases or decreases, which denotes the maximums and minimums. Is found from checking *critical points* where $f'(x) = 0$. Maximums and minimums occur where $f'(x)$ changes sign, which must be around a critical point.



The second derivative test determines where the function is **concave up** or **concave down**, or where $f''(x) > 0$ or < 0 respectively.



4.7 Second Derivative Test for Local Extrema

If $f'(x)=0$ and $f''(x) < 0$, then $f(x)$ is the location of a local maxima. Similarly if $f'(x)=0$ and $f''(x) > 0$, then it is a local minima. Can be visualized through a concave up/down image, where the apex of the curve determines the extrema required.

4.8 Absolute Maximums and Minimums (Candidates Test)

Uses a chart like the following with x values determined from endpoints and critical points of $f(x)$.

x	$f(x)$
a	$f(a)$
x_1	$f(x_1)$
x_2	$f(x_2)$
b	$f(b)$

4.9 Derivatives of Inverse Functions

Let $g(x)$ be $f^{-1}(x)$ and (x, a) be the point.

$$g'(a) = \frac{1}{f'(g(a))}$$

4.10 2D Particle Motion

Fundamental equations are position, velocity, and acceleration.

$$\begin{aligned} x(t) \\ v(t) &= x'(t) \\ a(t) &= v'(t) = x''(t) \end{aligned}$$

4.10.1 Velocity

A particle changes direction when $v(t)$ goes from $+$ to $-$. The particle is moving to the left when $v(t) < 0$ and to the right when $v(t) > 0$.

4.10.2 Speed

Speed is denoted as $|v(t)|$. A particle is speeding up when $a(t)$ and $v(t)$ have the same sign and slowing down when they have different signs.

4.10.3 Find Next Position and Min/Max

Uses FTC.

$$x(t_f) = x(t_i) + \int_{t_i}^{t_f} v(t) dt$$

Min/max problems are formatted as farthest to left or right. Uses the candidates test for abs. max/min. Use FTC to find position values. t -values are where $v(t) = 0$.

Farthest to left \Rightarrow abs. min., for right \Rightarrow abs. max.

4.10.4 Distance and Displacement

Distance (d) is total distance traveled, ignores a change in direction, displacement (D) accounts for this.

$$d = \int_a^b |v(t)| dt$$
$$D = \int_a^b v(t) dt$$

4.10.5 Average Speed and Velocity

Average speed (S) is related to distance. Average velocity (V) accounts for displacement.

$$S = \frac{\int_a^b |v(t)| dt}{b - a}$$
$$V = \frac{\int_a^b v(t) dt}{b - a}$$

4.11 Differentiability and Continuity

A function is not necessarily differentiable if it is continuous, but if it is differentiable, then it must be continuous. Types of non-differentiable points include corner points, cusps, and vertical tangents.

4.11.1 Corner Points

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = \text{DNE}$$

4.11.2 Vertical Tangents

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = \pm\infty$$

4.11.3 Cusps

$$\lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} = \pm\infty$$
$$\lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} = \mp\infty$$

5 Integration

5.1 Basic Rules

1. FTC: $F(b) = F(a) + \int_a^b f(x)dx$
2. Power rule: $\int x^n dx = \frac{x^{n+1}}{n+1}$

5.1.1 U-Substitution

$$\begin{aligned}\int_2^5 x(3x^2)dx &\Rightarrow u = 3x^2, du = 6xdx \\ &\Rightarrow \frac{1}{6} \int_{12}^{75} udu \\ &\frac{1}{6} \left[\frac{u^2}{2} \Big|_{12}^{75} \right]\end{aligned}$$

5.1.2 Integration by Partial Fractions

Factor denominator, separate into partial fractions, then integrate.

$$\begin{aligned}\int \frac{1}{x^2 - 4} dx &\Rightarrow \frac{1}{x^2 - 4} = \frac{1}{(x+2)(x-2)} \\ \frac{1}{(x+2)(x-2)} &\Rightarrow \frac{A}{x+2} + \frac{B}{x-2} \Rightarrow A = -\frac{1}{4}, B = \frac{1}{4} \\ \int \frac{-\frac{1}{4}}{x+2} + \frac{\frac{1}{4}}{x-2} dx &= -\frac{1}{4} \ln|x+2| + \frac{1}{4} \ln|x-2| + C\end{aligned}$$

