Logic and Set Theory

Sidharth Baskaran

March 22, 2022

1 Common Sense Reasoning (Bonevac)

- Thoughts are organized by general principle, e.g. acids being corrosive, what goes up must come down.
- Are not **universal or necessary** as they have exceptions. Thus, inference from principle to conclusions are not deductively valid.
- To formalize principles, need to be able to draw conclusions from them but also withdraw these with further information. Classical logic does not allow for widthdrawable conclusions, for a valid argument does not need to revalidated-**guarantees** truth of its conclusion.
- Thus, need a system of logic to reach *defeasible* conclusions-reasonable but defeatable with further info
- Classical logic is monotonic-if set S implies conclusion $\mathscr{B}$, then any set with all of S also implies $\mathscr{B}$, so adding premises cannot make valid argument invalid
- New system of logic is non-monotonic, adding premises can sometimes make invalid argument

1.1 Good arguments going bad

Can add a premise to a deductively invalid argument to make it valid. A conditional always works.

Example 1. Invalid: Mt. Everest is there. So Mt. Everest should be climbed.
With premise: Mt. Everest is there. **If it is there, it should be climbed.** So Mt. Everest should be climbed.

- A deductively valid argument cannot be invalidated with additional premises
- An argument is valid if the truth of premises guarantees truth of conclusion, so if the premises are true the conclusion **must** be true, *but also if* falsity of premises implies falsity of conclusions
- In real-world arguments, premise cannot guarantee truth of conclusion. Additional information may cause withdrawal of inference.
  - Validity doesn't require the truth of the premises, instead it merely necessitates that conclusion follows from the formers without violating the correctness of the logical form

- An argument can be deductively invalid but inductively valid, or *defeasibly valid/allowed*, so premises *defeasibly imply* conclusion.
  - Argument is defeasibly valid iff premises are true **and** reasonable to expect conclusion to be true.
- Precisely term this as a **counterexample**, circumstance in which premises are true but conclusion false

2 Abstraction and truth functional logic

- Use *atomic sentences* (e.g. A or B) to express information as a statement, whereas the statement is its *extension*. Always are capital letters and are simplest way to express information.
- Use parentheses in truth-functional expressions to clearly convey information

Definition 2.1 (Case)**.** A case is a particular assignment of truth values to atomic sentences. n atomic sentences correspond to 2^n cases.

Definition 2.2 (Principle of Excluded Middle)**.** The principle of the excluded middle says that given a sentence, either it or the negation is true in any case.

Definition 2.3 (Principle of Non-Contradiction)**.** A sentence and its negation can't be true in the same case.

2.1 Logical Operators

Operator	Name	Vernacular equivalent
$\neg$	negation	not
$\wedge$	conjunction	and, but, although
$\vee$	disjunction	or
$\rightarrow$	conditional	if-then, only if
$\leftrightarrow$	biconditional	iff, necessary + sufficient, exactly when

The English word *unless* can be translated as a conditional, but there is a trick in interpretation. Take example of The patient will die (A) unless we operate (B). The correct translation is $\neg B \rightarrow A$, **not** $B \rightarrow \neg A$. This is due to the definition of the conditional; we cannot say that $B \leftrightarrow \neg A$ or the inverse isn't true. Surgery does not *guarantee* the patient living, but we know that if we don't operate, the patient will surely die.

Further note that using the equivalence conditional disjunction equivalence theorem, $(\neg B \rightarrow A) \leftrightarrow (B \vee A)$. So we can also interpret unless as a simple disjunction.

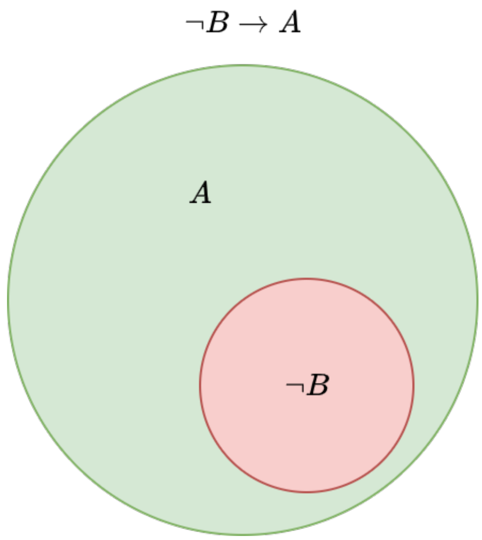

Figure 1: Visualization of conditional

3 Arguments

- Consists of a non-negative number of premises along with one conclusion
- Premise is a sentence, just like conclusion
- Standard form lists premises, followed by horizontal line (therefore) and the conclusion
- Logic assesses *validity* of arguments and we can see if the argument is *sound*

Definition 3.1 (Validity)**.** An argument is valid iff every case in which all its premises are true forces the conclusion to be true.

