Linear Algebra Reference

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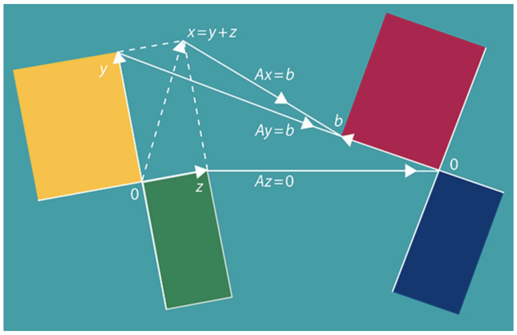

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1 Row and Column Picture

1.1 Row picture

Involves viewing matrix as linear equations graphed on a line or plane. Take the example $A\vec{x} = \vec{b}$ below:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ 10 \end{bmatrix}$$

This can be viewed as the following system:

$$\begin{cases} 1x + 2y + 3z = 8 \\ 3x + 4y + 5z = 9 \\ 4 + 5y + 6z = 10 \end{cases}$$

1.2 Column picture

Involves viewing this setup as a linear combination of column vectors. Take $A\vec{x} = \vec{b}$ again:

$$x \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + y \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} + z \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ 10 \end{bmatrix}.$$

1.3 Visualization in Space and Solutions

1.3.1 2D space

In $\mathbb{R}^2$, the equations form a line. Independent column vectors means infinite linear combinations of these to get a set of $\vec{b}$ in $\mathbb{R}^2$. If one column vector is dependent on another, they are parallel and various combinations of $\vec{b}$ are on a line.

1.3.2 3D space

The equations form a plane in $\mathbb{R}^3$. If column vectors independent, infinite linear combination of $\vec{b}$ exist in 3D space. If one vector is a scaled combination of another and the third is independent, then solutions lie on a line. If all three are interdependent, the solution is on a line.

2 Matrix Multiplication

2.1 Row and Column Swapping

Can define elementary row operations in the identity matrix.

2.1.1 Swapping rows

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Modifier B is always on **left**.

2.1.2 Swapping columns

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Modifier B is always on **right**.

2.2 Elimination steps

Performing elimination:

$$E_{2,1}A + E_{3,2}A = (E_{2,1}E_{3,2})A = U$$

Elimination algorithm:

- $E_{2,1}$ is the pivot. Swap with R_2 if 0 (and $(2,1)$ is nonzero).
- $E_{3,2}$ involves getting $(2,2)$ as a pivot assuming nonzero to get $(3,2)$ as 0
- Result is invertible and non-singular, where U is upper-triangular

Matrix multiplication is not necessarily commutative but always associative.

2.3 Matrix Multiplication Facts

If A is an $m \times n$ matrix and B is $n \times p$, then $AB = C$ must be $m \times p$. Standard method would be to take dot products by row and column. By column: Columns of C are combinations of columns of A. By row: Rows of C are combinations of rows of B.

2.4 Example (Row)

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

3 Factorization into A=LU

3.1 Notation

E_{21} is the location at row 2 and column 1, used to eliminate this value.

3.2 Inverse

$$AA^{-1} = I = A^{-1}A$$

Matrix multiplication is not commutative:

$$\left(AB^{-1}\right)\left(BA^{-1}\right) = AIA^{-1} = I$$

Transpose inverse fact:

$$\boxed{(A^{-1})^T A^T = I}$$

3.3 Concept

Given $E_{21}A = U$, where U is upper-triangular, $E_{21}^{-1}A = E_{21}^{-1}U$ gives:

$$\boxed{A = LU \text{ where } L = E_{21}^{-1}}$$

4 Linear Transformations

4.1 Rules and Notation

Domain is the input space and codomain is the output space.

- $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$
- $T(c\vec{v}) = cT(\vec{v})$

Thus,

$$\boxed{T(c\vec{v} + d\vec{w}) = cT(\vec{v}) + dT(\vec{w})}$$

where $c, d \in \mathbb{R}$ and $\vec{v}, \vec{w} \in \mathbb{R}^n$.

Notation given from $\mathbb{R}^m$ to $\mathbb{R}^n$:

$$T : \mathbb{R}^m \to \mathbb{R}^n.$$

4.2 Nonlinear examples

- $S : R^2 \to \mathbb{R}^2$ where $S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$
- $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = mx + b$

A transformation represented by the product of some matrix A and the column vector input $\vec{x}$ is always a linear transformation.

5 Inverse Matrices

5.1 Basic Facts

- If square matrix A is invertible (or inverse exists), then $\boxed{A^{-1}A = AA^{-1} = I}$.
- Can test invertibility of matrix using elimination, i.e. the $n \times n$ matrix A must have n nonzero pivots.
- If $\det(A) \neq 0$, then A is invertible.

5.2 Computing inverses

Can compute inverses with Gauss-Jordan, eliminating $[A\ I]$ to $[I\ A^{-1}]$. If a matrix is invertible, then solution to $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$.