5.1.3 Trigonometric Integrals

Trigonometric Integrals

- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \tan x \, dx = -\ln |\cos x| + C$
- $\int \cot x \, dx = \ln |\sin x| + C$
- $\int \sec x \, dx = \ln |\sec x + \tan x| + C$
- $\int \csc x \, dx = \ln |\csc x - \cot x| + C$
- $\int \sec x \tan x \, dx = \sec x + C$

- $\int -\csc x \cot x \, dx = \csc x + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int -\csc^2 x \, dx = \cot x + C$

Inverse Trigonometric Integrals

- $\int \frac{dx}{f(x)^2 + a^2} = \frac{1}{a} \arctan \frac{f(x)}{a} + C$
- $\int \frac{dx}{\sqrt{a^2 - f(x)^2}} = \arcsin \frac{f(x)}{a} + C$

5.2 Properties of Integrals

1. $\int_a^a f(x) dx = 0$
2. If $f(x)$ is odd, $\int_{-a}^a f(x) dx = 0$
3. If $f(x)$ is even and $\int_0^a f(x) dx = k$, then $\int_{-a}^a f(x) dx = 2k$
4. $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$
5. If $\int_a^b f(x) dx = k$, then $\int_b^a f(x) dx = -k$
6. If $f(x) \leq g(x) \, \forall x \in [a, b]$ then $\int_a^b f(x) dx \leq \int_a^b g(x) dx$
7. $|\int_a^b f(x) dx| \leq \int_a^b |f(x)| dx$

5.3 Riemann Sums

5.3.1 Riemann Sum Notation and Definite Integrals

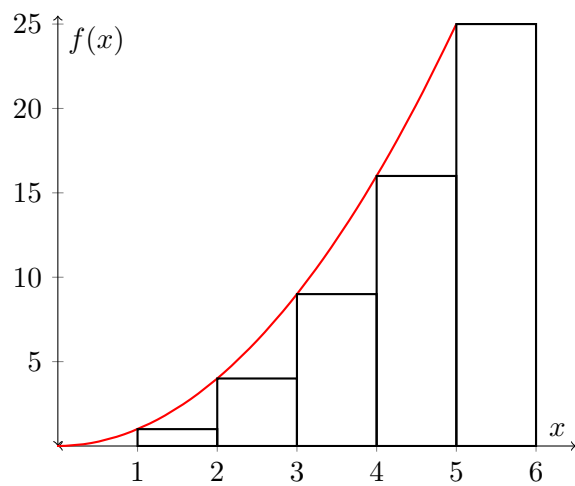
$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(a + k\Delta x) \Delta x$$

In $\Delta x = \frac{b-a}{n}$ (the subintervals), b is the end limit of the integral which is the same as $a + n\Delta x$. This can be used to solve for b given a . x can be represented as $a + k\Delta x$.

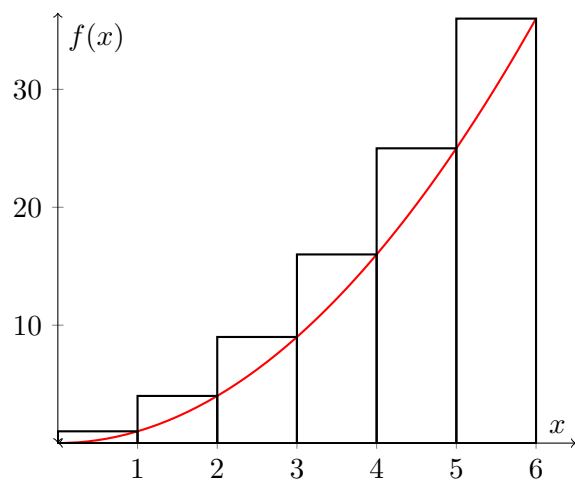
$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f\left(a + \frac{k(b-a)}{n}\right) \frac{b-a}{n}$$

