Spacecraft Motion with Thrust Normal to Velocity Vector

June 15-29 2021

Weekly Progress

- June 10-15
  - Found expression for $w_1(\phi)$ from $w(r)$ in terms of ϕ and β
  - Found an expression for $y_1(\phi)$ given $y(r)$
  - Found discrepancy between plots of $w(r)$ and $w_1(\phi)$ using Desmos and Octave
- June 15-29
  - Checked work for $w_1(\phi)$ and adjusted expression to account for newly provided definition of β
  - Found an expression for $y_1(\phi)$ given $y(r)$ using new definition
  - Numerically minimized $w_1(\phi)$ after obtaining a complicated relation (simplified to best means)
  - Work was checked thoroughly using Desmos and Octave plots

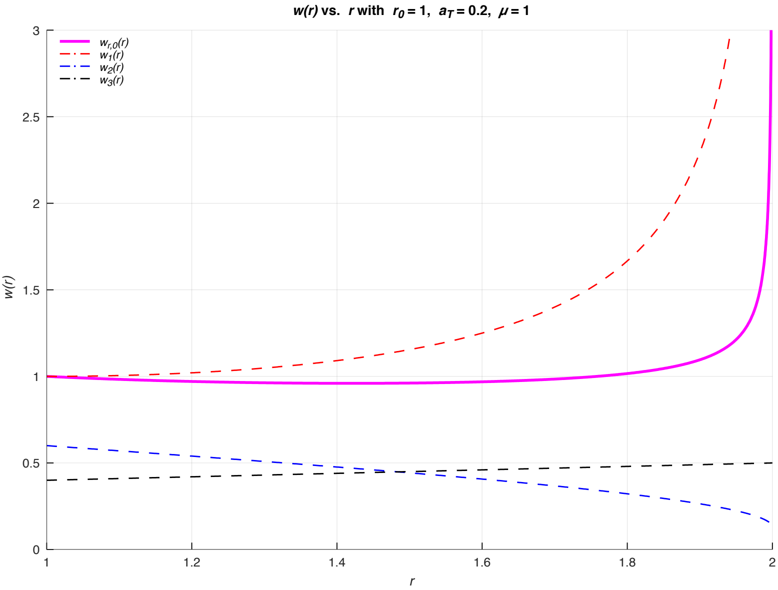
Expressing $w(r)$ in terms of ϕ and β

Approach: reexpress components of $w(r)$ and substitute $r(\phi)$

$$w(r) = \underbrace{\frac{r_0}{\sqrt{r\,(2r_0 - r)}}}_{w_1(r)} \underbrace{\left[1 - \frac{a_T r_0^2}{\mu}\left(3\sin^{-1}\left(\sqrt{\frac{r}{2r_0}}\right) - \frac{3\pi}{4} + 2\right)\right]}_{w_2(r)} + \underbrace{\frac{a_T r_0}{2\mu}(r + 3r_0)}_{w_3(r)}$$

For plots: $r_0 = 1, \mu = 1, a_T = 0.2$
Definitions: $k(\phi) = \phi + \pi/4$, $r(\phi) = 2r_0 \sin^2(k)$, $\beta = \sqrt{\frac{4\mu}{(3\pi+8)a_T r_0^2}}$
Subscript p denotes function of ϕ.

Plotting $w_1(r), w_2(r), w_3(r)$



Reexpressing $w_1(r)$

$$
\begin{aligned}
w_{1p}(\phi) &= \frac{r_0}{\sqrt{r\left(2r_0 - r\right)}}\bigg|_{r(\phi)} &(1)\\
&= \frac{r_0}{\sqrt{2r_0\sin^2(k)(2r_0 - 2r_0\sin^2(k))}} &(2)\\
&= \frac{r_0}{\sqrt{4r_0^2\sin^2(k)\cos^2(k)}} &(3)\\
&= \frac{1}{\sin(2k)} &(4)
\end{aligned}
$$

Obtain 3 using $\cos^2(k) = 1 - \sin^2(k)$ and 4 by $\sin(2k) = 2\sin(k)\cos(k)$.

