

Multivariable Calculus Reference

Sidharth Baskaran

March 28, 2021

Table of Contents

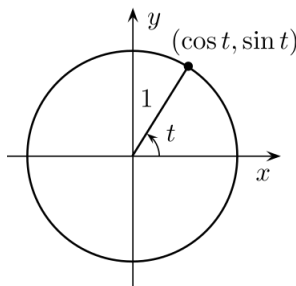
1	Paths whose image curve is a circle	3
1.1	Unit Circle	3
1.2	Observations	3
2	Paths whose image is a line or line segment in the plane	3
2.1	Line Parametrics	3
2.2	General Forms	4
3	Paths whose image curve is a line in \mathbb{R}^3	4
3.1	\mathbb{R}^3 parameterization	4
4	Cycloid Problem	5
4.1	Hypercycloid Derivation	6
4.2	Hypocycloid Derivation	6
5	Velocity Vector	6
5.1	Definitions	6
5.2	Tangent Line	7
6	Space Curves	7
7	Speed and Arclength	7
7.1	Speed	7
7.2	Arclength	8
7.3	Arclength Parameterization	9
8	Curvature	9
8.1	Proofs	9
8.2	Definition	10
9	Motion in 3D space	10
10	Derivatives of parameterized curves	11
10.1	Arclength parameterization derivation	11

10.2	Orthogonal derivative and position vectors	12
11	Planetary motion	12
11.1	Cross-product identities	12
12	Planes	13
12.1	Cross-product rules and identities	14
13	Graphs	14
13.1	Multivariable functions	14
13.2	Graphing multivariable functions	14
13.3	Contour maps and level curves	15
14	Partial Derivatives	16
14.1	Definition	16
14.2	Linear approximation with planes	16
14.3	Higher-order derivatives	17
15	Extrema	17
15.1	Definition and proofs	17
15.2	Second-derivative test	18
15.3	Global extrema	19
16	Lagrange Multipliers	19
16.1	Theory	19
16.2	Proof	20

1 Paths whose image curve is a circle

1.1 Unit Circle

Unit circle is set of points in \mathbb{R}^2 defined as $C = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$. Ellipse is $C = \{(x, y) \in \mathbb{R}^2 | \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$. Has standard parameterization of $\vec{c}(t) = (\cos(t), \sin(t))$. When parameterizing, always start from $t = 0$ reference unless otherwise given.



Properties

- Image of \vec{c} is a closed curve (has no endpoints, plane is divided into ≥ 2 disjoint regions)
- Image of \vec{c} is a simple curve; no self-intersection
- $\vec{c}(t)$ is an **injective path**; path is considered injective if $\vec{c}(t_1) = \vec{c}(t_2)$, which implies that $t_1 = t_2$ where these are on the open interval (a, b) even if $a = b$
- Orientation of \vec{c} is counter-clockwise in traversal

1.2 Observations

$$\vec{p}(t) = (a \cos(\pm(nt \pm \theta)) + x_0, b \sin(\pm(nt \pm \theta)) + y_0)$$

If $-t$ for t , orientation is CW, CCW is t . If $a = b$, then curve is a circle of radius a or b , else an ellipse with horizontal and vertical radii. x_0 and y_0 simply shift the center coordinate. $n > 0 \in \mathbb{R}$ determines how many times the circle is traversed given $t \in [0, 2\pi]$, for example. θ is the phase shift. When changing direction of traversal, cannot have $a > b$ for $[a, b]$ so to decrease argument of sin or cos must have $-t$ for t . Starting out, $-t$ goes through the angle range and t is just a sign flip.

2 Paths whose image is a line or line segment in the plane

2.1 Line Parametrics

A line is a 1D subspace of \mathbb{R}^2 , so $L = \{t\vec{m} | t \in \mathbb{R}\}$ for $\vec{m} \in \mathbb{R}^2$. $\vec{m} = \begin{bmatrix} m_x \\ m_y \end{bmatrix}$ is the **slope vector**. Path given by image of L :

$$\vec{c}(t) = (m_x t, m_y t), t \in \mathbb{R}$$

Can represent $\vec{c}(t) = t\vec{m}$ as well.

Lines Main Ideas

- Image of a line is a curve (e.g. $y = x$ represents image curve of $\vec{c}(t) = (t, t)$)
- Lines can have nonzero intercepts, so $\vec{c}(t) = t\vec{m}$ represents $y = 2x + 1$. Line that has intercept vector $P_0 = (x_0, y_0) \parallel \vec{m} = (m_x, m_y)$ can be expressed as:

$$\vec{c}(t) = (x_0 + tm_x, y_0 + m_y t) = \vec{P}_0 + t\vec{m}$$

Note endpoint of $\vec{c}(t)$ is on image line (curve).

2.2 General Forms

2 parametric lines **collide** if they intersect and the point of intersection corresponds to the same t in both curves. If you set the parameter vector coordinates equal to each other and solve for t , a solution indicates they collide. Intersection is found by **eliminating** the parameter (solve for t in terms of either x or y and plug into the other).

General form of parameterized curve can be expressed as the following:

$$\vec{c}(t) = \left(\frac{m_x}{\Delta t}(t - a) + x_0, \frac{m_y}{\Delta t}(t - a) + y_0 \right)$$

where Δt is the domain interval over $[a, b]$ and (x_0, y_0) represents the desired **starting coordinate**. This is important as when going in reverse, other coordinate can be used and slope might be negative. a is used in $(t - a)$ because everything is conventionally done with respect to starting coordinate.

