Differential Equations

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1 Definitions, Families of Curves

1.1 Definitions

Definition 1.1 (Order)**.** Order of a DE is the highest-ordered derivative appearing in it. So

$$\frac{d^2y}{dx^2} + 2b(\frac{dy}{dx})^3 + y = 0$$

is a 2nd order DE. In general,

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0.$$

is an n-th order DE. Under restrictions on F, can find a solution in terms of the other $n+1$ variables

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}).$$

Definition 1.2 (Solution)**.** A function ϕ on interval $x \in (a, b)$ is a solution to the DE if the n derivatives exist on $x \in (a, b)$ and $\phi^{(n)}(x) = f(x, \phi(x), \ldots, \phi^{(n-1)}(x))$.

Definition 1.3 (First order DE)**.** A first order DE is of the form

$$\frac{dy}{dx} = f(x, y)$$

with solution of the form $y = f(x)$. Can be rewritten for convenience in the form

$$M(x, y)dx + N(x, y)dy = 0$$

Definition 1.4 (Linear ODE)**.** An ODE of order n is linear if it can be written in the form

$$b_0(x)\frac{d^ny}{dx^n} + b_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + b_{n-1}(x)\frac{dy}{dx} + b_n(x)y = R(x)$$

Definition 1.5 (Partial DE)**.** Is of the form, for example

$$b_0(x, y)\frac{\partial w}{\partial x} + b_x(x, y)\frac{\partial w}{\partial y} = R(x, y)$$

1.2 Families of Solutions

Solutions to the DE

$$\frac{dy}{dx} = f(x, y) \Leftrightarrow y = \int f(x)dx + c$$

exist as one-parameter families with parameter c.

1.3 Isoclines

Let there be the DE

$$\frac{dy}{dx} = y$$

Isoclines are lines $f(x, y) = y = c$. Example:

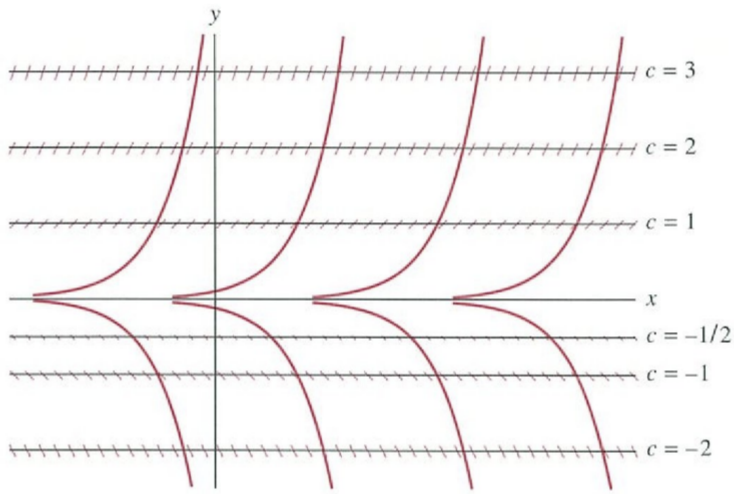


Figure 1: Isoclines of $\frac{dy}{dx} = y$

1.4 Existence Theorem

Consider equation*

$$\frac{dy}{dx} = f(x, y)$$

Further, let T denote the rectangle defined by

$$|x - x_0| \leq a \quad (1)$$
$$|y - y_0| \leq b \quad (2)$$

with the point (x_0, y_0) as the center. Also let $f, \frac{\partial f}{\partial y}$ be continuous functions of x, y in T.

With these conditions an interval exists for x_0 where $|x - x_0| \leq h$, and function $y(x)$ which has properties

1. $y = y(x)$ is a sol'n of the DE on interval $|x - x_0| \leq h$
2. On this interval, $|y(x) - y_0| \leq b$
3. $y = y(x_0) = y_0$ at $x = x_0$
4. $y(x)$ is unique on interval $|x - x_0| \leq h$ where it is the only function with above 3 properties

2 Equations of Order One

2.1 Separation of Variables

We begin with equation*s of the form

$$Mdx + Ndy = 0$$

where M and N can be multivariate of x, y.

It is separable iff

$$A(x)dx + B(y)dy = 0.$$

Then find a function F with total differential being the LHS of above, so $F = c$.

2.2 Homogeneity

Definition 2.1 (Homogeneity of polynomials)**.** Polynomials where all terms are of the same degree are homogeneous.

Homoegeneity of functions is analogous to assigning physical dimensions (e.g. length) to all of the variables. If the function has the length dimension to the kth power, then it is homogeneous of degree k.

Example 2.2. If x, y are lengths, then the following is homogeneous of degree 3.

$$f(x, y) = 2y^3 \exp(\frac{y}{x}) - \frac{x^4}{x + 3y}$$

Alternate definition also suffices for generality.

Definition 2.3 (Homogeneous function)**.** $f(x, y)$ is homogeneous of degree k iff $f(\lambda x, \lambda y) = \lambda^k f(x, y)$.

Definition 2.4 (Alternate definition of homogeneity)**.** If $f(x, y)$ can be rewritten as $f(\frac{y}{x})$ or $f(\frac{x}{y})$ then it is homogeneous.

2.3 Homogeneous Differential Equations

Theorem 2.5 (Homogeneous DEs)**.** If $M(x, y)$ and $N(x, y)$ are homogeneous and of same degree, then $M(x, y)dx + N(x, y)dy = 0$ is a homogeneous DE.

Theorem 2.6 (Homogeneous DEs)**.** $M(x, y)/N(x, y)$ is homogeneous of degree 0.

Proof. If M, N are homogeneous of some degree n, then

$$M(x, y) = M(\lambda x, \lambda y) = \lambda^n M(x, y) \tag{3}$$
$$M(x, y) = N(\lambda x, \lambda y) = \lambda^n N(x, y) \tag{4}$$

So for M/N,

$$\frac{M(x, y)}{N(x, y)} = \lambda^0 \frac{M(1, v)}{N(1, v)} \tag{5}$$

which is homogeneous of degree 0. □

Theorem 2.7 (Homogeneous DEs)**.** If $f(x, y)$ is homogeneous of degree 0 in x, y, then $f(x, y)$ is a function of y/x alone.

Proof. Let $y = vx$. Then, f must be proven to be of v alone. Substitute so that

$$f(x, y) = f(x, vx) = x^0 f(1, v) = f(1, v)$$

□

2.4 Homogeneous Coefficients

Suppose coefficients M, N in the equation*

$$M(x,y)dx + N(x,y)dy = 0$$

are both homogeneous of same degree. So M/N is a function of y/x alone. Can put in form using $y = vx$.

$$\frac{dy}{dx} + g(\frac{y}{x}) = 0 \tag{6}$$

$$x\frac{dv}{dx} + g(v) = 0 \tag{7}$$

This last equation* is separable.

