AP Calculus BC Reference

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1 Limits and Continuity

1.1 Evaluating Limits

1. To simply evaluate $\lim_{x \to c} f(x)$, plug in c such that $f(c) = L$, the value of the limit.
2. If $f(c) = \frac{0}{0}$, factor numerator and denominator, then cancel terms.

$$lim_{x \to 0} \frac{x^4 + x^2}{x^3 + 3x^2} = \lim_{x \to 0} \frac{x^2 + 1}{x + 3} = \frac{1}{3}$$

3. If $f(c) = \frac{0}{0}$ and radicals are involved, then rationalize using conjugate and resubstitute.

$$lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$$

4. For limits that approach $\pm\infty$, cancel everything but greatest degree terms from numerator and denominator, then re-evaluate.

Given the problem to find the value of constant c as below, observe that when the limit is indeterminate, it is in the form $\frac{0}{0}$. This is because a limit with just denominator as 0 does not exist, so the indeterminate form is required.

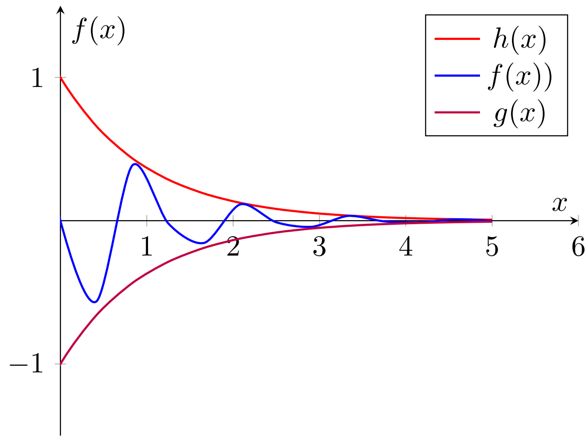
$$\begin{gathered}
\lim_{x \to 2} \frac{x^2 + cx + c - 10}{x^2 - 3x + 2} \\
x^2 - 3x + 2|_{x=2} = 0 \Rightarrow \lim_{x \to 2} \frac{x^2 + cx + c - 10}{x^2 - 3x + 2} = \frac{0}{0} \\
x^2 + cx + c - 10|_2 = 3c - 6 = 0 \\
c = 2
\end{gathered}$$

1.2 Horizontal Asymptotes

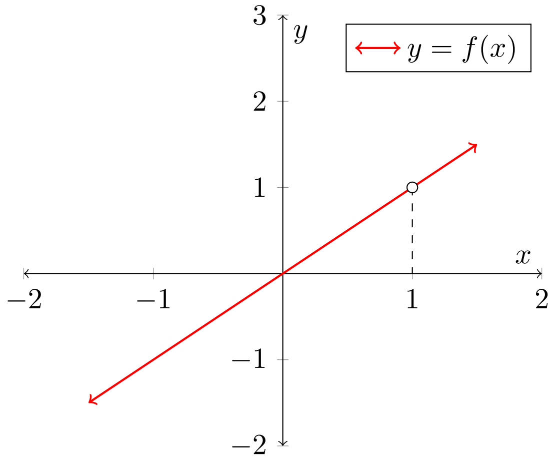
Take limit to $\pm\infty$ (end behavior).

1.3 Squeeze Theorem

Near $x = c$, let $g(x) \leq f(x) \leq h(x)\ \forall x$, if $\lim_{x \to c} g(x) = L$ and $\lim_{x \to c} h(x) = L$, then $\lim_{x \to c} f(x) = L$ must be true.



1.4 Graphical Limits



Notice that the limit of $f(x)$ exists at $x = 1$ though it is undefined at $(1, 1)$. Formally, $\lim_{x\to 1} f(x) = 1$ where $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$.

1.5 Continuity

A function is continuous at an interior point if $\lim_{x\to c} f(x) = f(c)$. It is continuous at a left endpoint if $\lim_{x\to a^+} f(x) = f(a)$ and at a right endpoint if $\lim_{x\to b^-} f(x) = f(b)$.

2 Derivatives

2.1 Derivative Rules

1. Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$
2. Product Rule: $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + g'(x)f(x)$
3. Quotient Rule: $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}$

4. Chain Rule: $\frac{d}{dx}f(g(x)) = g'(x)f'(g(x))$

5. Logarithmic Derivatives

   (a) $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$

   (b) $\frac{d}{dx}\log_a(f(x)) = \frac{f'(x)}{\ln a f(x)}$

6. Trigonometric Derivatives

   (a) $\frac{d}{dx}\sin x = \cos x$

   (b) $\frac{d}{dx}\cos x = -\sin x$

   (c) $\frac{d}{dx}\tan x = \sec^2 x$

   (d) $\frac{d}{dx}\cot x = -\csc^2 x$

   (e) $\frac{d}{dx}\sec x = \sec x \tan x$

   (f) $\frac{d}{dx}\csc x = -\csc x \cot x$

7. Inverse Trigonometric Derivatives

   (a) $\frac{d}{dx}\arcsin(f(x)) = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$

   (b) $\frac{d}{dx}\arccos(f(x)) = \frac{-f'(x)}{\sqrt{1-(f(x))^2}}$

   (c) $\frac{d}{dx}\arctan(f(x)) = \frac{f'(x)}{1+(f(x))^2}$

   (d) $\frac{d}{dx}\text{arccot}(f(x)) = \frac{-f'(x)}{1+(f(x))^2}$

   (e) $\frac{d}{dx}\text{arcsec}(f(x)) = \frac{f'(x)}{|f(x)|\sqrt{(f(x))^2-1}}$

   (f) $\frac{d}{dx}\text{arccsc}(f(x)) = \frac{-f'(x)}{|f(x)|\sqrt{(f(x))^2-1}}$

2.1.1 Example of logarithmic differentiation

Take ln of both sides, differentiate, then get in terms of $f'(x)$ and simplify.