A 2×2 matrix is only invertible if $ad - bc \neq 0$:

$$\boxed{A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}.$$

Matrix inversion occurs in reverse order:

$$\boxed{(ABC)^{-1} = C^{-1}B^{-1}A^{-1}}.$$

6 Linear Transformations and Inverse Matrices

6.1 Example with transformation

The 2×2 matrix A with property $R_\theta(\vec{v}) = A\vec{v}$ rotates the vector by θ. Using the unit circle to find the coordinates using the basis vectors $\vec{e}_1$ and $\vec{e}_2$:

$$R_\theta\left(\vec{e}_1\right) = \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array}\right]$$

$$R_\theta\left(\vec{e}_2\right) = \left[\begin{array}{c} -\sin\theta \\ \cos\theta \end{array}\right]$$

This results in A:

$$\left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$$

Finding the inverse of this is simply rotating back by θ, so finding R_θ^{-1}:

$$\left[\begin{array}{cc} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{array}\right] = \left[\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array}\right]$$

7 Linear Transformations in Geometry

7.1 Rotations

Any matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a^2 + b^2 = 1$. Thus, $\theta = \tan^{-1}(\frac{b}{a})$, or by any other trigonometric relation.

7.2 Scaling and dilation

Horizontal scaling affects the x-component:

$$T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}kx\\y\end{bmatrix}\implies A=\begin{bmatrix}k&0\\0&1\end{bmatrix}$$

Vertical scaling affects the y-component:

$$T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\ky\end{bmatrix}\implies A=\begin{bmatrix}1&0\\0&k\end{bmatrix}$$

Dilation is scaling by k for both x and y:

$$T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}kx\\ky\end{bmatrix}\implies A=\begin{bmatrix}k&0\\0&k\end{bmatrix}$$

7.3 Normalizing a vector

Can make any vector into a unit vector parallel to the original:

$$\boxed{\vec{u}=\frac{\vec{v}}{||\vec{v}||}}$$

The magnitude of a unit vector is always 1 ($||\vec{u}||=1$).

7.4 Projections

x-axis:

$$T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\0\end{bmatrix}$$

y-axis:

$$T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\y\end{bmatrix}$$

Mathematically:

$$||\text{proj}_l(\vec{v})||=||\vec{v}||\cos\theta$$

The dot product:

$$\boxed{\vec{a}\cdot\vec{b}=||\vec{a}||||\vec{b}||\cos\theta}$$

Unit vector $\vec{u}$ given by the following because the line l can be represented by $\begin{bmatrix}1\\m\end{bmatrix}$:

$$\vec{u}=\frac{1}{\sqrt{1+m^2}}\begin{bmatrix}1\\m\end{bmatrix}$$

The projection matrix, onto a line of slope m:

$$\text{proj}_l(\vec{v}) = (\vec{u} \cdot \vec{v})\vec{u} = \begin{bmatrix} \frac{v_1+v_2m}{1+m^2} \\ \frac{v_1m+v_2m^2}{1+m^2} \end{bmatrix}$$

Thus,

$$A = \begin{bmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} \end{bmatrix}$$

General projection matrix given $a^2 + b^2 = 1$:

$$\boxed{A = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}}$$

7.5 Reflections

Given by:

$$\text{refl}_l(\vec{v}) = 2\text{proj}_l(\vec{v}) - \vec{v}$$

Has the matrix A:

$$A = \begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{bmatrix}$$

If $a^2 + b^2 = 1$:

$$\boxed{A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}}$$

7.6 Shear

Horizontal:

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Vertical:

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

8 Inverse of a Linear Transformation

8.1 Definition in I/O space

- Each item in output receives **at most** 1 input $\implies$ injectivity
- Each item in output receives **at least** 1 input $\implies$ surjectivity

- If **both conditions are satisfied** $\implies$ bijectivity

Invertibility is therefore synonymous with bijectivity.

8.2 Conclusions

Injectivity concludes that $\text{rank}(A) = m$, where A is $n \times m$. This is because there must be a leading one in each column.

Surjectivity concludes that the last row in $\text{rref}(A)$ is $0\ 0 \cdots 0\ 1$. Thus there must be no rows of 0 in $\text{rref}(A)$, so all invertible matrices are square. Also an invertible matrix is **nonsingular** and an invertible matrix is **singular**.

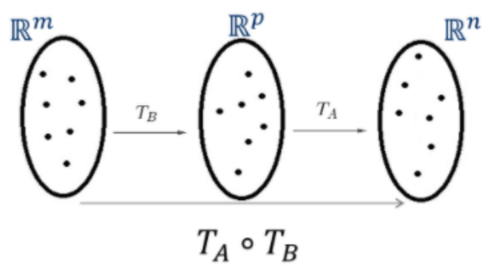
9 The Matrix Product

9.1 Composition

Can define the following (linear) transformation:

$$\boxed{T_C(\vec{x}) = T_A(T_B(\vec{x})) = (T_A \circ T_B)(\vec{x})}$$

Following diagram represents the composition:



Would imply that A is $n \times p$, B is $p \times m$, and AB is $n \times m$. Can define the following:

$$\boxed{\text{The } i^{th} \text{ column of the matrix } AB \text{ is the matrix-vector product } A(i^{th} \text{ column of the matrix } B)}$$

9.2 Proofs

Claim: The product of 2 invertible matrices must be an invertible matrix.
Proof: Given that $(AB)(AB)^{-1} = I_n$:

$$
\begin{aligned}
&(AB)(AB)^{-1} = I_n \\
&A(B(AB)^{-1}) = I_n \\
&A^{-1}A(B(AB)^{-1}) = A^{-1}I_n \\
&B(AB)^{-1} = A^{-1} \\
&B^{-1}B(AB)^{-1} = B^{-1}A^{-1} \\
&(AB)^{-1} = B^{-1}A^{-1}
\end{aligned}
$$