3 Paths whose image curve is a line in R3

3.1 R3 parameterization

If \vec{m} is a nonzero vector along L through origin in \mathbb{R}^3 , then $L = \{t\vec{m} | t \in \mathbb{R}\}$; follows that $\vec{m} = (m_x, m_y, m_z)$, the slope or direction vector of the line. The basic parameterization is:

$$\vec{c}(t) = (m_x t, m_y t, m_z t)$$

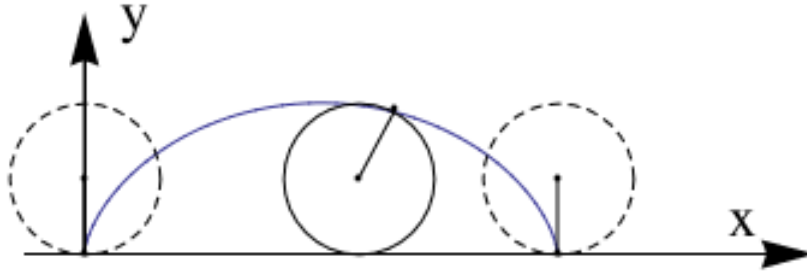
Basis vectors in \mathbb{R}^3 are $\vec{i}, \vec{j}, \vec{k}$. Rewriting parameterization:

$$\vec{c}(t) = (x_0 + m_x t)\vec{i} + (y_0 + m_y t)\vec{j} + (z_0 + m_z t)\vec{k}$$

2 lines $\vec{c}_1(t) = P_0 + \vec{m}_1 t$ and $\vec{c}_2(t) = Q_0 + \vec{m}_2 t$ are parallel if direction vectors are parallel ($\vec{m}_1 = k\vec{m}_2$). Collisions still exist. If neither parallel nor intersecting, considered as skew.

To determine skew, parallel, or coincide, use parameters s, t for each line and solve SOE. If same slope, rule out skew clearly, then check if $s, t \in \mathbb{R}$: if not, then parallel, if so, then they coincide. If intersecting and want to check if collide, some t must satisfy all relations.

4 Cycloid Problem



With radius 1 and passing through the origin:

$$\vec{c}(t) = (t - \sin t, 1 - \cos t)$$

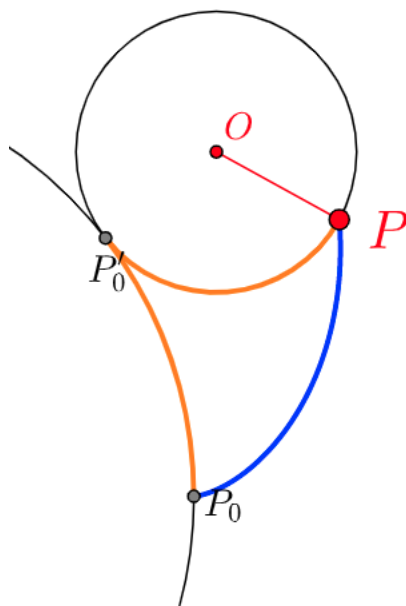
Observe that:

$$\vec{c}'(t) = (1 - \cos t, \sin t)$$

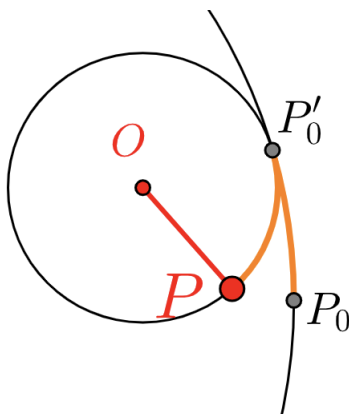
Can define the vector $\vec{u} = \begin{pmatrix} x'(t) \\ 0 \end{pmatrix}$ such that \vec{u} is always horizontal and $||\vec{u}|| = |x'(t)|$. Reaches maximum value at $t \in [k\pi | k \in \mathbb{R}]$ and is has minimum cusp where it is 0 at $t \in [2k\pi | k \in \mathbb{R}]$. Thus, $x'(t) \geq 0$ always, as the x-coordinate is never decreasing.

Can also define the vector $\vec{v} = \begin{pmatrix} 0 \\ y'(t) \end{pmatrix}$ with the same properties. Reaches maximum value when $t \in [k\frac{\pi}{2} | k \in \mathbb{R}]$. Can change, as observe t when $\sin t < 0$ or > 0 .

4.1 Hypercycloid Derivation



4.2 Hypocycloid Derivation



5 Velocity Vector

5.1 Definitions

Vector $\vec{u}(t_0) + \vec{v}(t_0)$ is the velocity vector to the curve $\vec{c}(t)$ at $t = t_0$.