$$f(x) = 2^x$$
$$\ln(f(x)) = x\ln(2)$$
$$\frac{f'(x)}{f(x)} = \ln 2$$
$$f'(x) = \ln 2 \cdot f(x)$$
$$f'(x) = \ln 2 \cdot 2^x$$

3 Applications of Derivatives

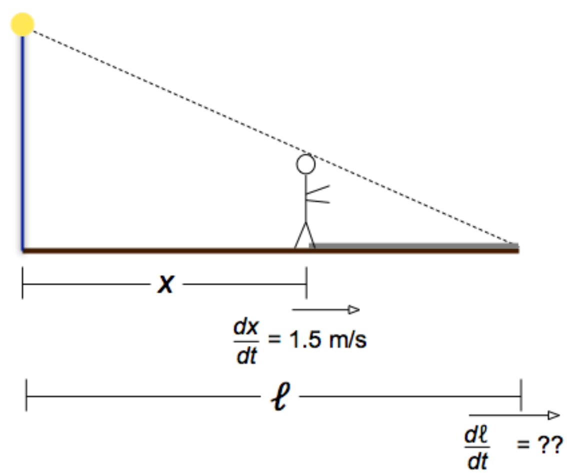
3.1 Implicit Differentiation

Explained in the following example.

$$\frac{dy}{dx}(x^2 + y^2 + y = 25)$$
$$2x + 2yy' + y' = 0$$
$$y'(2y + 1) = -2x$$
$$\frac{dy}{dx} = y' = \frac{-2x}{2y + 1}$$

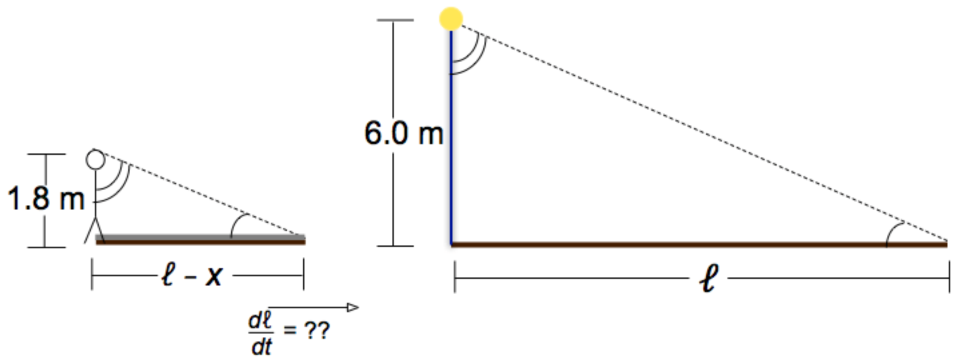
3.2 Related Rates

3.2.1 Shadow Problem



A 1.8-meter tall man walks away from a 6.0-meter lamp post at the rate of 1.5 m/s. The light at the top of the post casts a shadow in front of the man. How fast is the "head" of his shadow moving along the ground?

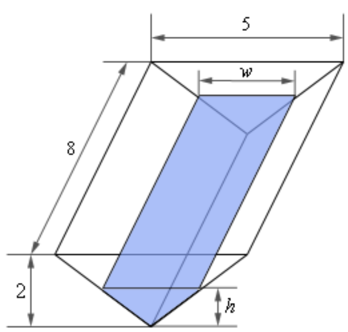
Must find $\frac{dl}{dt}$. The strategy is to use similar triangles to relate x and l.



$$\frac{l-x}{l} = \frac{1.8}{6.0}$$
$$l - x = 0.3l$$
$$x - l = 0.30l$$
$$x = 0.7l$$

$$0.7\frac{dl}{dt} = \frac{dx}{dt}$$
$$\frac{dl}{dt} = \frac{1.5}{0.7}$$
$$\frac{dl}{dt} = 2.1\frac{m}{s}$$

3.2.2 Trough Problem



A trough of water is 8 meters in length and its ends are in the shape of isosceles triangles whose width is 5 meters and height is 2 meters. If water is being pumped in at a constant rate of 6 m^3/sec, at what rate is the height of the water changing when the water has a height of 120 cm? At what rate is the width of the water changing when the water has a height of 120cm?

It is known that $V' = 6m^3/\text{sec}$. Need h' when $h = 1.2$.

The volume of the water in the tank is given by:

$$\begin{aligned} V &= \frac{1}{2} base \times height \times depth \\ &= \frac{1}{2} hw(8) \\ &= 4hw \end{aligned}$$

Need to eliminate w as target is h'. Using similar triangles.

$$\frac{w}{5} = \frac{h}{2} \Rightarrow w = \frac{5h}{2} \Rightarrow V = 10h^2$$
$$V' = 20hh' \Rightarrow 6 = 20(1.2)h' \Rightarrow h' = 0.25m/sec$$

R.O.C of width can be found by manipulating similar triangles to get V in terms of w only.

$$h = \frac{2w}{5} \Rightarrow V = \frac{8w^2}{5}$$

Differentiate and substitute as before.

3.3 L'Hopital's Rule

3.3.1 Indeterminate Forms: $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

Suppose that $f(x) = 0$ and $g(x) = 0$ and that $c \in \mathbb{R}$.

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Process can be repeated.

