# Multivariable Calculus Reference

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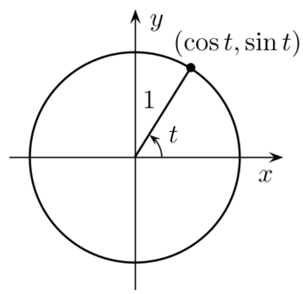
# 1 Paths whose image curve is a circle

## 1.1 Unit Circle

Unit circle is set of points in $\mathbb{R}^2$ defined as $C = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$. Ellipse is $C = \{(x, y) \in \mathbb{R}^2 | \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$ Has standard parameterization of $\vec{c}(t) = \big(\cos(t), \sin(t)\big)$. When parameterizing, always start from $t = 0$ reference unless otherwise given.




Properties

- Image of $\vec{c}$ is a closed curve (has no endpoints, plane is divided into $\geq 2$ disjoint regions)
- Image of $\vec{c}$ is a simple curve; no self-intersection
- $\vec{c}(t)$ is an **injective path**; path is considered injective if $\vec{c}(t_1) = \vec{c}(t_2)$, which implies that $t_1 = t_2$ where these are on the open interval $(a, b)$ even if $a = b$
- Orientation of $\vec{c}$ is counter-clockwise in traversal

## 1.2 Observations

$$\boxed{\vec{p}(t) = (a\cos(\pm n(t \pm \theta)) + x_0, b\sin(\pm n(t \pm \theta)) + y_0)}$$

If $-t$ for $t$, orientation is CW, CCW is $t$. If $a = b$, then curve is a circle of radius $a$ or $b$, else an ellipse with horizontal and vertical radii. $x_0$ and $y_0$ simply shift the center coordinate. $n > 0 \in \mathbb{R}$ determines how many times the circle is traversed given $t \in [0, 2\pi]$, for example. $\theta$ is the phase shift. When changing direction of traversal, cannot have $a > b$ for $[a, b]$ so to decrease argument of sin or cos must have $-t$ for $t$. Starting out, $-t$ goes through the angle range and $t$ is just a sign flip.

# 2 Paths whose image is a line or line segment in the plane

## 2.1 Line Parametrics

A line is a 1D subspace of $\mathbb{R}^2$, so $L = \{t\vec{m} | t \in \mathbb{R}\}$ for $\vec{m} \in \mathbb{R}^2$. $\vec{m} = \begin{bmatrix} m_x \\ m_y \end{bmatrix}$ is the **slope vector**. Path given by image of $L$:

$$\vec{c}(t) = (m_x t, m_y t), t \in \mathbb{R}$$

Can represent $\vec{c}(t) = t\vec{m}$ as well.

Lines Main Ideas

- Image of a line is a curve (e.g. $y = x$ represents image curve of $\vec{c}(t) = (t, t)$)
- Lines can have nonzero intercepts, so $\vec{c}(t) = t\vec{m}$ represents $y = 2x+1$. Line that has intercept vector $P_0 = (x_0, y_0) \parallel \vec{m} = (m_x, m_y)$ can be expressed as:

$$\boxed{\vec{c}(t) = (x_0 + tm_x, y_0 + m_y t) = \vec{P_0} + t\vec{m}}$$

Note endpoint of $\vec{c}(t)$ is on image line (curve).

### 2.2 General Forms

2 parametric lines **collide** if they intersect and the point of intersection corresponds to the same $t$ in both curves. If you set the parameter vector coordinates equal to each other and solve for $t$, a solution indicates they collide. Intersection is found by **eliminating** the parameter (solve for $t$ in terms of either $x$ or $y$ and plug into the other).

General form of parameterized curve can be expressed as the following:

$$\boxed{\vec{c}(t) = (\frac{m_x}{\Delta t}(t-a) + x_0, \frac{m_y}{\Delta t}(t-a) + y_0)}$$

where $\Delta t$ is the domain interval over $[a, b]$ and $(x_0, y_0)$ represents the desired **starting coordinate**. This is important as when going in reverse, other coordinate can be used and slope might be negative. $a$ is used in $(t-a)$ because everything is conventionally done with respect to starting coordinate.

## 3 Paths whose image curve is a line in R3

### 3.1 R3 parameterization

If $\vec{m}$ is a nonzero vector along $L$ through origin in $\mathbb{R}^3$, then $L = \{t\vec{m} | t \in \mathbb{R}\}$; follows that $\vec{m} = (m_x, m_y, m_z)$, the slope or direction vector of the line. The basic parameterization is:

$$\vec{c}(t) = (m_x t, m_y t, m_z t)$$

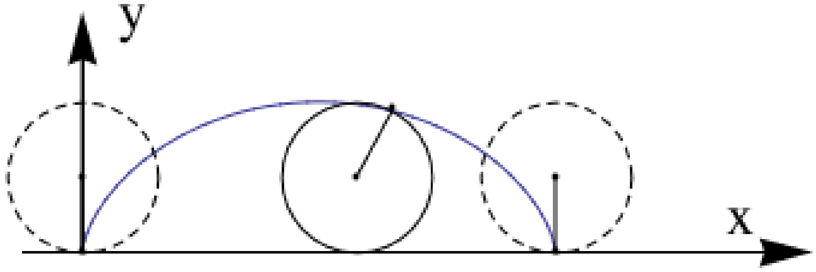
Basis vectors in $\mathbb{R}^3$ are $\vec{i}, \vec{j}, \vec{k}$. Rewriting parameterization:

$$\vec{c}(t) = (x_0 + m_x t)\vec{i} + (y_0 + m_y t)\vec{j} + (z_0 + m_z t)\vec{k}$$

