

Multivariable Calculus Reference

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January 21, 2021

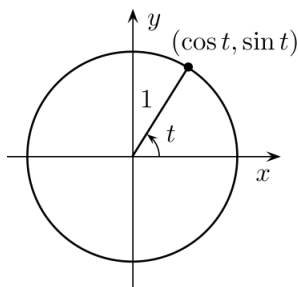
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1 Paths whose image curve is a circle

1.1 Unit Circle

Unit circle is set of points in \mathbb{R}^2 defined as $C = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$. Ellipse is $C = \{(x, y) \in \mathbb{R}^2 | \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$. Has standard parameterization of $\vec{c}(t) = (\cos(t), \sin(t))$. When parameterizing, always start from $t = 0$ reference unless otherwise given.



Properties

- Image of \vec{c} is a closed curve (has no endpoints, plane is divided into ≥ 2 disjoint regions)
- Image of \vec{c} is a simple curve; no self-intersection
- $\vec{c}(t)$ is an **injective path**; path is considered injective if $\vec{c}(t_1) = \vec{c}(t_2)$, which implies that $t_1 = t_2$ where these are on the open interval (a, b) even if $a = b$
- Orientation of \vec{c} is counter-clockwise in traversal

1.2 Observations

$$\vec{p}(t) = (a \cos(\pm n(t \pm \theta)) + x_0, b \sin(\pm n(t \pm \theta)) + y_0)$$

If $-t$ for t , orientation is CW, CCW is t . If $a = b$, then curve is a circle of radius a or b , else an ellipse with horizontal and vertical radii. x_0 and y_0 simply shift the center coordinate. $n > 0 \in \mathbb{R}$ determines how many times the circle is traversed given $t \in [0, 2\pi]$, for example. θ is the phase shift. When changing direction of traversal, cannot have $a > b$ for $[a, b]$ so to decrease argument of sin or cos must have $-t$ for t . Starting out, $-t$ goes through the angle range and t is just a sign flip.

2 Paths whose image is a line or line segment in the plane

2.1 Line Parametrics

A line is a 1D subspace of \mathbb{R}^2 , so $L = \{t\vec{m} | t \in \mathbb{R}\}$ for $\vec{m} \in \mathbb{R}^2$. $\vec{m} = \begin{bmatrix} m_x \\ m_y \end{bmatrix}$ is the **slope vector**.

Path given by image of L :

$$\vec{c}(t) = (m_x t, m_y t), t \in \mathbb{R}$$

Can represent $\vec{c}(t) = t\vec{m}$ as well.

Lines Main Ideas

- Image of a line is a curve (e.g. $y = x$ represents image curve of $\vec{c}(t) = (t, t)$)
- Lines can have nonzero intercepts, so $\vec{c}(t) = t\vec{m}$ represents $y = 2x + 1$. Line that has intercept vector $P_0 = (x_0, y_0) \parallel \vec{m} = (m_x, m_y)$ can be expressed as:

$$\vec{c}(t) = (x_0 + tm_x, y_0 + tm_y) = \vec{P}_0 + t\vec{m}$$

Note endpoint of $\vec{c}(t)$ is on image line (curve).

2.2 General Forms

2 parametric lines **collide** if they intersect and the point of intersection corresponds to the same t in both curves. If you set the parameter vector coordinates equal to each other and solve for t , a solution indicates they collide. Intersection is found by **eliminating** the parameter (solve for t in terms of either x or y and plug into the other).

General form of parameterized curve can be expressed as the following:

$$\vec{c}(t) = \left(\frac{m_x}{\Delta t}(t - a) + x_0, \frac{m_y}{\Delta t}(t - a) + y_0 \right)$$

where Δt is the domain interval over $[a, b]$ and (x_0, y_0) represents the desired **starting coordinate**. This is important as when going in reverse, other coordinate can be used and slope might be negative. a is used in $(t - a)$ because everything is conventionally done with respect to starting coordinate.