Reexpressing $w_2(r)$

$$w_{2p}(\phi) = 1 - \frac{a_T r_0^2}{\mu}\left(3\arcsin(\sqrt{\frac{r}{2r_0}}) - \frac{3\pi}{4} + 2\right)\Big|_{r(\phi)} \tag{5}$$

$$= 1 - \frac{a_T r_0^2}{\mu}\underbrace{\left(3k - \frac{3\pi}{4} + 2\right)}_{3\phi+2} \tag{6}$$

$$= 1 - \frac{a_T r_0^2}{\mu}(3\phi + 2) \tag{7}$$

Reexpressing $w_3(r)$

$$w_{3p}(\phi) = \frac{a_T r_0}{2\mu}(r + 3r_0)\Big|_{r(\phi)} \tag{8}$$

$$= \frac{a_T r_0}{2\mu}(r_0(2\sin^2(k) + 3)) \tag{9}$$

$$= \frac{a_T r_0^2}{2\mu}(2\sin^2(k) + 3) \tag{10}$$

$$= \frac{2}{\beta^2(3\pi + 8)}(2\sin^2(k) + 3) \tag{11}$$

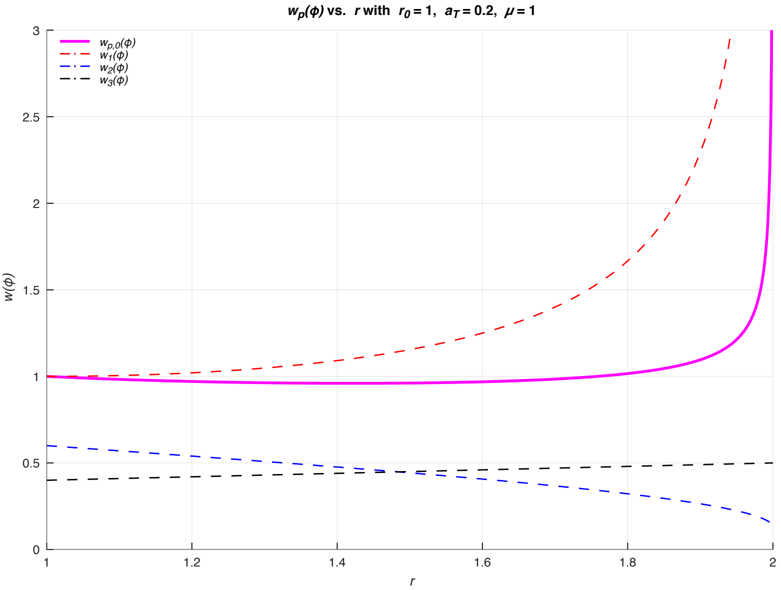
Obtain 11 by $\beta = \sqrt{\frac{4\mu}{(3\pi+8)a_T r_0^2}} \implies \frac{1}{\beta^2} = \frac{(3\pi+8)a_T r_0^2}{4\mu}$

Expression for $w_p(\phi)$

$$w_p(\phi) = \frac{1}{\sin(2k)} \left[1 - \frac{a_T r_0^2}{\mu} (3\phi + 2)\right] \frac{2}{\beta^2(3\pi + 8)} (2\sin^2(k) + 3) \tag{12}$$

$$= w_{1p}(\phi) w_{2p}(\phi) + w_{3p}(\phi) \tag{13}$$

Checking answer with plots



Expressing $y(r)$ in terms of ϕ and β

Approach similar to $w(r)$:

$$y(r) = \underbrace{\sqrt{\frac{\mu(2r_0 - r)}{r r_0}}}_{y_1(r)} \underbrace{\sqrt{1 - w(r)^2}}_{y_2(r)}$$