3.3.2 Indeterminate Forms: $\infty \cdot 0$ or $\infty - \infty$

Requires that the limit be rewritten in fractional form for differentiation.

$$\lim_{x \to 0^+} x \csc x$$
$$x \csc x = \frac{x}{\sin x}$$
$$\lim_{x \to 0} \frac{x}{\sin x} = \frac{1}{\cos x}|_0 = \frac{1}{1} = 1$$

3.3.3 Indeterminate Forms: 0^0, 1^∞, or ∞^0

Incorporates changes using logarithms in order to get limit in to the form as shown before.

$$L = \lim_{x\to 0^+} x^x$$
$$\ln L = x\ln x \Rightarrow -\infty \cdot 0$$
$$\ln L = \lim_{x\to 0^+} = \frac{\ln x}{\frac{1}{x}} = \frac{\frac{1}{x}}{\frac{-1}{x^2}}$$
$$\ln L = \lim_{x\to 0^+} -x = 0$$
$$e^{\ln L} = e^0$$
$$L = 1$$

Special example in the following form:

$$\lim_{x\to c}(d + \frac{a}{x})^{bx} = e^{ab}$$

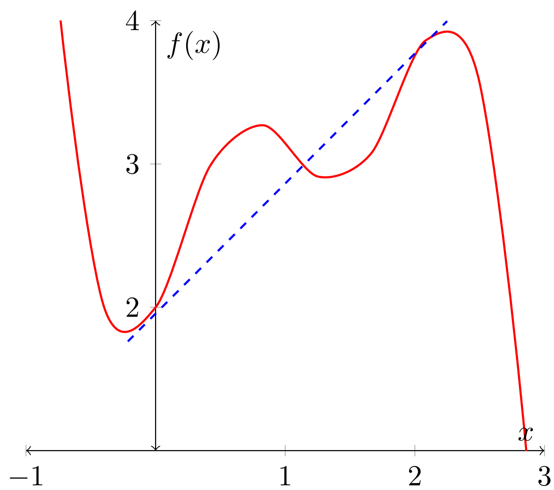
3.3.4 Limit Definition of Derivative

In the form $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, where the derivative is taken with respect to h. By recognizing this form, the answer would be $\frac{d}{dx}f(x)$.

4 Graphing and Analytical Applications

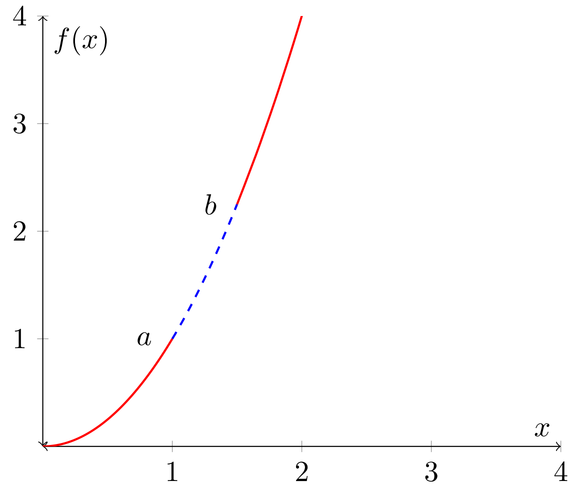
4.1 Mean Value Theorem (MVT)

Given that $f(x)$ is continuous $\forall x \in [a,b]$ and differentiable $\forall x \in (a,b)$, $\exists c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.



4.2 Intermediate Value Theorem (IVT)

Let $f(x)$ be continuous $\forall x \in [a, b]$. Let m between $f(a)$ and $f(b)$. $\exists c \in (a, b)$ such that $f(c) = m$.

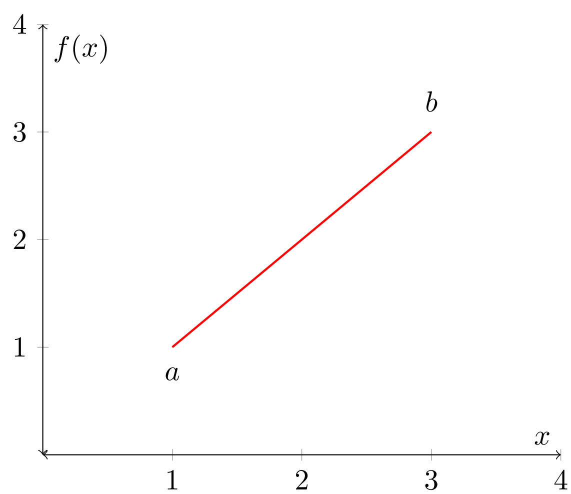


4.3 Rolles' Theorem

Given that $f(x)$ is continuous $\forall x \in [a, b]$ and differentiable $\forall x \in (a, b)$ and $f(a) = f(b)$, then $\exists c \in (a, b)$ such that $f'(c) = 0$.

4.4 Extreme Value Theorem (EVT)

If $f(x)$ is continuous $\forall x \in [a, b]$ then $f(x)$ has a max and min value on $[a, b]$.

