Differential Equations

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1 Definitions, Families of Curves

1.1 Definitions

Definition 1.1 (Order)**.** Order of a DE is the highest-ordered derivative appearing in it. So

$$\frac{d^2y}{dx^2} + 2b(\frac{dy}{dx})^3 + y = 0 \tag{1}$$

is a 2nd order DE. In general,

$$F(x, y, y', y'', \dots, y^{(n)}) = 0. \tag{2}$$

is an n-th order DE. Under restrictions on F, can find a solution in terms of the other $n+1$ variables

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}). \tag{3}$$

Definition 1.2 (Solution)**.** A function ϕ on interval $x \in (a, b)$ is a solution to the DE (3) if the n derivatives exist on $x \in (a, b)$ and $\phi^{(n)}(x) = f(x, \phi(x), \dots, \phi^{(n-1)}(x))$.

Definition 1.3 (First order DE)**.** A first order DE is of the form

$$\frac{dy}{dx} = f(x, y) \tag{4}$$

with solution of the form $y = f(x)$. Can be rewritten for convenience in the form

$$M(x, y)dx + N(x, y)dy = 0 \tag{5}$$

Definition 1.4 (Linear ODE)**.** An ODE of order n is linear if it can be written in the form

$$b_0(x)\frac{d^ny}{dx^n} + b_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + b_{n-1}(x)\frac{dy}{dx} + b_n(x)y = R(x) \tag{6}$$

Definition 1.5 (Partial DE)**.** Is of the form, for example

$$b_0(x, y)\frac{\partial w}{\partial x} + b_x(x, y)\frac{\partial w}{\partial y} = R(x, y) \tag{7}$$

1.2 Families of Solutions

Solutions to the DE

$$\frac{dy}{dx} = f(x, y) \Leftrightarrow y = \int f(x)dx + c \tag{8}$$

exist as one-parameter families with parameter c.

1.3 Isoclines

Let there be the DE

$$\frac{dy}{dx} = y \tag{9}$$

Isoclines are lines $f(x, y) = y = c$. Example:

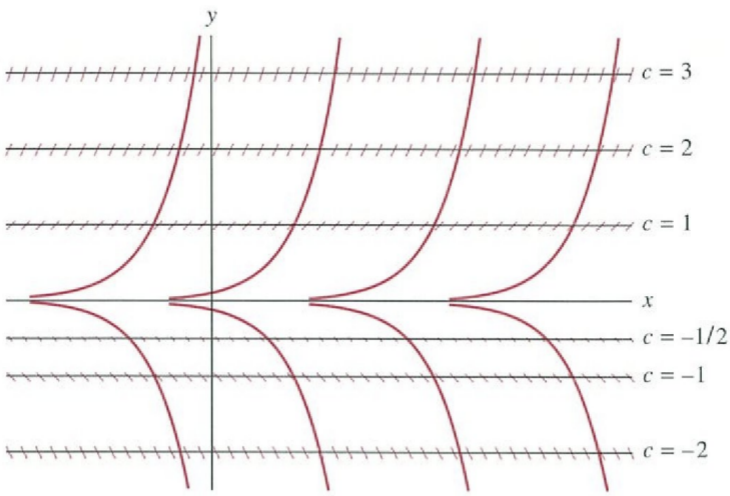

Figure 1: Isoclines of $\frac{dy}{dx} = y$

1.4 Existence Theorem

Consider equation

$$\frac{dy}{dx} = f(x, y) \tag{10}$$

Further, let T denote the rectangle defined by

$$|x - x_0| \leq a \tag{11}$$

$$|y - y_0| \leq b \tag{12}$$

with the point (x_0, y_0) as the center. Also let $f, \frac{\partial f}{\partial y}$ be continuous functions of x, y in T.

With these conditions an interval exists for x_0 where $|x - x_0| \leq h$, and function $y(x)$ which has properties

1. $y = y(x)$ is a sol'n of the DE on interval $|x - x_0| \leq h$
2. On this interval, $|y(x) - y_0| \leq b$
3. $y = y(x_0) = y_0$ at $x = x_0$
4. $y(x)$ is unique on interval $|x - x_0| \leq h$ where it is the only function with above 3 properties

2 Equations of Order One

2.1 Separation of Variables

We begin with equations of the form

$$Mdx + Ndy = 0 \tag{13}$$

where M and N can be multivariate of x, y.

It is separable iff

$$A(x)dx + B(y)dy = 0. \tag{14}$$

Then find a function F with total differential being the LHS of above, so $F = c$.

