Multivariable Calculus Reference

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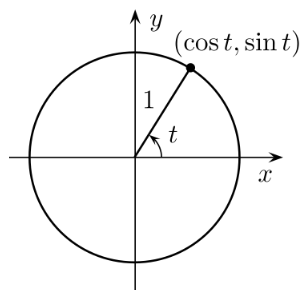
1 Paths whose image curve is a circle

1.1 Unit Circle

Unit circle is set of points in $\mathbb{R}^2$ defined as $C = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$. Ellipse is $C = \{(x, y) \in \mathbb{R}^2 | \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$ Has standard parameterization of $\vec{c}(t) = \big(\cos(t), \sin(t)\big)$. When parameterizing, always start from $t = 0$ reference unless otherwise given.



Properties

- Image of $\vec{c}$ is a closed curve (has no endpoints, plane is divided into ≥ 2 disjoint regions)
- Image of $\vec{c}$ is a simple curve; no self-intersection
- $\vec{c}(t)$ is an **injective path**; path is considered injective if $\vec{c}(t_1) = \vec{c}(t_2)$, which implies that $t_1 = t_2$ where these are on the open interval (a, b) even if $a = b$
- Orientation of $\vec{c}$ is counter-clockwise in traversal

1.2 Observations

$$\boxed{\vec{p}(t) = (a\cos(\pm(nt \pm \theta)) + x_0, b\sin(\pm(nt \pm \theta)) + y_0)}$$

If $-t$ for t, orientation is CW, CCW is t. If $a = b$, then curve is a circle of radius a or b, else an ellipse with horizontal and vertical radii. x_0 and y_0 simply shift the center coordinate. $n > 0 \in \mathbb{R}$ determines how many times the circle is traversed given $t \in [0, 2\pi]$, for example. θ is the phase shift. When changing direction of traversal, cannot have $a > b$ for $[a, b]$ so to decrease argument of sin or cos must have $-t$ for t. Starting out, $-t$ goes through the angle range and t is just a sign flip.

2 Paths whose image is a line or line segment in the plane

2.1 Line Parametrics

A line is a 1D subspace of $\mathbb{R}^2$, so $L = \{t\vec{m} | t \in \mathbb{R}\}$ for $\vec{m} \in \mathbb{R}^2$. $\vec{m} = \begin{bmatrix} m_x \\ m_y \end{bmatrix}$ is the **slope vector**. Path given by image of L:

$$\vec{c}(t) = (m_x t, m_y t), t \in \mathbb{R}$$

Can represent $\vec{c}(t) = t\vec{m}$ as well.

Lines Main Ideas

- Image of a line is a curve (e.g. $y = x$ represents image curve of $\vec{c}(t) = (t, t)$)
- Lines can have nonzero intercepts, so $\vec{c}(t) = t\vec{m}$ represents $y = 2x+1$. Line that has intercept vector $P_0 = (x_0, y_0) \parallel \vec{m} = (m_x, m_y)$ can be expressed as:

$$\boxed{\vec{c}(t) = (x_0 + tm_x, y_0 + m_y t) = \vec{P_0} + t\vec{m}}$$

  Note endpoint of $\vec{c}(t)$ is on image line (curve).

2.2 General Forms

2 parametric lines **collide** if they intersect and the point of intersection corresponds to the same t in both curves. If you set the parameter vector coordinates equal to each other and solve for t, a solution indicates they collide. Intersection is found by **eliminating** the parameter (solve for t in terms of either x or y and plug into the other).

General form of parameterized curve can be expressed as the following:

$$\boxed{\vec{c}(t) = (\frac{m_x}{\Delta t}(t-a) + x_0, \frac{m_y}{\Delta t}(t-a) + y_0)}$$

where Δt is the domain interval over $[a, b]$ and (x_0, y_0) represents the desired **starting coordinate**. This is important as when going in reverse, other coordinate can be used and slope might be negative. a is used in $(t-a)$ because everything is conventionally done with respect to starting coordinate.