Definition 3.2 (Soundness)**.** An argument is sound iff it is valid and premises are actually true (i.e. in real life).

Definition 3.3 (Counterexample)**.** A case in which an argument's premises are all true but the conclusion is false.

A valid argument therefore has a relaxed definition; it is only invalid if there exists a counterexample to it.

3.1 Arguments and theorems

Definition 3.4 (Tautology)**.** A sentence if a tautology iff it is true in all cases.

Definition 3.5 (Contingent)**.** A contingent sentence is true in some cases and false in others.

Definition 3.6 (Contradiction)**.** Sentence of the form $A \wedge \neg A$, so it is always false. The sentences A and $\neg A$ are **contradictory**.

Definition 3.7 (Consistent)**.** A set of sentences is consistent iff there is a case where all of them are true. So for n sentences $\{A_1, A_2, \ldots A_n\}$, $A_1 \wedge A_2 \wedge A_3 \ldots \wedge A_n$ is true for **at least one case**.

Definition 3.8 (Argument-theorem exchange)**.** The argument

$$\begin{array}{c} P_1 \\ P_2 \\ P_3 \\ \vdots \\ \underline{P_n} \\ \\ \text{C} \end{array}$$

is valid iff $(P_1 \wedge P_2 \wedge P_3 \ldots \wedge P_n) \rightarrow C$ is a tautology.

4 Common Fallacies (Bennett)

- **Error of Conversion**. Can't affirm the antecedent given the premise that affirms the consequent. In other words, given $\{p \rightarrow q, q\}$ as premises, cannot conclude that p is true.
- **Error of Inversion**. Denying the antecedent and incorrectly denying the consequent. In the premises p is not true then q is true, p is not true, incorrect to conclude that q is not true.

5 Informal proof

Formal proofs are computationally verified (e.g. truth table) and informal proofs fall in one of below categories. The below methodology is to prove validity of an argument.

5.1 Conditional proof

Assume premises are true. Use light of reasoning to show that conclusion is also true in such a case. Conclude that argument is valid. Is sufficient to prove validity because by assuming the premises, we show that the conclusion must follow, or be implied.

5.2 Proof by contradiction

Assume there exists a counterexample (true premises and a false conclusion). Show this assumption leads to a contradiction. Then conclude that the argument is valid. Notably, this method is effective at proving **disproving** arguments, because when assuming the existence of a counterexample, you either reach a contradiction or affirm its existence–sufficient for a disproof.

6 Logical Equivalences and Inference

Name	Tautology	Code
Conditional Disjunction	$(A \rightarrow B) \leftrightarrow (\neg A \vee B)$	CDis
Contraposition	$(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A)$	ContraPos
Definition of Equivalence	$(A \leftrightarrow B) \leftrightarrow ((A \rightarrow B) \wedge (B \rightarrow A))$	Equiv
DeMorgan's Law (1)	$\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$	DeM
DeMorgan's Law (2)	$\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$	DeM
Distribution of And over Or	$(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$	Distr
Distribution of Or over And	$(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$	Distr
Double Negation	$\neg\neg A \leftrightarrow A$	DN
Repetition	$(A \vee A) \leftrightarrow A$	Rep

Also, commute sentences over $\wedge$,V, and $\leftrightarrow$. Final operator applied in evaluation is the **main operator**.

Name	Inference	Code
Addition	From A, infer $A \vee B$(B may be any sentence)	Add
Conjunction	From $\{A,B\}$, infer $A \wedge B$	Conj
Constructive Dilemma	From $\{A \vee B, A \rightarrow C, B \rightarrow D\}$, infer $C \vee D$	CD
Contradictory Premises	From $\{A, \neg A\}$, infer B(B may be any sentence)	ContraPrm
Disjunctive Syllogism	From $\{A \vee B, \neg A\}$, infer B	DS
Hypothetical Syllogism	From $\{A \rightarrow B, B \rightarrow C\}$, infer $A \rightarrow C$	HS
Modus Ponens	From $\{A \rightarrow B, A\}$, infer B	MP
Simplification	From $A \wedge B$, infer A	Simp
Tautology	Infer $A \vee \neg A$(A may be any sentence)	Taut

7 Natural deduction

Begin by listing **premises**, then apply set of logical equivalences or rules of inference to arrive at the **conclusion**. See 1.6 of Textbook for equivalences and inference rules.