Claim: If $(AB)^{-1}$ exists, then A and B are both invertible.
Proof: Given that $(AB)(AB)^{-1} = I_n$ and $(AB)^{-1}(AB) = I_n$:

$$
\begin{aligned}
&A(B(AB)^{-1}) = I_n \\
&((AB)^{-1}A)B = I_n \\
&\boxed{\therefore \ \exists \ A^{-1},\ B^{-1} \in \mathbb{R}^n}
\end{aligned}
$$

9.3 Properties

- Associativity: $(AB)C = A(BC)$
- Distribution: $A(B + C) = AB + BC$
- Respects scalar multiplication: $(kA)B = k(AB) = A(kB)$

10 Transposes, Permutations, Spaces

10.1 Permutations

Function to make row exchanges. Elimination with row exchanges:

$$A = LU \implies PA = LU$$

Works for any invertible A.

$$P = \text{identity with reordered rows (exchanges)}$$

Count of possible reorderings ($n \times n$ permutations): $n! = n(n-1)\cdots 3(2)(1)$.

$$\boxed{P^{-1} = P^T \text{ and } P^TP = I}$$

Defining a transpose, or flip over diagonal:

$$(A^T)_{ij} = A_{ji}$$

For symmetric matrices, transpose does not cause change; $A^T = A$. If two rectangular matrices R^T and R give a square matrix, then R^TR is always symmetric.

$$\boxed{(R^TR)^T = R^TR^{TT} = R^TR}$$

10.2 Vector Spaces and Subspaces

Examples: $\mathbb{R}^2$ is all vectors in 2D space, $x-y$ plane: $\begin{bmatrix}3\\2\end{bmatrix}$. $\mathbb{R}^3$ is all vectors with 3 components. All combinations of vectors in $\mathbb{R}^n$ yield a result in that space $\mathbb{R}^n$.

$$\boxed{\mathbb{R}^n \text{ is all column vectors with } n \text{ components.}}$$

The origin exists to allow for scalar multiplication and addition of vectors. Every vector space has a $\vec{0}$.

10.2.1 Subspaces

If a vector space is defined as 1st quadrant in $\mathbb{R}^2$, then multiplying by a negative scalar k removes the result from that space, so it is not **closed** under that operation, so this is not a vector space. Vector space must be closed under linear combinations. Thus, subspace in $\mathbb{R}^2$ is all multiples of that vector, a line and the line must go through $\vec{0}$. Every subspace must contain $\vec{0}$.

Subspaces of $\mathbb{R}^2$:

- All of $\mathbb{R}^2$
- Any line through $\vec{0}_2$ or L
- Just $\vec{0}_2$ or Z

Similarly, for $\mathbb{R}^3$ can have $\mathbb{R}^3$, plane, line, $\vec{0}_3$.

Given $A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$, all linear combinations of these columns form a subspace. This is called **column space**, $C(A)$. This would form a plane in $\mathbb{R}^3$. Thus, the column space is a subspace.

11 Image and Kernel

11.1 Defining Image and Kernel

11.1.1 Image

Of a function, the set of vectors in the codomain hit by the domain. The image of a function $f: \mathbb{R}^m \to \mathbb{R}^n$ is:

$$\boxed{\text{Im}(f) = \{\vec{y} \in \mathbb{R}^n | \exists\, \vec{x} \in \mathbb{R}^m \text{ s.t. } f(\vec{x}) = \vec{y}\}}$$

Similar in concept to the range of a function in non-linear context.

11.1.2 Kernel

Set of vectors in the domain that are mapped to $\vec{0}$ in the codomain. Kernel of a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is:

$$\boxed{\text{Ker}(f) = \{\vec{x} \in \mathbb{R}^m | f(\vec{x}) = \vec{0}\}}$$

Analogous to the roots/zeros of a polynomial.

11.2 Examples

Following linear transformation's image lives in $\mathbb{R}^2$:

$$T(\vec{x}) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \vec{x}$$

Following linear transformation's image forms a plane that is $\mathbb{R}^2$:

$$T(\vec{x}) = \begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 2 \end{pmatrix} \vec{x}$$

11.3 Span

If A is an $n \times m$ matrix, then image of $T(\vec{x}) = A\vec{x}$ is set of all vectors in $\mathbb{R}^n$ that are linear combinations of column vectors of A.

Thus, the span of a set of n vectors is all linear combinations of those vectors:

$$\boxed{\text{span}(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n) = \{\sum c_i \vec{v}_i | c_i \in \mathbb{R}\}}$$

So the span of column vectors of A is the image of the associated linear transformation.

11.4 Kernel

Kernel amounts to finding solutions to $A\vec{x} = \vec{0}$. Kernels are closed under linear combinations. Kernel can never be empty set, it always holds true that $T(\vec{0}) = \vec{0}$.

11.5 Invertible Linear Transformations

Main conclusions about image and kernel:

- Kernel is always (trivially) $\{\vec{0}\}$, else would imply dependance and therefore singularity in the associated matrix A so $\boxed{\ker(T) = \{\vec{0}\}}$
- Image is always the space $\mathbb{R}^n$ if associated matrix A is $n \times n$, so $\boxed{\text{Im}(T) = \mathbb{R}^n}$

12 Subspaces and Basis

12.1 Image and Kernel

If $T:\ \mathbb{R}^m \to \mathbb{R}^n$ then $\text{Im}(T) \subset \mathbb{R}^n$ and $\ker(T) \subset \mathbb{R}^m$ because the associated matrix A is $n \times m$ in dimension.

