

# Multivariable Calculus Reference

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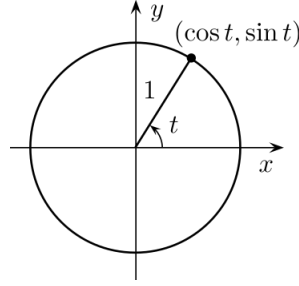
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# 1 Paths whose image curve is a circle

## 1.1 Unit Circle

Unit circle is set of points in  $\mathbb{R}^2$  defined as  $C = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ . Ellipse is  $C = \{(x, y) \in \mathbb{R}^2 | \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$  Has standard parameterization of  $\vec{c}(t) = (\cos(t), \sin(t))$ . When parameterizing, always start from  $t = 0$  reference unless otherwise given.



Properties

- Image of  $\vec{c}$  is a closed curve (has no endpoints, plane is divided into  $\geq 2$  disjoint regions)
- Image of  $\vec{c}$  is a simple curve; no self-intersection
- $\vec{c}(t)$  is an **injective path**; path is considered injective if  $\vec{c}(t_1) = \vec{c}(t_2)$ , which implies that  $t_1 = t_2$  where these are on the open interval  $(a, b)$  even if  $a = b$
- Orientation of  $\vec{c}$  is counter-clockwise in traversal

## 1.2 Observations

$$\vec{p}(t) = (a \cos(\pm n(t \pm \theta)) + x_0, b \sin(\pm n(t \pm \theta)) + y_0)$$

If  $-t$  for  $t$ , orientation is CW, CCW is  $t$ . If  $a = b$ , then curve is a circle of radius  $a$  or  $b$ , else an ellipse with horizontal and vertical radii.  $x_0$  and  $y_0$  simply shift the center coordinate.  $n > 0 \in \mathbb{R}$  determines how many times the circle is traversed given  $t \in [0, 2\pi]$ , for example.  $\theta$  is the phase shift. When changing direction of traversal, cannot have  $a > b$  for  $[a, b]$  so to decrease argument of sin or cos must have  $-t$  for  $t$ . Starting out,  $-t$  goes through the angle range and  $t$  is just a sign flip.

# 2 Paths whose image is a line or line segment in the plane

## 2.1 Line Parametrics

A line is a 1D subspace of  $\mathbb{R}^2$ , so  $L = \{t\vec{m} | t \in \mathbb{R}\}$  for  $\vec{m} \in \mathbb{R}^2$ .  $\vec{m} = \begin{bmatrix} m_x \\ m_y \end{bmatrix}$  is the **slope vector**.

Path given by image of  $L$ :

$$\vec{c}(t) = (m_x t, m_y t), t \in \mathbb{R}$$

Can represent  $\vec{c}(t) = t\vec{m}$  as well.

Lines Main Ideas

- Image of a line is a curve (e.g.  $y = x$  represents image curve of  $\vec{c}(t) = (t, t)$ )
- Lines can have nonzero intercepts, so  $\vec{c}(t) = t\vec{m}$  represents  $y = 2x + 1$ . Line that has intercept vector  $P_0 = (x_0, y_0) \parallel \vec{m} = (m_x, m_y)$  can be expressed as:

$$\vec{c}(t) = (x_0 + tm_x, y_0 + tm_y) = \vec{P}_0 + t\vec{m}$$

Note endpoint of  $\vec{c}(t)$  is on image line (curve).

## 2.2 General Forms

2 parametric lines **collide** if they intersect and the point of intersection corresponds to the same  $t$  in both curves. If you set the parameter vector coordinates equal to each other and solve for  $t$ , a solution indicates they collide. Intersection is found by **eliminating** the parameter (solve for  $t$  in terms of either  $x$  or  $y$  and plug into the other).