2.2 Homogeneity

Definition 2.1 (Homogeneity of polynomials)**.** Polynomials where all terms are of the same degree are homogeneous.

Homoegeneity of functions is analogous to assigning physical dimensions (e.g. length) to all of the variables. If the function has the length dimension to the kth power, then it is homogeneous of degree k.

Example 2.2. If x, y are lengths, then the following is homogeneous of degree 3.

$$f(x, y) = 2y^3 \exp(\frac{y}{x}) - \frac{x^4}{x + 3y} \tag{15}$$

Alternate definition also suffices for generality.

Definition 2.3 (Homogeneous function)**.** $f(x, y)$ is homogeneous of degree k iff $f(\lambda x, \lambda y) = \lambda^k f(x, y)$.

Definition 2.4 (Alternate definition of homogeneity)**.** If $f(x, y)$ can be rewritten as $f(\frac{y}{x})$ or $f(\frac{x}{y})$ then it is homogeneous.

2.3 Homogeneous Differential Equations

Theorem 2.5 (Homogeneous DEs)**.** If $M(x, y)$ and $N(x, y)$ are homogeneous and of same degree, then $M(x, y)dx + N(x, y)dy = 0$ is a homogeneous DE.

Theorem 2.6 (Homogeneous DEs)**.** $M(x, y)/N(x, y)$ is homogeneous of degree 0.

Proof. If M, N are homogeneous of some degree n, then

$$M(x, y) = M(\lambda x, \lambda y) = \lambda^n M(x, y) \tag{16}$$

$$M(x, y) = N(\lambda x, \lambda y) = \lambda^n N(x, y) \tag{17}$$

So for M/N,

$$\frac{M(x,y)}{N(x,y)} = \lambda^0 \frac{M(1,v)}{N(1,v)} \tag{18}$$

which is homogeneous of degree 0. $\square$

Theorem 2.7 (Homogeneous DEs)**.** If $f(x,y)$ is homogeneous of degree 0 in x, y, then $f(x,y)$ is a function of y/x alone.

Proof. Let $y = vx$. Then, f must be proven to be of v alone. Substitute so that

$$f(x,y) = f(x,vx) = x^0 f(1,v) = f(1,v) \tag{19}$$

$\square$

2.4 Homogeneous Coefficients

Suppose coefficients M, N in the equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{20}$$

are both homogeneous of same degree. So M/N is a function of y/x alone. Can put in form using $y = vx$.

$$\frac{dy}{dx} + g(\frac{y}{x}) = 0 \tag{21}$$

$$x\frac{dv}{dx} + g(v) = 0 \tag{22}$$

This last equation is separable.

2.5 Exact Equations

If there exists an equation of the form $A(x)dx + B(y)dy = 0$, the solution is a function with differential $A(x)dx + B(y)dy$. Idea works for equations of form

$$dF = Mdx + Ndy. \tag{23}$$

So, $F(x,y) = c \implies dF = 0$ and

$$Mdx + Ndy = 0. \tag{24}$$

If there's a function F such that $Mdx + Ndy$ is the **total differential** of F, then Eq. 5 is an *exact equation* by definition. Can rewrite the total differential from the chain rule:

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy. \tag{25}$$

So $M = \frac{\partial F}{\partial x}, N = \frac{\partial F}{\partial y}$. We can take 2nd derivative to show these are equal because the partials are continuous (Clairaut's theorem).

$$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} \tag{26}$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial y \partial x}. \tag{27}$$

Definition 2.8 (Exactness)**.**

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{28}$$

Proof. Let $\phi(x, y)$ be a function where $\frac{\partial \phi}{\partial x} = M$. ϕ is the function you get from integrating Mdx wrt x and holding y. Then

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{29}$$

Integrating both sides wrt x:

$$\frac{\partial \phi}{\partial x} = N + B'(y) \tag{30}$$

where $B'(y)$ is the integration constant. Let

$$F = \phi(x, y) - B(y) \tag{31}$$

such that

$$dF = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy - B'(y)dy \tag{32}$$

$$= Mdx + \left[N + B'(y)\right] dy - B'(y)dy \tag{33}$$

$$= Mdx + Ndy \tag{34}$$

□

Example 2.9. We have the DE

$$3x(xy - 2)dx + (x^3 + 2y)dy = 0. \tag{35}$$

Then,

$$\frac{\partial M}{\partial y} = 3x^2, \frac{\partial N}{\partial y} = 3x^2 \tag{36}$$

The DE is exact, and $F = c$ is the solution.