Let $\vec{c}: [a, b] \rightarrow \mathbb{R}^n$ have a path $\vec{c}(t) = (x_1(t), x_2(t), x_3(t), \dots, x_n(t))$ (let $x_i(t): [a, b] \rightarrow \mathbb{R}$ for each i)

- If $t_0 \in [a, b]$, then $\vec{c}'(t_0) := (x_1'(t_0), x_2'(t_0), x_3'(t_0), \dots, x_n'(t_0))$; the velocity vector to \vec{c} at t_0

- The path $\vec{c}'(t_0) := (x_1'(t_0), x_2'(t_0), x_3'(t_0), \dots, x_n'(t_0))$; the velocity vector to \vec{c} is referred to as velocity of $\vec{c}(t)$

Recall chain rule: if $y = f(x)$ where x is a function of t , $y'(t) = x'f'(x)$, not to be confused with product rule. Can write $f'(x) = \frac{y'(t)}{x'(t)}$

- If $\vec{p}(t) = \vec{c}(t) + \vec{r}(t)$, then $\vec{p}'(t) = \vec{c}'(t) + \vec{r}'(t)$
- If $g(t) = \vec{c}(t) \cdot \vec{r}(t)$, then $g'(t) = \vec{c}'(t) \cdot \vec{r}(t) + \vec{c}(t) \cdot \vec{r}'(t)$
- If $\vec{p}(t) = f(t)\vec{c}(t)$, then $\vec{p}'(t) = f'(t)\vec{c}(t) + f(t)\vec{c}'(t)$
- If $\vec{p}(t) = \vec{c}(t) \times \vec{r}(t)$, then $\vec{p}'(t) = \vec{c}'(t) \times \vec{r}(t) + \vec{c}(t) \times \vec{r}'(t)$
- If $\vec{p}(t) = \vec{c}(f(t))$, then $\vec{p}'(t) = f'(t)\vec{c}'(f(t))$
- If $g(t) = \|\vec{c}(t)\|$, then $g'(t) = \frac{\vec{c}(t) \cdot \vec{c}'(t)}{\|\vec{c}(t)\|}$

5.2 Tangent Line

Tangent line can be visualized as a base vector in standard position plus a velocity vector tangent to the tip which traces a shifted line in some interval. General formula with base vector $\vec{c}(t_0)$ and slope $\vec{c}'(t_0)$:

$$\ell(t) = \vec{c}(t_0) + (t - t_0)\vec{c}'(t_0)$$

6 Space Curves

- Projection into the xy plane is the path $(x(t), y(t), 0)$.
- Projection into the xz - plane is the path $(x(t), 0, z(t))$.
- Projection into the yz plane is the path $(0, y(t), z(t))$.

7 Speed and Arclength

7.1 Speed

Speed of a parametric function in \mathbb{R}^n is given by:

$$\|\vec{c}'(t)\| = \sqrt{\sum_{i=1}^n c_i'(t)^2}$$

(being the magnitude of the velocity vector)

7.2 Arclength

Arclength of a parametric function is given by:

$$S = \int_a^b \|\vec{c}'(t)\| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 + \dots} dt$$

Can approximate arclength as a sum of the lengths of secant vector approximations $\vec{s}_i = \vec{c}(t_i) - \vec{c}(t_{i-1})$:

$$\text{arclength} \approx \sum_{i=1}^n \|\vec{s}_i\|$$

According to the MVT, there exists a \hat{t}_i in (t_{i-1}, t_i) (open interval due to differentiability requirement) such that:

$$\begin{aligned} x'(\hat{t}_i) &= \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \\ y'(\hat{t}_i) &= \frac{y(t_i) - y(t_{i-1})}{t_i - t_{i-1}} \end{aligned}$$

This means that, since \vec{s}_i is given as the difference between 2 points, being a secant:

$$\begin{aligned} \vec{s}_i &= ((t_i - t_{i-1})x(\hat{t}_i), (t_i - t_{i-1})y(\hat{t}_i)) \\ \vec{s}_i &= (t_i - t_{i-1}) (x'(\hat{t}_i), y'(\hat{t}_i)) \\ \vec{s}_i &= (t_i - t_{i-1})\vec{c}'(\hat{t}_i) \end{aligned}$$

Thus,

$$\begin{aligned} \text{arclength} &\approx \sum_{i=1}^n \|\vec{s}_i\| \\ \text{arclength} &\approx \sum_{i=1}^n \|\Delta t \vec{c}'(\hat{t}_i)\| \\ \text{arclength} &\approx \sum_{i=1}^n \Delta t \|\vec{c}'(\hat{t}_i)\| \end{aligned}$$

Can define the arclength differential as follows:

$$ds = \sqrt{dx^2 + dy^2}$$

Can just define arclength as $\text{arclength} = \int ds$

7.3 Arclength Parameterization

Higher the speed of a curve, farther the points are spaced apart. An arclength parametrization of a curve is a path whose image is the desired curve and whose speed is constantly one. Or, $\vec{c}: [a, b] \rightarrow \mathbb{R}^n$ with $\|\vec{c}'(t)\| = 1$ for $t \in [a, b]$. If a curve is not an arclength parameterization, then can do $\frac{\vec{c}(t)}{\|\vec{c}'(t)\|}$ but only dividing the coefficients (slopes).