Both are closed under linear combinations:

- If $\vec{y_1}, \vec{y_2} \in \text{Im}(T)$ then $a\vec{y_1} + b\vec{y_2} \in \text{Im}(T)$ as well
- If $\vec{x_1}, \vec{x_2} \in \text{Ker}(T)$ then $a\vec{x_1} + b\vec{x_2} \in \text{Ker}(T)$ as well

12.2 Subspaces

Collection of vectors in $\mathbb{R}^n$ is called a subspace in $\mathbb{R}^n$ if collection is nonempty and closed under linear combinations. Examples (and counterexamples):

- $W = \left\{ \begin{pmatrix} 3s \\ 2+5s \end{pmatrix} \mid s \in \mathbb{R} \right\} \subset \mathbb{R}^2$ is not a subspace because $\vec{0}$ is not contained within the set, so not closed under linear combinations
- $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix} \mid 2x_1 + x_2 - x_3 = 0 \right\}$ is a subspace due to matrix representation and the image of this matrix containing $\vec{0}$ due to $T(\vec{0} = \vec{0})$

$\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ because though a plane can be drawn in $\mathbb{R}^3$, its components will be of the form $\begin{bmatrix} x \\ y \\ k \end{bmatrix}$, where k is fixed. Since $\mathbb{R}^3$ vectors always have 3 coordinates, they can't represent $\mathbb{R}^2$. $\mathbb{R}^2$ can only be represented by $\mathbb{R}^2$ vectors. Thus, R^n is not a subspace of $\mathbb{R}^{n+1}$.

Claim: Span of a set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.
Proof:
Let $S = \{\vec{v_1}, \vec{v_2}, \cdots \vec{v_m}\}$. Let $\vec{w}, \vec{y} \in \text{span}(S)$. Thus, $\vec{w} = \sum c_i \vec{v_i}$ and $\vec{y} = \sum d_i \vec{v_i}$ where $d_i, c_i \in \mathbb{R}$.

$$\begin{aligned} a\vec{w} + b\vec{y} &= a\sum c_i\vec{v_i} + b\sum d_i\vec{v_i} \\ &= \sum ac_i\vec{v_i} + \sum bd_i\vec{v_i} \\ &= \sum (ac_i + bd_i)\,\vec{v_i} \in \text{span}(S) \end{aligned}$$

List of subspaces in $\mathbb{R}^2$ would be $\mathbb{R}^2$, $\{t\vec{v} \mid t \in \mathbb{R}\}$, $\{\vec{0}\}$.

12.3 Intersection and Union

If V and W are collections of vectors in $\mathbb{R}^n$:

- $V \cap W = \{\vec{x} \mid \vec{x} \in V \text{ and } \vec{x} \in W\}$ is the intersection
- $V \cup W = \{\vec{x} \mid \vec{x} \in V \text{ or } \vec{x} \in W\}$ is the union

12.4 Redundant Vectors

If for some transformation T there exists the following:

$$\text{im}(T) = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\2\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\3\\4\end{pmatrix}\right\}$$

There are redundant vectors in this case. The minimum number of vectors in the span is 2, for $\vec{0}$ cannot be produced then. With 3 vectors in $\mathbb{R}^3$, any one can be the result of linear combinations of the other 2. So, it would be appropriate to say that:

$$\text{im}(T) = \left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$$

These are then **linearly independent**. This set forms a **basis** for that set of vectors. Thus, the basis can be found for any matrix. The basis of I_n is then $\{\vec{e}_1, \vec{e}_2 \cdots \vec{e}_n\}$.

12.5 Intersection and Union

If V and W are subspaces of $\mathbb{R}^n$, then $V \cap W$ is a subspace of $\mathbb{R}^n$ and $V \cup W$ is **not** a subspace of $\mathbb{R}^n$.

An intersection is the items contained in both sets, so $\vec{0} \in V \cap W$. If $\vec{v}, \vec{w} \in V \cap W$, then $\vec{v}, \vec{w} \in V$ and $\vec{v}, \vec{w} \in W$. This means that $\vec{v} + \vec{w} \in V$ and $\vec{v} + \vec{w} \in W$ so $\vec{v} + \vec{w} \in V \cap W$. Similarly, if some $k\vec{v} \in V \cap W$ where $k \in \mathbb{R}$ then $k\vec{v} \in V$ and $k\vec{v} \in V$. Thus, $V \cap W$ is a subspace of $\mathbb{R}^n$.

The union is the items contained in either set. If $V = \text{span}(\vec{e}_2)$ and $W = \text{span}(\vec{e}_1)$, then let $\vec{y} = \begin{bmatrix}0\\y\end{bmatrix} \in V$ and $\vec{x} = \begin{bmatrix}x\\0\end{bmatrix} \in W$. Thus, $\vec{y}, \vec{x} \in V \cup W$. However, $a\vec{x} + b\vec{y} \notin V \cup W$ where $a, b \in \mathbb{R}$.