$$\frac{\partial F}{\partial x} = M = 3x^2y - 6x \tag{37}$$

$$\frac{\partial F}{\partial y} = N = x^3 + 2y \tag{38}$$

Try to find F from 18, integrate both sides wrt x with an integration constant $T(y)$.

$$F = x^3y - 3x^2 + T(y) \tag{39}$$

Using Eq. 19, can can find $\frac{\partial F}{\partial y}$ from Eq. 20 and equate:

$$x^3 + T'(y) = x^3 + 2y \implies T'(y) = 2y \tag{40}$$

Because $F = c$ is the I.C., can conclude

$$T(y) = y^2 \tag{41}$$

Thus,

$$F = x^3y - 3x^2 + y^2 \Leftrightarrow x^3y - 3x^2 + y^2 = c \tag{42}$$

2.6 Linear Equations of Order 1

If an equation is not exact, can attempt to do so by multiplying DE by an integrating factor.

Definition 2.10 (Linear DE of order 1)**.**

$$A(x)\frac{dy}{dx} + B(x)y = C(x) \tag{43}$$

Divide each side by $A(x)$ to obtain

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{44}$$

Suppose there exists for Eq. 25 a I.F. $v(x) > 0$. Then,

$$v(x)\left[\frac{dy}{dx} + P(x)y\right] = v(x)Q(x) \tag{45}$$

becomes exact, or of form $Mdx + Ndy = 0$. Here,

$$M = vPy - vQ \tag{46}$$
$$N = v \tag{47}$$

Because the requirement is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,

$$vP = \frac{dv}{dx} \tag{48}$$
$$Pdx = \frac{dv}{v} \tag{49}$$
$$\ln v = \int Pdx \tag{50}$$
$$v = \exp(\int Pdx) \tag{51}$$

We can then multiply both sides of the DE by this I.F. One side of this eqn will be of the product rule form, the derivative of $y \exp(\int Pdx)$:

$$\exp(\int Pdx)\frac{dy}{dx} + P\exp(\int Pdx)y = Q\exp(\int Pdx) \tag{52}$$

2.7 General Solution of a Linear Equation

Given the original form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{53}$$

suppose P and Q are continuous on $x \in (a, b)$ and $x = x_0$ is such a number. $y = y_0$ satisfies the initial condition. This sol'n satisfies Eq. 34 for all x in the interval. Multiplying Eq. 34 by integrating factor $\exp(\int Pdx)$ gives

$$yv = \int vQ\, dx + c \tag{54}$$

Because $v \neq 0$,

$$y = v^{-1} \int vQ\, dx + cv^{-1} \tag{55}$$

Given any x_0, y_0 in the interval, can find c s.t. the DE is satisfied. Every eqn of above form will have P, Q with common interval of continuity and a unique set of solutions with one I.C. obtained by using the integrating factor. These solutions are unique, so any other method yields a solution that aligns with the general solution–all possible solutions satisfying the DE on $x \in (a, b)$.

2.8 Application of Mixing Problem

Strategy is to determine the differential equation describing rate of change of a certain quantity, then finding the particular solution with some trivial IC.

Example 2.11. 100 liter tank contains 10 kg salt mixed with 60 liter water. Sol'n with concentration 0.1 $\frac{\text{kg}}{\text{liter}}$ flows in at rate 5 liters/min. Solution is well stirred (asume equal distribution), outflow rate of 3 liters/min. Need to find salt in tank when it is full.

Note that the tank will become full, as in − out > 0. Let x be kg of salt. Then, inflow rate is $0.1\frac{\text{kg}}{\text{liter}} \cdot 5\frac{\text{liter}}{\text{min}} = 0.5\frac{\text{kg}}{\text{min}}$. Out is $x\frac{\text{kg}}{60 \text{ liter}} \cdot 3\frac{\text{liter}}{\text{min}} = \frac{x}{20}\frac{\text{kg}}{\text{min}}$. We then express the DE as

$$\frac{dx}{dt} = 0.5 - \frac{x}{20} \tag{56}$$

Then just express in linear form, solve with I.F. method.

Example 2.12. Initially 50 gallons of brine, 10 lb dissolved salt. Inflow of 2 lb salt/gal at 5 gal/min, outflow of 3 gal/min, but **mixture kept uniform**.

Inflow is thus 10 lb/min, outflow is $\frac{3x}{50+2t}$ lb/min. Key here is that mixture concentration on outflow does not change, so the volume dynamically adapts for changing amount of salt. 50 gallons initially, influx of 5 gal - 3 gal out $\implies 50 + (5-3)t$. DE is thus

$$\frac{dx}{dt} = 10 - \frac{3x}{50+2t} \tag{57}$$

2.9 Integrating Factor by Inspection

By recognizing differentials in a problem, can find the integrating factor by inspection.