3 Paths whose image curve is a line in R3

3.1 R3 parameterization

If $\vec{m}$ is a nonzero vector along L through origin in $\mathbb{R}^3$, then $L = \{t\vec{m} | t \in \mathbb{R}\}$; follows that $\vec{m} = (m_x, m_y, m_z)$, the slope or direction vector of the line. The basic parameterization is:

$$\vec{c}(t) = (m_x t, m_y t, m_z t)$$

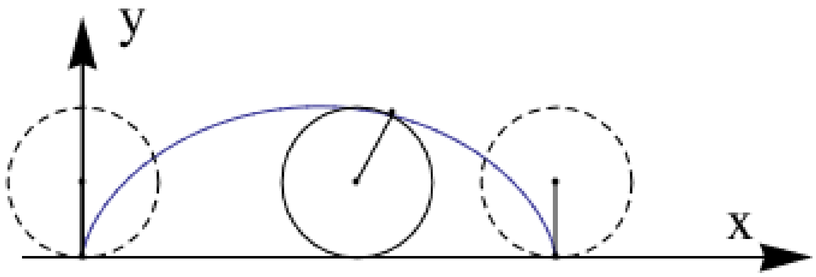
Basis vectors in $\mathbb{R}^3$ are $\vec{i}, \vec{j}, \vec{k}$. Rewriting parameterization:

$$\vec{c}(t) = (x_0 + m_x t)\vec{i} + (y_0 + m_y t)\vec{j} + (z_0 + m_z t)\vec{k}$$

2 lines $\vec{c}_1(t) = P_0 + \vec{m}_1 t$ and $\vec{c}_2(t) = Q_0 + \vec{m}_2 t$ are parallel if direction vectors are parallel ($\vec{m}_1 = k\vec{m}_2$). Collisions still exist. If neither parallel nor intersecting, considered as skew.

To determine skew, parallel, or coincide, use parameters s, t for each line and solve SOE. If same slope, rule out skew clearly, then check if $s, t \in \mathbb{R}$: if not, then parallel, if so, then they coincide. If intersecting and want to check if collide, some t must satisfy all relations.

4 Cycloid Problem



With radius 1 and passing through the origin:

$$\vec{c}(t) = (t - \sin t, 1 - \cos t)$$

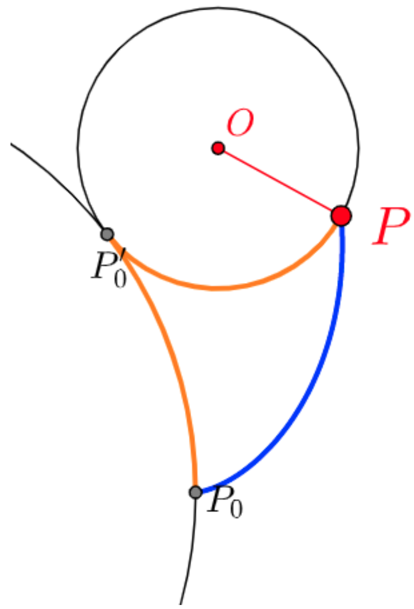
Observe that:

$$\vec{c}'(t) = (1 - \cos t, \sin t)$$

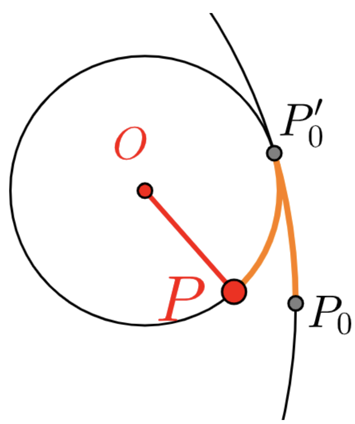
Can define the vector $\vec{u} = \begin{pmatrix} x'(t) \\ 0 \end{pmatrix}$ such that $\vec{u}$ is always horizontal and $||\vec{u}|| = |x'(t)|$. Reaches maximum value at $t \in [k\pi | k \in \mathbb{R}]$ and is has minimum cusp where it is 0 at $t \in [2k\pi | k \in \mathbb{R}]$. Thus, $x'(t) \geq 0$ always, as the x-coordinate is never decreasing.