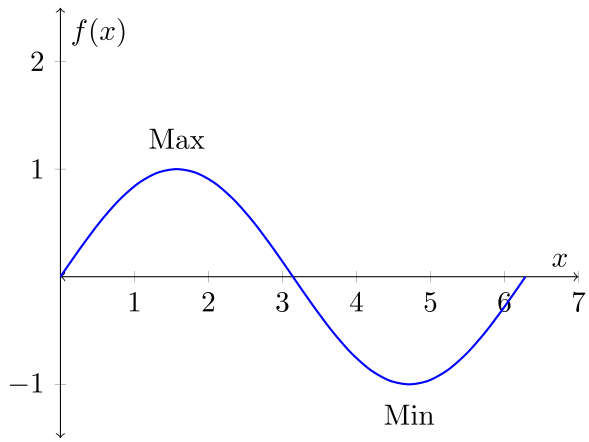


4.5 Fermat's theorem

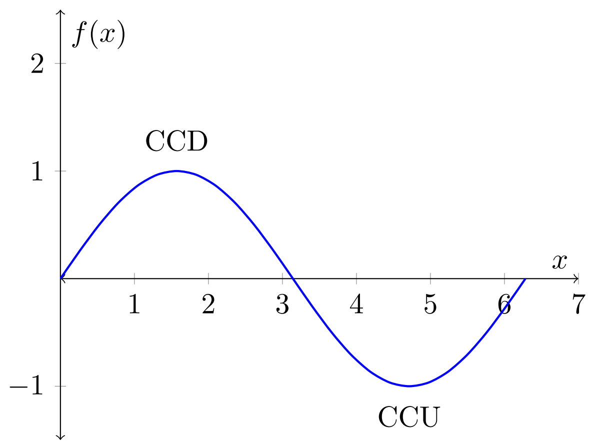
If $f(x)$ has a rel. max or min at $x = c$ and $f'(c)$ exists, then $f'(c) = 0$. However, there can be a rel. max/min when $f'(c)$ does not exist but implements a sign change.

4.6 First Derivative Test for Local Extrema and Second Derivative Test

Determines where a function increases or decreases, which denotes the maximums and mininums. Is found from checking *critical points* where $f'(x) = 0$. Maximums and mininums occur where $f'(x)$ changes sign, which must be around a critical point.



The second derivative test determines where the function is **concave up** or **concave down**, or where $f''(x) > 0$ or < 0 respectively.



4.7 Second Derivative Test for Local Extrema

If $f'(x)$=0 and $f''(x) < 0$, then $f(x)$ is the location of a local maxima. Similarly if $f'(x)$=0 and $f''(x) > 0$, then it is a local minima. Can be visualized through a concave up/down image, where the apex of the curve determines the extrema required.

4.8 Absolute Maximums and Minimums (Candidates Test)

Uses a chart like the following with x values determined from endpoints and critical points of $f(x)$.

x	$f(x)$
a	$f(a)$
x_1	$f(x_1)$
x_2	$f(x_2)$
b	$f(b)$

4.9 Derivatives of Inverse Functions

Let $g(x)$ be $f^{-1}(x)$ and (x, a) be the point.

$$g'(a) = \frac{1}{f'(g(a))}$$

4.10 2D Particle Motion

Fundamental equations are position, velocity, and acceleration.

$$\begin{gathered} x(t) \\ v(t) = x'(t) \\ a(t) = v'(t) = x''(t) \end{gathered}$$

4.10.1 Velocity

A particle changes direction when $v(t)$ goes from $+$ to $-$. The particle is moving to the left when $v(t) < 0$ and to the right when $v(t) > 0$.

4.10.2 Speed

Speed is denoted as $|v(t)|$. A particle is speeding up when $a(t)$ and $v(t)$ have the same sign and slowing down when they have different signs.

4.10.3 Find Next Position and Min/Max

Uses FTC.

$$x(t_f) = x(t_i) + \int_{t_i}^{t_f} v(t)dt$$

Min/max problems are formatted as farthest to left or right. Uses the candidates test for abs. max/min. Use FTC to find position values. t-values are where $v(t) = 0$.

Farthest to left $\Rightarrow$ abs. min., for right $\Rightarrow$ abs. max.

4.10.4 Distance and Displacement

Distance (d) is total distance traveled, ignores a change in direction, displacement (D) accounts for this.

$$d = \int_a^b |v(t)| dt$$
$$D = \int_a^b v(t) dt$$

4.10.5 Average Speed and Velocity

Average speed (S) is related to distance. Average velocity (V) accounts for displacement.

$$S = \frac{\int_a^b |v(t)| dt}{b-a}$$
$$V = \frac{\int_a^b v(t) dt}{b-a}$$

4.11 Differentiability and Continuity

A function is not neccesarily differentiable if it is continuous, but if it is differentiable, then it must be continuous. Types of non-differentiable points include corner points, cusps, and vertical tangents.

4.11.1 Corner Points

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \text{DNE}$$

4.11.2 Vertical Tangents

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \pm\infty$$

4.11.3 Cusps

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \pm\infty$$
$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \mp\infty$$

5 Integration

5.1 Basic Rules

1. FTC: $F(b) = F(a) + \int_a^b f(x)dx$
2. Power rule: $\int x^n dx = \frac{x^{n+1}}{n-1}$

5.1.1 U-Substitution

$$\int_2^5 x(3x^2)dx \Rightarrow u = 3x^2, du = 6xdx$$
$$\Rightarrow \frac{1}{6}\int_{12}^{75} udu$$
$$\frac{1}{6}[\frac{u^2}{2}|_{75} - \frac{u^2}{2}|_{12}]$$

5.1.2 Integration by Partial Fractions

Factor denominator, separate into partial fractions, then integrate.

$$\int \frac{1}{x^2-4}dx \Rightarrow \frac{1}{x^2-4} = \frac{1}{(x+2)(x-2)}$$
$$\frac{1}{(x+2)(x-2)} \Rightarrow \frac{A}{x+2} + \frac{B}{x-2} \Rightarrow A = -\frac{1}{4}, B = \frac{1}{4}$$
$$\int \frac{-\frac{1}{4}}{x+2} + \frac{\frac{1}{4}}{x-2}dx = -\frac{1}{4}\ln|x+2| + \frac{1}{4}\ln|x-2| + C$$