General form of parameterized curve can be expressed as the following:

$$\vec{c}(t) = \left( \frac{m_x}{\Delta t}(t - a) + x_0, \frac{m_y}{\Delta t}(t - a) + y_0 \right)$$

where  $\Delta t$  is the domain interval over  $[a, b]$  and  $(x_0, y_0)$  represents the desired **starting coordinate**. This is important as when going in reverse, other coordinate can be used and slope might be negative.  $a$  is used in  $(t - a)$  because everything is conventionally done with respect to starting coordinate.

## 3 Paths whose image curve is a line in R3

### 3.1 R3 parameterization

If  $\vec{m}$  is a nonzero vector along  $L$  through origin in  $\mathbb{R}^3$ , then  $L = \{t\vec{m} | t \in \mathbb{R}\}$ ; follows that  $\vec{m} = (m_x, m_y, m_z)$ , the slope or direction vector of the line. The basic parameterization is:

$$\vec{c}(t) = (m_x t, m_y t, m_z t)$$

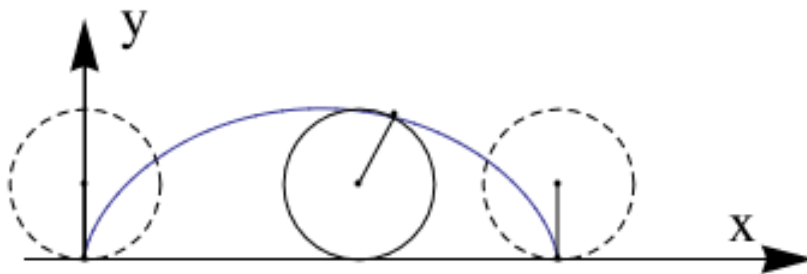
Basis vectors in  $\mathbb{R}^3$  are  $\vec{i}, \vec{j}, \vec{k}$ . Rewriting parameterization:

$$\vec{c}(t) = (x_0 + m_x t)\vec{i} + (y_0 + m_y t)\vec{j} + (z_0 + m_z t)\vec{k}$$

2 lines  $\vec{c}_1(t) = P_0 + \vec{m}_1 t$  and  $\vec{c}_2(t) = Q_0 + \vec{m}_2 t$  are parallel if direction vectors are parallel ( $\vec{m}_1 = k\vec{m}_2$ ). Collisions still exist. If neither parallel nor intersecting, considered as skew.

To determine skew, parallel, or coincide, use parameters  $s, t$  for each line and solve SOE. If same slope, rule out skew clearly, then check if  $s, t \in \mathbb{R}$ : if not, then parallel, if so, then they coincide. If intersecting and want to check if collide, some  $t$  must satisfy all relations.

## 4 Cycloid Problem



With radius 1 and passing through the origin:

$$\vec{c}(t) = (t - \sin t, 1 - \cos t)$$

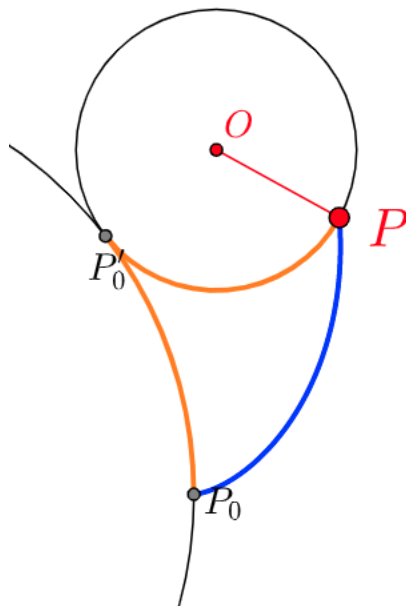
Observe that:

$$\vec{c}'(t) = (1 - \cos t, \sin t)$$

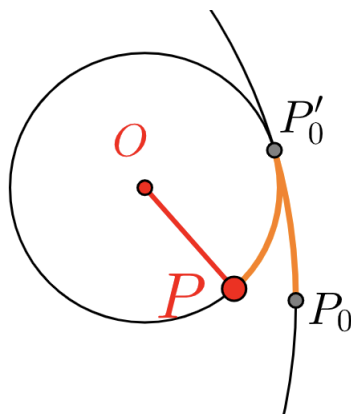
Can define the vector  $\vec{u} = \begin{pmatrix} x'(t) \\ 0 \end{pmatrix}$  such that  $\vec{u}$  is always horizontal and  $||\vec{u}|| = |x'(t)|$ . Reaches maximum value at  $t \in [k\pi | k \in \mathbb{R}]$  and is has minimum cusp where it is 0 at  $t \in [2k\pi | k \in \mathbb{R}]$ . Thus,  $x'(t) \geq 0$  always, as the x-coordinate is never decreasing.

Can also define the vector  $\vec{v} = \begin{pmatrix} 0 \\ y'(t) \end{pmatrix}$  with the same properties. Reaches maximum value when  $t \in [k\frac{\pi}{2} | k \in \mathbb{R}]$ . Can change, as observe  $t$  when  $\sin t < 0$  or  $> 0$ .

## 4.1 Hypercycloid Derivation



## 4.2 Hypocycloid Derivation



# 5 Velocity Vector

## 5.1 Definitions

Vector  $\vec{u}(t_0) + \vec{v}(t_0)$  is the velocity vector to the curve  $\vec{c}(t)$  at  $t = t_0$ .

Let  $\vec{c}: [a, b] \rightarrow \mathbb{R}^n$  have a path  $\vec{c}(t) = (x_1(t), x_2(t), x_3(t), \dots, x_n(t))$  (let  $x_i(t): [a, b] \rightarrow \mathbb{R}$  for each  $i$ )

- If  $t_0 \in [a, b]$ , then  $\vec{c}'(t_0) := (x'_1(t_0), x'_2(t_0), x'_3(t_0), \dots, x'_n(t_0))$ ; the velocity vector to  $\vec{c}$  at  $t_0$

- The path  $\vec{c}'(t_0) := (x'_1(t_0), x'_2(t_0), x'_3(t_0), \dots, x'_n(t_0))$ ; the velocity vector to  $\vec{c}$  is referred to as velocity of  $\vec{c}(t)$

Recall chain rule: if  $y = f(x)$  where  $x$  is a function of  $t$ ,  $y'(t) = x' f'(x)$ , not to be confused with product rule. Can write  $f'(x) = \frac{y'(t)}{x'(t)}$

- If  $\vec{p}(t) = \vec{c}(t) + \vec{r}(t)$ , then  $\vec{p}'(t) = \vec{c}'(t) + \vec{r}'(t)$
- If  $g(t) = \vec{c}(t) \cdot \vec{r}(t)$ , then  $g'(t) = \vec{c}'(t) \cdot \vec{r}(t) + \vec{c}(t) \cdot \vec{r}'(t)$
- If  $\vec{p}(t) = f(t)\vec{c}(t)$ , then  $\vec{p}'(t) = f'(t)\vec{c}(t) + f(t)\vec{c}'(t)$
- If  $\vec{p}(t) = \vec{c}(t) \times \vec{r}(t)$ , then  $\vec{p}'(t) = \vec{c}'(t) \times \vec{r}(t) + \vec{c}(t) \times \vec{r}'(t)$
- If  $\vec{p}(t) = \vec{c}(f(t))$ , then  $\vec{p}'(t) = f'(t)\vec{c}'(f(t))$
- If  $g(t) = \|\vec{c}(t)\|$ , then  $g'(t) = \frac{\vec{c}(t) \cdot \vec{c}'(t)}{\|\vec{c}(t)\|}$

## 5.2 Tangent Line

Tangent line can be visualized as a base vector in standard position plus a velocity vector tangent to the tip which traces a shifted line in some interval. General formula with base vector  $\vec{c}(t_0)$  and slope  $\vec{c}'(t_0)$ :

$$\ell(t) = \vec{c}(t_0) + (t - t_0)\vec{c}'(t_0)$$

## 6 Space Curves

- Projection into the  $xy$  plane is the path  $(x(t), y(t), 0)$ .
- Projection into the  $xz$ - plane is the path  $(x(t), 0, z(t))$ .
- Projection into the  $yz$  plane is the path  $(0, y(t), z(t))$ .