3 Paths whose image curve is a line in R3

3.1 R3 parameterization

If \vec{m} is a nonzero vector along L through origin in \mathbb{R}^3 , then $L = \{t\vec{m} | t \in \mathbb{R}\}$; follows that $\vec{m} = (m_x, m_y, m_z)$, the slope or direction vector of the line. The basic parameterization is:

$$\vec{c}(t) = (m_x t, m_y t, m_z t)$$

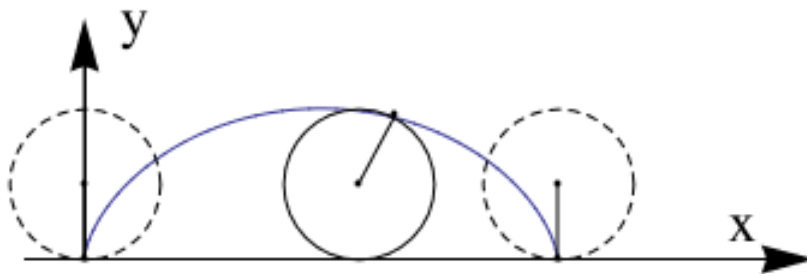
Basis vectors in \mathbb{R}^3 are $\vec{i}, \vec{j}, \vec{k}$. Rewriting parameterization:

$$\vec{c}(t) = (x_0 + m_x t)\vec{i} + (y_0 + m_y t)\vec{j} + (z_0 + m_z t)\vec{k}$$

2 lines $\vec{c}_1(t) = P_0 + \vec{m}_1 t$ and $\vec{c}_2(t) = Q_0 + \vec{m}_2 t$ are parallel if direction vectors are parallel ($\vec{m}_1 = k\vec{m}_2$). Collisions still exist. If neither parallel nor intersecting, considered as skew.

To determine skew, parallel, or coincide, use parameters s, t for each line and solve SOE. If same slope, rule out skew clearly, then check if $s, t \in \mathbb{R}$: if not, then parallel, if so, then they coincide. If intersecting and want to check if collide, some t must satisfy all relations.

4 Cycloid Problem



With radius 1 and passing through the origin:

$$\vec{c}(t) = (t - \sin t, 1 - \cos t)$$

Observe that:

$$\vec{c}'(t) = (1 - \cos t, \sin t)$$

Can define the vector $\vec{u} = \begin{pmatrix} x'(t) \\ 0 \end{pmatrix}$ such that \vec{u} is always horizontal and $||\vec{u}|| = |x'(t)|$. Reaches maximum value at $t \in [k\pi | k \in \mathbb{R}]$ and is has minimum cusp where it is 0 at $t \in [2k\pi | k \in \mathbb{R}]$. Thus, $x'(t) \geq 0$ always, as the x-coordinate is never decreasing.

Can also define the vector $\vec{v} = \begin{pmatrix} 0 \\ y'(t) \end{pmatrix}$ with the same properties. Reaches maximum value when $t \in [k\frac{\pi}{2} | k \in \mathbb{R}]$. Can change, as observe t when $\sin t < 0$ or > 0 .

5 Velocity Vector

5.1 Definitions

Vector $\vec{u}(t_0) + \vec{v}(t_0)$ is the velocity vector to the curve $\vec{c}(t)$ at $t = t_0$.

Let $\vec{c}: [a, b] \rightarrow \mathbb{R}^n$ have a path $\vec{c}(t) = (x_1(t), x_2(t), x_3(t), \dots, x_n(t))$ (let $x_i(t): [a, b] \rightarrow \mathbb{R}$ for each i)

- If $t_0 \in [a, b]$, then $\vec{c}'(t_0) := (x'_1(t_0), x'_2(t_0), x'_3(t_0), \dots, x'_n(t_0))$; the velocity vector to \vec{c} at t_0
- The path $\vec{c}'(t_0) := (x'_1(t_0), x'_2(t_0), x'_3(t_0), \dots, x'_n(t_0))$; the velocity vector to \vec{c} is referred to as velocity of $\vec{c}(t)$