5.1.3 Trigonometric Integrals

Trigonometric Integrals

- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \tan x \, dx = -\ln \cos x + C$
- $\int \cot x \, dx = \ln \sin x + C$
- $\int \sec x \, dx = \ln(\sec x + \tan x) + C$
- $\int \csc x \, dx = \ln(\csc x - \cot x) + C$
- $\int \sec x \tan x \, dx = \sec x + C$

- $\int -\csc x \cot x \, dx = \csc x + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int -\csc^2 x \, dx = \cot x + C$

Inverse Trigonometric Integrals

- $\int \frac{dx}{f(x)^2+a^2} = \frac{1}{a} \arctan \frac{f(x)}{a} + C$
- $\int \frac{dx}{\sqrt{a^2-f(x)^2}} = \arcsin \frac{f(x)}{a} + C$

5.2 Properties of Integrals

1. $\int_a^a f(x)dx = 0$
2. If $f(x)$ is odd, $\int_{-a}^a f(x)dx = 0$
3. If $f(x)$ is even and $\int_0^a f(x)dx = k$, then $\int_{-a}^a f(x)dx = 2k$
4. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$
5. If $\int_a^b f(x)dx = k$, then $\int_b^a f(x)dx = -k$
6. If $f(x) \leq g(x)\ \forall x \in [a, b]$ then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$
7. $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$

5.3 Riemann Sums

5.3.1 Riemann Sum Notation and Definite Integrals

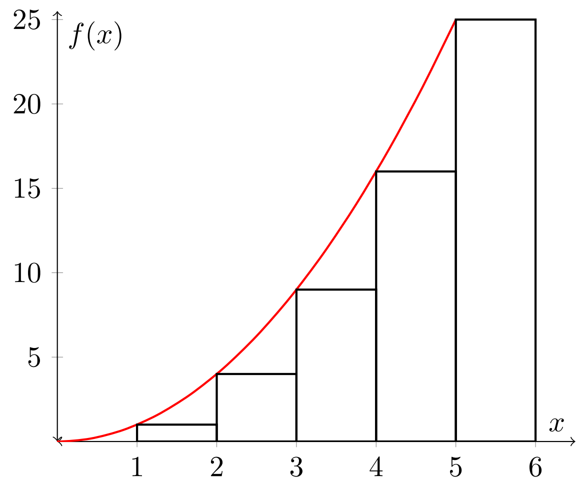
$$\lim_{n\to\infty} \sum_{k=1}^{n} f(a + k\Delta x)\Delta x$$

In $\Delta x = \frac{b-a}{n}$ (the subintervals), b is the end limit of the integral which is the same as $a + n\Delta x$. This can be used to solve for b given a. x can be represented as $a + k\Delta x$.

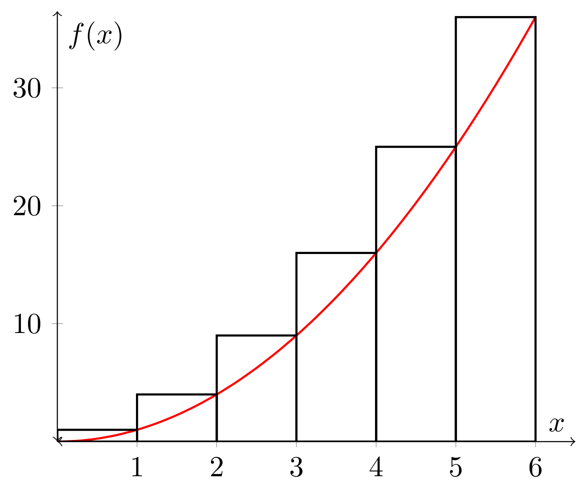
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(a + \frac{k(b-a)}{n}) \frac{b-a}{n}$$

5.3.2 Basic Riemann Sums

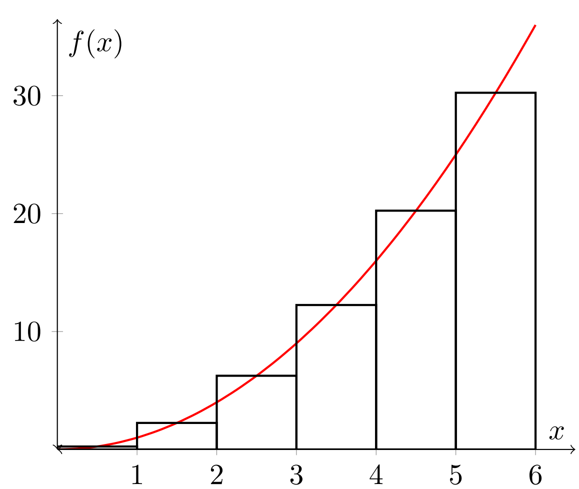
Left sum: $(x_R - x_L)f(x_L)$



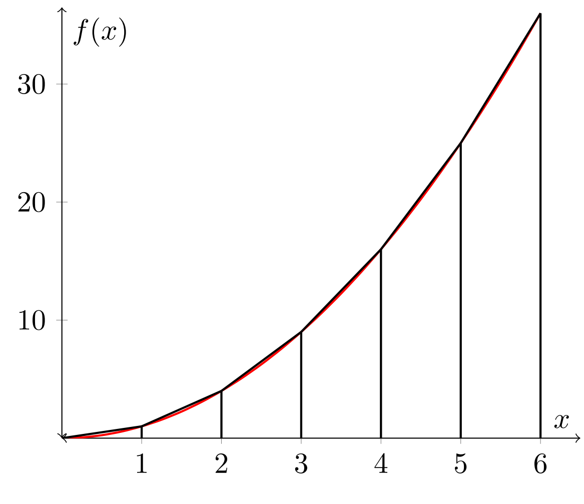
Right sum: $(x_R - x_L)f(x_R)$



Midpoint sum: $(x_R - x_L)f(x_L + \frac{x_R - x_L}{2})$



Trapezoidal sum: $\frac{1}{2}(f(x_R)+f(x_L))(x_R-x_L)$



5.4 Integration by Parts

$$\int f'(x)g(x)dx = f(x)g(x) - \int g'(x)f(x)dx$$

Choose $g(x)$ in order of **logs, inverse, algebraic, trig, exponential**.