## 7 Speed and Arclength

### 7.1 Speed

Speed of a parametric function in  $\mathbb{R}^n$  is given by:

$$\|\vec{c}'(t)\| = \sqrt{\sum_{i=1}^n c_i(t)^2}$$

(being the magnitude of the velocity vector)

## 7.2 Arclength

Arclength of a parametric function is given by:

$$S = \int_a^b \|\vec{c}(t)\| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 + \dots} dt$$

Can approximate arclength as a sum of the lengths of secant vector approximations  $\vec{s}_i = \vec{c}(t_i) - \vec{c}(t_{i-1})$ :

$$\text{arclength} \approx \sum_{i=1}^n \|\vec{s}_i\|$$

According to the MVT, there exists a  $\hat{t}_i$  in  $(t_{i-1}, t_i)$  (open interval due to differentiability requirement) such that:

$$\begin{aligned} x'(\hat{t}_i) &= \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \\ y'(\hat{t}_i) &= \frac{y(t_i) - y(t_{i-1})}{t_i - t_{i-1}} \end{aligned}$$

This means that, since  $\vec{s}_i$  is given as the difference between 2 points, being a secant:

$$\begin{aligned} \vec{s}_i &= ((t_i - t_{i-1})x(\hat{t}_i), (t_i - t_{i-1})y(\hat{t}_i)) \\ \vec{s}_i &= (t_i - t_{i-1}) (x'(\hat{t}_i), y'(\hat{t}_i)) \\ \vec{s}_i &= (t_i - t_{i-1})\vec{c}'(\hat{t}_i) \end{aligned}$$

Thus,

$$\begin{aligned} \text{arclength} &\approx \sum_{i=1}^n \|\vec{s}_i\| \\ \text{arclength} &\approx \sum_{i=1}^n \|\Delta t \vec{c}'(\hat{t}_i)\| \\ \text{arclength} &\approx \sum_{i=1}^n \Delta t \|\vec{c}'(\hat{t}_i)\| \end{aligned}$$

Can define the arclength differential as follows:

$$ds = \sqrt{dx^2 + dy^2}$$

Can just define arclength as  $\text{arclength} = \int ds$



### 7.3 Arclength Parameterization

Higher the speed of a curve, farther the points are spaced apart. An arclength parametrization of a curve is a path whose image is the desired curve and whose speed is constantly one. Or,  $\vec{c}: [a, b] \rightarrow \mathbb{R}^n$  with  $\|\vec{c}'(t)\| = 1$  for  $t \in [a, b]$ . If a curve is not an arclength parameterization, then can do  $\frac{\vec{c}(t)}{\|\vec{c}'(t)\|}$  but only dividing the coefficients (slopes).

When speed is variable, is difficult to define arclength parameterization. Thus, can define displacement to be  $s(t) = \int_a^t v(t) dt$ . If  $v(t) \neq 0$ , then  $s$  is injective because according to FTC,  $s'(t) = v(t)$ . By definition,  $v'(t) \geq 0$  always since it is composed of a radical, so it must be **increasing**. Thus, if  $t_1 = t_2$ ,  $s(t_1) \neq s(t_2)$ . Arclength parameterization:

$$s(t) = \int_0^t \|\vec{c}'(u)\| du$$

This means that  $s$  is invertible, so can solve for  $t$  to get  $t = \varphi(s)$ . An arclength parameterization can be found by:

$$\boxed{\vec{p}(s) = \vec{c}(\varphi(s))}$$

## 8 Curvature

### 8.1 Proofs

Recall that to make an arclength parameterization accumulate the magnitudes of infinitesimal velocity vectors:

$$s(t) = \int_a^t \|\vec{r}'(u)\| du$$

Given some curve  $\vec{r}(t)$ , define an arclength parameterization by  $\vec{r}(g(s)) \rightarrow \vec{r}_1(s)$ , so  $\vec{r}$  is defined in terms of  $s$ . The unit tangent vector  $\vec{T}_1(s)$  is then  $\frac{\vec{r}_1'(s)}{\|\vec{r}_1'(s)\|} = \vec{r}_1'(s)$ .