5.3.2 Basic Riemann Sums

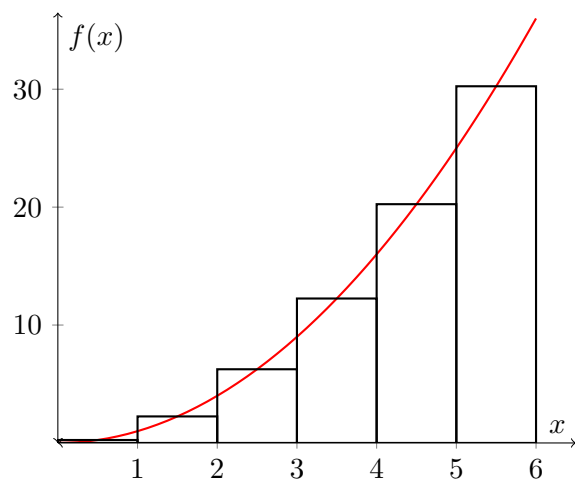
Left sum: $(x_R - x_L)f(x_L)$



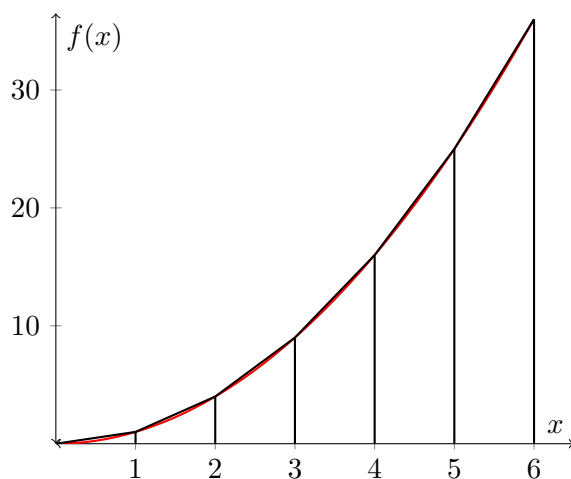
Right sum: $(x_R - x_L)f(x_R)$



Midpoint sum: $(x_R - x_L)f(x_L + \frac{x_R - x_L}{2})$



Trapezoidal sum: $\frac{1}{2}(f(x_R) + f(x_L))(x_R - x_L)$



5.4 Integration by Parts

$$\int f'(x)g(x)dx = f(x)g(x) - \int g'(x)f(x)dx$$

Choose $g(x)$ in order of **logs, inverse, algebraic, trig, exponential**.

5.4.1 Example

$$\begin{aligned} \int x\sqrt{x+1}dx &\Rightarrow g(x) = x, f'(x) = \sqrt{x+1} \\ &= \frac{x}{2}(x+1)^{3/2} - \int \frac{1}{2}(x+1)^{3/2}dx \\ &= \frac{x}{2}(x+1)^{3/2} - \frac{1}{5}(x+1)^{5/2} + C \end{aligned}$$

5.4.2 Tabular Method

Negate every second entry under derivative column.

$$\int x^2 \sin x \, dx \Rightarrow f(x) = x^2, \, g(x) = \sin x$$

$f(x)$	$g(x)$
x^2	$\sin x$
$-2x$	$-\cos x$
2	$-\sin x$
0	$\cos x$

$$\int x^2 \sin x \, dx = -x^2 \cos x - 2x \sin x + 2 \cos x + C$$

6 Differential Equations

6.1 Separation of Variables

MIT OCW Reference

$$\begin{aligned}\frac{dy}{dx} &= x(y-1) \\ \frac{dy}{y-1} &= xdx \\ \ln|y-1| + C &= \frac{x^2}{2} + C \\ |y-1| &= e^C e^{\frac{x^2}{2}} \Leftrightarrow |y-1| = C e^{\frac{x^2}{2}} \\ y-1 &= \pm C e^{\frac{x^2}{2}} \\ y &= 1 + C e^{\frac{x^2}{2}}\end{aligned}$$

6.2 Exponential growth

$$\frac{dP}{dt} = kP \Rightarrow P = P_0 e^{kt}$$

A negative growth constant k represents decay and a positive one represents growth. For half life problems, one is solving for half (or any fraction or multiple) of the initial population.

$$\begin{aligned}\frac{1}{2}P_0 &= P_0 e^{kt} \\ \frac{1}{2} &= e^{kt} \Rightarrow t = \frac{\ln \frac{1}{2}}{k}\end{aligned}$$

6.3 Euler's Method

When approximating a lower x -value than given, the procedure is still identical **but Δx is negative**.

$$y_{new} = y_{old} + \left. \frac{dy}{dx} \right|_{(x_{old}, y_{old})} \Delta x$$

6.4 Slope Fields

A differential equation in the form $\frac{dy}{dx} = F(x, y)$, where the slopes at each point in a 2D graph are plotted with line segments. The solution curves for a differential equation align with these segments, and the initial condition can be visualized.