5.4.1 Example

$$\begin{aligned}\int x\sqrt{x+1}dx &\Rightarrow g(x)=x, f'(x)=\sqrt{x+1}\\ &= \frac{x}{2}(x+1)^{3/2} - \int \frac{1}{2}(x+1)^{3/2}dx\\ &= \frac{x}{2}(x+1)^{3/2} - \frac{1}{5}(x+1)^{5/2}+C\end{aligned}$$

5.4.2 Tabular Method

Negate every second entry under derivative column.

$$\int x^2 \sin x \, dx \Rightarrow f(x)=x^2,\ g(x)=\sin x$$

$f(x)$	$g(x)$
x^2	$\sin x$
$-2x$	$-\cos x$
2	$-\sin x$
0	$\cos x$

$$\int x^2 \sin x \, dx = -x^2\cos x - 2x\sin x + 2\cos x + C$$

6 Differential Equations

6.1 Separation of Variables

MIT OCW Reference

$$\frac{dy}{dx} = x(y-1)$$
$$\frac{dy}{y-1} = xdx$$
$$\ln|y-1| + C = \frac{x^2}{2} + C$$
$$|y-1| = e^C e^{\frac{x^2}{2}} \Leftrightarrow |y-1| = Ce^{\frac{x^2}{2}}$$
$$y - 1 = \pm Ce^{\frac{x^2}{2}}$$
$$y = 1 + Ce^{\frac{x^2}{2}}$$

6.2 Exponential growth

$$\frac{dP}{dt} = kP \Rightarrow P = P_0 e^{kt}$$

A negative growth constant k represents decay and a positive one represents growth. For half life problems, one is solving for half (or any fraction or multiple) of the initial population.

$$\frac{1}{2}P_0 = P_0 e^{kt}$$
$$\frac{1}{2} = e^{kt} \Rightarrow t = \frac{\ln \frac{1}{2}}{k}$$

6.3 Euler's Method

When approximating a lower x-value than given, the procedure is still identical **but Δx is negative**.

$$y_{new} = y_{old} + \frac{dy}{dx}|_{(x_{old}, y_{old})} \Delta x$$

6.4 Slope Fields

A differential equation in the form $\frac{dy}{dx} = F(x, y)$, where the slopes at each point in a 2D graph are plotted with line segments. The solution curves for a differential equation align with these segments, and the initial condition can be visualized.

6.5 Logistic Growth

Growth is limited by the carrying capacity L, which is found by setting the differential equation to 0 since this is when the curve flattens and the carrying capacity is achieved. There is a positive growth constant k. $\lim_{t\to\infty} \frac{dP}{dt} = 0$ always because of the asymptote to L. However, $\frac{dP}{dt}$ **asymptotes to** 0 and is never exactly 0.

$$\frac{dP}{dt} = kP(L - P)$$

When derived, the population formula is as follows, with A being a constant.

$$P(t) = \frac{L}{1 + Ae^{-kt}}$$

The inflection point of the population (not growth) equation is when the population is growing the fastest. This occurs at the time t when $P = \frac{L}{2}$. Furthermore, the carrying capacity of this population is given by $\lim_{t\to\infty} P(t) = L$, which is always true due to form of $P(t)$.

7 Applications of Integration

7.1 In/Out Rates

When finding the *total* quantity at a certain time (not specific to how much going in/out), the net rate is used. This is given by $R(t) - E(t)$ where $R(t)$ is the rate in and $E(t)$ is the rate out. Thus, the quantity at a certain time is given by the following equation.

$$A(t_f) = A(t_i) + \int_{t_i}^{t_f} (R(t) - E(t))dt$$

7.2 Average Value and R.O.C of a Function

Average value involves the antiderivative of f while the average R.O.C involves f itself.

$$f_{avg} = \frac{\int_a^b f(x)dx}{b - a}$$
$$A = \frac{f(b) - f(a)}{b - a}$$

Arclength of a function also is given from an integral and is found from the following (useful in perimeter problems).

$$S = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

7.3 Accumulation Functions

Generally given in the following form as $F(x)$. The integral must use a different variable t as it is not dependent on x. a is a constant representing the lower integral limit.

$$F(x) = \int_a^x f(t)dt$$

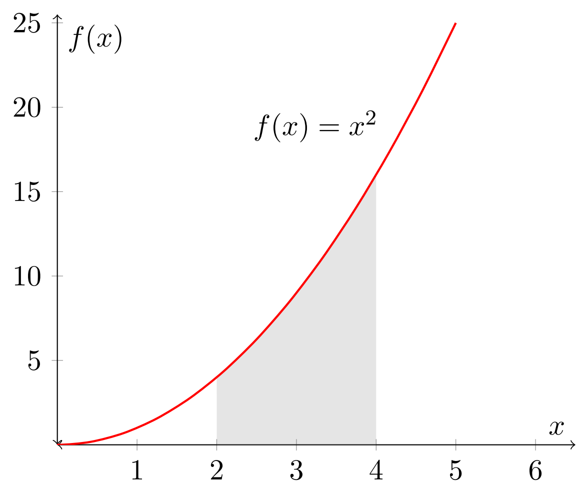
Differentiation is also applicable, with each progression a lower offset of a normal function's derivative, observable in the following expressions.