2.5 Exact Equations

If there exists an equation* of the form $A(x)dx + B(y)dy = 0$, the solution is a function with differential $A(x)dx + B(y)dy$. Idea works for equation*s of form

$$dF = Mdx + Ndy.$$

So, $F(x,y) = c \implies dF = 0$ and

$$Mdx + Ndy = 0.$$

If there's a function F such that $Mdx + Ndy$ is the **total differential** of F, then Eq. 5 is an *exact equation** by definition. Can rewrite the total differential from the chain rule:

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy.$$

So $M = \frac{\partial F}{\partial x}, N = \frac{\partial F}{\partial y}$. We can take 2nd derivative to show these are equal because the partials are continuous (Clairaut's theorem).

$$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} \tag{8}$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial y \partial x}. \tag{9}$$

Definition 2.8 (Exactness)**.**

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Proof. Let $\phi(x,y)$ be a function where $\frac{\partial\phi}{\partial x} = M$. ϕ is the function you get from integrating Mdx wrt x and holding y. Then

$$\frac{\partial^2\phi}{\partial y\partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Integrating both sides wrt x:

$$\frac{\partial\phi}{\partial x} = N + B'(y)$$

where $B'(y)$ is the integration constant. Let

$$F = \phi(x,y) - B(y)$$

such that

$$\begin{aligned} dF &= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy - B'(y)dy \\ &= Mdx + \left[N + B'(y)\right]dy - B'(y)dy \\ &= Mdx + Ndy \end{aligned}$$

$\square$

Example 2.9. We have the DE

$$3x(xy-2)dx + (x^3+2y)dy = 0.$$

Then,

$$\frac{\partial M}{\partial y} = 3x^2, \frac{\partial N}{\partial y} = 3x^2$$

The DE is exact, and $F = c$ is the solution.

$$\frac{\partial F}{\partial x} = M = 3x^2y - 6x \tag{10}$$

$$\frac{\partial F}{\partial y} = N = x^3 + 2y \tag{11}$$

Try to find F from 18, integrate both sides wrt x with an integration constant $T(y)$.

$$F = x^3y - 3x^2 + T(y)$$

Using Eq. 19, can can find $\frac{\partial F}{\partial y}$ from Eq. 20 and equate:

$$x^3 + T'(y) = x^3 + 2y \implies T'(y) = 2y$$

Because $F = c$ is the I.C., can conclude

$$T(y) = y^2$$

Thus,

$$F = x^3y - 3x^2 + y^2 \Leftrightarrow x^3y - 3x^2 + y^2 = c$$

2.6 Linear Equations of Order 1

If an equation* is not exact, can attempt to do so by multiplying DE by an integrating factor.

Definition 2.10 (Linear DE of order 1)**.**

$$A(x)\frac{dy}{dx} + B(x)y = C(x)$$

Divide each side by $A(x)$ to obtain

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Suppose there exists for Eq. 25 a I.F. $v(x) > 0$. Then,

$$v(x)\left[\frac{dy}{dx} + P(x)y\right] = v(x)Q(x)$$

becomes exact, or of form $Mdx + Ndy = 0$. Here,

$$M = vPy - vQ \tag{12}$$
$$N = v \tag{13}$$

Because the requirement is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,

$$\begin{aligned} vP &= \frac{dv}{dx} \\ Pdx &= \frac{dv}{v} \\ \ln v &= \int Pdx \\ v &= \exp(\int Pdx) \end{aligned}$$

We can then multiply both sides of the DE by this I.F. One side of this eqn will be of the product rule form, the derivative of $y \exp(\int P dx)$:

$$\exp(\int P dx)\frac{dy}{dx} + P\exp(\int P dx)y = Q\exp(\int P dx)$$

2.7 General Solution of a Linear Equation

Given the original form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

suppose P and Q are continuous on $x \in (a,b)$ and $x = x_0$ is such a number. $y = y_0$ satisfies the initial condition. This sol'n satisfies Eq. 34 for all x in the interval. Multiplying Eq. 34 by integrating factor $\exp(\int P dx)$ gives

$$yv = \int vQ\,dx + c$$

Because $v \neq 0$,

$$y = v^{-1}\int vQ\,dx + cv^{-1}$$

Given any x_0, y_0 in the interval, can find c s.t. the DE is satisfied. Every eqn of above form will have P, Q with common interval of continuity and a unique set of solutions with one I.C. obtained by using the integrating factor. These solutions are unique, so any other method yields a solution that align*s with the general solution–all possible solutions satisfying the DE on $x \in (a,b)$.

2.8 Application of Mixing Problem

Strategy is to determine the differential equation* describing rate of change of a certain quantity, then finding the particular solution with some trivial IC.

Example 2.11. 100 liter tank contains 10 kg salt mixed with 60 liter water. Sol'n with concentration 0.1 $\frac{\text{kg}}{\text{liter}}$ flows in at rate 5 liters/min. Solution is well stirred (asume equal distribution), outflow rate of 3 liters/min. Need to find salt in tank when it is full.

Note that the tank will become full, as in $-$ out > 0. Let x be kg of salt. Then, inflow rate is $0.1\frac{\text{kg}}{\text{liter}} \cdot 5\frac{\text{liter}}{\text{min}} = 0.5\frac{\text{kg}}{\text{min}}$. Out is $x\frac{\text{kg}}{60\text{ liter}} \cdot 3\frac{\text{liter}}{\text{min}} = \frac{x}{20}\frac{\text{kg}}{\text{min}}$. We then express the DE as

$$\frac{dx}{dt} = 0.5 - \frac{x}{20}$$

Then just express in linear form, solve with I.F. method.

Example 2.12. Initially 50 gallons of brine, 10 lb dissolved salt. Inflow of 2 lb salt/gal at 5 gal/min, outflow of 3 gal/min, but **mixture kept uniform**.

Inflow is thus 10 lb/min, outflow is $\frac{3x}{50+2t}$ lb/min. Key here is that mixture concentration on outflow does not change, so the volume dynamically adapts for changing amount of salt. 50 gallons initially, influx of 5 gal - 3 gal out $\implies 50 + (5-3)t$. DE is thus

$$\frac{dx}{dt} = 10 - \frac{3x}{50+2t}$$

2.9 Integrating Factor by Inspection

By recognizing differentials in a problem, can find the integrating factor by inspection.