8 Quantifiers (Bonevac)

- Aristotle logic gives arguments restricted form. Every sentence is of form {some,all,no}F{are,not}G
- Syllogistic argument has two such setences as premises and one as a conclusion. The meshing of these is specific, limited.
- *Sentential logic* takes sentences as basic analytical units, covers broader realm.
- Sentential logic does not solve problems of syllogistic, e.g. cannot explain why an argument is valid
- Divergence b/w syllogistic and sentential resolved by Friege and Peirce
  - Introduced determiners (e.g. all, some, no, every, any, etc.)
  - Universal quantifier $\forall$ and existential quantifier $\exists$

8.1 Constants and Quantifiers

Atomic sentences consist of a main subject/noun phrase and verb phrase. Examples of verb phrases:

1. is a man
2. sleeps very soundly
3. kicked the ball into the end zone

Verb phrases are terms in syllogistic logic, are **true or false** of individual objects. E.g. a man can sleep soundly or not. Objects of which verb phrase **satisfy** it, phrase will **apply** to them. The set of objects which make a verb phrase true are called **extensions**.

Verb phrases combine with noun phrases to form *sentences*. Noun phrases specify an object or groups of objects, since verb phrases describe their truth values. Following examples are complete sentences.

- Socrates is a man
- Mr. Hendley sleeps very soundly
- Nate have Fred a copy of the letter

Upper-case alphabet letters w/wout subscripts are **predicates**. Each predicate has an assigned number. Predicate with number n is n-ary. Predicate yields a truth value when combined with certain number of objects. Assigned value n indicates of how many objects the predicate takes on this truth value.

For example, man has singulary predicates (true false of a single object). Binary predicates take on two objects. Example is Person 1 respects Person 2, but Person 1 respects makes no sense.

To structure sentences, take example.

Example 2. Something is missing

Missing applies to an object.

Example 3. Missing(something)

We use lowercase letters to denote constants, capital for predicates as discussed. Socrates is a man can be translated to Ma where a is a constant symbolizing Socrates, M means man.

Example 4. Translation
(for some x)(x is missing)
$\exists x Mx$

9 Quantificational Logic

Main idea is that we use predicates to describe properties of variables, which we say are *quantified*. Say we define Lxy to be that x likes y. L is our predicate, and x, y are variables, of which we can pass any number to an (appropriate) predicate.

Idea of quantifiers

- $\exists x$: there is some individual x such that
- $\forall x$: every individual x such that

9.1 Subtlety with quantifier order

Difference between

$$1.\ \exists y \forall x(Lxy)$$
$$2.\ \forall x \exists y(Lxy)$$

First: there exists someone who likes all individuals.
Second: for every individual, there exists someone who likes them.

9.2 Rules and properties

An identity predicate exists to test equality of two subjects.

Definition 9.1. (Identity predicate)

$$x = y : x \text{ is equal to } y$$

We can obviously tell that identical subjects have the same properties, so

Definition 9.2. (Leibniz' law)

$$\forall x \forall y(x = y \rightarrow (Px \leftrightarrow Py))$$

We can't "for loop" over all properties of subjects, so that must be done manually.

9.3 Describing quantities

Let Px mean that x has property P. Be aware that we can trivially extend the below equivalences to n subjects.

Statement	QL
There is exactly one thing that is P	$\exists x(Px \wedge \forall y(Py \rightarrow x = y))$
There are exactly two things that are P	$\exists x \exists y(Px \wedge Py \wedge x \neq y \wedge \forall z(Pz \rightarrow (z = x \vee z = y))$
There are at least two things that are P	$\exists x \exists y(Px \wedge Py \wedge x \neq y)$
There are less than two things that are P	$\forall x \forall y(\neg Px \vee \neg Py \vee x = y)$ (this is not identical to the first)

9.4 Interpretations

To make interpretations of sentences (i.e. constant values using names) we set $UD = \{a, b, c\}$ or whatever, then interpret the predicates with these names Pa, Pb, Pc, etc. Here we can cherry pick a, b, c to arbitrarily give truth values to P when interpreted. Useful for informal proofs.

9.5 Natural deduction

Below is a list of inferences or tautologies for natural deduction.

Name	Tautology or Inference	Code
Quantifier Exchange (occurs both ways)	$\exists x(\ldots x \ldots) \leftrightarrow \forall x \neg(\ldots x \ldots)$	QEx
Universal Instantiation	$\forall x(\ldots x \ldots) \because \ldots a \ldots$ where a is a new name in proof	UI
Universal Generalization	$\ldots a \ldots \therefore \forall x(\ldots x \ldots)$ iff a was declared by UI	UG
Existential Instantiation	$\exists x(\ldots x \ldots) \because \ldots a \ldots$ where a is any name	EI
Existential Generalization	$\ldots a \ldots \therefore \exists x(\ldots x \ldots)$	EG
Leibniz Law	$\forall x \forall y(x = y \rightarrow (Px \leftrightarrow Py))$	Leibniz