Reexpressing $y_1(r)$

$$y_{1p}(\phi) = \sqrt{\frac{\mu(2r_0 - r)}{rr_0}}\Bigg|_{r(\phi)} \tag{14}$$

$$= \sqrt{\frac{\mu 2r_0(1 - \cos^2(k))}{2r_0^2 \sin^2(k)}} \tag{15}$$

$$= \sqrt{\frac{\mu 2r_0 \cos^2(k)}{2r_0^2 \sin^2(k)}} \tag{16}$$

$$= \sqrt{\frac{\mu}{r_0}} \cot(k) \tag{17}$$

Reexpressing $y_2(r)$

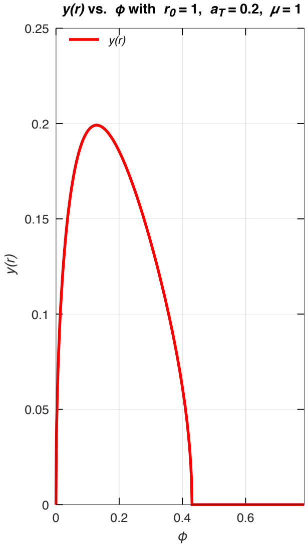
Given $y_{2p}(\phi) = \sqrt{1 - w_p(\phi)^2}$.
Define $\varphi(\phi)$ such that $\frac{\varphi(\phi)}{(3\pi+8)^2\beta^4\sin^2(2k)} = 1 - w_p(\phi)^2$.
Then, by squaring 12,

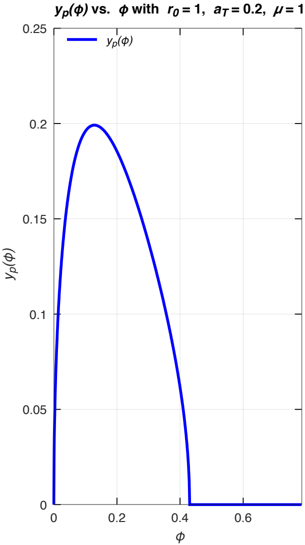
$$
\begin{aligned}
1 - w_p(\phi)^2 = {} & \frac{(3\pi+8)^2\beta^4\sin^2(2k) - \left[(3\pi+8)^2\beta^4 - 8(3\phi+2)(3\pi+8)\beta^2 + 16(3\phi+2)^2\right]}{(3\pi+8)^2\beta^4\sin^2(2k)} \\
& - \frac{2\sin(2k)(4\sin^2(k)+6)((3\pi+8)\beta^2 - 4(3\phi+2))}{(3\pi+8)^2\beta^4\sin^2(2k)} \\
& - \frac{\sin^2(2k)(16\sin^4(k)+48\sin^2(k)+36)}{(3\pi+8)^2\beta^4\sin^2(2k)}
\end{aligned}
$$

Extracting the denominator,

$$
y_p(\phi) = \frac{\sqrt{\varphi(\phi)}}{\beta^2(3\pi+8)\sin(2k)}\sqrt{\frac{\mu}{r_0}}\cot(k)
$$

Plots to verify $y_p(\phi)$





Minimizing $w(r)$ or $w_p(\phi)$

- Began by differentiating $w(r)$ and simplifying
- Substituted $r(\phi)$ to find $w_p{}'(\phi)$
- Solving for ϕ in $w_p{}'(\phi)$ would not be feasible by hand since there were trigonometric and linear terms of ϕ
  - Obtained a greatly simplified expression that was used for minimization
- Used Octave to minimize the expression numerically

First derivative test on $w(r)$

$$
\begin{aligned}
{w_1}'(r) &= \frac{d}{dr} r_0 (2r_0 r - r^2)^{-1/2} && (18)\\
&= -\frac{1}{2} r_0 (2r_0 - 2r)(2r_0 r - r^2)^{-3/2} && (19)\\
&= \frac{-r_0(r_0 - r)}{(2r_0 r - r^2)^{3/2}} && (20)\\
{w_2}'(r) &= \frac{-3a_T r_0^2}{\mu} \frac{d}{dr} \operatorname{arcsin} \sqrt{\frac{r}{2r_0}} && (21)\\
&= \frac{-3a_T r_0^2}{\mu} \frac{\frac{1}{4r_0}\left(\frac{r}{2r_0}\right)^{-\frac{1}{2}}}{\sqrt{1 - \frac{r}{2r_0}}} && (22)\\
&= \frac{-3a_T r_0}{\mu} \frac{1}{4\sqrt{\frac{r}{2r_0}\left(\frac{2r_0 - r}{2r_0}\right)}} && (23)\\
&= -\frac{3a_T r_0}{2\mu\sqrt{2r_0 r - r^2}} && (24)\\
{w_3}'(r) &= \frac{d}{dr} \frac{a_T r_0}{2\mu}(r + 3r_0) && (25)\\
&= \frac{a_T r_0}{2\mu} && (26)
\end{aligned}
$$