Example 2.13. Given

$$ydx + (x + x^3y^2)dy = 0$$

the terms can be grouped by like degree so

$$(ydx + xdy) + x^3y^2dy = 0.$$

Can be rewritten as

$$d(xy) + x^3y^2dy = 0$$

then divide by $(xy)^3$ for it does not affect integrability of $d(xy)$ term but keeps function of y with dy term,

$$\frac{d(xy)}{(xy)^3} + \frac{dy}{y} = 0.$$

Integrating:

$$\begin{aligned} \int (xy)^{-3}d(xy) + \int \frac{dy}{y} &= 0 \\ \frac{(xy)^{-2}}{-2} + \ln|y| &= C \\ Cy &= \frac{(xy)^{-2}}{2} \\ Cy(xy)^2 &= 1 \end{aligned}$$

2.10 Determining Complex Integrating Factors

Let there be the DE

$$Mdx + Ndy = 0.$$

Suppose $\exists u$, possibly of both x, y that is an integrating factor such that

$$uMdx + uNdy = 0$$

and for it to be exact,

$$\frac{\partial}{\partial y}(uM) = \frac{\partial}{\partial x}(uN)$$

so u satisfies

$$u\frac{\partial M}{\partial y}+M\frac{\partial u}{\partial y}=u\frac{\partial N}{\partial x}+N\frac{\partial u}{\partial x} \tag{14}$$

$$u(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})=N\frac{\partial u}{\partial x}-M\frac{\partial u}{\partial y}. \tag{15}$$

This does not lead anywhere, so let u be a function of x. Thus, $\partial u/\partial y=0, \partial u/\partial x=du/dx$. So the above reduces to

$$u(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})=N\frac{du}{dx}\Leftrightarrow\frac{1}{N}(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})dx=\frac{du}{u}$$

and integrating factor is the following, assuming LHS of above is a function of x or y alone

$$u=\exp\left[\int f(x)dx\right] \text{ (for } x\text{)} \tag{16}$$

$$u=\exp\left[\int -g(y)dy\right] \text{ (for } y\text{)} \tag{17}$$

3 Linear Equations with Constant Equations

3.1 Auxiliary Equation

Linear homogeneous DE with constant coefficients can be expressed as

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy}{dx}+a_ny=0$$

Can be written in the form

$$f(D)y=0$$

where $f(D)$ is a linear differential operator. If the algebraic eqn $f(m)=0$ then we know $f(D)e^{mx}=0 \implies y=e^{mx}$ is a solution to the form above. $f(m)=0$ is the auxiliary equation* associated with the DE. Since the DE is of order n, the auxiliary equation* is of degree n with roots $m_1,\ldots,m_n$.

Thus we have n solutions $y_1=\exp(m_1x),\ldots,y_n=\exp(m_nx)$ assuming the roots are **real** and **distinct** are then **linearly independent**. The general solution is thus

$$y=c_1\exp(m_1x)+\cdots+c_n\exp(m_nx)$$

with arbitrary constants $c_1,\ldots,c_n$.

3.1.1 Derivation

We can say that $y^{(k)}$ is the kth derivative of y, so say the general form is

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

If we take $y = e^{rx}$, then observe $y^{(n)} = r^n e^{rx}$. So rewrite the general form as

$$\begin{aligned} a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_1 r e^{rx} + a_0 e^{rx} &= 0 \\ a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 &= 0 \end{aligned}$$

Solving for the roots r in this characteristic equation* helps us obtain the general solution.

3.2 Auxiliary Equation Repeated Roots

Need method for obtaining n linearly independent solutions for n equal roots of auxiliary equation*. Suppose auxiliary equation* $f(m) = 0$ has n roots $m_1 = m_2 = \cdots = m_n = b$. Thus, the operator function $f(D)$ has a factor $(D-b)^n$. Want to find n linearly independent y for which $(D-b)^n y = 0$. Use the substitution $y_k = x^k e^{bx}$ such that

$$(D-b)^n (x^k e^{bx}) = 0, \ k = 0, 1, 2, \ldots, n-1$$

The functions $y_k = x^k e^{bx}$ are linearly independent because the respective powers $x^0, \ldots, x^k$ are linearly independent. So the general solution takes form

$$y = c_1 e^{bx} + c_2 x e^{bx} + \cdots + c_n x^{n-1} e^{bx}$$

3.3 Complex roots for the auxiliary equation*

Complex roots typically come in conjugate pairs of the form $a \pm bi$. So using the general form of the auxiliary solution and the Euler identity $e^{i\theta} = \sin\theta + i\cos\theta$ and the fact that $\cos(-x) = \cos x, \sin(-x) = -\sin x$, we find that

$$\begin{aligned} y_1 &= c_1 e^{(a+bi)x} = c_1 e^{ax} e^{bxi} = c_1 e^{ax} \left(\sin bx + i \cos bx\right) \\ y_2 &= c_2 e^{(a-bi)x} = c_2 e^{ax} e^{-bxi} = c_2 e^{ax} \left(-\sin bx + i \cos bx\right) \\ y_1 + y_2 &= c_1 e^{ax} \left(\sin bx + i \cos bx\right) + c_2 e^{ax} \left(-\sin bx + i \cos bx\right) \\ &= e^{ax}(c_1 - c_2) \sin bx + e^{ax}(c_1 + c_2) i \cos bx \\ &= c_3 e^{ax} \sin bx + c_4 e^{ax} \cos bx \end{aligned}$$

3.4 Linear Independence

Linear independence of two functions f_1, f_2 is calculated with a determinant called the Wronskian $W(f_1, f_2)$.

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ {f_1}' & {f_2}' \end{vmatrix}$$

Let f_1, f_2 be differentiable on $[a, b]$. if $W(f_1, f_2)(t_0) \neq 0$ for some $t \in [a, b]$ then f_1, f_2 are linearly independent on $[a, b]$. If they are linearly dependent, $W(f_1, f_2)(t) = 0 \; \forall t$.

3.5 Review of rational roots, synthetic division

For rational roots test, take factors of constant coeff. c divided by factors of leading coefficient a to get possible roots.

3.6 Undetermined Coefficients

Can apply to the general form of a problem where

$$f(D)y = R(x)$$

Roots of the aux. eqn $f(m) = 0$ are $m = m_1, \ldots, m_n$. General sol'n is $y = y_c + y_p$. We suppose that $R(x)$ is a particular solution of some equation* equation* $g(D)R = 0$. This equation* has its own set of roots $m' = m_1', \ldots, m_n'$, which you can find from inspection of R.