Example 2.13. Given

$$ydx + (x + x^3y^2)dy = 0 \tag{58}$$

the terms can be grouped by like degree so

$$(ydx + xdy) + x^3y^2dy = 0. \tag{59}$$

Can be rewritten as

$$d(xy) + x^3y^2dy = 0 \tag{60}$$

then divide by $(xy)^3$ for it does not affect integrability of $d(xy)$ term but keeps function of y with dy term,

$$\frac{d(xy)}{(xy)^3} + \frac{dy}{y} = 0. \tag{61}$$

Integrating:

$$\int (xy)^{-3} d(xy) + \int \frac{dy}{y} = 0 \tag{62}$$

$$\frac{(xy)^{-2}}{-2} + \ln|y| = C \tag{63}$$

$$Cy = \frac{(xy)^{-2}}{2} \tag{64}$$

$$Cy(xy)^2 = 1 \tag{65}$$

2.10 Determining Complex Integrating Factors

Let there be the DE

$$Mdx + Ndy = 0. \tag{66}$$

Suppose $\exists u$, possibly of both x, y that is an integrating factor such that

$$uMdx + uNdy = 0 \tag{67}$$

and for it to be exact,

$$\frac{\partial}{\partial y}(uM) = \frac{\partial}{\partial x}(uN) \tag{68}$$

so u satisfies

$$u\frac{\partial M}{\partial y} + M\frac{\partial u}{\partial y} = u\frac{\partial N}{\partial x} + N\frac{\partial u}{\partial x} \tag{69}$$

$$u(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y}. \tag{70}$$

This does not lead anywhere, so let u be a function of x. Thus, $\partial u/\partial y = 0, \partial u/\partial x = du/dx$. So the above reduces to

$$u(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = N\frac{du}{dx} \Leftrightarrow \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})dx = \frac{du}{u} \tag{71}$$

and integrating factor is the following, assuming LHS of above is a function of x or y alone

$$u = \exp\left[\int f(x)dx\right] \text{ (for } x\text{)} \tag{72}$$

$$u = \exp\left[\int -g(y)dy\right] \text{ (for } y\text{)} \tag{73}$$

3 Linear Equations with Constant Equations

3.1 Auxiliary Equation

Linear homogeneous DE with constant coefficients can be expressed as

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \tag{74}$$

Can be written in the form

$$f(D)y = 0 \tag{75}$$

where $f(D)$ is a linear differential operator. If the algebraic eqn $f(m) = 0$ then we know $f(D)e^{mx} = 0 \implies y = e^{mx}$ is a solution to the form above. $f(m) = 0$ is the auxiliary equation associated with the DE. Since the DE is of order n, the auxiliary equation is of degree n with roots $m_1, \ldots, m_n$.

Thus we have n solutions $y_1 = \exp(m_1 x), \ldots, y_n = \exp(m_n x)$ assuming the roots are **real** and **distinct** are then **linearly independent**. The general solution is thus

$$y = c_1 \exp(m_1 x) + \cdots + c_n \exp(m_n x) \tag{76}$$

with arbitrary constants $c_1, \ldots, c_n$.

3.1.1 Derivation

We can say that $y^{(k)}$ is the kth derivative of y, so say the general form is

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

If we take $y = e^{rx}$, then observe $y^{(n)} = r^n e^{rx}$. So rewrite the general form as

$$\begin{aligned} a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_1 r e^{rx} + a_0 e^{rx} &= 0 \\ a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 &= 0 \end{aligned}$$

Solving for the roots r in this characteristic equation helps us obtain the general solution.

3.2 Auxiliary Equation Repeated Roots

Need method for obtaining n linearly independent solutions for n equal roots of auxiliary equation. Suppose auxiliary equation $f(m) = 0$ has n roots $m_1 = m_2 = \cdots = m_n = b$. Thus, the operator function $f(D)$ has a factor $(D-b)^n$. Want to find n linearly independent y for which $(D-b)^n y = 0$. Use the substitution $y_k = x^k e^{bx}$ such that

$$(D-b)^n (x^k e^{bx}) = 0, \ k = 0, 1, 2, \ldots, n-1 \tag{77}$$

The functions $y_k = x^k e^{bx}$ are linearly independent because the respective powers $x^0, \dots, x^k$ are linearly independent. So the general solution takes form