6.5 Logistic Growth

Growth is limited by the carrying capacity L , which is found by setting the differential equation to 0 since this is when the curve flattens and the carrying capacity is achieved. There is a positive growth constant k . $\lim_{t \rightarrow \infty} \frac{dP}{dt} = 0$ always because of the asymptote to L . However, $\frac{dP}{dt}$ **asymptotes to 0** and is never exactly 0.

$$\frac{dP}{dt} = kP(L - P)$$

When derived, the population formula is as follows, with A being a constant.

$$P(t) = \frac{L}{1 + Ae^{-kt}}$$

The inflection point of the population (not growth) equation is when the population is growing the fastest. This occurs at the time t when $P = \frac{L}{2}$. Furthermore, the carrying capacity of this population is given by $\lim_{t \rightarrow \infty} P(t) = L$, which is always true due to form of $P(t)$.

7 Applications of Integration

7.1 In/Out Rates

When finding the *total* quantity at a certain time (not specific to how much going in/out), the net rate is used. This is given by $R(t) - E(t)$ where $R(t)$ is the rate in and $E(t)$ is the rate out. Thus, the quantity at a certain time is given by the following equation.

$$A(t_f) = A(t_i) + \int_{t_i}^{t_f} (R(t) - E(t))dt$$

7.2 Average Value and R.O.C of a Function

Average value involves the antiderivative of f while the average R.O.C involves f itself.

$$f_{avg} = \frac{\int_a^b f(x)dx}{b-a}$$
$$A = \frac{f(b) - f(a)}{b-a}$$

Arclength of a function also is given from an integral and is found from the following (useful in perimeter problems).

$$S = \int_a^b \sqrt{1 + f'(x)^2} dx$$

7.3 Accumulation Functions

Generally given in the following form as $F(x)$. The integral must use a different variable t as it is not dependent on x . a is a constant representing the lower integral limit.

$$F(x) = \int_a^x f(t)dt$$

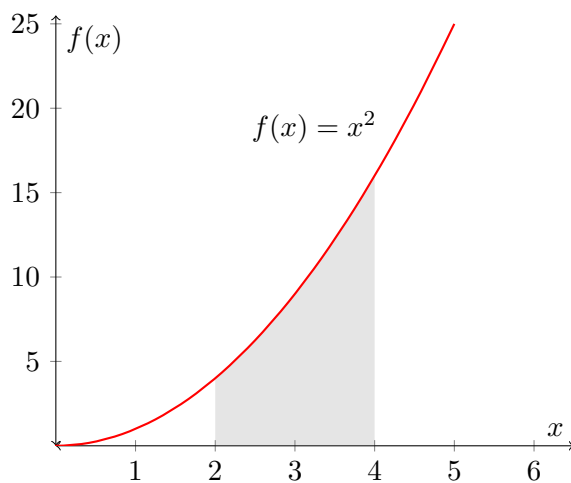
Differentiation is also applicable, with each progression a lower offset of a normal function's derivative, observable in the following expressions.

$$\begin{aligned}F'(x) &= 1 \cdot f(x) - 0 = f(x) \\F''(x) &= f'(x)\end{aligned}$$

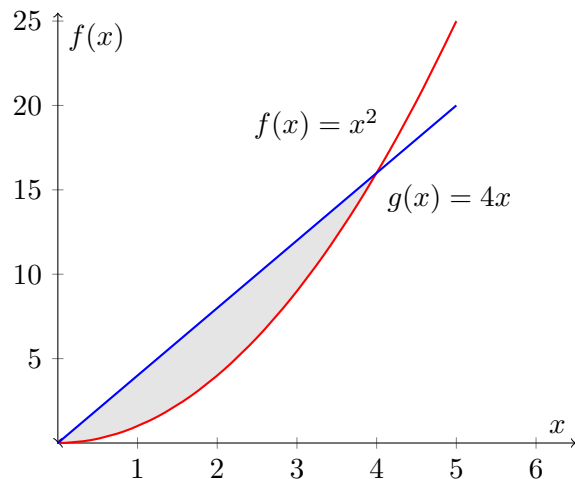
7.4 Area and Volume

7.4.1 Area

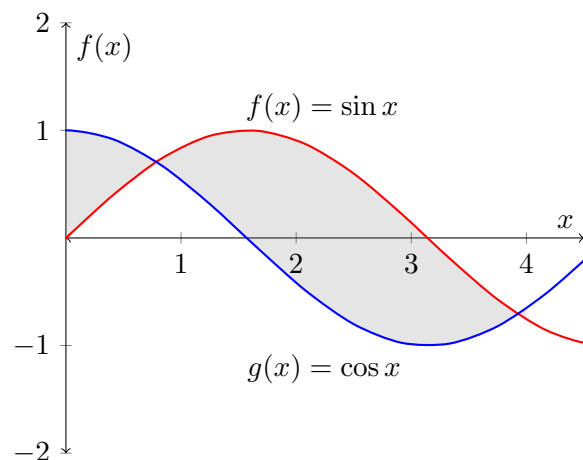
Area with respect to a curve $f(x)$ and the x -axis is given by $\int_a^b f(x)dx$. If the curve from a to b is below the x -axis, then it is negative in value, **but the area is not negative**.