Can also define the vector $\vec{v} = \begin{pmatrix} 0 \\ y'(t) \end{pmatrix}$ with the same properties. Reaches maximum value when $t \in [k\frac{\pi}{2} | k \in \mathbb{R}]$. Can change, as observe t when $\sin t < 0$ or > 0.

4.1 Hypercycloid Derivation



4.2 Hypocycloid Derivation



5 Velocity Vector

5.1 Definitions

Vector $\vec{u}(t_0) + \vec{v}(t_0)$ is he velocity vector to the curve $\vec{c}(t)$ at $t = t_0$.

Let $\vec{c}:\ [a,b] \to \mathbb{R}^n$ have a path $\vec{c}(t) = (x_1(t), x_2(t), x_3(t), \ldots, x_n(t))$ (let $x_i(t) : [a,b] \to \mathbb{R}$ for each i)

- If $t_0 \in [a,b]$, then $\vec{c}'\left(t_0\right) := \left(x_1'\left(t_0\right), x_2'\left(t_0\right), x_3'\left(t_0\right), \ldots, x_n'\left(t_0\right)\right)$; the velocity vector to $\vec{c}$ at t_0

- The path $\vec{c}'(t_0) := (x_1'(t_0), x_2'(t_0), x_3'(t_0), \ldots, x_n'(t_0))$; the velocity vector to $\vec{c}$ is referred to as velocity of $\vec{c}(t)$

Recall chain rule: if $y = f(x)$ where x is a function of t, $y'(t) = x'f'(x)$, not to be confused with product rule. Can write $f'(x) = \frac{y'(t)}{x'(t)}$

- If $\vec{p}(t) = \vec{c}(t) + \vec{r}(t)$, then $\vec{p}'(t) = \vec{c}'(t) + \vec{r}'(t)$
- If $g(t) = \vec{c}(t) \cdot \vec{r}(t)$, then $g'(t) = \vec{c}'(t) \cdot \vec{r}(t) + \vec{c}(t) \cdot \vec{r}'(t)$
- If $\vec{p}(t) = f(t)\vec{c}(t)$, then $\vec{p}'(t) = f'(t)\vec{c}(t) + f(t)\vec{c}'(t)$
- If $\vec{p}(t) = \vec{c}(t) \times \vec{r}(t)$, then $\vec{p}'(t) = \vec{c}'(t) \times \vec{r}(t) + \vec{c}(t) \times \vec{r}'(t)$
- If $\vec{p}(t) = \vec{c}(f(t))$, then $\vec{p}'(t) = f'(t)\vec{c}'(f(t))$
- If $g(t) = \|\vec{c}(t)\|$, then $g'(t) = \frac{\vec{c}(t) \cdot \vec{c}'(t)}{\|\vec{c}(t)\|}$

5.2 Tangent Line

Tangent line can be visualized as a base vector in standard position plus a velocity vector tangent to the tip which traces a shifted line in some interval. General formula with base vector $\vec{c}(t_0)$ and slope $\vec{c}'(t_0)$:

$$\boxed{\ell(t) = \vec{c}(t_0) + (t - t_0)\vec{c}'(t_0)}$$

6 Space Curves

- Projection into the xy plane is the path $(x(t), y(t), 0)$.
- Projection into the $xz-$ plane is the path $(x(t), 0, z(t))$.
- Projection into the yz plane is the path $(0, y(t), z(t))$.

7 Speed and Arclength

7.1 Speed

Speed of a parametric function in $\mathbb{R}^n$ is given by:

$$||\vec{c}'(t)|| = \sqrt{\sum_{i=1}^{n} c_i(t)^2}$$

(being the magnitude of the velocity vector)

7.2 Arclength

Arclength of a parametric function is given by:

$$S = \int_a^b ||\vec{c}'(t)||dt = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2 + \cdots}\, dt$$

Can approximate arclength as a sum of the lengths of secant vector approximations $\vec{s}_i = \vec{c}(t_i) - \vec{c}(t_{i-1})$:

$$\text{arclength} \approx \sum_{i=1}^{n} ||\vec{s}_i||$$

According to the MVT, there exists a $\hat{t}_i$ in (t_{i-1}, t_i) (open interval due to differentiability requirement) such that:

$$x'(\hat{t}_i) = \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}$$
$$y'(\hat{t}_i) = \frac{y(t_i) - y(t_{i-1})}{t_i - t_{i-1}}$$

