

# Spacecraft Motion with Thrust Normal to Velocity Vector

June 15-29 2021

# Weekly Progress

- ▶ June 10-15
  - ▶ Found expression for  $w_1(\phi)$  from  $w(r)$  in terms of  $\phi$  and  $\beta$
  - ▶ Found an expression for  $y_1(\phi)$  given  $y(r)$
  - ▶ Found discrepancy between plots of  $w(r)$  and  $w_1(\phi)$  using Desmos and Octave
- ▶ June 15-29
  - ▶ Checked work for  $w_1(\phi)$  and adjusted expression to account for newly provided definition of  $\beta$
  - ▶ Found an expression for  $y_1(\phi)$  given  $y(r)$  using new definition
  - ▶ Numerically minimized  $w_1(\phi)$  after obtaining a complicated relation (simplified to best means)
  - ▶ Work was checked thoroughly using Desmos and Octave plots

## Expressing $w(r)$ in terms of $\phi$ and $\beta$

Approach: reexpress components of  $w(r)$  and substitute  $r(\phi)$

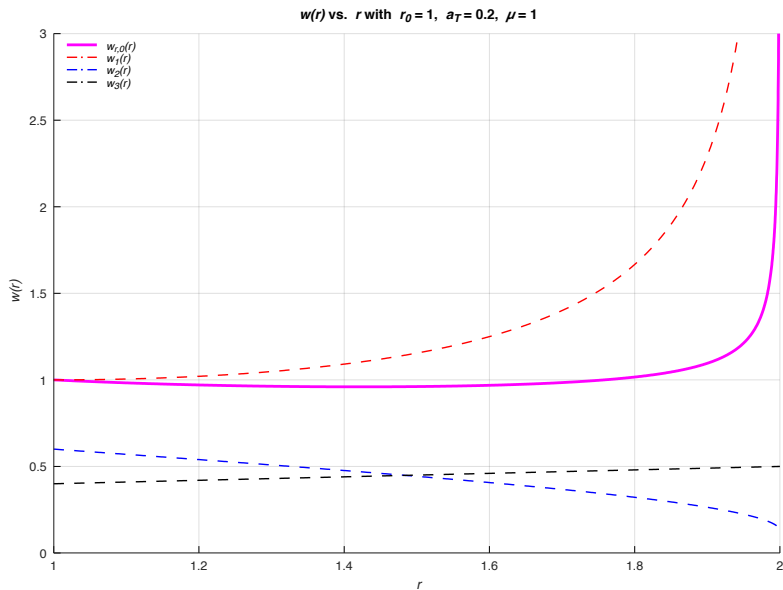
$$w(r) = \underbrace{\frac{r_0}{\sqrt{r(2r_0 - r)}}}_{w_1(r)} \underbrace{\left[ 1 - \frac{a_T r_0^2}{\mu} \left( 3 \sin^{-1} \left( \sqrt{\frac{r}{2r_0}} \right) - \frac{3\pi}{4} + 2 \right) \right]}_{w_2(r)} + \underbrace{\frac{a_T r_0}{2\mu} (r + 3r_0)}_{w_3(r)}$$

For plots:  $r_0 = 1, \mu = 1, a_T = 0.2$

Definitions:  $k(\phi) = \phi + \pi/4$ ,  $r(\phi) = 2r_0 \sin^2(k)$ ,  $\beta = \sqrt{\frac{4\mu}{(3\pi+8)a_T r_0^2}}$

Subscript  $p$  denotes function of  $\phi$ .

# Plotting $w_1(r)$ , $w_2(r)$ , $w_3(r)$



## Reexpressing $w_1(r)$

$$w_{1\rho}(\phi) = \frac{r_0}{\sqrt{r(2r_0 - r)}} \Big|_{r(\phi)} \quad (1)$$

$$= \frac{r_0}{\sqrt{2r_0 \sin^2(k)(2r_0 - 2r_0 \sin^2(k))}} \quad (2)$$

$$= \frac{r_0}{\sqrt{4r_0^2 \sin^2(k) \cos^2(k)}} \quad (3)$$

$$= \frac{1}{\sin(2k)} \quad (4)$$

Obtain 3 using  $\cos^2(k) = 1 - \sin^2(k)$  and 4 by  $\sin(2k) = 2 \sin(k) \cos(k)$ .