$$\begin{aligned} \vec{T}_1(s) &= \vec{r}_1'(s) \\ &= \frac{d}{ds} \vec{r}_1(s) \\ &= \frac{d}{ds} \vec{r}(g(s)) \\ &= \vec{r}'(g(s)) \cdot g'(s) = \vec{r}'(t) \cdot \frac{dt}{ds} \\ &= \frac{\vec{r}'(t)}{\frac{ds}{dt}} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \end{aligned}$$

This means that  $\vec{T}_1(s) = T(t)$

Continuing, to find curvature  $\kappa(t)$ :

$$\begin{aligned}
\vec{T}'_1(s) &= \frac{d}{ds} \vec{T}(t) \\
&= \frac{d}{ds} \vec{T}(g(s)) \\
&= \vec{T}'(t) \cdot \frac{dt}{ds} \\
&= \frac{\vec{T}'(t)}{\frac{ds}{dt}} \\
&= \frac{\vec{T}'(t)}{\|\vec{r}'(t)\|}
\end{aligned}$$

Thus,  $\boxed{\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}}$

## 8.2 Definition

Given a curve  $C$  parameterized with arclength by the path  $\vec{c}: [a, b] \rightarrow \mathbb{R}^n$ , curvature is defined as:

$$\boxed{\kappa(s) = \|\vec{T}'(s)\|}$$

where  $\vec{c}'(s) \neq 0$  and  $\vec{T}(s) = \frac{\vec{c}'(s)}{\|\vec{c}'(s)\|}$  (normalized slope vector).

A loose geometric interpretation is that a greater  $\kappa(s)$  implies more curvature, that is, the curve is changing at a greater rate there. When  $\vec{c}'(s) \neq 0$  is always true for a curve, it is **regular**. Is defined in terms of arclength parameterization so curvature is an intrinsic property of the curve independent of parameterization.

Formula for curvature at the point  $\vec{c}(t)$ :

$$\boxed{\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{c}'(t)\|} = \frac{\|\vec{c}'(t) \times \vec{c}''(t)\|}{\|\vec{c}'(t)\|^3}}$$

## 9 Motion in 3D space

Given a path  $\vec{c}: \mathbb{R} \rightarrow \mathbb{R}^3$  with  $\vec{c}(t) = (x(t), y(t), z(t))$ , then we have defined:

- $\vec{v}(t) = \vec{c}'(t) = (x'(t), y'(t), z'(t))$  is also path in  $\mathbb{R}^3$  called the velocity of  $\vec{c}$
- $\vec{a}(t) = \vec{v}'(t) = \vec{c}''(t) = (x''(t), y''(t), z''(t))$  is also a path in  $\mathbb{R}^3$  called the acceleration of  $\vec{c}$
- $v(t) = \|\vec{v}(t)\| = \|\vec{c}'(t)\|$  is a scalar valued function on  $\mathbb{R}$  (that's a fancy way of saying the domain and codomain of this function are both  $\mathbb{R}$ ) called the speed of  $\vec{c}$

- $\vec{T}(t) = \frac{\vec{c}'(t)}{v(t)}$  is also a path in  $\mathbb{R}^3$  called the unit tangent to  $\vec{c}$
- $\kappa(t) = \frac{\vec{T}'(t)}{v(t)} = \frac{\|\vec{c}'(t) \times \vec{c}''(t)\|}{\|\vec{c}'(t)\|^3}$  is a scalar valued function on  $\mathbb{R}$  called the curvature of  $\vec{c}$

Note that  $\vec{T} \cdot \vec{T} = \|\vec{v}\|^2 = 1$ . Computing the derivative,  $\frac{d}{dt} \vec{T} \cdot \vec{T} = 2\vec{T} \cdot \vec{T}' = 0$  This means that  $\vec{T} \perp \vec{T}'$ .