So the DE $g(D)R = 0$, or

$$g(D)f(D)y = 0$$

has the roots from m, m'. So the solution is of the form $y = y_c + y_q$ where y_q satisfied $g(D) = 0$. We now observe that a particular solution of the original DE $f(D)y = R(x)$ must also satisfy the above DE. Thus,

$$f(D)(y_c + y_q) = R(x)$$

Since $f(D)y_c = 0$,

$$f(D)y_q = R(x)$$

So we can find coefficients to satisfy $y_q = y_p$. Method can only be applied when $R(x)$ is a solution to homogeneous linear DE with constant coefficients.

3.7 Variation of Parameters

We start with the general equation*

$$ay'' + by' + cy = g(x)$$

First, find complementary solution y_c from $y = y_p + y_c$. This is simply the solution of the homogeneous equation*

$$ay'' + by' + cy = 0$$

Then, find the particular solution $y_p = u_1 y_1 + u_2 y_2$ using Cramer's rule. First, we differentiate y_p and impose the condition $\dot{u}_1 y_1 + \dot{u}_2 y_2 = 0$.

$$\begin{aligned}
y_p &= u_1 y_1 + u_2 y_2 \\
\dot{y}_p &= \dot{u}_1 y_1 + u_1 \dot{y}_1 + \dot{u}_2 y_2 + u_2 \dot{y}_2 \\
&= u_1 \dot{y}_1 + u_2 \dot{y}_2 \\
\ddot{y}_p &= \dot{u}_1 \dot{y}_1 + u_1 \ddot{y}_1 + \dot{u}_2 \dot{y}_2 + u_2 \ddot{y}_2
\end{aligned}$$

Plugging into the original equation*,

$$\begin{aligned}
&a(\dot{u}_1 \dot{y}_1 + u_1 \ddot{y}_1 + \dot{u}_2 \dot{y}_2 + u_2 \ddot{y}_2) \\
&+ b(u_1 \dot{y}_1 + u_2 \dot{y}_2) \\
&+ c(u_1 y_1 + u_2 y_2) \\
&= a\dot{u}_1 \dot{y}_1 + a\dot{u}_2 \dot{y}_2 = g(x)
\end{aligned}$$

So the conditions we have to solve for $\dot{u}_1, \dot{u}_2$ are

$$\begin{cases} \dot{u}_1 y_1 + \dot{u}_2 y_2 = 0 \\ \dot{u}_1 \dot{y}_1 + \dot{u}_2 \dot{y}_2 = \frac{g(x)}{a} \end{cases}$$

When we set up the matrix-vector equation* $\mathrm{A}\mathbf{x} = \mathbf{b}$, it corresponds to the following:

$$\begin{bmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{g(x)}{a} \end{bmatrix}$$

Following Cramer's rule, to find the solutions $1, \ldots, n$, we create A_n by replacing the nth row of A with b.

Strategy is to find u_1', u_2' then integrate to find the coefficients and thus the eventual solution.

$$u_1' = \frac{W_1}{W} \tag{18}$$

$$u_2' = \frac{W_2}{W} \tag{19}$$

3.7.1 Determinants

Can find the determinant of a $n \times n$ matrix X by finding the sum of its cofactors along a row or column. For the below definition, assume we fix the row to be some i.

$$\det(X) = \sum_{j=1}^{n} A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} \det(M_{ij})$$

where M_{ij} is the $n-1 \times n-1$ matrix determined by excluding values along the axes originating from the element of X at ij.

3.8 Reduction of Order

Method is derived from the general 2nd order linear DE. Useful when y_p does not correspond to a template solution class.

$$y'' + py' + qy = R$$

Assume there exists a solution $y = y_1$ of the homogeneous equation* (e.g. can pick a solution from y_c).

$$y'' + py" + qy = 0 \tag{20}$$

We then introduce a dependent variable from $y = y_1 v$ as v. It follows that

$$y' = y_1 v' + y_1' v \tag{21}$$
$$y'' = y_1 v'' + 2y_1' v' + y_1'' v \tag{22}$$

So substituting into 3.8, we get

$$y_1 v'' + 2y_1' v' + y_1'' v + py_1 v' + py_1' v + qy_1 v = R$$

This rearranges to

$$y_1 v'' + \left(2y_1' + py_1\right) v' + \left(y_1'' + py_1' + qy_1\right) v = R.$$

We use the definition of the homogeneous equation*, since $y = y_1$ is a solution of the above.

$$y_1 v'' + (2y_1' + py_1) v' = R$$

If we let $v' = w$ and solve for w, then integrate, it is simply a first order problem with integrating factor

$$k(x) = \exp \int \frac{2y_1' + py_1}{y_1} dx$$

Another way to simplify this problem is to derive the general form of the reduction of order coefficient v, from which solution $y = vy_1$ follows trivially.

3.9 Spring Oscillation

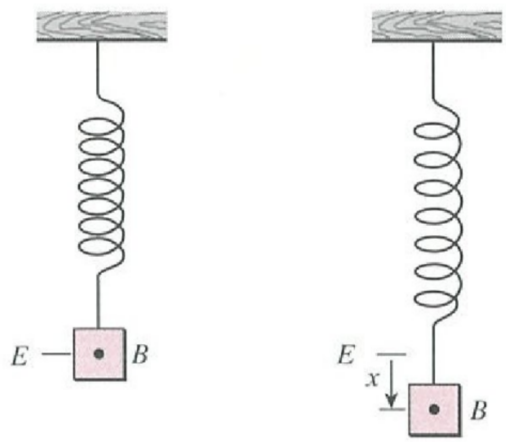


Figure 2: Spring at equilibrium and stretched positions

By setting E as the gravitational equilibrium position with some mass B, can avoid consideration of the original length of the spring. Mass of the spring comes from $w = mg \implies m = \frac{w}{g}$. In addition to the restoring force which causes the block B to accelerate, there is a damping force proportional to velocity that acts in opposition to the direction of motion. Taking downwards to be conventionally positive, net force equation* is

$$\frac{w}{g}\ddot{x} = -kx - b\dot{x} + w$$

Notice that at the equilibrium position, some s below the horizontal, $w - ks = 0 \implies ks = w$. When the mass is further stretched by x downwards and released, $\frac{w}{g}\ddot{x} = -k(s + x) - b\dot{x} + w$. But $ks = w$ so this reduces to the above.