## Reexpressing $w_2(r)$

$$w_{2p}(\phi) = 1 - \frac{a_T r_0^2}{\mu} \left( 3 \arcsin\left(\sqrt{\frac{r}{2r_0}}\right) - \frac{3\pi}{4} + 2 \right) \Big|_{r(\phi)} \quad (5)$$

$$= 1 - \frac{a_T r_0^2}{\mu} \underbrace{\left( 3k - \frac{3\pi}{4} + 2 \right)}_{3\phi+2} \quad (6)$$

$$= 1 - \frac{a_T r_0^2}{\mu} (3\phi + 2) \quad (7)$$

## Reexpressing $w_3(r)$

$$w_{3p}(\phi) = \frac{a_T r_0}{2\mu} (r + 3r_0) \Big|_{r(\phi)} \quad (8)$$

$$= \frac{a_T r_0}{2\mu} (r_0(2 \sin^2(k) + 3)) \quad (9)$$

$$= \frac{a_T r_0^2}{2\mu} (2 \sin^2(k) + 3) \quad (10)$$

$$= \frac{2}{\beta^2(3\pi + 8)} (2 \sin^2(k) + 3) \quad (11)$$

Obtain 11 by  $\beta = \sqrt{\frac{4\mu}{(3\pi+8)a_T r_0^2}} \implies \frac{1}{\beta^2} = \frac{(3\pi+8)a_T r_0^2}{4\mu}$

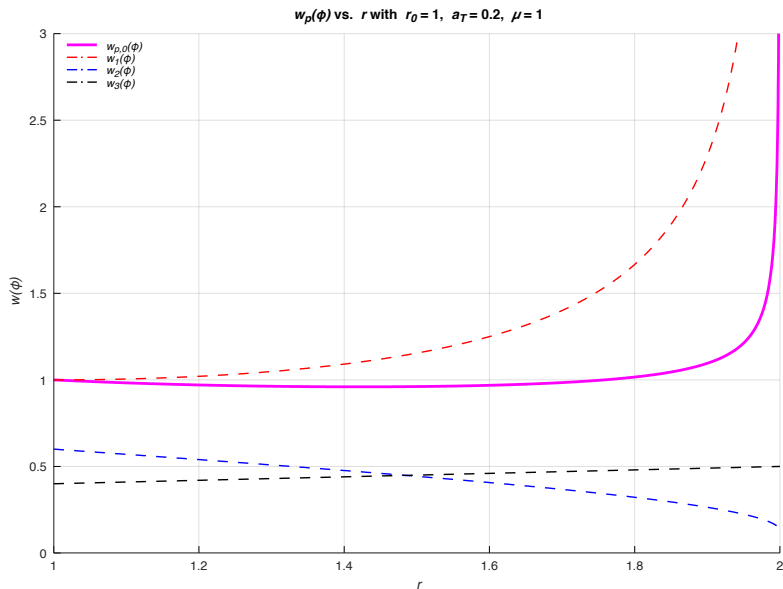
## Expression for $w_p(\phi)$

$$w_p(\phi) = \frac{1}{\sin(2k)} \left[ 1 - \frac{a_T r_0^2}{\mu} (3\phi + 2) \right] + \frac{2}{\beta^2(3\pi + 8)} (2 \sin^2(k) + 3) \quad (12)$$

$$= w_{1p}(\phi) w_{2p}(\phi) + w_{3p}(\phi) \quad (13)$$



# Checking answer with plots



## Expressing $y(r)$ in terms of $\phi$ and $\beta$

Approach similar to  $w(r)$ :

$$y(r) = \underbrace{\sqrt{\frac{\mu(2r_0 - r)}{rr_0}}}_{y_1(r)} \underbrace{\sqrt{1 - w(r)^2}}_{y_2(r)}$$