13 Basis of a Kernel

13.1 Example

Let $A = \begin{bmatrix}1 & 4 & 7\\2 & 5 & 8\\3 & 6 & 9\end{bmatrix}$. Finding the basis:

$$\text{rref}(A) = \begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}$$

Since there is redundance, a possible expression of the basis of A:

$$\mathfrak{B} = \left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right\}$$

When finding kernel, must solve $A\vec{x} = \vec{0}$. So with $[\text{rref}(A)|\vec{0}]$:

$$\vec{x} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Thus,

$$\ker(A) = \left\{ t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

Subsequently, the basis of the kernel of A can be represented as $\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Equivalent statements for $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_m\}$ being linearly independent:

- None of the vectors are redundant
- Only relation is trivial
- Kernel of $\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_m \\ | & | & & | \end{pmatrix}$ is trivial
- Rank of $\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_m \\ | & | & & | \end{pmatrix}$ is m
- If $m = n$ then $\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_m \\ | & | & & | \end{pmatrix}$ reduces to I_n

14 Dimension

14.1 Rank and independence

If $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\}$ is a collection if independent vectors then

$$\begin{pmatrix} | & | & | & & | \\ \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \ldots & \bar{v}_m \\ | & | & | & & | \end{pmatrix}$$

must have a rank of m. This is because row reducing the matrix corresponds to the following relation:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \cdots + c_m\vec{v}_m = \vec{0}$$

Also, $m \leq n$ where n is the number of rows in each column vector, in order to have linear independence for this set.

14.2 Dimension

Considering an xy-plane in $\mathbb{R}^3$:

$$V = \left\{ \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}$$

The basis of this set contains 2 vectors (e.g. dimension of 2), with example being:

$$\mathfrak{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

> If V is a subspace of $\mathbb{R}^n$ and $\mathfrak{B}$ and $\mathfrak{C}$ are two bases of V, then $\mathfrak{B}$ and $\mathfrak{C}$ contain the same number of vectors.

Dimension of a subspace is number of vectors in the basis.

14.2.1 Example

Considering the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 & 5 \end{pmatrix}$$

By discounting the redundant vectors, a possible basis for $\mathrm{Im}(A)$:

$$\mathfrak{B}_{\text{image}} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$$

So the dimension of $\mathrm{Im}(A)$ is 2. Finding a basis for $\ker(A)$ is the same as solving $A\vec{x} = \vec{0}$:

$$\ker(A) = \left\{ \begin{pmatrix} -2s - w \\ s \\ t \\ -w \\ w \end{pmatrix} \middle| s, t, w \in \mathbb{R} \right\} = \left\{ s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \middle| s, t, w \in \mathbb{R} \right\}$$

So the basis for $\ker(A)$:

$$\mathfrak{B}_{\text{kernel}} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

And dimension of $\ker(A)$ is 3. However, it is shown that $\text{rref}(A)$ gives dimension of **image and kernel**.

14.3 Rank-Nullity Theorem

- If T is a linear transformation, then
  $\dim(\text{Im}(T)) + \dim(\ker(T)) = \text{dimension of domain of } T$
- If A is a matrix, then $\text{rank}(A) + \text{nullity}(A) = \text{number of columns of } A$
- In a linear system,
  number of leading variables + number of free variables = total number of variables

Considering non-invertible matrices A and B, let AB be invertible. It must hold true that $\ker(B) = \{\vec{0}\}$. If the dimensions of B are $p \times n$, $\text{Im}(B)$ is a subspace of $\mathbb{R}^p$ has dimension n. This means that it is a vertically rectangular matrix with $n \leq p$. Thus, A is $n \times p$ so it is horizontally rectangular.

15 Coordinates

15.1 Coordinate vectors

For example, the basis of xy plane can be:

$$\mathfrak{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

To form $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ with this basis, can do $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. The coefficients used form the following vector:

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Known as **$\mathfrak{B}$-coordinate vector**. Notation:

$$\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]_{\mathfrak{B}} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Generally, given $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_m\} \subset \mathbb{R}^n$ is linearly independent, then $[\vec{v}_i]_{\mathfrak{B}} = \vec{e}_i \in \mathbb{R}^m$.

This is because row-reducing the matrix of $\mathfrak{B}$ gives rref(A) where A is this matrix. Given the same $\mathfrak{B}$, can find the components of $\vec{w}$:

$$[\vec{w}]_{\mathfrak{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$$

Thus,

$$\vec{w} = \begin{pmatrix} | & | & \dots & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ | & | & \dots & | \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

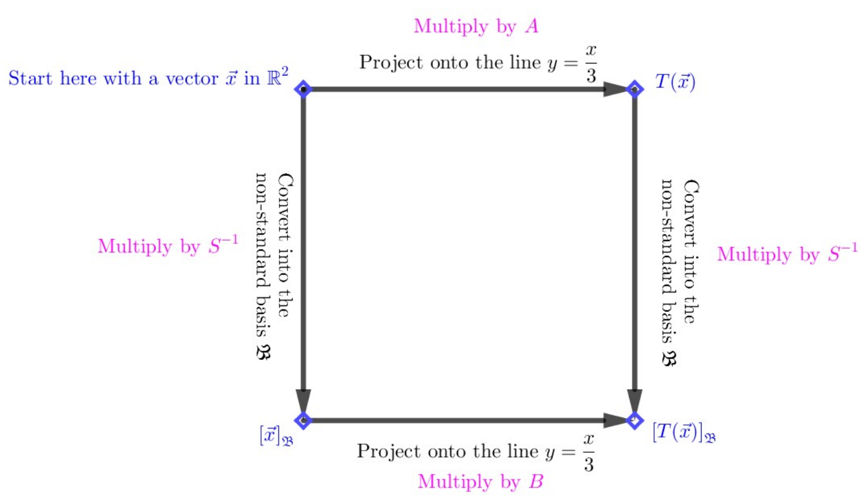
Matrix is called change of basis matrix S. A standard basis is given as $\vec{e}_1, \vec{e}_2, \cdots$. A nonstandard basis is not of this form.