Recall chain rule: if $y = f(x)$ where x is a function of t , $y'(t) = x'f'(x)$, not to be confused with product rule. Can write $f'(x) = \frac{y'(t)}{x'(t)}$

- If $\vec{p}(t) = \vec{c}(t) + \vec{r}(t)$, then $\vec{p}'(t) = \vec{c}'(t) + \vec{r}'(t)$
- If $g(t) = \vec{c}(t) \cdot \vec{r}(t)$, then $g'(t) = \vec{c}'(t) \cdot \vec{r}(t) + \vec{c}(t) \cdot \vec{r}'(t)$

- If $\vec{p}(t) = f(t)\vec{c}(t)$, then $\vec{p}'(t) = f'(t)\vec{c}(t) + f(t)\vec{c}'(t)$
- If $\vec{p}(t) = \vec{c}(t) \times \vec{r}(t)$, then $\vec{p}'(t) = \vec{c}'(t) \times \vec{r}(t) + \vec{c}(t) \times \vec{r}'(t)$
- If $\vec{p}(t) = \vec{c}(f(t))$, then $\vec{p}'(t) = f'(t)\vec{c}'(f(t))$
- If $g(t) = \|\vec{c}(t)\|$, then $g'(t) = \frac{\vec{c}(t) \cdot \vec{c}'(t)}{\|\vec{c}(t)\|}$

5.2 Tangent Line

Tangent line can be visualized as a base vector in standard position plus a velocity vector tangent to the tip which traces a shifted line in some interval. General formula with base vector $\vec{c}(t_0)$ and slope $\vec{c}'(t_0)$:

$$\ell(t) = \vec{c}(t_0) + (t - t_0)\vec{c}'(t_0)$$

6 Space Curves

- Projection into the xy plane is the path $(x(t), y(t), 0)$.
- Projection into the xz - plane is the path $(x(t), 0, z(t))$.
- Projection into the yz plane is the path $(0, y(t), z(t))$.

7 Speed and Arclength

7.1 Speed

Speed of a parametric function in \mathbb{R}^n is given by:

$$\|\vec{c}'(t)\| = \sqrt{\sum_{i=1}^n c_i(t)^2}$$

(being the magnitude of the velocity vector)

7.2 Arclength

Arclength of a parametric function is given by:

$$S = \int_a^b \|\vec{c}'(t)\| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 + \dots} dt$$

Can approximate arclength as a sum of the lengths of secant vector approximations $\vec{s}_i = \vec{c}(t_i) - \vec{c}(t_{i-1})$:

$$\text{arclength} \approx \sum_{i=1}^n \|\vec{s}_i\|$$

According to the MVT, there exists a \hat{t}_i in (t_{i-1}, t_i) (open interval due to differentiability requirement) such that:

$$\begin{aligned}x'(\hat{t}_i) &= \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \\y'(\hat{t}_i) &= \frac{y(t_i) - y(t_{i-1})}{t_i - t_{i-1}}\end{aligned}$$

This means that, since \vec{s}_i is given as the difference between 2 points, being a secant:

$$\begin{aligned}\vec{s}_i &= ((t_i - t_{i-1})x(\hat{t}_i), (t_i - t_{i-1})y(\hat{t}_i)) \\ \vec{s}_i &= (t_i - t_{i-1}) (x'(\hat{t}_i), y'(\hat{t}_i)) \\ \vec{s}_i &= (t_i - t_{i-1})\vec{c}'(\hat{t}_i)\end{aligned}$$

Thus,

$$\begin{aligned}\text{arclength} &\approx \sum_{i=1}^n \|\vec{s}_i\| \\ \text{arclength} &\approx \sum_{i=1}^n \|\Delta t \vec{c}'(\hat{t}_i)\| \\ \text{arclength} &\approx \sum_{i=1}^n \Delta t \|\vec{c}'(\hat{t}_i)\|\end{aligned}$$