This means that, since $\vec{s}_i$ is given as the difference between 2 points, being a secant:

$$\vec{s}_i = \big((t_i - t_{i-1})x(\hat{t}_i), (t_i - t_{i-1})y(\hat{t}_i)\big)$$
$$\vec{s}_i = (t_i - t_{i-1})\left(x'(\hat{t}_i), y'(\hat{t}_i)\right)$$
$$\vec{s}_i = (t_i - t_{i-1})\vec{c}\,'(\hat{t}_i)$$

Thus,

$$\text{arclength} \approx \sum_{i=1}^{n} ||\vec{s}_i||$$
$$\text{arclength} \approx \sum_{i=1}^{n} ||\Delta t\, \vec{c}\,'(\hat{t}_i)||$$
$$\text{arclength} \approx \sum_{i=1}^{n} \Delta t||\vec{c}\,'(\hat{t}_i)||$$

Can define the arclength differential as follows:

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$$

Can just define arclength as arclength $= \int \mathrm{ds}$

7.3 Arclength Parameterization

Higher the speed of a curve, farther the points are spaced apart. An arclength parametrization of a curve is a path whose image is the desired curve and whose speed is constantly one. Or, $\vec{c}\colon [a,b] \to \mathbb{R}^n$ with $||\vec{c}\,'(t)|| = 1$ for $t \in [a,b]$. If a curve is not an arclength parameterization, then can do $\frac{\vec{c}(t)}{||\vec{c}'(t)||}$ but only dividing the coefficients (slopes).

When speed is variable, is difficult to define arclength parameterization. Thus, can define displacement to be $s(t) = \int_a^b (t)dt$. If $v(t)! \neq 0$, then s is injective because according to FTC, $s\,'(t) = v(t)$. By definition, $v\,'(t) \geq 0$ always since it is composed of a radical, so it must be **increasing**. Thus, if $t_1 = t_2$, $s(t_1) \neq s(t_2)$. Arclength parameterization:

$$s(t) = \int_0^t ||\vec{c}'(u)||du$$

This means that s is invertible, so can solve for t to get $t = \varphi(s)$. An arclength parameterization can be found by:

$$\boxed{\vec{p}(s) = \vec{c}(\varphi(s))}$$

8 Curvature

8.1 Proofs

Recall that to make an arclength parameterization accumulate the magnitudes of infinitesimal velocity vectors:

$$s(t) = \int_a^t ||\vec{r}'(u)||du$$

Given some curve $\vec{r}(t)$, define an arclength parameterization by $\vec{r}(g(s)) \to \vec{r}_1(s)$, so $\vec{r}$ is defined in terms of s. The unit tangent vector $\vec{T}_1(s)$ is then $\frac{\vec{r}_1\,'(s)}{||\vec{r}_1\,'(s)||} = \vec{r}_1\,'(s)$.

$$\begin{aligned}
\vec{T}_1(s) &= \vec{r}_1\,'(s) \\
&= \frac{d}{ds}\vec{r}_1(s) \\
&= \frac{d}{ds}\vec{r}(g(s)) \\
&= \vec{r}\,'(g(s)) \cdot g\,'(s) = \vec{r}\,'(t) \cdot \frac{dt}{ds} \\
&= \frac{\vec{r}\,'(t)}{\frac{ds}{dt}} = \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||}
\end{aligned}$$

This means that $\vec{T}_1(s) = T(t)$

Continuing, to find curvature $\kappa(t)$:

$$
\begin{aligned}
\vec{T_1}'(s) &= \frac{d}{ds}\vec{T}(t) \\
&= \frac{d}{ds}\vec{T}(g(s)) \\
&= \vec{T}'(t) \cdot \frac{dt}{ds} \\
&= \frac{\vec{T}'(t)}{\frac{ds}{dt}} \\
&= \frac{\vec{T}'(t)}{||\vec{r}'(t)||}
\end{aligned}
$$

Thus, $\boxed{\kappa(t) = \frac{\vec{T}'(t)||}{||\vec{r}'(t)||}}$

8.2 Definition

Given a curve C parameterized with arclength by the path $\vec{c}\colon [a,b] \to \mathbb{R}^n$, curvature is defined as:

$$\boxed{\kappa(s) = ||\vec{T}'(s)||}$$

where $\vec{c}'(s) \neq 0$ and $\vec{T}(s) = \frac{\vec{c}'(s)}{||\vec{c}'(s)||}$ (normalized slope vector).