Area of two intersecting regions is given by $\int_a^b (f(x) - g(x))dx$.



Area of regions with multiple intersections is given by $\int_a^b |f(x) - g(x)| dx$, ignoring the central intersection point.

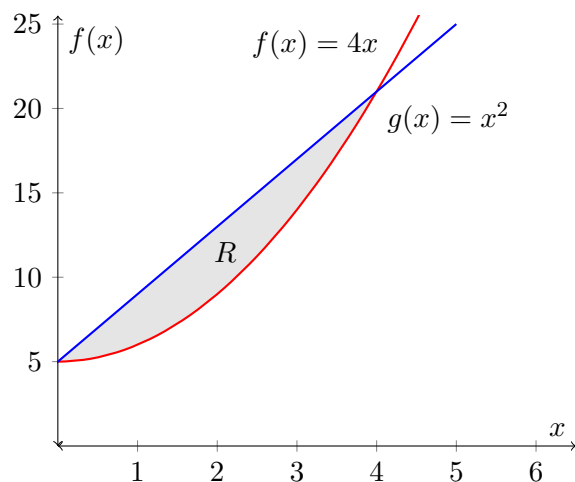


dy integration is similar to normal integration, but uses the y -axis for reference. The following example uses $f(y) = \sin(y)$ and $g(y) = \cos(y)$, with the integral being $\int_0^{\pi/4} (g(y) - f(y)) dy$.

7.5 Volume around horizontal axes

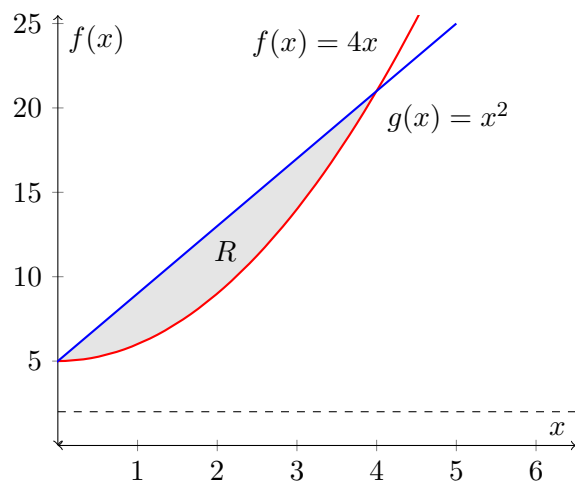
Given that $f(x) \geq g(x) \forall x \in [a, b]$, the integral is as follows for the region R revolved around $x = 0$.

$$V_x = \pi \int_a^b (f(x)^2 - g(x)^2) dx$$

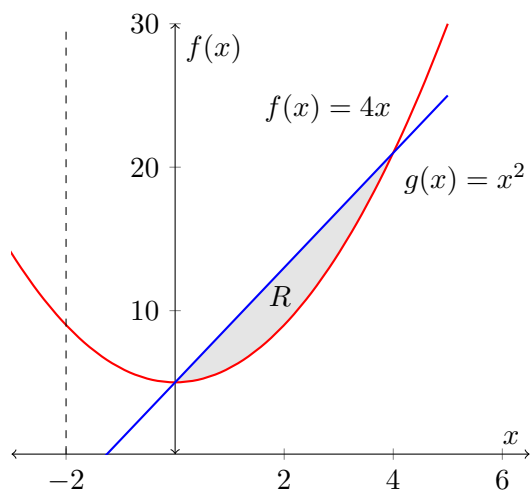


For the same region R revolving around $x = 2$, the radius of rotation (washer space) is reduced, so 2 is subtracted. This would give the integral equation as the following. If $x = -2$, for instance, 2 would be added as it increases the washer radius.