This is better illustrated as follows (for now $F = 0$)

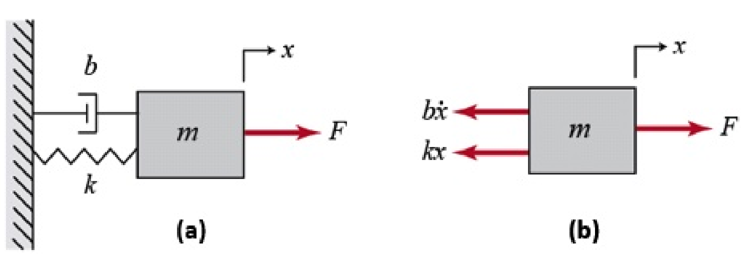


Figure 3: Damped spring oscillator

So the form of the DE becomes

$$\frac{w}{g}\ddot{x} + b\dot{x} + kx = 0$$

This is a second order homogeneous equation*. If we impose another driving force (as shown above, but with F as a function of t) that acts on the mass DE becomes

$$\frac{w}{g}\ddot{x} + b\dot{x} + kx = F(t)$$

The initial conditions are $x(0) = x_0$, $x'(0) = v_0$. Convenient to rewrite in the form

$$\ddot{x} + 2\gamma\dot{x} + \beta^2 x = F$$

where

$$\frac{bg}{w} = 2\gamma,\ \frac{kg}{w} = \beta^2$$

We choose $\beta > 0$ and $\gamma \geq 0$ because $\gamma = 0 \implies b = 0$.

3.9.1 Undamped

Case of $\gamma = 0$

Results in DE

$$\ddot{x} + \beta^2 x = F$$

Solve aux. eqn $m^2 + \beta^2 = 0$ to get $m = \pm\beta i$ to get complementary solution

$$x_c = c_1 \cos\beta t + c_2 \sin\beta t$$

Use method such as VOP, undetermined coefficient, or reduction of order to find x_p such that

$$x = c_1 \cos\beta t + c_2 \sin\beta t + x_p$$

Note that $\beta = \sqrt{\frac{kg}{w}}$ is the angular frequency, so $T = \frac{2\pi}{\omega}$. Want to find amplitude in the form

$$x = A\sin(\beta t + \phi)$$

Apply sine angle sum identity

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

such that

$$\begin{aligned} c_1 \cos \beta t + c_2 \sin \beta t &= A(\sin \beta t \cos \phi + \sin \phi \cos \beta t) \\ &= (A \cos \phi) \sin \beta t + (A \sin \phi) \cos \beta t \end{aligned}$$

Thus it follows that

$$\begin{aligned} c_1 &= A \cos \phi \\ c_2 &= A \sin \phi \end{aligned}$$

Using Pythagorean identity,

$$c_1^2 + c_2^2 = A^2 \implies A = \sqrt{c_1^2 + c_2^2}$$

Note that must appropriately add/subtract π to ϕ when considering sign of $\tan \phi = \frac{c_2}{c_1}$.

- If $c_1, c_2 < 0$, we use $\phi + \pi$ as the angle in Q3.
- If $c_1 < 0, c_2 > 0$, we use $\phi - \pi$ as the angle in Q2.
- If $c_1, c_2 > 0$, we use ϕ as the angle in Q1.
- If $c_1 > 0, c_2 < 0$, we use ϕ as the angle in Q4.

3.9.2 Resonance

Resonance case involves an increasing amplitude that moves to instability of the system. Consider the undamped force motion case (where $2\gamma \ddot{x} = 0$ term) with initial conditions $x(0) = 0, \dot{x}(0) = 0$.

$$\ddot{x} + \omega^2 x = F_0 \sin \gamma t$$

Using quadratic formula for $m^2 + \omega^2 m = 0$,

$$\begin{aligned} m &= \frac{\pm\sqrt{-4\omega^2}}{2} \\ &= \pm \omega i \end{aligned}$$

Thus,

$$x_c(t) = c_1 \cos \omega t + c_2 \sin \omega t \tag{23}$$
$$x_p(t) = A \cos \gamma t + B \sin \gamma t \tag{24}$$

We observe that

$$\ddot{x} + \omega^2 x_p = A(\omega^2 - \gamma^2)\cos\gamma t + B(\omega^2 - \gamma^2)\sin\gamma t = F_0 \sin\gamma t$$

We get the equation*s

$$A(\omega^2 - \gamma^2) = 0 \implies A = 0 \tag{25}$$
$$B(\omega^2 - \gamma^2) = F_0 \implies B = \frac{F_0}{\omega^2 - \gamma^2} \tag{26}$$

Thus,

$$x(t) = c_1 \cos\omega t + c_2 \sin\omega t + \frac{F_0}{\omega^2 - \gamma^2}\sin\gamma t$$

We skip the IC application, but it ends up being that $c_1 = 0, c_2 = \frac{\gamma F_0}{\omega^2 - \gamma^2}$. Then,

$$x(t) = \frac{F_0}{\omega(\omega^2 - \gamma^2)}\left(-\gamma\sin\omega t + \omega\sin\gamma t\right)$$

Resonance is found when we take the limit from $\gamma \to \omega$.

$$\begin{aligned}
\lim_{\gamma\to\omega}\frac{F_0(-\gamma\sin\omega t + \omega\sin\gamma t)}{\omega(\omega^2 - \gamma^2)} &= F_0 \lim_{\gamma\to\omega}\frac{\frac{d}{d\gamma}(-\gamma\sin\omega t + \omega\sin\gamma t)}{\frac{d}{d\gamma}\omega(\omega^2 - \gamma^2)} \\
&= F_0 \lim_{\gamma\to\omega}\frac{-\sin\omega t + \omega t\cos\gamma t}{-2\omega\gamma} \\
&= F_0 \frac{-\sin\omega t + \omega t\cos\gamma t}{-2\omega^2}
\end{aligned}$$

3.9.3 Damped Motion Cases

The complimentary auxiliary equation* is

$$m^2 + 2\gamma m + \beta^2 = 0$$

with roots $-\gamma \pm \sqrt{\gamma^2 - \beta^2}$. We see the three cases arise, $\beta > \gamma, \beta = \gamma, \beta < \gamma$.

Examining the first case, $\beta > \gamma \implies \beta^2 - \gamma^2 = \delta^2$ (**damped oscillatory**), so $m = -\gamma \pm \delta i$. General solution is

$$x(t) = e^{-\gamma t}(c_1 \cos\delta t + c_2 \sin\delta t)$$

assuming $F(t) = 0$.