A loose geometric interpretation is that a greater $\kappa(s)$ implies more curvature, that is, the curve is changing at a greater rate there. When $\vec{c}'(s) \neq 0$ is always true for a curve, it is **regular**. Is defined in terms of arclength parameterization so curvature is an intrinsic property of the curve independent of parameterization.

Formula for curvature at the point $\vec{c}(t)$:

$$\boxed{\kappa(t) = \frac{||T'(t)||}{||\vec{c}\,'(t)||} = \frac{||\vec{c}\,'(t) \times \vec{c}\,''(t)||}{||\vec{c}\,'(t)||^3}}$$

9 Motion in 3D space

Given a path $\vec{c}\colon \mathbb{R} \to \mathbb{R}^3$ with $\vec{c}(t) = (x(t), y(t), z(t))$, then we have defined:

- $\cdot\vec{v}(t) = \vec{c}'(t) = (x'(t), y'(t), z'(t))$ is also path in $\mathbb{R}^3$ called the velocity of $\vec{c}$
- $\vec{a}(t) = \vec{v}'(t) = \vec{c}''(t) = (x''(t), y''(t), z''(t))$ is also a path in $\mathbb{R}^3$ called the acceleration of $\vec{c}$
- $v(t) = \|\vec{v}(t)\| = \|\vec{c}'(t)\|$ is a scalar valued function on $\mathbb{R}$ (that's a fancy way of saying the domain and codomain of this function are both $\mathbb{R}$) called the speed of $\vec{c}$

- $\vec{T}(t) = \frac{\vec{c}\,'(t)}{v(t)}$ is also a path in $\mathbb{R}^3$ called the unit tangent to $\vec{c}$
- $\kappa(t) = \frac{\vec{T}'(t)}{v(t)} = \frac{\|\vec{c}\,'(t)\times\vec{c}\,''(t)\|}{\|\vec{c}\,'(t)\|^3}$ is a scalar valued function on $\mathbb{R}$ called the curvature of $\vec{c}$

Note that $\vec{T}\cdot\vec{T} = ||\vec{v}||^2 = 1$. Computing the derivative, $\frac{\mathrm{d}}{\mathrm{dt}}\vec{T}\cdot\vec{T} = 2\vec{T}\cdot\vec{T}' = 0$ This means that $\vec{T}\perp\vec{T}\,'$.

Define $\boxed{\vec{N}(t) = \dfrac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|}}$ as the unit normal vector, which is the unit tangent to the unit tangent.
From observation, $\vec{T}\perp\vec{N}$. Fact: the acceleration vector always lies in the plane spanned by $\vec{N}$ and $\vec{T}$.

Acceleration $\vec{a}(t)$ is thus split component-wise into a_T from $\vec{T}$ and a_N from $\vec{N}$:

$$\boxed{a_T = v\,'(t) = \frac{a(\vec{t})\cdot\vec{v}(t)}{v(t)}}$$

$$\boxed{a_N = \kappa(t)v(t)^2 = \frac{||\vec{a}(t)\times\vec{v}(t)||}{v(t)} = \sqrt{\|\vec{a}(t)\|^2 - |a_T|^2}}$$