Can define the arclength differential as follows:

$$ds = \sqrt{dx^2 + dy^2}$$

Can just define arclength as $\text{arclength} = \int ds$

7.3 Arclength Parameterization

Higher the speed of a curve, farther the points are spaced apart. An arclength parametrization of a curve is a path whose image is the desired curve and whose speed is constantly one. Or, $\vec{c}: [a, b] \rightarrow \mathbb{R}^n$ with $\|\vec{c}'(t)\| = 1$ for $t \in [a, b]$. If a curve is not an arclength parameterization, then can do $\frac{\vec{c}(t)}{\|\vec{c}'(t)\|}$ but only dividing the coefficients (slopes).

When speed is variable, is difficult to define arclength parameterization. Thus, can define displacement to be $s(t) = \int_a^b v(t)dt$. If $v(t) \neq 0$, then s is injective because according to FTC, $s'(t) = v(t)$. By definition, $v'(t) \geq 0$ always since it is composed of a radical, so it must be **increasing**. Thus, if $t_1 = t_2$, $s(t_1) \neq s(t_2)$. Arclength parameterization:

$$s(t) = \int_0^t \|\vec{c}'(u)\| du$$

This means that s is invertible, so can solve for t to get $t = \varphi(s)$. An arclength parameterization can be found by:

$$\boxed{\vec{p}(s) = \vec{c}(\varphi(s))}$$

8 Curvature

9 Proofs

Recall that to make an arclength parameterization accumulate the magnitudes of infinitesimal velocity vectors:

$$s(t) = \int_a^t \|\vec{r}'(u)\| du$$

Given some curve $\vec{r}(t)$, define an arclength parameterization by $\vec{r}(g(s)) \rightarrow \vec{r}_1(s)$, so \vec{r} is defined in terms of s . The unit tangent vector $\vec{T}_1(s)$ is then $\frac{\vec{r}'_1(s)}{\|\vec{r}'_1(s)\|} = \vec{r}'_1(s)$.

$$\begin{aligned} \vec{T}_1(s) &= \vec{r}'_1(s) \\ &= \frac{d}{ds} \vec{r}_1(s) \\ &= \frac{d}{ds} \vec{r}(g(s)) \\ &= \vec{r}'(g(s)) \cdot g'(s) = \vec{r}'(t) \cdot \frac{dt}{ds} \\ &= \frac{\vec{r}'(t)}{\frac{ds}{dt}} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \end{aligned}$$

This means that $\vec{T}_1(s) = T(t)$

Continuing, to find curvature $\kappa(t)$:

$$\begin{aligned} \vec{T}'_1(s) &= \frac{d}{ds} \vec{T}(t) \\ &= \frac{d}{ds} \vec{T}(g(s)) \\ &= \vec{T}'(t) \cdot \frac{dt}{ds} \\ &= \frac{\vec{T}'(t)}{\frac{ds}{dt}} \\ &= \frac{\vec{T}'(t)}{\|\vec{r}'(t)\|} \end{aligned}$$

Thus, $\boxed{\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}}$

9.1 Definition

Given a curve C parameterized with arclength by the path $\vec{c}: [a, b] \rightarrow \mathbb{R}^n$, curvature is defined as:

$$\kappa(s) = \|\vec{T}'(s)\|$$

where $\vec{c}'(s) \neq 0$ and $\vec{T}(s) = \frac{\vec{c}'(s)}{\|\vec{c}'(s)\|}$ (normalized slope vector).

A loose geometric interpretation is that a greater $\kappa(s)$ implies more curvature, that is, the curve is changing at a greater rate there. When $\vec{c}'(s) \neq 0$ is always true for a curve, it is **regular**. Is defined in terms of arclength parameterization so curvature is an intrinsic property of the curve independent of parameterization.

Formula for curvature at the point $\vec{c}(t)$:

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{c}'(t)\|} = \frac{\|\vec{c}'(t) \times \vec{c}''(t)\|}{\|\vec{c}'(t)\|^3}$$