Define  $\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$  as the unit normal vector, which is the unit tangent to the unit tangent.

From observation,  $\vec{T} \perp \vec{N}$ . Fact: the acceleration vector always lies in the plane spanned by  $\vec{N}$  and  $\vec{T}$ .

Acceleration  $\vec{a}(t)$  is thus split component-wise into  $a_T$  from  $\vec{T}$  and  $a_N$  from  $\vec{N}$ :

$$a_T = v'(t) = \frac{\vec{a}(t) \cdot \vec{v}(t)}{v(t)}$$

$$a_N = \kappa(t)v(t)^2 = \frac{\|\vec{a}(t) \times \vec{v}(t)\|}{v(t)} = \sqrt{\|\vec{a}(t)\|^2 - |a_T|^2}$$

## 10 Derivatives of parameterized curves

### 10.1 Arclength parameterization derivation

Take the following function:

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

An arclength parameterization is achieved with the following computation:

$$s = \int_0^t \sqrt{x'(u)^2 + y'(u)^2} du$$

Can say that  $t = g(s)$ , so the arclength parameterization, which is the path in terms of  $s$ :

$$\vec{r}_1(s) = \langle x(g(s)), y(g(s)) \rangle$$

Taking the derivative by the chain rule:

$$\begin{aligned} \vec{r}_1'(s) &= \langle x'(g(s)) \cdot g'(s), y'(g(s)) \cdot g'(s) \rangle \\ &= g'(s) \langle x'(t), y'(t) \rangle \end{aligned}$$

Note that  $g'(s) = \frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{\|\vec{r}'(t)\|}$  by taking the derivative of the integral for arclength:

$$\vec{r}'_1(s) = \frac{1}{\|\vec{r}'(t)\|} \langle x'(t), y'(t) \rangle = \vec{T}(t)$$

Following from this,  $g'(s) = \frac{1}{\|\vec{r}'(g(s))\|}$  so  $g''(s) = -\frac{1}{\|\vec{r}'(g(s))\|^2} \cdot g'(s) = -\frac{1}{\|\vec{r}'(t)\|^3}$

## 10.2 Orthogonal derivative and position vectors

Observe that  $\vec{r}' \cdot \vec{r}' = \|\vec{r}'\|^2$ . Thus,  $\frac{d}{dt}[\vec{r}' \cdot \vec{r}'] = 2\vec{r}' \cdot \vec{r}'' = 2\|\vec{r}'\|\|\vec{r}''\|$ . Rearranging:  $\frac{\vec{r}' \cdot \vec{r}''}{\|\vec{r}'\|} = \|\vec{r}'\|'$ . Means that magnitude of position vector has to be a constant value in order for it to be  $\perp$  to derivative.

## 11 Planetary motion

- Law of ellipses – orbit of planet is ellipse with sun as focus
- Law of equal area in equal time – position vector pointing from sun to planet sweeps out equal area in equal time (so speed must increase/decrease)

Can approximate the area swept in time by  $\frac{dA}{dt} = \frac{1}{2}\|\vec{r}(t) \times \vec{r}'(t)\| = \frac{1}{2}\|\vec{J}\|$ . The differential equation for each of Kepler's laws is:  $\vec{r}''(t) = -\frac{k}{\|\vec{r}(t)\|^3}\vec{r}(t)$ , so it is in the direction of  $\vec{r}(t)$ . Thus, differentiating  $\frac{d\vec{J}}{dt} = \frac{d}{dt}(\vec{r}'(t) \times \vec{r}''(t)) = 0$ .

### 11.1 Cross-product identities

Cross product identities:

- $\vec{u} \times (\vec{v} \times \vec{w}) = (u \cdot w)\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$
- $u \cdot (\vec{v} \times \vec{w}) = v \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v})$