## Reexpressing $y_1(r)$

$$y_{1p}(\phi) = \sqrt{\frac{\mu(2r_0 - r)}{rr_0}} \Big|_{r(\phi)} \quad (14)$$

$$= \sqrt{\frac{\mu 2r_0(1 - \cos^2(k))}{2r_0^2 \sin^2(k)}} \quad (15)$$

$$= \sqrt{\frac{\mu 2r_0 \cos^2(k)}{2r_0^2 \sin^2(k)}} \quad (16)$$

$$= \sqrt{\frac{\mu}{r_0}} \cot(k) \quad (17)$$

## Reexpressing $y_2(r)$

Given  $y_{2p}(\phi) = \sqrt{1 - w_p(\phi)^2}$ .

Define  $\varphi(\phi)$  such that  $\frac{\varphi(\phi)}{(3\pi+8)^2\beta^4 \sin^2(2k)} = 1 - w_p(\phi)^2$ .

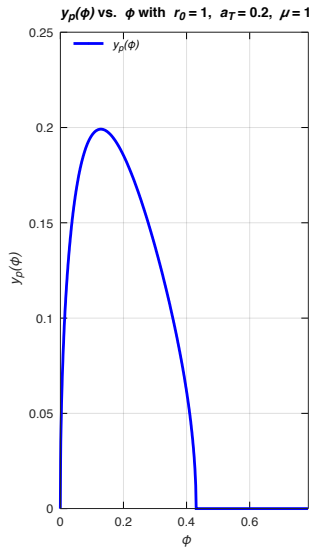
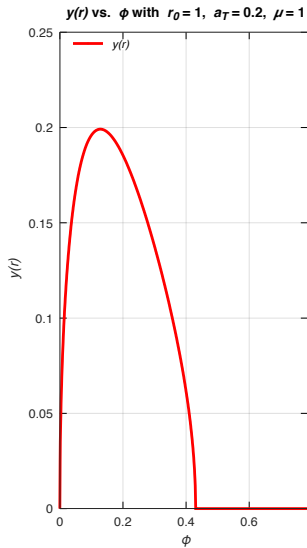
Then, by squaring 12,

$$\begin{aligned} 1 - w_p(\phi)^2 &= \frac{(3\pi + 8)^2 \beta^4 \sin^2(2k) - [(3\pi + 8)^2 \beta^4 - 8(3\phi + 2)(3\pi + 8)\beta^2 + 16(3\phi + 2)^2]}{(3\pi + 8)^2 \beta^4 \sin^2(2k)} \\ &\quad - \frac{2 \sin(2k)(4 \sin^2(k) + 6)((3\pi + 8)\beta^2 - 4(3\phi + 2))}{(3\pi + 8)^2 \beta^4 \sin^2(2k)} \\ &\quad - \frac{\sin^2(2k)(16 \sin^4(k) + 48 \sin^2(k) + 36)}{(3\pi + 8)^2 \beta^4 \sin^2(2k)} \end{aligned}$$

Extracting the denominator,

$$y_p(\phi) = \frac{\sqrt{\varphi(\phi)}}{\beta^2(3\pi + 8) \sin(2k)} \sqrt{\frac{\mu}{r_0}} \cot(k)$$

# Plots to verify $y_p(\phi)$



## Minimizing $w(r)$ or $w_p(\phi)$

- ▶ Began by differentiating  $w(r)$  and simplifying
- ▶ Substituted  $r(\phi)$  to find  $w_p'(\phi)$
- ▶ Solving for  $\phi$  in  $w_p'(\phi)$  would not be feasible by hand since there were trigonometric and linear terms of  $\phi$ 
  - ▶ Obtained a greatly simplified expression that was used for minimization
- ▶ Used Octave to minimize the expression numerically