15.2 B-matrix

If A is $n \times n$ and $T(\vec{x}) = A\vec{x}$ where $T : \mathbb{R}^n \to \mathbb{R}^n$, then there exists a matrix B such that $[T(\vec{x})]_{\mathfrak{B}} = B\,[\vec{x}]_{\mathfrak{B}}$. This is called the $\mathfrak{B} - matrix$. If $\vec{v}_i \in \mathfrak{B}$, then $[\vec{v}_i]_{\mathfrak{B}} = \vec{e}_i$.

This means that $[T(\vec{v}_i)]_{\mathfrak{B}} = B\,[\vec{v}_i]_{\mathfrak{B}} = B\vec{e}_i$, so **the i^{th}column of B must be $[T(\vec{v}_i)]_{\mathfrak{B}}$.**

Multiple ways to calculate $\mathfrak{B}$-matrix of T, considering T to be a projection onto $y = \frac{x}{3}$:



Means that multiple ways to get to $[T(\vec{x})]_{\mathfrak{B}}$. When following $\vec{x}$ and going right and down:

$$S^{-1}(A\vec{x}) = S^{-1}A\vec{x} = [T(\vec{x})]_{\mathfrak{B}}$$

Going down and right:

$$B\left(S^{-1}\vec{x}\right) = BS^{-1}\vec{x} = [T(\vec{x})]_{\mathfrak{B}}$$

Thus,

$$S^{-1}A = BS^{-1}$$
$$\boxed{S^{-1}AS = B}$$

If this is satisfied, then A is similar to B or $A \sim B$.

16 Determinants

16.1 Introduction to Determinant

Can define the 2×2 determinant as a function $D : \mathbb{M}_{2\times 2} \to R$. It can be observed that 2×2 matrix A is only invertible if $D(A) = ad - bc \neq 0$.

16.2 Cross-Product

Given $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$, the cross product is defined as the $\mathbb{R}^3$ vector $D(A)\vec{e}_3 = (ad - bc)\vec{e}_3$. The direction of this vector is the sign of $\det(A)$.
Can visualize using right hand rule: if sweeping index into middle is appropriate for the vectors, then the direction of thumb is cross-product direction (positive). Otherwise, sign is negative.

16.2.1 Algorithm

For $\vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$:

$$\boxed{\vec{v} \times \vec{w} = c_z\vec{e}_1 + c_y\vec{e}_2 + c_z\vec{e}_z}$$

Following through, to calculate each component ignore the desired row and perform cross-product on remaining matrix:

$$c_x = \begin{bmatrix} v_y \\ v_z \end{bmatrix} \times \begin{bmatrix} w_y \\ w_z \end{bmatrix} = v_y w_z - v_z w_z$$

The y component is done as $bc - ad$ compared to $ad - bc$.

$$c_y = \begin{bmatrix} v_x \\ v_z \end{bmatrix} \times \begin{bmatrix} w_x \\ w_z \end{bmatrix} = w_x v_z - w_z v_x$$

$$c_z = \begin{bmatrix} v_y \\ v_z \end{bmatrix} \times \begin{bmatrix} w_y \\ w_z \end{bmatrix} = v_x w_y - v_y w_x$$

16.3 Determinant Theory

Considering $A = \begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix}$, say it is singular such that $\vec{v}_3 \in \text{span}\{\vec{v}_1, \vec{v}_2\}$. Because it is assumed that $\vec{v}_1, \vec{v}_2$ are linearly independent, then $\text{span}\{\vec{v}_1, \vec{v}_2\}$ is perpendicular to $\vec{v}_1 \times \vec{v}_2$ by definition. Thus, $\boxed{(\vec{v}_1 \times \vec{v}_2) \cdot \vec{v}_3 = 0}$. If $\vec{v}_1, \vec{v}_2$ are not linearly independent, then this is still 0 because the cross-product (area of parallelogram made by vectors) is still 0.
If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent, then $(\vec{v}_1 \times \vec{v}_2) \cdot \vec{v}_3 \neq 0$.

$$\boxed{D(A) = \det(A) = (\vec{v}_1 \times \vec{v}_2) \cdot \vec{v}_3}$$

The sub-matrices used when computing cross-products are **minors**. Can rewrite determinant:

$$\boxed{\det A = a_{1,3}\,|A_{1,3}| - a_{2,3}\,|A_{2,3}| + a_{3,3}\,|A_{3,3}|}$$

Must use following rule for the sign of constant terms $a_{m,n}$ (dot product):

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

16.4 Rules

Determinant of $n \times n$ follows recursively:

$$\boxed{\det A = a_{1,1}\,|A_{1,1}| - a_{1,2}\,|A_{1,2}| + a_{1,3}\,|A_{1,3}| + \cdots \pm a_{1,n}\,|A_{1,n}|}$$