2 lines $\vec{c}_1(t) = P_0 + \vec{m}_1 t$ and $\vec{c}_2(t) = Q_0 + \vec{m}_2 t$ are parallel if direction vectors are parallel ($\vec{m}_1 = k\vec{m}_2$). Collisions still exist. If neither parallel nor intersecting, considered as skew.

To determine skew, parallel, or coincide, use parameters $s, t$ for each line and solve SOE. If same slope, rule out skew clearly, then check if $s, t \in \mathbb{R}$: if not, then parallel, if so, then they coincide. If intersecting and want to check if collide, some $t$ must satisfy all relations.

# 4 Cycloid Problem




With radius 1 and passing through the origin:

$$\vec{c}(t) = (t - \sin t, 1 - \cos t)$$

Observe that:

$$\vec{c}'(t) = (1 - \cos t, \sin t)$$

Can define the vector $\vec{u} = \begin{pmatrix} x'(t) \\ 0 \end{pmatrix}$ such that $\vec{u}$ is always horizontal and $||\vec{u}|| = |x'(t)|$. Reaches maximum value at $t \in [k\pi | k \in \mathbb{R}]$ and is has minimum cusp where it is 0 at $t \in [2k\pi | k \in \mathbb{R}]$. Thus, $x'(t) \geq 0$ always, as the x-coordinate is never decreasing.

Can also define the vector $\vec{v} = \begin{pmatrix} 0 \\ y'(t) \end{pmatrix}$ with the same properties. Reaches maximum value when $t \in [k\frac{\pi}{2} | k \in \mathbb{R}]$. Can change, as observe $t$ when $\sin t < 0$ or $> 0$.

# 5 Velocity Vector

Vector $\vec{u}(t_0) + \vec{v}(t_0)$ is he velocity vector to the curve $\vec{c}(t)$ at $t = t_0$.

Let $\vec{c}: [a, b] \to \mathbb{R}^n$ have a path $\vec{c}(t) = (x_1(t), x_2(t), x_3(t), \ldots, x_n(t))$ (let $x_i(t) : [a, b] \to \mathbb{R}$ for each $i$)

- If $t_0 \in [a, b]$, then $\vec{c}'(t_0) := (x_1'(t_0), x_2'(t_0), x_3'(t_0), \ldots, x_n'(t_0))$; the velocity vector to $\vec{c}$ at $t_0$
- The path $\vec{c}'(t_0) := (x_1'(t_0), x_2'(t_0), x_3'(t_0), \ldots, x_n'(t_0))$; the velocity vector to $\vec{c}$ is referred to as velocity of $\vec{c}(t)$

Recall chain rule: if $y = f(x)$ where $x$ is a function of $t$, $y'(t) = x' f'(x)$, not to be confused with product rule. Can write $f'(x) = \frac{y'(t)}{x'(t)}$

- If $\vec{p}(t) = \vec{c}(t) + \vec{r}(t)$, then $\vec{p}'(t) = \vec{c}'(t) + \vec{r}'(t)$
- If $g(t) = \vec{c}(t) \cdot \vec{r}(t)$, then $g'(t) = \vec{c}'(t) \cdot \vec{r}(t) + \vec{c}(t) \cdot \vec{r}'(t)$
- If $\vec{p}(t) = f(t)\vec{c}(t)$, then $\vec{p}'(t) = f'(t)\vec{c}(t) + f(t)\vec{c}'(t)$

- If $\vec{p}(t) = \vec{c}(t) \times \vec{r}(t)$, then $\vec{p}'(t) = \vec{c}'(t) \times \vec{r}(t) + \vec{c}(t) \times \vec{r}'(t)$
- If $\vec{p}(t) = \vec{c}(f(t))$, then $\vec{p}'(t) = f'(t)\vec{c}'(f(t))$
- If $g(t) = \|\vec{c}(t)\|$, then $g'(t) = \frac{\vec{c}(t)\cdot\vec{c}'(t)}{\|\vec{c}(t)\|}$

# 6 Space Curves

- Projection into the $xy$ plane is the path $(x(t), y(t), 0)$.
- Projection into the $xz-$ plane is the path $(x(t), 0, z(t))$.
- Projection into the $yz$ plane is the path $(0, y(t), z(t))$.