Competitive Math Notes

Sidharth Baskaran

September 4, 2021

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1 Quadratics and Polynomials

1.1 Basic factoring

Given some ax^2+bx+c, try and find the factorization $(sx+u)(tx+v)$ by expanding this template expression.

Remark. If one root is 0, the product of roots is 0 so the equation is of form $ax^2+bx=0$.

Remark. Difference of squares are of the form $x^2-a^2=0 \implies (x-a)(x+a)=0$.

Remark. A perfect square is of the form $(x+a)^2=x^2+2ax+a^2$. This is a case of a "double root."

1.2 Quadratic Formula

By manipulating $ax^2+bx+c=0$ by completing the square, the quadratic formula can be found.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{1}$$

1.3 Expansions

$$(a+b)^2=a^2+2ab+b^2 \tag{2}$$

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3 \tag{3}$$

1.4 Factoring

Definition 1.1 (Difference of squares)**.**

$$a^2-b^2=(a+b)(a-b) \tag{4}$$

Definition 1.2 (Sum of squares)**.**

$$a^2+b^2=(a+b)^2-2ab \tag{5}$$

Definition 1.3 (Sum of cubes)**.**

$$a^3+b^3=(a+b)(a^2-ab+b^2) \tag{6}$$

Definition 1.4 (Difference of cubes)**.**

$$a^3-b^3=(a-b)(a^2+ab+b^2) \tag{7}$$

Definition 1.5 (Some cube identity)**.**

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-ac-bc) \tag{8}$$

Definition 1.6 (Simon factoring trick)**.** Used to factor in a diophantine equation. If $ab+ka+nb=c$, then $(a+n)(b+k)=c+nk$.

1.5 Vieta's Formulas

$$x^2 + ax + b = (x - p)(x - q) \tag{9}$$
$$x^2 + ax + b = x^2 - (p + q)x + pq \tag{10}$$

Thus, $a = p + q$ and $b = pq$. Generally,

$$\begin{aligned}
s_1 &= & r_1 + r_2 + r_3 + \cdots + r_n & &= -\frac{a_{n-1}}{a_n} \\
s_2 &= & r_1r_2 + r_1r_3 + r_1r_4 + \cdots + r_{n-2}r_{n-1} & &= \frac{a_{n-2}}{a_n} \\
s_3 &= & r_1r_2r_3 + r_1r_2r_4 + \cdots + r_{n-2}r_{n-1}r_n & &= -\frac{a_{n-3}}{a_n} \\
& & \vdots & & \\
s_n &= & r_1r_2r_3 \cdots r_n & &= (-1)^n \frac{a_0}{a_n}.
\end{aligned}$$

Definition 1.7 (AM-GM Inequality)**.**

$$\frac{a_1 + \ldots + a_n}{n} \geq \sqrt[n]{a_1a_2 \ldots a_n} \tag{11}$$

2 Systems

Definition 2.1 (Useful fraction property)**.** If there exists some $k \in \mathbb{R}$ such that $k = \frac{a}{b} = \frac{c}{d}$, then $\frac{a+c}{b+d} = k$ as well. Can be extended to n equal fractions of this form.

3 Sequences

Sum for finite arithmetic series with starting value a:

$$\frac{2a + (n-1)d}{2} \cdot n \tag{12}$$

Sum for infinite geometric series:

$$\frac{a}{1 - r} \tag{13}$$

Sum for finite geometric series:

$$\frac{a(1 - r^n)}{1 - r} \tag{14}$$

4 Functions and Polynomials

4.1 Floor function

The floor function $\lfloor x \rfloor$ yields greatest integer leq to argument. For positive values, equivalent to rounding down (truncating decimals). For negative values, equivalent to next lowest negative integer.

Example 1.

$$\lfloor -3.2 \rfloor = -4$$

A useful simplification is

$$\lfloor x \rfloor = \lfloor y + k \rfloor \tag{15}$$

where y is an integer and $0 \leq k < 1$. An alternate definition is

$$\lfloor x \rfloor = x - \{x\} \tag{16}$$

where $\{x\}$ is the fractional component of x.

4.2 Change of base formula

Express a logarithm in base b using logarithms of base d. Let $d, a, b \in \mathbb{R}$ s.t $d, b \neq 1$.

$$\log_b a = \frac{\log_d a}{\log_d b} \tag{17}$$

4.3 Polynomial division

Definition 4.1 (Polynomial remainder theorem)**.** Upon dividing any polynomial $P(x)$ by linear polynomial $x - a$, the remainder is $P(a)$.

We can express $P(x)$ as the following

$$P(x) = (x - a)Q(x) + R(x) \tag{18}$$

where $x - a$ is the dividend, $Q(x)$ is the quotient and $R(x)$ is the remainder. Also, $\deg R(x) < \deg(x - a)$, hence $R(x) \in \mathbb{R}$.

The general approach is $P(x) = D(x)Q(x) - R(x) \implies R(x) = D(x)Q(x) - P(x)$. Find zeros of $D(x)$ to eliminate $Q(x)$ and thus find $R(x)$ through substitution into $-P(x)$.

4.4 Inverse functions

Definition 4.2 (Inverse of a function)**.** Let $f(x) : A \to B$ with range C. The inverse is $f^{-1}(x) : C \to A$ iff f is injective (i.e. a distinct one-to-one mapping from every value in A to C). Can verify via horizontal line test.

The properties are $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$