Rules:

- Swapping rows multiplies determinant by -1
- Multiplying row by m scales determinant by m
- Replacing row with sum of row and multiple of another does not change determinant
- If A and B are $n \times n$, then $\det(AB) = \det(A)\det(B)$
- Cramer's rule: If $A\vec{x} = \vec{b}$ is a linear system with invertible A then $\vec{x}$ components can be determined from $x_i = \frac{\det(A-_{b,i})}{\det(A)}$ where $A-_{b,i}$ replaces i^{th} column of A with $\vec{b}$

17 Intro to Dynamical Systems

17.1 Dynamical Systems and Eigenvectors

In general, a discrete dynamical system can be modeled as:

$$\boxed{\vec{x}(t+1) = A\vec{x}(t)}$$

where the transformation undergone by the system is $\vec{x}(t) \to \vec{x}(t+1)$ with matrix A. Additionally, note that $\vec{x}(t) = \begin{bmatrix} c(t) \\ r(t) \end{bmatrix}$ where $c(t)$ and $r(t)$ are some closed formulas.

Finding $\vec{x}(t)$ for an arbitrary integer $t > 0$:

$$\boxed{\vec{x}(t) = A^t \vec{x}(0) = A^t \vec{x}_0}$$

Repeat definition of eigenvector from below:

> If A is an $n \times n$ matrix, an eigenvector of A is a nonzero vector $\vec{v}$ that has the property that $\vec{v}$ and $A\vec{v}$ are parallel. Same as saying that $A\vec{v} = \lambda\vec{v}$, so λ is an eigenvalue.

17.2 Dynamical Systems Example

Following equations model transformation from t to $t+1$:

$$\boxed{\begin{array}{l} c(t+1) = 0.86c(t) + 0.08r(t) \\ r(t+1) = -0.12c(t) + 1.14r(t) \end{array}}$$

Is discrete dynamical linear system: changed over discrete time interval and dynamic as variables change according to t. As a matrix-vector equation:

$$\begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix} \begin{pmatrix} c(t) \\ r(t) \end{pmatrix} = \begin{pmatrix} c(t+1) \\ r(t+1) \end{pmatrix}$$

$\vec{x}(t) = \begin{pmatrix} c(t) \\ r(t) \end{pmatrix}$ is the **state vector** at time t. $\vec{x}(0) = \begin{pmatrix} c_0 \\ r_0 \end{pmatrix}$ is the **initial state vector**. Calculating arbitrary state vector:

$$\begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix}^t \vec{x}_0 = \vec{x}(t)$$

In this example, $c(t) = (100)1.1^t$ and $r(t) = (300)1.1^t$, so next state vector is 1.1 times the current. However, for $\vec{x}_0 = \begin{pmatrix} 1000 \\ 1000 \end{pmatrix}$ there exists no such scalar pattern. Can use the basis of 2 (scalar pattern respected) vectors:

$$\mathfrak{B} = \left\{ \begin{pmatrix} 100 \\ 300 \end{pmatrix}, \begin{pmatrix} 200 \\ 100 \end{pmatrix} \right\}$$

Writing this state vector as a lin. combination:

$$\begin{pmatrix} 1000 \\ 1000 \end{pmatrix} = 2 \begin{pmatrix} 100 \\ 300 \end{pmatrix} + 4 \begin{pmatrix} 200 \\ 100 \end{pmatrix}$$

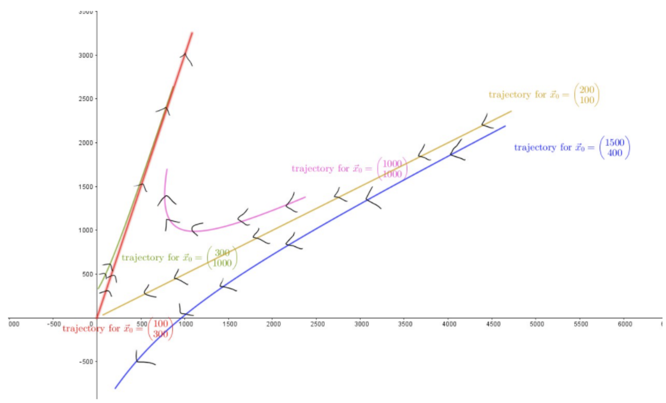
After applying coefficient matrix to both sides and simplifying:

$$\begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix}^t \begin{pmatrix} 1000 \\ 1000 \end{pmatrix} = 2(1.1)^t \begin{pmatrix} 100 \\ 300 \end{pmatrix} + 4(0.9)^t \begin{pmatrix} 200 \\ 100 \end{pmatrix}$$

Thus,

$$c(t) = 200(1.1)^t + 800(0.9)^t$$
$$r(t) = 600(1.1)^t + 400(0.9)^t$$

Different trajectories for various initial state vectors:



Called a **phase portrait** for a discrete dynamical system. Indicates performing of system based on initial states. The 2 state vectors in the basis are **eigenvectors**.

> If A is an $n \times n$ matrix, an eigenvector of A is a nonzero vector $\vec{v}$ that has the property that $\vec{v}$ and $A\vec{v}$ are parallel. Same as saying that $A\vec{v} = \lambda\vec{v}$, so λ is an eigenvalue.