# First derivative test on $w(r)$

$$w_1'(r) = \frac{d}{dr} r_0(2r_0r - r^2)^{-1/2} \quad (18)$$

$$= -\frac{1}{2} r_0(2r_0 - 2r)(2r_0r - r^2)^{-3/2} \quad (19)$$

$$= \frac{-r_0(r_0 - r)}{(2r_0r - r^2)^{3/2}} \quad (20)$$

$$w_2'(r) = \frac{-3a_T r_0^2}{\mu} \frac{d}{dr} \arcsin \sqrt{\frac{r}{2r_0}} \quad (21)$$

$$= \frac{-3a_T r_0^2}{\mu} \frac{\frac{1}{4r_0} \left(\frac{r}{2r_0}\right)^{-\frac{1}{2}}}{\sqrt{1 - \frac{r}{2r_0}}} \quad (22)$$

$$= \frac{-3a_T r_0}{\mu} \frac{1}{4\sqrt{\frac{r}{2r_0} \left(\frac{2r_0 - r}{2r_0}\right)}} \quad (23)$$

$$= -\frac{3a_T r_0}{2\mu \sqrt{2r_0r - r^2}} \quad (24)$$

$$w_3'(r) = \frac{d}{dr} \frac{a_T r_0}{2\mu} (r + 3r_0) \quad (25)$$

$$= \frac{a_T r_0}{2\mu} \quad (26)$$

## First derivative test on $w(r)$ (continued)

$$w'(r) = \frac{d}{dr} [w_1(r)w_2(r) + w_3(r)] \quad (27)$$

$$= w_1'(r)w_2(r) + w_1(r)w_2'(r) + w_3'(r) \quad (28)$$

Substituting expressions found prior,

$$w'(r) = \underbrace{\frac{-r_0(r_0 - r)}{(2r_0r - r^2)^{3/2}}}_{w_1'(r)} \underbrace{\left[ 1 - \frac{a_T r_0^2}{\mu} \left( 3 \arcsin\left(\sqrt{\frac{r}{2r_0}}\right) - \frac{3\pi}{4} + 2 \right) \right]}_{w_2(r)} \quad (29)$$

$$\underbrace{-\frac{3a_T r_0}{2\mu(2r_0r - r^2)}}_{w_1(r)w_2'(r)} + \underbrace{\frac{a_T r_0}{2\mu}}_{w_3(r)} \quad (30)$$



## Reubstituting with $r(\phi)$ and simplifying

First,  $2r_0r - r^2 = 4r_0^2 \sin^2(k) - 4r_0 \sin^4(k) = 4r_0^2 \sin^2(k) \cos^2(k)$ .

$$w_{1p}'(\phi) = \frac{2r_0^2 \sin^2(k) - r_0^2}{(2r_0 \sin(k) \cos(k))^3} = \frac{2 \sin^2(k) - 1}{r_0 \sin^3(2k)} \quad (31)$$

$$w_{2p}(\phi) = 1 - \frac{a_T r_0^2}{\mu} (3\phi + 2) \quad (32)$$

$$w_1(r)w_2'(r) = -\frac{3a_T r_0}{2\mu(4r_0^2 \sin^2(k) \cos^2(k))} = -\frac{3a_T r_0}{2\mu \sin^2(2k)} \quad (33)$$

Then,

$$w_p'(\phi) = \frac{2 \sin^2(k) - 1}{r_0 \sin^3(2k)} \left( 1 - \frac{a_T r_0^2}{\mu} (3\phi + 2) \right) - \frac{3a_T r_0}{2\mu \sin^2(2k)} + \frac{a_T r_0}{2\mu} \quad (34)$$

## Reubstituting with $r(\phi)$ and simplifying (continued)

$$w_p'(\phi) = \underbrace{\frac{2\sin^2(k) - 1}{r_0 \sin^3(2k)} \left(1 - \frac{a_T r_0^2}{\mu} (3\phi + 2)\right)}_{\frac{-2\mu \cos(2k) \left(1 - \frac{a_T r_0^2}{\mu} (3\phi + 2)\right)}{2\mu r_0 \sin^3(2k)}} - \underbrace{\frac{3a_T r_0}{2\mu \sin^2(2k)} + \frac{a_T r_0}{2\mu}}_{\frac{a_T r_0^2 \sin^3(2k) - 3a_T r_0^2 \sin(2k)}{2\mu r_0 \sin^3(2k)}} \quad (35)$$

$$= \frac{a_T r_0^2 (\sin^3(2k) - 3) \sin(2k) - 2\mu \cos(2k) \left(1 - \frac{a_T r_0^2}{\mu} (3\phi + 2)\right)}{2\mu r_0 \sin^3(2k)} \quad (36)$$