When speed is variable, is difficult to define arclength parameterization. Thus, can define displacement to be $s(t) = \int_a^b(t)dt$. If $v(t) \neq 0$, then s is injective because according to FTC, $s'(t) = v(t)$. By definition, $v'(t) \geq 0$ always since it is composed of a radical, so it must be **increasing**. Thus, if $t_1 = t_2$, $s(t_1) \neq s(t_2)$. Arclength parameterization:

$$s(t) = \int_0^t \|\vec{c}'(u)\| du$$

This means that s is invertible, so can solve for t to get $t = \varphi(s)$. An arclength parameterization can be found by:

$$\boxed{\vec{p}(s) = \vec{c}(\varphi(s))}$$

8 Curvature

8.1 Proofs

Recall that to make an arclength parameterization accumulate the magnitudes of infinitesimal velocity vectors:

$$s(t) = \int_a^t \|\vec{r}'(u)\| du$$

Given some curve $\vec{r}(t)$, define an arclength parameterization by $\vec{r}(g(s)) \rightarrow \vec{r}_1(s)$, so \vec{r} is defined in terms of s . The unit tangent vector $\vec{T}_1(s)$ is then $\frac{\vec{r}_1'(s)}{\|\vec{r}_1'(s)\|} = \vec{r}_1'(s)$.

$$\begin{aligned} \vec{T}_1(s) &= \vec{r}_1'(s) \\ &= \frac{d}{ds} \vec{r}_1(s) \\ &= \frac{d}{ds} \vec{r}(g(s)) \\ &= \vec{r}'(g(s)) \cdot g'(s) = \vec{r}'(t) \cdot \frac{dt}{ds} \\ &= \frac{\vec{r}'(t)}{\frac{ds}{dt}} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \end{aligned}$$

This means that $\vec{T}_1(s) = T(t)$

Continuing, to find curvature $\kappa(t)$:

$$\begin{aligned}\vec{T}_1'(s) &= \frac{d}{ds}\vec{T}(t) \\ &= \frac{d}{ds}\vec{T}(g(s)) \\ &= \vec{T}'(t) \cdot \frac{dt}{ds} \\ &= \frac{\vec{T}'(t)}{\frac{ds}{dt}} \\ &= \frac{\vec{T}'(t)}{\|\vec{r}'(t)\|}\end{aligned}$$

Thus, $\boxed{\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}}$

8.2 Definition

Given a curve C parameterized with arclength by the path $\vec{c} : [a, b] \rightarrow \mathbb{R}^n$, curvature is defined as:

$$\boxed{\kappa(s) = \|\vec{T}'(s)\|}$$

where $\vec{c}'(s) \neq 0$ and $\vec{T}(s) = \frac{\vec{c}'(s)}{\|\vec{c}'(s)\|}$ (normalized slope vector).

A loose geometric interpretation is that a greater $\kappa(s)$ implies more curvature, that is, the curve is changing at a greater rate there. When $\vec{c}'(s) \neq 0$ is always true for a curve, it is **regular**. Is defined in terms of arclength parameterization so curvature is an intrinsic property of the curve independent of parameterization.

Formula for curvature at the point $\vec{c}(t)$:

$$\boxed{\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{c}'(t)\|} = \frac{\|\vec{c}'(t) \times \vec{c}''(t)\|}{\|\vec{c}'(t)\|^3}}$$

9 Motion in 3D space

Given a path $\vec{c} : \mathbb{R} \rightarrow \mathbb{R}^3$ with $\vec{c}(t) = (x(t), y(t), z(t))$, then we have defined:

- $\vec{v}(t) = \vec{c}'(t) = (x'(t), y'(t), z'(t))$ is also path in \mathbb{R}^3 called the velocity of \vec{c}
- $\vec{a}(t) = \vec{v}'(t) = \vec{c}''(t) = (x''(t), y''(t), z''(t))$ is also a path in \mathbb{R}^3 called the acceleration of \vec{c}
- $v(t) = \|\vec{v}(t)\| = \|\vec{c}'(t)\|$ is a scalar valued function on \mathbb{R} (that's a fancy way of saying the domain and codomain of this function are both \mathbb{R}) called the speed of \vec{c}
- $\vec{T}(t) = \frac{\vec{c}'(t)}{v(t)}$ is also a path in \mathbb{R}^3 called the unit tangent to \vec{c}
- $\kappa(t) = \frac{\vec{T}'(t)}{v(t)} = \frac{\|\vec{c}'(t) \times \vec{c}''(t)\|}{\|\vec{c}'(t)\|^3}$ is a scalar valued function on \mathbb{R} called the curvature of \vec{c}

Note that $\vec{T} \cdot \vec{T} = \|\vec{v}\|^2 = 1$. Computing the derivative, $\frac{d}{dt} \vec{T} \cdot \vec{T} = 2\vec{T} \cdot \vec{T}' = 0$ This means that $\vec{T} \perp \vec{T}'$.

Define $\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$ as the unit normal vector, which is the unit tangent to the unit tangent.

From observation, $\vec{T} \perp \vec{N}$. Fact: the acceleration vector always lies in the plane spanned by \vec{N} and \vec{T} .