For the case $\gamma = \beta$ (**critically damped**), roots are equal so

$$x(t) = e^{-\gamma t}(c_1 + c_2 t)$$

If $\beta < \gamma$ (**overdamped**), then $\gamma^2 - \beta^2 = \sigma^2$ for $\sigma > 0$ so

$$x(t) = c_1 e^{(-\gamma-\sigma)t} + c_2 e^{(-\gamma+\sigma)t}$$

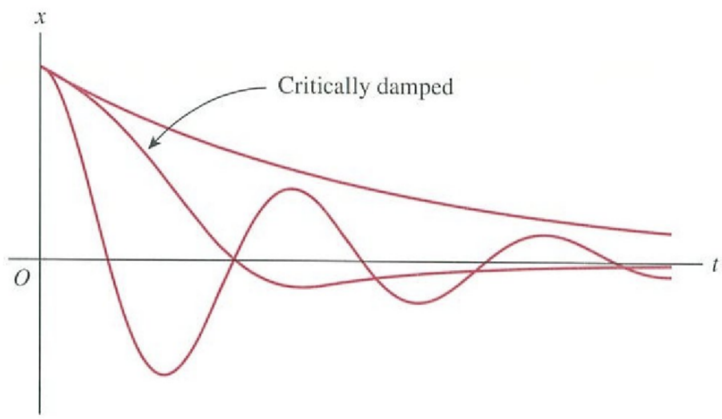


Figure 4: Graphical spring oscillatory cases

3.10 Circuit Application

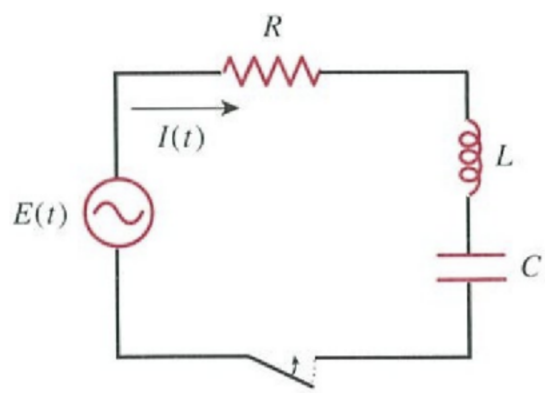


Figure 5: Basic RLC circuit

Using Kirchoff's law (KVL), the sum of drops is equal to the introduced voltage $E(t)$.

$$RI + L\dot{I} + \frac{q}{C} = E(t)$$
$$R\dot{q} + L\ddot{q} + \frac{1}{C}q = E(t)$$

Thus, the auxiliary equation* is

$$Lm^2 + Rm + \frac{1}{C} = 0$$

with solutions

$$m = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$$

Thus, the cases that arise are

- **Damped Oscillatory**: $R^2 - \frac{4L}{C} < 0$
- **Critically Damped**: $R^2 - \frac{4L}{C} = 0$
- **Overdamped**: $R^2 - \frac{4L}{C} > 0$

3.11 Dependent Variable Missing

Let there be a DE of form that doesn't contain DV y explicitly.

$$f(\ddot{y}, \dot{y}, x) = 0$$

Can use a substitution $\dot{y} = p$ in order to reduce order so that we get

$$f(\frac{dp}{dx}, p, x).$$

Solve for p, then y using integration.

4 Laplace Transform

Laplace transform of a function $F(t)$ denoted by $\mathcal{L}\{F(t)\}$. The Laplace operator $\mathcal{L}$ is linear, so

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}$$

Defined as

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t) dt$$

Integral is a function of parameter s, say $f(s)$ such that $\mathcal{L}\{F(t)\} = f(s)$. Improper integral will always converge, and can be written as a limit definition:

$$\int_0^\infty e^{-st} F(t) dt = \lim_{A \to \infty} \int_0^A e^{-st} F(t) dt$$

We can derive some general forms based on this idea.

4.1 Laplace transform definitions

Definition 4.1. ($\mathcal{L}\{e^{kt}\}$)
Finding $\mathcal{L}\{e^{kt}\}$ for $s > k$ (condition for convergence), we see that

$$\mathcal{L}\{e^{kt}\} = \lim_{A\to\infty} \int_0^A e^{-st} e^{kt} dt = \lim_{A\to\infty} \frac{1}{k-s} e^{(k-s)t} \mathcal{L}\Big|_0^A = \frac{1}{s-k}$$

Definition 4.2. ($\mathcal{L}\{\sin kt\}$)
Evaluating the integral:

$$\begin{aligned}
\int e^{-st} \sin kt &= -\frac{1}{k} e^{-st} \cos kt - \int s e^{-st} \frac{1}{k} \cos kt \; dt \\
&= -\frac{1}{k} e^{-st} \cos kt - \frac{s}{k} \int e^{-st} \cos kt \; dt \\
&= -\frac{1}{k} e^{-st} \cos kt - \frac{s}{k} \left(\frac{1}{k} e^{-st} \sin kt - \int -\frac{s}{k} e^{-st} \sin kt \; dt \right) \\
&= -\frac{1}{k} e^{-st} \cos kt - \frac{s}{k} \left(\frac{1}{k} e^{-st} \sin kt + \int \frac{s}{k} e^{-st} \sin kt \; dt \right) \\
(1 + \frac{s^2}{k^2}) \int e^{-st} \sin kt &= -\frac{1}{k} e^{-st} \cos kt - \frac{s}{k^2} e^{-st} \sin kt \\
\frac{k^2 + s^2}{k^2} \int e^{-st} \sin kt &= \frac{1}{k} e^{-st} \cos kt - \frac{s}{k^2} e^{-st} \sin kt \\
\int e^{-st} \sin kt &= -\frac{k e^{-st} \cos kt}{k^2 + s^2} - \frac{s e^{-st} \sin kt}{k^2 + s^2} \\
&= -\frac{e^{-st}}{k^2 + s^2} \left(k \cos kt - s \sin kt \right)
\end{aligned}$$

So evaluating the improper integral,

$$\begin{aligned}
\left[-\frac{e^{-st}}{k^2 + s^2} \left(k \cos kt - s \sin kt \right) \right]_0^\infty &= 0 + \frac{1(k-0)}{s^2 + k^2} \\
&= \frac{k}{s^2 + k^2}
\end{aligned}$$

It can be similarly obtained that $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$ (for $s > 0$ in both cases).