$$V_x = \pi \int_0^4 ((f(x) - 2)^2 - (g(x) - 2)^2) dx$$



7.6 Volume around vertical axes

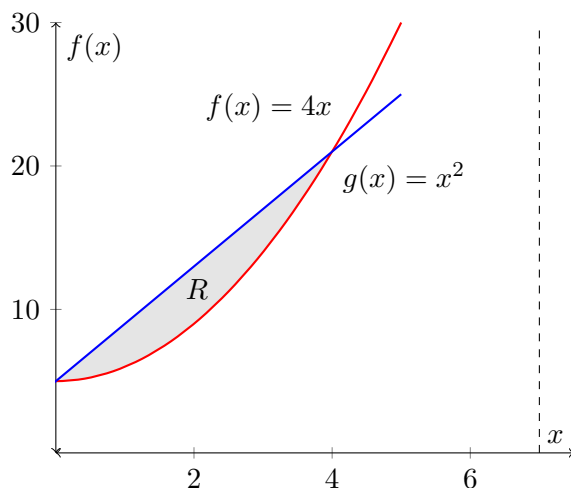


Is given in the general form $V_y = 2\pi \int_a^b (\text{radius})(\text{height}) dx$. The height is the value of $f(x)$ and the radius is some value of x since this is with respect to the y -axis. The preceding example is a rotation around $y = -2$, and the integral would be given by the following.

$$V_y = 2\pi \int_0^4 (x + 2)(f(x) - g(x)) dx$$

The procedure is the opposite when given a vertical axis on the other side of the graph in quadrant I, namely $x = 7$. This is because the radius would be given by $7 - x$ since that difference is the distance between each varying point and $x = 7$. The formula would be the following.

$$V_y = 2\pi \int_0^4 (7 - x)(f(x) - g(x)) dx$$



Notably, the last two examples can also be computed using dy integration in a similar way to horizontal axis-based solids. However, the bottom function would become the top one and the limits would become the y -coordinates. They are below. Let $g(y) = \sqrt{y}$ and $f(y) = \frac{y}{4}$.

$$A_y = \pi \int_5^{21} ((g(y) + 2)^2 - (f(y) + 2)^2) dy$$

$$A_y = \pi \int_5^{21} ((7 - g(y))^2 - (7 - f(y))^2) dy$$

7.7 Volume with known cross-sections

Let $s = f(x) - g(x)$ and $g(x) \leq f(x) \forall x \in [a, b]$.

- Squares: $\int_a^b s^2 dx$
- Rectangles (with the length being n times the width): $\int_a^b ns^2 dx$
- Equilateral triangles: $\int_a^b \frac{\sqrt{3}s^2}{4} dx$
- Semi-circles: $\int_a^b \frac{1}{8}\pi s^2 dx$
- Right isosceles triangle: $\int_a^b \frac{1}{4}s^2 dx$

8 Infinite Series

8.1 Fundamental Series

As listed below the harmonic series 1, diverging due to P-series, and alternating series 2, which converges according to AST.

$$\sum_{n=1}^{\infty} \frac{1}{n} \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \tag{2}$$

8.2 P-series test

A series in the form of $\sum_{n=1}^{\infty} \frac{1}{n^P}$. If $P \leq 1$, the series diverges, if $P > 1$, it converges.

8.3 n-th term test (divergence only)

If $\lim_{n \rightarrow \infty} |a_n| \neq 0$, the series diverges, else if it is 0, it is **inconclusive**. But if a series diverges, *it is not necessarily due to the n-th term test*.

8.4 Geometric series

$\sum_{n=1}^{\infty} a_1 r^{n-1}$ converges if and only if $|r| < 1$. A power series is a form of a geometric one.

$$\Rightarrow \sum_{n=1}^{\infty} a_1 r^{n-1} = \frac{a_1}{1-r}$$

8.5 Ratio test

$\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$.

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$.

The test if inconclusive is the result is 1.

8.6 Alternating series test

A decreasing alternating series, where $|a_{n+1}| < |a_n|$, converges if $\lim_{n \rightarrow \infty} a_n = 0$.

8.7 Taylor/Maclaurin Series

$$\sum_{n=0}^{\infty} \frac{f^n(c)(x-c)^n}{n!}$$

Since the coefficient of each term is the n -th derivative, if given a term T , the derivative can be found by setting $\frac{f^n(c)(x-c)^n}{n!} = T$ in order to get an expression.

A Maclaurin polynomial is a Taylor polynomial centered at $x = 0$. Here are some common Maclaurin polynomial for function:

- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$
- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} x^n$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} x^n$
- $\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} x^n$

8.8 Trigonometric Identities

Pythagorean Identities

- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $1 + \cot^2 x = \csc^2 x$

Double-Angle Identities

- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$
- $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$