Acceleration $\vec{a}(t)$ is thus split component-wise into a_T from \vec{T} and a_N from \vec{N} :

$$a_T = v'(t) = \frac{\vec{a}(t) \cdot \vec{v}(t)}{v(t)}$$

$$a_N = \kappa(t)v(t)^2 = \frac{\|\vec{a}(t) \times \vec{v}(t)\|}{v(t)} = \sqrt{\|\vec{a}(t)\|^2 - |a_T|^2}$$

10 Derivatives of parameterized curves

10.1 Arclength parameterization derivation

Take the following function:

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

An arclength parameterization is achieved with the following computation:

$$s = \int_0^t \sqrt{x'(u)^2 + y'(u)^2} du$$

Can say that $t = g(s)$, so the arclength parameterization, which is the path in terms of s :

$$\vec{r}_1(s) = \langle x(g(s)), y(g(s)) \rangle$$

Taking the derivative by the chain rule:

$$\begin{aligned}\vec{r}_1'(s) &= \langle x'(g(s)) \cdot g'(s), y'(g(s)) \cdot g'(s) \rangle \\ &= g'(s) \langle x'(t), y'(t) \rangle\end{aligned}$$

Note that $g'(s) = \frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{\|\vec{r}'(t)\|}$ by taking the derivative of the integral for arclength:

$$\vec{r}_1'(s) = \frac{1}{\|\vec{r}'(t)\|} \langle x'(t), y'(t) \rangle = \vec{T}(t)$$

Following from this, $g'(s) = \frac{1}{\|\vec{r}'(g(s))\|}$ so $g''(s) = -\frac{1}{\|\vec{r}'(g(s))\|^2} \cdot g'(s) = -\frac{1}{\|\vec{r}'(t)\|^3}$

10.2 Orthogonal derivative and position vectors

Observe that $\vec{r} \cdot \vec{r} = \|\vec{r}\|^2$. Thus, $\frac{d}{dt}[\vec{r} \cdot \vec{r}] = 2\vec{r} \cdot \vec{r}' = 2\|\vec{r}\| \|\vec{r}'\|$. Rearranging: $\frac{\vec{r} \cdot \vec{r}'}{\|\vec{r}\|} = \|\vec{r}'\|$. Means that magnitude of position vector has to be a constant value in order for it to be \perp to derivative.

11 Planetary motion

- Law of ellipses – orbit of planet is ellipse with sun as focus
- Law of equal area in equal time – position vector pointing from sun to planet sweeps out equal area in equal time (so speed must increase/decrease)

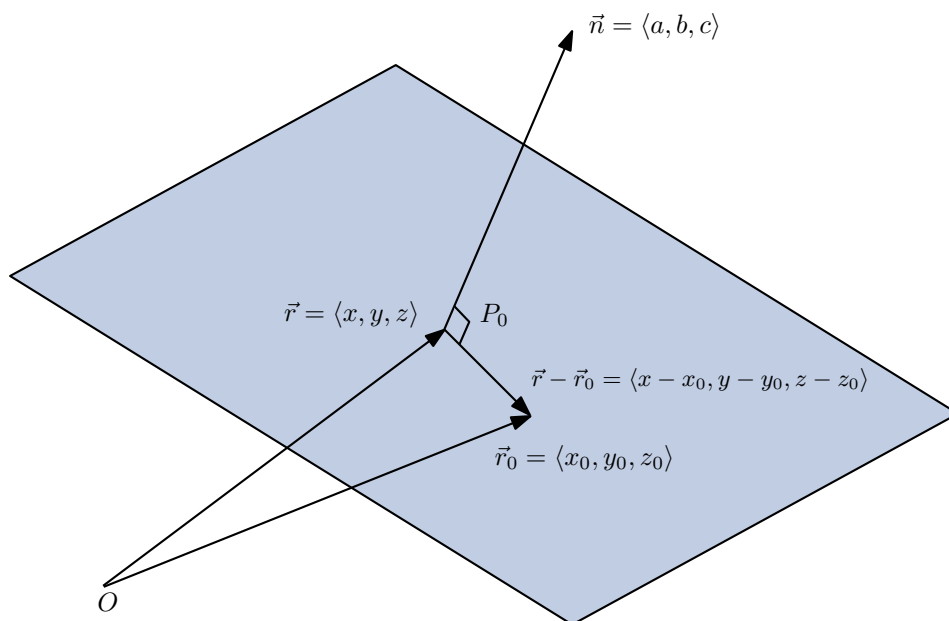
Can approximate the area swept in time by $\frac{dA}{dt} = \frac{1}{2} \|\vec{r}(t) \times \vec{r}'(t)\| = \frac{1}{2} \|\vec{J}\|$. The differential equation for each of Kepler's laws is: $\vec{r}''(t) = -\frac{k}{\|\vec{r}(t)\|^3} \vec{r}(t)$, so it is in the direction of $\vec{r}(t)$. Thus, differentiating $\frac{d\vec{J}}{dt} = \frac{d}{dt}(\vec{r}'(t) \times \vec{r}''(t)) = 0$.

11.1 Cross-product identities

Cross product identities:

- $\vec{u} \times (\vec{v} \times \vec{w}) = (u \cdot w)\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$
- $u \cdot (\vec{v} \times \vec{w}) = v \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v})$

12 Planes



If (x, y, z) is a point on the plane, then given $\vec{P}_0 = (x_0, y_0, z_0)$, $(x - x_0, y - y_0, z - z_0)$ is a vector on the plane perpendicular to \vec{n} , the normal vector. Thus, $(A, B, C) \cdot (x - x_0, y - y_0, z - z_0) = 0$ where A, B, C are vector coordinates of \vec{n} . With expansion:

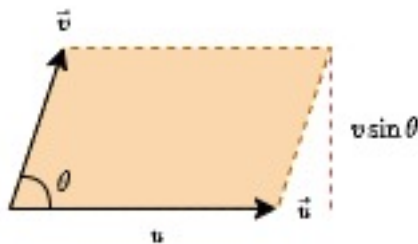
$$\begin{aligned} A(x - x_0) + B(y - y_0) + C(z - z_0) &= 0 \\ Ax + By + Cz = Ax_0 + By_0 + Cz_0 &= 0 \\ Ax + By + Cz &= \vec{n} \cdot \vec{P}_0 \end{aligned}$$

Note that (A, B, C) form coordinates of \vec{n} .