Definition 4.3. ($\mathcal{L}\{t\}$)

$$\begin{aligned}
\mathcal{L}\{t\} &= \int_0^\infty e^{-st} t dt \\
&= \lim_{A\to\infty} \int_0^A \underbrace{e^{-st}}_{dv} \underbrace{t}_{u} dt \\
&= \lim_{A\to\infty} \left[-\frac{1}{s} e^{-st} t \Big|_0^A - \int_0^A -\frac{1}{s} e^{-st} dt \right] \\
&= \lim_{A\to\infty} \left[-\frac{1}{s} e^{-st} t - \frac{1}{s^2} e^{-st} \right]_0^A \\
&= - \left[-\frac{1}{s^2} \right] = \frac{1}{s^2}
\end{aligned}$$

Definition 4.4. ($\mathcal{L}\{\sin^2 at\}$)

$$\mathcal{L}\{\sin^2 at\} = \mathcal{L}\{\frac{1}{2}(1 - \cos 2at)\} = \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos 2at\}$$

Definition 4.5. (Transforms of Derivatives) If $F(t)$ is continuous for $t \geq 0$ and of exponential order as $t \to \infty$, then we can simplify

$$\begin{aligned}
\mathcal{L}\{F'(t)\} &= \int_0^\infty e^{-st} F'(t)\, dt \\
&= \left[e^{-st} F(t) \right]_0^\infty + s \int_0^\infty e^{-st} F(t)\, dt \\
&= -F(0) + s\mathcal{L}\{F(t)\}
\end{aligned}$$

Useful fact that can be resubstituted for

$$\mathcal{L}\{F''(t)\} = -F'(0) + s\mathcal{L}\{F'(t)\} = -F'(0) - sF(0) + s^2\mathcal{L}\{F(t)\}.$$

General form:

$$\mathcal{L}\{f^n(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - f^{n-1}(0)$$

Definition 4.6. (Translation property)

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t) dt = \int_0^\infty e^{-(s-a)t} f(t) dt = F(s-a)$$

Definition 4.7. ($\mathcal{L}\{t^n f(t)\}$)

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \mathcal{L}\{f(t)\}$$

4.2 Solving DEs

Approach to solving DEs involves applying Laplace to both sides of some DE

$$F(y) = f(t)$$

and isolating $Y(s) = \mathcal{L}\{y(t)\}$ using the derivative transform. Then, use the inverse Laplace transform and linearity property to find $y(t)$ from $Y(s)$.

4.3 Step Function

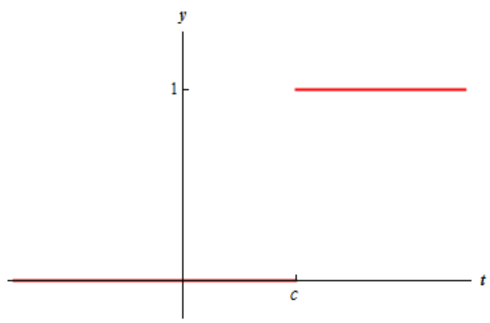


Figure 6: Unit step function

The unit step function centered at a is defined as

$$u_a(t) = u(t-a) = \begin{cases} 0, t < a \\ 1, t \leq a \end{cases}$$

Can define rectangular step function with pulse width $b - a$ where $b > a$:

$$r(t) = u(t-a) - u(t-b)$$

Calculating the Laplace transform is simple:

$$\begin{aligned} \mathcal{L}\{u_a(t)\} &= \int_0^a e^{-st}(0)dt + \int_a^\infty e^{-st}dt \\ &= -\frac{e^{-st}}{s}\bigg|_a^\infty = \frac{e^{-sa}}{s} \end{aligned}$$

Definition 4.8. (Step Function Translation)

Let there be some arbitrary curve $f(t)$ defined for all t and the step function $u(t-a)$. The new function $u(t-a)f(t-a)$ gives the step function with shape $f(t-a)$ for $t > a$. Applying the Laplace transform, we get

$$\begin{aligned}\mathcal{L}\{u(t-a)f(t-a)\} &= \int_0^\infty e^{-st}u(t-a)f(t-a)dt \\ &= \int_0^a e^{-st}(0)dt + \int_a^\infty e^{-st}f(t-a)dt\end{aligned}$$

Let $x = t - a \implies t = x + a$. Further note that when differentiating, $dx = dt$, making this substitution useful.

$$\begin{aligned}\int_0^a e^{-st}(0)dt + \int_a^\infty e^{-st}f(t-a)dt &= \int_a^\infty e^{-s(x+a)}f(x)dt \\ &= e^{-sa}\int_a^\infty e^{-sx}f(x)dx \\ &= e^{-sa}\mathcal{L}\{f(t)\}\end{aligned}$$

4.4 Periodic Functions

Definition 4.9. (Periodic Laplace Transform) If $f(t)$ is periodic, then $f(t-P) = f(t)$ (for $t \geq P$) and

$$\mathcal{L}\{f(t)\} = \frac{\int_0^P e^{-st}f(t)dt}{1-e^{-Ps}} \tag{27}$$

Proof. We can write the transform as two integrals,

$$\begin{aligned}F(s) &= \int_0^P e^{-st}f(t)dt + \int_P^\infty e^{-st}f(t)dt &(28)\\ &= \int_0^P e^{-st}f(t)dt + \int_P^\infty e^{-st}f(t+P)dt &(29)\end{aligned}$$

Note that $\tau = t - P \implies t = \tau + P$ so $dt = d\tau$.

$$\begin{aligned}\int_P^\infty e^{-st}f(t+P)dt &= \int_{2P}^\infty e^{-s(\tau+P)}f(\tau)d\tau &(30)\\ &= e^{-sP}\int_0^\infty e^{-s\tau}f(\tau)d\tau &(31)\\ &= e^{-sP}F(s) &(32)\end{aligned}$$

Substituting into 29, we get

$$F(s) = \int_0^P e^{-st}f(t)dt + e^{-st}F(s) \implies (1-e^{-sP})F(s) = \int_0^P e^{-st}f(t)dt \tag{33}$$

□

4.5 Convolution Operation

Definition 4.10. (Convolution Theorem)

$$f * g = \int_0^t f(r)g(t-r)dr = g * f \tag{34}$$

Applying the Laplace operator,

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s) \tag{35}$$

A more useful property is of the inverse,

$$f * g = \mathcal{L}^{-1}\{F(s)G(s)\} \tag{36}$$

Example 4.11. Finding the the inverse transform, we can apply the Laplace convolution definition,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{a}{s^2(a^2+s^2)}\right\} &= \int_0^t (t-u)\sin au\,du \\
&= t\int_0^t \sin au\,du - \int_0^t u\sin au\,du \\
&= -t(\cos at - 1) - \left(-u\cos au\Big|_0^t + \int_0^t \cos au\,du\right) \\
&= t(1-\cos at) + t\cos at + \sin at
\end{aligned}$$