10 Derivatives of parameterized curves

10.1 Arclength parameterization derivation

Take the following function:

$$\vec{r}(t) = \langle x(t), y(t)\rangle$$

An arclength parameterization is achieved with the following computation:

$$s = \int_0^t \sqrt{x\,'(u)^2 + y\,'(u)^2}\, du$$

Can say that $t = g(s)$, so the arclength parameterization, which is the path in terms of s:

$$\vec{r}_1(s) = \langle x(g(s)), y(g(s))\rangle$$

Taking the derivative by the chain rule:

$$\begin{aligned}\vec{r}_1\,'(s) &= \langle x\,'(g(s))\cdot g\,'(s), y\,'(g(s))\cdot g\,'(s)\rangle \\ &= g\,'(s)\langle x\,'(t), y\,'(t)\rangle\end{aligned}$$

Note that $g\,'(s) = \frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{||\vec{r}\,'(t)||}$ by taking the derivative of the integral for arclength:

$$\vec{r}_1{}'(s) = \frac{1}{||\vec{r}'(t)||}\langle x'(t), y'(t)\rangle = \vec{T}(t)$$

Following from this, $g'(s) = \frac{1}{||\vec{r}'(g(s))||}$ so $g''(s) = -\frac{1}{||\vec{r}'(g(s))||^2} \cdot g'(s) = -\frac{1}{||\vec{r}'(t)||^3}$

10.2 Orthogonal derivative and position vectors

Observe that $\vec{r} \cdot \vec{r} = ||\vec{r}||^2$. Thus, $\frac{d}{dt}[\vec{r} \cdot \vec{r}] = 2\vec{r} \cdot \vec{r}' = 2||\vec{r}||||\vec{r}||'$. Rearranging: $\frac{\vec{r} \cdot \vec{r}}{||\vec{r}||} = ||\vec{r}||'$. Means that magnitude of position vector has to be a constant value in order for it to be $\perp$ to derivative.

11 Planetary motion

- Law of ellipses – orbit of planet is ellipse with sun as focus
- Law of equal area in equal time – position vector pointing from sun to planet sweeps out equal area in equal time (so speed must increase/decrease)

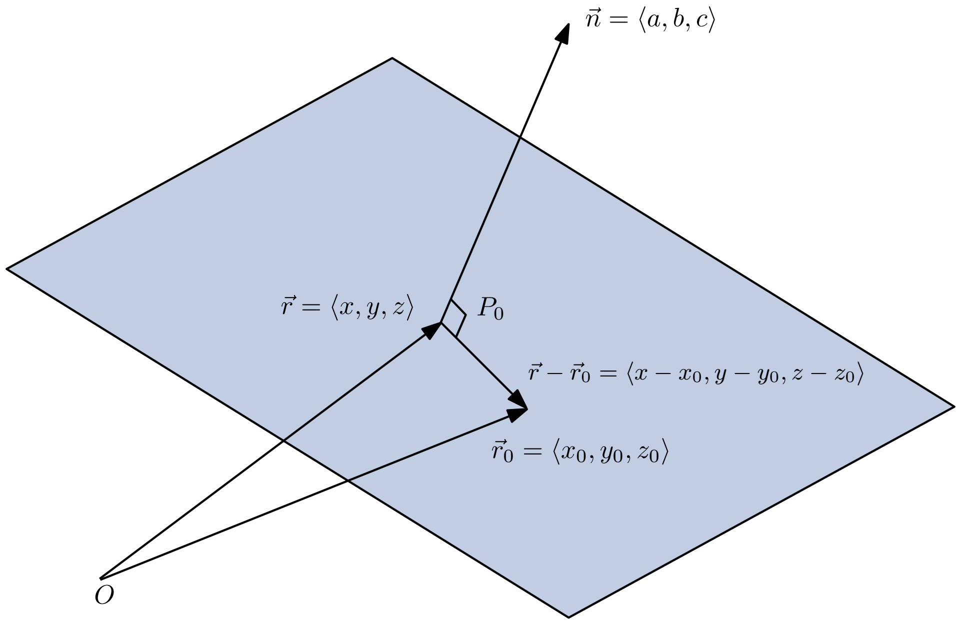
Can approximate the area swept in time by $\frac{dA}{dt} = \frac{1}{2}||\vec{r}(t) \times \vec{r}'(t)|| = \frac{1}{2}||\vec{J}||$. The differential equation for each of Kepler's laws is: $\vec{r}''(t) = -\frac{k}{||\vec{r}(t)||^3}\vec{r}(t)$, so it is in the direction of $\vec{r}(t)$. Thus, differentiating $\frac{d\vec{J}}{dt} = \frac{d}{dt}(\vec{r}'(t) \times \vec{r}''(t)) = 0$.