To find a plane containing 3 points $\vec{v}_1, \vec{v}_2, \vec{v}_3$, compute, for example $\vec{c}_1 = \vec{v}_3 - \vec{v}_1$ and $\vec{c}_2 = \vec{v}_2 - \vec{v}_1$. This finds 2 vectors in the plane. Then compute $\vec{c}_1 \times \vec{c}_2 = \vec{n}$.

The trace of a plane is the intersection of a plane \mathcal{P} with xy , xz , or yz coordinate planes. Can be found by setting respective variable to 0.

12.1 Cross-product rules and identities



Overview

- $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta$
- $\vec{a}, \vec{b} \perp \vec{a} \times \vec{b}$

Algebraic

- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $\vec{a} \times \vec{a} = \vec{0}$
- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- Distributive properties hold – preserve direction however
- $(\alpha \vec{a}) \times \vec{b} = \alpha(\vec{a} \times \vec{b})$

13 Graphs

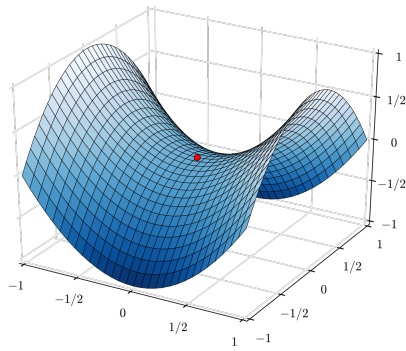
13.1 Multivariable functions

Function of n -variables is real-valued function with $f(x_1, \dots, x_n)$ with domain \mathcal{D} being a set of n -tuples (x_1, \dots, x_n) in \mathbb{R}^n , or where f is defined. Range of f is all values $f(x_1, \dots, x_n)$ for (x_1, \dots, x_n) in the domain.

13.2 Graphing multivariable functions

Traces are 2D curves obtained by intersection with planes parallel to coordinate plane.

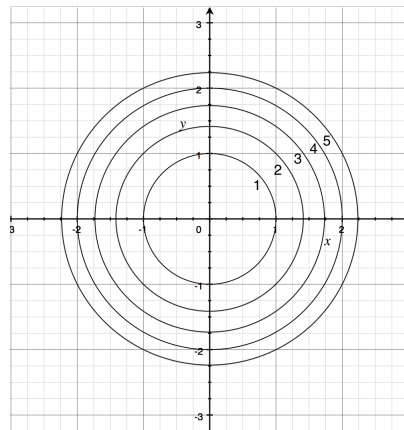
- Horizontal trace at height c – intersection of graph with plane $z = c$, so points (x, y, c) such that $f(x, y) = c$
- Vertical trace in plane $x = a$ – intersection of graph with vertical plane $x = a$ for all points $(a, y, f(a, y))$
- Vertical trace in plane $y = b$ – intersection of graph with vertical plane $y = b$ for all points $(x, b, f(x, b))$



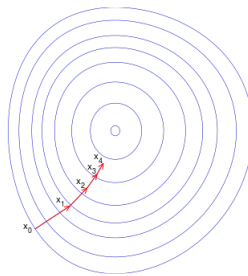
Saddle surface general form is $f(x, y) = x^2 - y^2$. The horizontal traces are hyperbolas of the form $c = x^2 - y^2$. Vertical traces are parabolas, as either x, y set to 0.

Linear functions in 2 variables are of the form $f(x, y) = mx + ny + r | m, n, r \in \mathbb{R}$.

13.3 Contour maps and level curves



Can specify a contour interval for each $z = c$ value. Is a 2D representation of level curves of $f(x, y)$ at an interval. Going along level curve means change in altitude is 0. Altitude has change of $\pm m$ (contour interval) when going up/down contour levels. Average ROC is $\Delta \text{elevation} / \Delta \text{distance}$. Path of steepest ascent follows the shortest possible segment from one contour line to another and always points in steepest direction.



14 Partial Derivatives

14.1 Definition

If $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by $f(x, y) = z$ and $P_0 = (a, b)$ is a point in the domain of f , then the partial derivative are:

- If $h : \mathbb{R} \rightarrow \mathbb{R}$ by $h(t) = f(t, b)$, then partial derivative with respect to x at P_0 is $h'(a)$ with following limit definition

$$\left. \frac{\partial f}{\partial x} \right|_{(a,b)} = f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(x+h, b) - f(x, b)}{h} = \lim_{x \rightarrow a} \frac{f(x, b) - f(a, b)}{x - a}$$

- If $g : \mathbb{R} \rightarrow \mathbb{R}$ by $g(t) = f(a, t)$ then partial derivative with respect to y at P_0 is $g'(b)$ with following limit definition

$$\left. \frac{\partial f}{\partial y} \right|_{(a,b)} = f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, y+h) - f(a, y)}{h} = \lim_{y \rightarrow b} \frac{f(a, y) - f(a, b)}{y - b}$$

Can be thought of as the intersection of the plane shifted by b with f , and the derivative of the resulting trace.