$$y = c_1 e^{bx} + c_2 x e^{bx} + \cdots + c_n x^{n-1} e^{bx} \tag{78}$$

3.3 Complex roots for the auxiliary equation

Complex roots typically come in conjugate pairs of the form $a \pm bi$. So using the general form of the auxiliary solution and the Euler identity $e^{i\theta} = \sin\theta + i\cos\theta$ and the fact that $\cos(-x) = \cos x, \sin(-x) = -\sin x$, we find that

$$\begin{align}
y_1 &= c_1 e^{(a+bi)x} = c_1 e^{ax} e^{bxi} = c_1 e^{ax} (\sin bx + i\cos bx) \tag{79}\\
y_2 &= c_2 e^{(a-bi)x} = c_2 e^{ax} e^{-bxi} = c_2 e^{ax} (-\sin bx + i\cos bx) \tag{80}\\
y_1 + y_2 &= c_1 e^{ax} (\sin bx + i\cos bx) + c_2 e^{ax} (-\sin bx + i\cos bx) \tag{81}\\
&= e^{ax}(c_1 - c_2)\sin bx + e^{ax}(c_1 + c_2) i \cos bx \tag{82}\\
&= c_3 e^{ax} \sin bx + c_4 e^{ax} \cos bx \tag{83}
\end{align}$$

3.4 Linear Independence

Linear independence of two functions f_1, f_2 is calculated with a determinant called the Wronskian $W(f_1, f_2)$.

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ {f_1}' & {f_2}' \end{vmatrix} \tag{84}$$

Let f_1, f_2 be differentiable on $[a, b]$. if $W(f_1, f_2)(t_0) \neq 0$ for some $t \in [a, b]$ then f_1, f_2 are linearly independent on $[a, b]$. If they are linearly dependent, $W(f_1, f_2)(t) = 0 \ \forall t$.

3.5 Review of rational roots, synthetic division

For rational roots test, take factors of constant coeff. c divided by factors of leading coefficient a to get possible roots.

3.6 Undetermined Coefficients

3.7 Variation of Parameters

We start with the general equation

$$ay'' + by' + cy = g(x) \tag{85}$$

First, find complementary solution y_c from $y = y_p + y_c$. This is simply the solution of the homogeneous equation

$$ay'' + by' + cy = 0 \tag{86}$$

Then, find the particular solution $y_p = u_1 y_1 + u_2 y_2$ using Cramer's rule. Define

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \; W_1 = \begin{vmatrix} 0 & y_2 \\ g(x)' & y_2' \end{vmatrix} \; W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x)' \end{vmatrix} \tag{87}$$

When we set up the matrix-vector equation $A\mathbf{x} = \mathbf{b}$, it corresponds to the following:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ g(x) \end{bmatrix} \tag{88}$$

Following Cramer's rule, to find the solutions $1, \ldots, n$, we create A_n by replacing the nth row of A with b.

Strategy is to find u_1', u_2' then integrate to find the coefficients and thus the eventual solution.

$$u_1' = \frac{W_1}{W} \tag{89}$$

$$u_2' = \frac{W_2}{W} \tag{90}$$

3.8 Reduction of Order

Method is derived from the general 2nd order linear DE. Useful when y_p does not correspond to a template solution class.

$$y'' + py' + qy = R \tag{91}$$

Assume there exists a solution $y = y_1$ of the homogeneous equation (e.g. can pick a solution from y_c).

$$y'' + py" + qy = 0 \tag{92}$$

We then introduce a dependent variable from $y = y_1 v$ as v. It follows that

$$y' = y_1 v' + y_1' v \tag{93}$$

$$y'' = y_1 v'' + 2y_1' v' + y_1'' v \tag{94}$$

So substituting into 91, we get

$$y_1 v'' + 2y_1' v' + y_1'' v + p y_1 v' + p y_1' v + q y_1 v = R \tag{95}$$

This rearranges to

$$y_1 v'' + \left(2y_1' + p y_1\right) v' + \left(y_1'' + p y_1' + q y_1\right) v = R. \tag{96}$$

We use the definition of the homogeneous equation, since $y = y_1$ is a solution of the above.

$$y_1 v'' + (2y_1' + p y_1) v' = R \tag{97}$$

If we let $v' = w$ and solve for w, then integrate, it is simply a first order problem with integrating factor

$$k(x) = \exp \int 2y_1' + p y_1 dx \tag{98}$$

3.9 Spring Oscillation

3.9.1 Derivation and Amplitude Form

3.9.2 Solution Cases