If there exists an $n \times n$ matrix A with $\lambda = 0$, then kernel of A must be nontrivial because $A\vec{v} = \vec{0} = 0\vec{v}$, therefore A is singular.

18 Eigenvalue of a Matrix

18.1 Eigenvalue for rotation transformation

Claim: If $0 < \theta < 2\pi$ then transformation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ only has an eigenvector when $\theta = \pi$ (when $\lambda = -1$).

Proof: The matrix is $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Following by the definition of an eigenvector:

$$\begin{aligned} A\vec{v} = \lambda v &\iff \\ A\vec{v} - \lambda\vec{v} = \overrightarrow{0} &\iff \\ A\vec{v} - \lambda(I\vec{v}) = \overrightarrow{0} &\iff \\ A\bar{v} - (\lambda I)\vec{v} = \overrightarrow{0} &\iff \\ (A - \lambda I)\vec{v} = \overrightarrow{0} &\iff \\ \det(A - \lambda I) = 0 & \end{aligned}$$

Thus,

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\det \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix} &= 0 \\
\lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta &= 0 \\
\lambda^2 - 2\lambda\cos\theta + 1 &= 0
\end{aligned}$$

The discriminant of this quadratic ($b^2 - 4ac$) is $4\cos^2\theta - 4$, so for a real solution $4\cos^2\theta - 4 \geq 0$. It then follows that:

$$\begin{aligned}
4\cos^2\theta - 4 &\geq 0 \\
\cos^2\theta &\geq 1 \\
\cos^2\theta &= \pm 1 \\
\theta &= \pi
\end{aligned}$$

Note that because $(A - \lambda I)\vec{v} = \vec{0}$ implies a nontrivial kernel for $A - \lambda I$, $\det(A - \lambda I) = 0$.

18.2 Characteristic Polynomials

Characteristic polynomial is for $\det(A - \lambda I)$ with variable λ:

$$\boxed{P_A(\lambda) = \det(A - \lambda I)}$$

General polynomial for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$\begin{aligned}
p_A(\lambda) &= \det(A - \lambda I) \\
&= \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \\
&= (a - \lambda)(d - \lambda) - bc \\
&= \lambda^2 - (a + b)\lambda + (ad - bc)
\end{aligned}$$

Ends up that $\operatorname{tr}(A) = a + d$ and $\det(A) = ad - bc$:

$$\boxed{p_A(\lambda) = \lambda^2 - \operatorname{tr} A\lambda + \det A}$$

18.2.1 General formula

In general, if A is an $n \times n$ matrix, then

$$\boxed{p_A(\lambda) = (-1)^n\lambda^n + (-1)^{n-1}\operatorname{tr} A\lambda^{n-1} + \cdots + \det A}$$

Conjectures:

- By FTLA, degree n polynomial will have n complex roots so at least n real eigenvalues

- If all n roots are real, then $\text{tr}(A)$ is sum of eigenvalues and determinant is the product of them
- Since roots are either real or in complex conjugate pairs, ($a + bi$ or $a - bi$) then when n is odd A has at least 1 real eigenvalue

19 Eigenvector of a Matrix

19.1 Eigenspace

Kernel of a matrix always forms subspace of domain. If λ is an eigenvalue for A, kernel of $A - \lambda I$ is the **eigenspace** associated with λ and this is denotes as $E_\lambda = \ker(A - \lambda I)$.

If $A - \lambda I$ has column vectors $\vec{v}_1$ and $\vec{v}_2$, then $E_\lambda = \text{span}\left\{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right\}$ where $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$.

19.2 Multiplicity

Dimension of eigenspace E_λ is **geometric multiplicity** of λ. Multiplicity of root λ is **algebraic multiplicity** in characteristic polynomial $p_A(\lambda)$. Therefore, geometric multiplicity $\leq$ algebraic multiplicity, considering the case of $p_A(\lambda) = (\lambda - \lambda_0)^2$.

Thus, this represents a 2×2 matrix which fixes one line and moves every other line, known as a shear. All lines but x-axis move. Has characteristic polynomial $p_A(\lambda) = (\lambda - 1)^2$, so $E_1 = \ker\begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

19.3 Eigenbasis

Consists of the eigenvectors of the coefficient matrix. An $n \times n$ matrix needs n linearly independent eigenvectors to have an eigenbasis. This means that if A has eigenvectors $\lambda_1 \neq \lambda_2$, then $E_{\lambda_1} \cap E_{\lambda_2} = \left\{\vec{0}\right\}$. This is because if some $\vec{v} \in E_{\lambda_1}$, then $A\vec{v} = \lambda_1\vec{v}$ and $A\vec{v} = \lambda_2\vec{v}$. Thus, $(\lambda_1 - \lambda_2)\vec{v} = \vec{0}$, so $\vec{v} = \vec{0}$.

Furthermore, if E_{λ_1} has basis $\mathfrak{E}_{\lambda_1}$ and E_{λ_2} with $\mathfrak{E}_{\lambda_2}$, $\mathfrak{E}_{\lambda_1} \cup \mathfrak{E}_{\lambda_2}$ is linearly independent as well with the total elements being the sum of the number of elements in each individual basis. Can then be concluded that:

$$\boxed{\text{An } n \times n \text{ matrix has eigenbasis iff sum of geometric multiplicities of eigenvalues is } n.}$$