14.2 Linear approximation with planes

Let $z = f(x, y)$ be a scalar-valued function in \mathbb{R}^2 and $P_0 = (a, b)$ be a point in domain of f . Can have 2 slope vectors representing partial derivatives: $(1, 0, f_x(a, b))$ and $(0, 1, f_y(a, b))$. Can find a linear approximation by finding set of points in plane spanned by these vectors passing through $(a, b, f(a, b))$.

$$\vec{n} = (1, 0, f_x(a, b)) \times (0, 1, f_y(a, b)) = (-f_x(a, b), -f_y(a, b), 1)$$

Building the plane:

$$\begin{aligned} (x - a, y - b, z - f(a, b)) \cdot \vec{n} &= 0 \\ (x - a, y - b, z - f(a, b)) \cdot (-f_x(a, b), -f_y(a, b), 1) &= 0 \\ -f_x(a, b)(x - a) - f_y(a, b)(y - b) + z - f(a, b) &= 0 \end{aligned}$$

Thus,

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

14.3 Higher-order derivatives

Can be calculated using derivatives of f_x and f_y . Notation:

$$f_{xx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right), \quad f_{yy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right)$$

Can also have mixed partials (read as with respect to x or y):

$$f_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right), \quad f_{yx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$$

By Clairaut's Theorem, if f_{xy} and f_{yx} are both continuous functions on a disk D , then $f_{xy}(a, b) = f_{yx}(a, b) \forall (a, b) \in D$. Means that $f_{xyxy} = f_{xxyy} = f_{yyxx} = f_{yxyx}$.

15 Extrema

15.1 Definition and proofs

A function f has:

- local maximum at P_0 in domain if $f(P_0) > f(x, y) \forall (x, y)$ sufficiently near P_0
- local minimum at P_0 in domain if $f(P_0) < f(x, y) \forall (x, y)$ sufficiently near P_0

Sufficiently near: positive radius R used to build a circle centered at P_0 that traps points in domain with desired property (min or max). Global extrema redefine sufficiently near as in the domain of f .

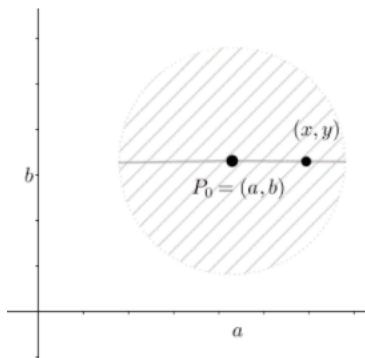
Critical point is defined as either of the following:

- $f_x(P_0) = f_y(P_0) = 0$
- either $f_x(P), f_y(P)$ does not exist

Method: find critical value y from f_y and x from f_x through cross-substitution.

Proof that if $f_x(P), f_y(P)$ both exist and there is a local max at P , then both partials are 0:

Define $g : \mathbb{R} \rightarrow \mathbb{R}$ to be single variable function from holding $y = b$ in f . Considering points sufficiently near:



Plugging into function g : $g(x) < f(x, b) \leq f(P_0) = g(a)$, so for x -values sufficiently near a , $g(x) \leq g(a)$, so it is a local max of g . Invoking theorem of extrema, either $g'(a) = 0$ or DNE. As $g'(a) = f_x(P_0)$, demonstrates that $f_x = 0$ or DNE for a local max. Same can be done for f_y .

Can minimize a function's distance to origin through distance formula. As a square root minimizes at its argument, can just concentrate on minimizing the argument.

Reiteration: If $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ has continuous first and second order partial derivatives then $f_{xy}(x, y) = f_{yx}(x, y)$. Continuous 2nd order partials: C^2 ; continuous n order partials: C^∞ .

Fermat's theorem proof (same as above): If $f(x, y)$ has a local min/max at $P = (a, b)$, then $P = (a, b)$ is a critical point of $f(x, y)$.

Assuming $f(x, y)$ has a local min at P , then this means that $f(x, y) \geq f(a, b)$ for (x, y) in the surrounding disk $D(r, P)$. For some $y = b$, the distance between any 2 x -values must be contained in the disk: $|x - a| < r$. Shows that $g(x) = f(x, b)$ has a local min at $x = a$, so $g'(a) = 0$ or DNE. Because $g'(a) = f_x(a, b)$, $f_x(a, b)$ is either 0 or DNE. As the same can be said for f_y , P is a critical point.

15.2 Second-derivative test

Given a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ that has continuous 2nd partials near a critical point P , define the discriminant of f at P to be:

$$D(P) = f_{xx}(P)f_{yy}(P) - (f_{xy}(P))^2$$

Then:

- If $D > 0$, P is a local extreme of f
 - $f_{xx}(P) > 0$ implies a local min
 - $f_{xx}(P) < 0$ implies a local max
- If $D < 0$ then there is neither a min or max at P , but an inflection point (saddle point)
- If $D = 0$ then there is no information about P
- If $D > 0$ and $f_{xx}(a, b) > 0$, then local minimum