$$F'(x) = 1 \cdot f(x) - 0 = f(x)$$
$$F''(x) = f'(x)$$

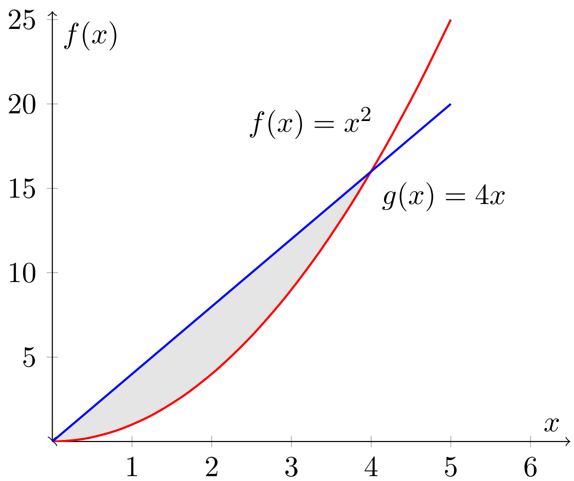
7.4 Area and Volume

7.4.1 Area

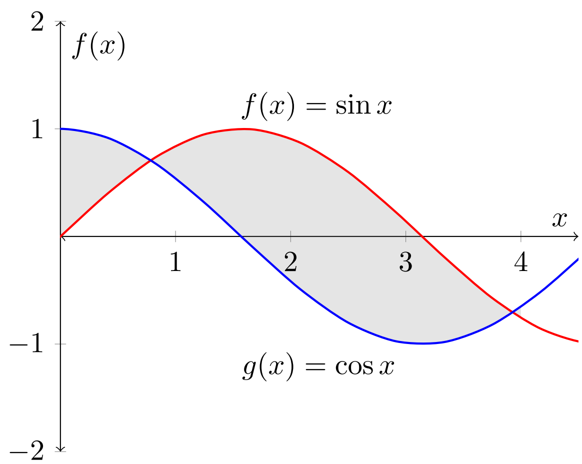
Area with respect to a curve $f(x)$ and the x-axis is given by $\int_a^b f(x)dx$. If the curve from a to b is below the x-axis, then it is negative in value, **but the area is not negative**.



Area of two intersecting regions is given by $\int_a^b (f(x) - g(x))dx$.



Area of regions with multiple intersections is given by $\int_a^b |f(x) - g(x)| dx$, ignoring the central intersection point.

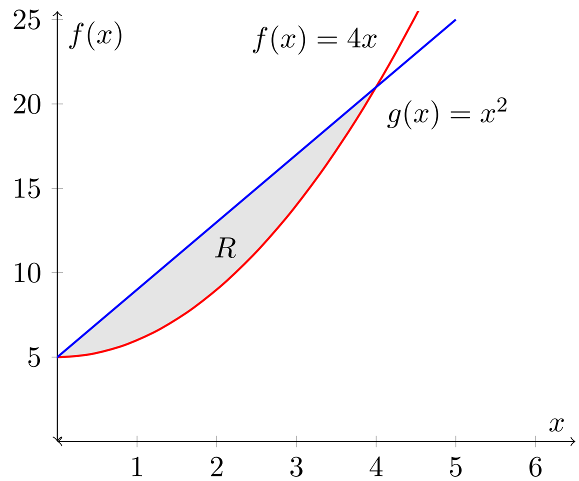


dy integration is similar to normal integration, but uses the y-axis for reference. The following example uses $f(y) = sin(y)$ and $g(y) = cos(y)$, with the integral being $\int_0^{\pi/4} (g(y) - f(y))\, dy$.

7.5 Volume around horizontal axes

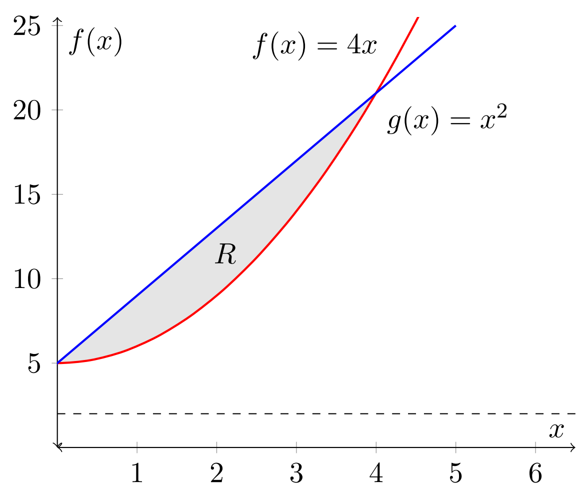
Given that $f(x) \geq g(x)\ \forall x \in [a, b]$, the integral is as follows for the region R revolved around $x = 0$.