## Minimization of $w_p(\phi)$

Note that  $2\mu r_0 \sin^3(2k) \neq 0 \implies \sin(2k) \neq 0$ . So  $2(\phi + \pi/4) \notin \{0, \pi\}$  means  $\phi \neq \pi/4$  since  $\phi \in [0, \pi/4]$ .

$$w_p'(\phi) = \frac{a_T r_0^2 (\sin^3(2k) - 3) \sin(2k) - 2\mu \cos(2k) \left(1 - \frac{a_T r_0^2}{\mu} (3\phi + 2)\right)}{2\mu r_0 \sin^3(2k)} = 0 \quad (37)$$

$$\implies a_T r_0^2 (\sin^3(2k) - 3) \sin(2k) - 2\mu \cos(2k) \left(1 - \frac{a_T r_0^2}{\mu} (3\phi + 2)\right) = 0 \quad (38)$$

$$\frac{\sin(2k)}{2\mu \cos(2k)} = \underbrace{\frac{\tan(2k)}{2\mu}}_{d_1(\phi)} = \underbrace{\frac{1 - \frac{a_T r_0^2}{\mu} (3\phi + 2)}{a_T r_0^2 (\sin^3(2k) - 3)}}_{d_2(\phi)} \quad (39)$$

Numerically finding the intersection of  $d_1(\phi)$  and  $d_2(\phi)$  yields the value of  $\phi$  at  $\min(w_p(\phi))$ , which can then be used to find  $r$  for  $\min(w(r))$ .

# Numerical minimization of $w_p(\phi)$

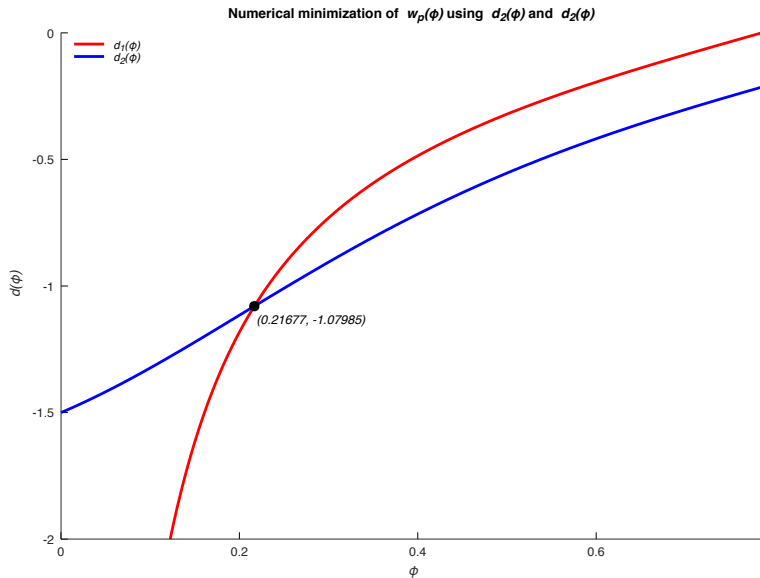
Code:

```
d1 = tan(2*k)/(2*mu);  
d2 = (1-a_T*r_0^2/mu * (3*phi+2))./(a_T*r_0^2*(sin(2*k).^2-3));  
  
% minimization  
intersect = find(abs(d1 - d2) <= min(abs(d1 - d2)));  
  
ix = phi(intersect)  
iy = mean([d1(intersect) d2(intersect)])  
r_min = r(intersect)
```

Output:

```
ix = 0.2168  
iy = -1.0799  
r_min = 1.4201
```

# Intersection of $d_1(\phi)$ and $d_2(\phi)$



# Visual of $\min(w_p(\phi))$