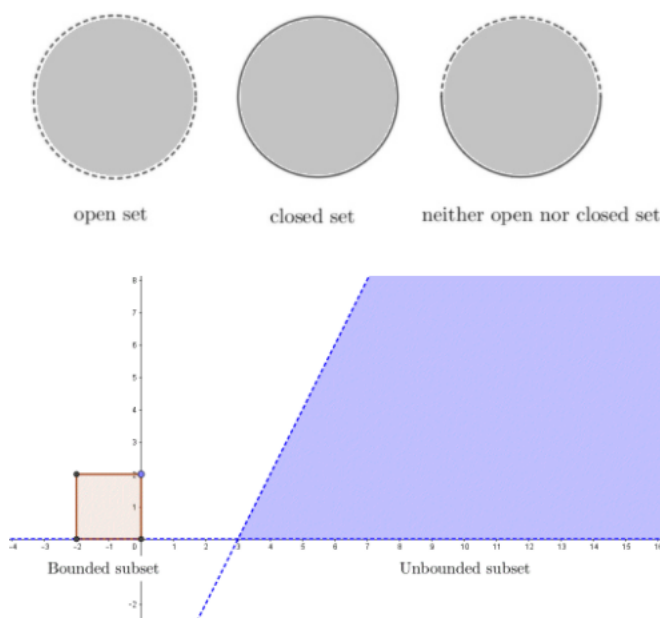
- If $D > 0$ and $f_{xx}(a, b) < 0$, then local maximum

Observe that $D(P) = \det \begin{pmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P) \end{pmatrix}$

15.3 Global extrema

If $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous on a closed and bounded subset of \mathbb{R}^2 then it has a global max/min on the subset (at critical point or along boundary)

A closed subset of \mathbb{R}^2 is one that contains the boundary. Boundary points have the property that any circle centered around them with positive radius will contain points in and out of the subset. Bounded subset is where any distance between 2 points in the set never exceeds some fixed bound $M \in \mathbb{R}$.



The boundary curve is denoted as ∂D where D is a disk. By parameterizing ∂D and using function composition to take this curve $\vec{c}(t)$ into f : $h(t) = f(\vec{c}(t))$. A min/max for f along boundary curve is a min/max of h . Plug resulting coordinates into f and determine global extrema.

16 Lagrange Multipliers

16.1 Theory

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\vec{c} : \mathbb{R} \rightarrow \mathbb{R}^2$ such that there is a composite function $h(t) = f(\vec{c}(t))$. Then, $h'(t) = f_x(\vec{c}(t))x'(t) + f_y(\vec{c}(t))y'(t) = (f_x(\vec{c}(t)), f_y(\vec{c}(t))) \cdot \vec{c}'(t)$. This brings definition of gradient vector, so that for $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, each point in domain is in domain of f the vector $(f_x(P_0), f_y(P_0))$:

$$\nabla f(P_0) = (f_x(P_0), f_y(P_0))$$

Rewriting chain rule:

$$\frac{d}{dt}f(\vec{c}(t_0)) = \nabla f(\vec{c}(t_0)) \cdot \vec{c}'(t_0)$$

Let $g(x, y)$ be boundary curve function and $f(x, y)$ be original.

Use a boundary curve which is level curve $c = 0$ which represents the boundary of the subset where global extrema can be found. If f achieves maximum at point P along curve, then either $g_x(P) = g_y(P) = 0$ ($\nabla g(P) = \vec{0}$) or there is a scalar λ (Lagrange multiplier) such that $f_x(P) = \lambda g_x(P)$ and $f_y(P) = \lambda g_y(P)$ ($\nabla f(P) = \lambda \nabla g(P)$). Can find locations of global extreme without parameterizing.

16.2 Proof

Suppose $f(P) \geq f(x, y) \forall (x, y)$ satisfying $g(x, y) = 0$. Parameterizing $g(x, y) = 0$ with $\vec{c}(t)$ such that $\vec{c}(0) = P$: Observe that $h(t) = f(\vec{c}(t))$ is from $\mathbb{R} \rightarrow \mathbb{R}$ with local max at $t = 0$. Thus, $h'(t) = 0$ or DNE. Taking derivative, $h'(t) = \nabla f(\vec{c}(t)) \cdot \vec{c}'(t)$. Then, $h'(0) = \nabla f(\vec{c}(0)) \cdot \vec{c}'(0) = 0$ or DNE. This dot product is either 0 or DNE because 1 or both vectors could not exist, or if they are \perp . Thus, $\nabla f(P) \perp \vec{c}'(0)$ and $\vec{c}'(0) \neq \vec{0}$ (because \vec{c} is assumed to be regular).

Showing $\nabla g(P) \perp \vec{c}'(0)$. Define $j(t) = g(\vec{c}(t))$. Thus, $j'(t) = \nabla g(\vec{c}(t)) \cdot \vec{c}'(t)$. Because $\vec{c}(t)$ is a parameterization of level curve $g(x, y) = 0$, it always outputs 0, so $j(t) = g(\vec{c}(t)) = 0$ always. Can then conclude that $j'(t) = \nabla g(\vec{c}(t)) \cdot \vec{c}'(t) = 0$. Plugging in $t = 0$ gives $\nabla g(\vec{c}(0)) \cdot \vec{c}'(0) = 0$ so $\nabla g(P) \perp \vec{c}'(0)$.

Because both vectors are perpendicular to $\vec{c}'(0)$, they are parallel – $\nabla f(P) = \lambda \nabla g(P)$. Or:

$$\begin{aligned} f_x(P) &= \lambda g_x(P) \\ f_y(P) &= \lambda g_y(P) \end{aligned}$$