$$V_x = \pi \int_a^b (f(x)^2 - g(x)^2)\, dx$$

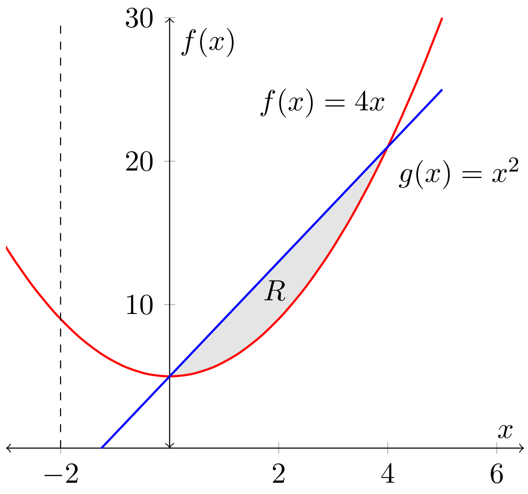


For the same region R revolving around $x = 2$, the radius of rotation (washer space) is reduced, so 2 is subtracted. This would give the integral equation as the following. If $x = -2$, for instance, 2 would be added as it increases the washer radius.

$$V_x = \pi \int_0^4 ((f(x) - 2)^2 - (g(x) - 2)^2)\, dx$$



7.6 Volume around vertical axes

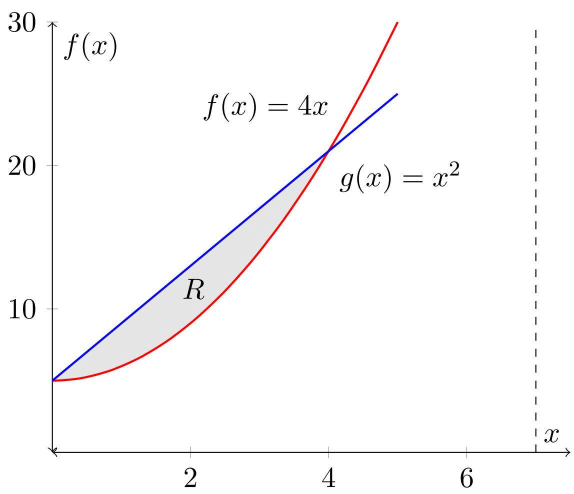


Is given in the general form $V_y = 2\pi \int_a^b (radius)(height)\, dx$. The height is the value of $f(x)$ and the radius is some value of x since this is with respect to the y-axis. The preceding example is a rotation around $y = -2$, and the integral would be given by the following.

$$V_y = 2\pi \int_0^4 (x+2)(f(x) - g(x))\, dx$$

The procedure is the opposite when given a vertical axis on the other side of the graph in quadrant I, namely $x = 7$. This is because the radius would be given by $7 - x$ since that difference is the distance between each varying point and $x = 7$. The formula would be the following.

$$V_y = 2\pi \int_0^4 (7-x)(f(x) - g(x))\, dx$$



Notably, the last two examples can also be computed using dy integration in a similar way to horizontal axis-based solids. However, the bottom function would become the top one and the limits would become the y-coordinates. They are below. Let $g(y) = \sqrt{y}$ and $f(y) = \frac{y}{4}$.

$$A_y = \pi \int_5^{21} ((g(y) + 2)^2 - (f(y) + 2)^2) dy$$
$$A_y = \pi \int_5^{21} ((7 - g(y))^2 - (7 - f(y))^2) dy$$

7.7 Volume with known cross-sections

Let $s = f(x) - g(x)$ and $g(x) \leq f(x)\ \forall x \in [a, b]$.

- Squares: $\int_a^b s^2 dx$
- Rectangles (with the length being n times the width): $\int_a^b ns^2 dx$
- Equilateral triangles: $\int_a^b \frac{\sqrt{3}s^2}{4} dx$
- Semi-circles: $\int_a^b \frac{1}{8}\pi s^2 dx$
- Right isosceles triangle: $\int_a^b \frac{1}{4}s^2 dx$

8 Infinite Series

8.1 Fundamental Series

As listed below the harmonic series 1, diverging due to P-series, and alternating series 2, which converges according to AST.

$$\sum_{n=1}^{\infty} \frac{1}{n} \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \tag{2}$$

8.2 P-series test

A series in the form of $\sum_{n=1}^{\infty} \frac{1}{n^P}$. If $P \leq 1$, the series diverges, if $P > 1$, it converges.

8.3 n-th term test (divergence only)

If $\lim_{n\to\infty} |a_n| \neq 0$, the series diverges, else if it is 0, it is **inconclusive**. But if a series diverges, *it is not neccesarily due to the n-th term test.*

8.4 Geometric series

$\sum_{n=1}^{\infty} a_1 r^{n-1}$ converges if and only if $|r| < 1$. A power series is a form of a geometric one.

$$\Rightarrow \sum_{n=1}^{\infty} a_1 r^{n-1} = \frac{a_1}{1-r}$$

8.5 Ratio test

$\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$.
$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1$.
The test if inconclusive is the result is 1.

8.6 Alternating series test

A decreasing alternating series, where $|a_{n+1}| < |a_n|$, converges if $\lim_{n\to\infty} a_n = 0$.

8.7 Taylor/Maclaurin Series

$$\sum_{n=0}^{\infty} \frac{f^n(c)(x-c)^n}{n!}$$

Since the coefficient of each term is the n-th derivative, if given a term T, the derivative can be found by setting $\frac{f^n(c)(x-c)^n}{n!} = T$ in order to get an expression.

A Maclaurin polynomial is a Taylor polynomial centered at $x = 0$. Here are some common Maclaurin polynomial for function:

- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$
- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} x^n$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} x^n$
- $\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} x^n$

8.8 Trigonometric Identities

Pythagorean Identities

- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = sec^2 x$
- $1 + \cot^2 x = \csc^2 x$

Double-Angle Identities

- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$
- $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$