11.1 Cross-product identities

Cross product identities:

- $\boxed{\vec{u} \times (\vec{v} \times \vec{w}) = (u \cdot w)\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}}$
- $\boxed{u \cdot (\vec{v} \times \vec{w}) = v \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v})}$

12 Planes



If (x, y, z) is a point on the plane, then given $\vec{P}_0 = (x_0, y_0, z_0)$, $(x - x_0, y - y_0, z - z_0)$ is a vector on the plane perpendicular to $\vec{n}$, the normal vector. Thus, $(A, B, C) \cdot (x - x_0, y - y_0, z - z_0) = 0$ where A, B, C are vector coordinates of $\vec{n}$. With expansion:

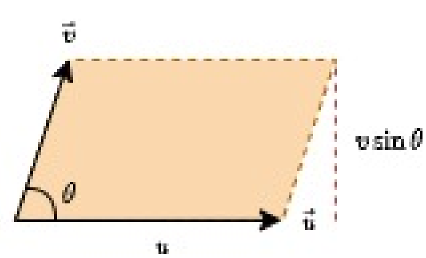
$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$
$$Ax + By + Cz = Ax_0 + By_0 + Cz_0 = 0$$
$$Ax + By + Cz = \vec{n} \cdot \vec{P}_0$$

Note that (A, B, C) form coordinates of $\vec{n}$.

To find a plane containing 3 points $\vec{v}_1, \vec{v}_2, \vec{v}_3$, compute, for example $\vec{c}_1 = \vec{v}_3 - \vec{v}_1$ and $\vec{c}_2 = \vec{v}_2 - \vec{v}_1$. This finds 2 vectors in the plane. Then compute $\vec{c}_1 \times \vec{c}_2 = \vec{n}$.

The trace of a plane is the intersection of a plane $\mathcal{P}$ with xy, xz, or yz coordinate planes. Can be found by setting respective variable to 0.

12.1 Cross-product rules and identities



Overview

- $||\vec{a} \times \vec{b}|| = ||\vec{a}|| ||\vec{b}|| \sin \theta$
- $\vec{a}, \vec{b} \perp \vec{a} \times \vec{b}$

Algebraic

- $\vec{a} \times \vec{b} = \vec{0}$
- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- Distributive properties hold – preserve direction however
- $(\alpha\vec{a}) \times \vec{b} = \alpha(\vec{a} \times \vec{b})$

13 Graphs

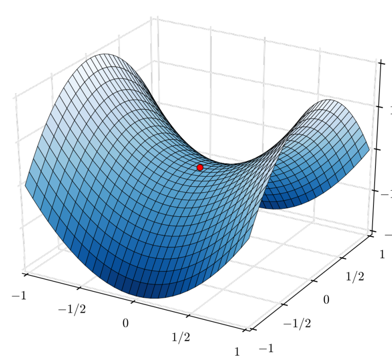
13.1 Multivariable functions

Function of n-variables is real-valued function with $f(x_1, \cdots, x_n)$ with domain $\mathcal{D}$ being a set of n-tuples $(x_1, \cdots, x_n)$ in $\mathbb{R}^n$, or where f is defined. Range of f is all values $f(x_1, \cdots, x_n)$ for $(x_1, \cdots, x_n)$ in the domain.

13.2 Graphing multivariable functions

Traces are 2D curves obtained by intersection with planes parallel to coordinate plane.