4.6 More on Step Functions

Definition 4.12. (Step function)

$$\alpha(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases} \tag{37}$$

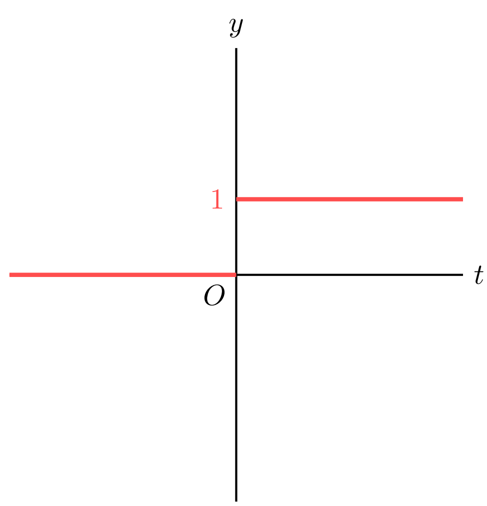


Figure 7: Unit step function

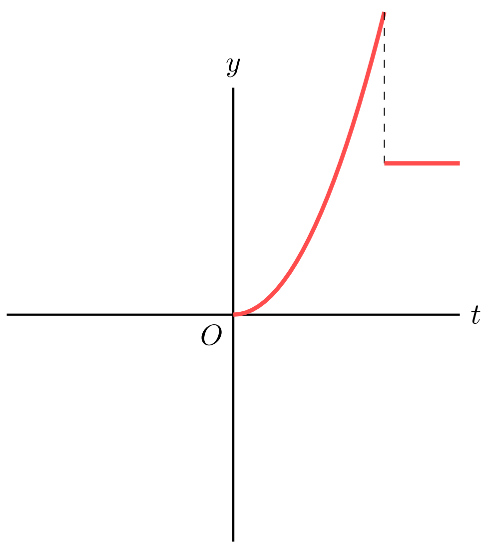
We can apply this to a function $F(t)$ such that $y = \alpha(t-c)F(t)$ turns off F for $t \leq c$.

Applying the Laplace transform and the substitution $t - c = v$ (note that the lower bound changes from c to 0), we get

$$\begin{aligned}
\mathcal{L}\left\{\alpha(t-c)f(t-c)\right\} &= \int_0^\infty e^{-st}\alpha(t-c)f(t-c)\,dt \\
&= \int_c^\infty e^{-st}f(t-c)\,dt \\
&= \int_0^\infty e^{-s(c+v)}f(v)dv \\
&= e^{-sc}\int_0^\infty e^{-sv}f(v)dv \\
&= e^{-sc}F(s)
\end{aligned}$$

Example 4.13. We want to model the function

$$y(t) = \begin{cases} t^2,\ 0 < t < 2 \\ 2,\ t > 2 \end{cases}$$



This can be written as $y = t^2(1 - \alpha(t-2)) + 6\alpha(t-2)$. However, need coefficient of $\alpha(t-2)$ as a function of $(t-2)$.

$$\begin{aligned}
y &= t^2(1 - \alpha(t-2)) + 6\alpha(t-2) \\
&= t^2 - t^2\alpha(t-2) + 6\alpha(t-2) \\
&= t^2 - (t-2)^2\alpha(t-2) + 6\alpha(t-2) - (4t-4)\alpha(t-2) \\
&= t^2 - (t-2)^2\alpha(t-2) - 2(t-2)\alpha(t-2) + 6\alpha(t-2)
\end{aligned}$$

Applying laplace,

$$F(s) = \frac{2}{s^3} - \frac{2e^{-2s}}{s^3} - \frac{2e^{-2s}}{s^2} + \frac{6e^{-2s}}{s^2}$$

4.7 Integral Equations

Defined as an equation containing dependent variable under integral sign. Can use the convolution theorem to solve a special class of integral equations. Recall that given

$$\mathcal{L}\{f(t)\} = F(s),\, \mathcal{L}\{g(t)\} = G(s) \tag{38}$$

then

$$\mathcal{L}\{\int_0^t g(t-r)f(r)dr\} = F(s)G(s) \tag{39}$$

Example 4.14. We are given

$$f(t) = 4t - 3\int_0^t f(r)\sin(t-r)dr$$

Thus,

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{4}{s^2} - \frac{3F(s)}{s^2+1} \\
F(s)\left(1 + \frac{3}{s^2+1}\right) &= \frac{4}{s^2} \\
F(s) &= \frac{4}{s^2} \cdot \frac{s^2+1}{s^2+4} \\
F(s) &= 4\frac{s^2+1}{s^2(s^2+4)}
\end{aligned}$$

Decomposing the partial fraction,

$$\begin{aligned}
\frac{s^2+1}{s^2(s^2+4)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4} \\
s^2+1 &= As(s^2+4) + B(s^2+4) + (Cs+D)s^2 \\
&= As^3 + 4As + Bs^2 + 4B + Cs^3 + Ds^2 \\
&= s^3(A+D) + s^2(B+D) + s(4A) + 4B
\end{aligned}$$

Then, $A+C=0, B+D=1, 4A=0, 4B=1 \implies B=\frac{1}{4} \implies D=\frac{3}{4}, A=0, C=0$. So the result is

$$F(s) = \frac{1}{4s^2} + \frac{\frac{3}{4}}{s^2+4}$$

4.8 Gamma Function

Definition 4.15. (Gamma Function)

$$\Gamma(x) = \int_0^\infty e^{-\beta}\beta^{x-1}d\beta\, x > 0 \tag{40}$$

To find $\Gamma(x+1)$,

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty e^{-\beta}\beta^x\, d\beta \\
&= \left[-e^{-\beta}\beta^x\right]_0^\infty + x\int_0^\infty e^{-\beta}\beta^{x-1}d\beta \\
&= x\Gamma(x)
\end{aligned}$$

In general,

$$\Gamma(x+1) = x\Gamma(x),\ x > 0 \tag{41}$$

If we iterate,

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n) \\ &= n(n-1)\Gamma(n-1) \\ &= n!\Gamma(1) \\ &= n!\end{aligned}$$

We can further substitute $\beta = st$ for $s > 0$ and t as new variable in $\Gamma(x+1)$:

$$\Gamma(x+1) = \int_0^\infty e^{-st} s^{x+1} t^x dt = s^{x+1} \int_0^\infty e^{-st} t^x dt$$

Thus,

$$\frac{\Gamma(x+1)}{s^{x+1}} = \int_0^\infty e^{-st} t^x dt,\ s > 0, x > -1 \tag{42}$$

This aligns with the form of a laplace transform, so

$$\mathcal{L}\{t^x\} = \frac{\Gamma(x+1)}{s^{x+1}} \tag{43}$$

For $x = -\frac{1}{2}$,

$$\mathcal{L}\{t^{-1/2}\} = \frac{\Gamma(\frac{1}{2})}{s^{1/2}}$$

From Laplace table, $\mathcal{L}\{t^{-1/2}\} = (\pi/s)^{1/2}$ so

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$