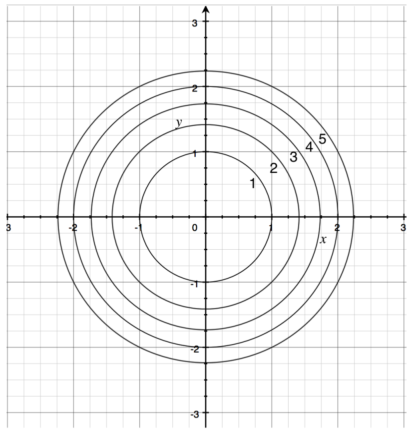
- Horizontal trace at height c – intersection of graph with plane $z = c$, so points (x, y, c) such that $f(x, y) = c$
- Vertical trace in plane $x = a$ – intersection of graph with vertical plane $x = a$ for all points $(a, y, f(a, y))$
- Vertical trace in plane $y = b$ – intersection of graph with vertical plane $y = b$ for all points $(x, b, f(x, b))$



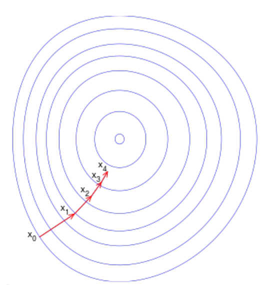
Saddle surface general form is $f(x, y) = x^2 - y^2$. The horizontal traces are hyperbolas of the form $c = x^2 - y^2$. Vertical traces are parabolas, as either x, y set to 0.

Linear functions in 2 variables are of the form $f(x, y) = mx + ny + r | m, n, r \in \mathbb{R}$.

13.3 Contour maps and level curves



Can specify a contour interval for each $z = c$ value. Is a 2D representation of level curves of $f(x, y)$ at an interval. Going along level curve means change in altitude is 0. Altitude has change of $\pm m$ (contour interval) when going up/down contour levels. Average ROC is Δelevation/Δdistance. Path of steepest ascent follows the shortest possible segment from one contour line to another and always points in steepest direction.



14 Partial Derivatives

14.1 Definition

If $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x, y) = z$ abd $P_0 = (a, b)$ is a point in the domain of f, then the partial derivative are:

- If $h : \mathbb{R} \to \mathbb{R}$ by $h(t) = f(t, b)$, then partial derivative with respect to x at P_0 is $h'(a)$ with following limit definition

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a, b) = \lim_{h \to 0} \frac{f(x + h, b) - f(x, b)}{h} = \lim_{x \to a} \frac{f(x, b) - f(a, b)}{x - a}$$

- If $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = g(t, b)$ then partial derivative with respect to y at P_0 is $g'(b)$ with following limit definition

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a,b) = \lim_{h\to 0}\frac{f(a,x+h)-f(a,b)}{h} = \lim_{y\to b}\frac{f(a,y)-f(a,b)}{y-b}$$

Can be thought of as the intersection of the plane shifted by b with f, and the derivative of the resulting trace.

14.2 Linear approximation with planes

Let $z = f(x,y)$ be a scalar-valued function in $\mathbb{R}^2$ and $P_0 = (a,b)$ be a point in domain of f. Can have 2 slope vectors representing partial derivatives: $(1,0,f_x(a,b))$ and $(0,1,f_y(a,b))$. Can find a linear approximation by finding set of points in plane spanned by these vectors passing through $(a,b,f(a,b))$.

$$\vec{n} = (1,0,f_x(a,b)) \times (0,1,f_y(a,b)) = (-f_x(a,b), -f_y(a,b), 1)$$

Building the plane:

$$\begin{aligned}(x-a, y-b, z-f(a,b)) \cdot \vec{n} &= 0\\ (x-a, y-b, z-f(a,b)) \cdot (-f_x(a,b), -f_y(a,b), 1) &= 0\\ -f_x(a,b)(x-a) - f_y(a,b)(y-b) + z - f(a,b) &= 0\end{aligned}$$

Thus,

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

14.3 Higher-order derivatives

Can be calculated using derivatives of f_x and f_y. Notation:

$$f_{xx} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}),\ f_{yy} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y})$$

Can also have mixed partials (read as with respect to x or y):

$$f_{xy} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}),\ f_{yx} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})$$

By Clairaut's Theorem, if f_{xy} and f_{yx} are both continuous functions on a disk D, then $f_{xy}(a,b) = f_{yx}(a,b)\ \forall (a,b) \in D$. Means that $f_{xyxy} = f_{xxyy} = f_{yyxx} = f_{yxyx}$.