First derivative test on $w(r)$ (continued)

$$w'(r) = \frac{d}{dr}\left[w_1(r)w_2(r) + w_3(r)\right] \tag{27}$$

$$= w_1{}'(r)w_2(r) + w_1(r)w_2{}'(r) + w_3{}'(r) \tag{28}$$

Substituting expressions found prior,

$$w'(r) = \underbrace{\frac{-r_0(r_0 - r)}{(2r_0 r - r^2)^{3/2}}}_{w_1{}'(r)} \underbrace{\left[1 - \frac{a_T r_0^2}{\mu}\left(3\arcsin\left(\sqrt{\frac{r}{2r_0}}\right) - \frac{3\pi}{4} + 2\right)\right]}_{w_2(r)} \tag{29}$$

$$\underbrace{- \frac{3a_T r_0}{2\mu\,(2r_0 r - r^2)}}_{w_1(r)w_2{}'(r)} + \underbrace{\frac{a_T r_0}{2\mu}}_{w_3(r)} \tag{30}$$

Reubstituting with $r(\phi)$ and simplifying

First, $2r_0 r - r^2 = 4r_0^2 \sin^2(k) - 4r_0 sin^4(k) = 4r_0^2 \sin^2(k)\cos^2(k)$.

$$w_{1p}{}'(\phi) = \frac{2r_0^2 \sin^2(k) - r_0^2}{(2r_0 \sin(k)\cos(k))^3} = \frac{2\sin^2(k) - 1}{r_0 \sin^3(2k)} \tag{31}$$

$$w_{2p}(\phi) = 1 - \frac{a_T r_0^2}{\mu}(3\phi + 2) \tag{32}$$

$$w_1(r)w_2{}'(r) = -\frac{3a_T r_0}{2\mu(4r_0^2 \sin^2(k)\cos^2(k))} = -\frac{3a_T r_0}{2\mu \sin^2(2k)} \tag{33}$$

Then,

$$w_p{}'(\phi) = \frac{2\sin^2(k) - 1}{r_0 \sin^3(2k)}\left(1 - \frac{a_T r_0^2}{\mu}(3\phi + 2)\right) - \frac{3a_T r_0}{2\mu \sin^2(2k)} + \frac{a_T r_0}{2\mu} \tag{34}$$

Reubstituting with $r(\phi)$ and simplifying (continued)

$$
\begin{aligned}
{w_p}'(\phi) &= \underbrace{\frac{2\sin^2(k)-1}{r_0\sin^3(2k)}\left(1-\frac{a_T r_0^2}{\mu}(3\phi+2)\right)}_{\frac{-2\mu\cos(2k)\left(1-\frac{a_T r_0^2}{\mu}(3\phi+2)\right)}{2\mu r_0\sin^3(2k)}} \underbrace{-\frac{3a_T r_0}{2\mu\sin^2(2k)}+\frac{a_T r_0}{2\mu}}_{\frac{a_T r_0^2\sin^3(2k)-3a_T r_0^2\sin(2k)}{2\mu r_0\sin^3(2k)}} && (35)\\
&= \frac{a_T r_0^2\left(\sin^3(2k)-3\right)\sin(2k)-2\mu\cos(2k)\left(1-\frac{a_T r_0^2}{\mu}(3\phi+2)\right)}{2\mu r_0\sin^3(2k)} && (36)
\end{aligned}
$$

Minimization of $w_p(\phi)$

Note that $2\mu r_0 \sin^3(2k) \neq 0 \implies \sin(2k) \neq 0$. So $2(\phi + \pi/4) \notin \{0, \pi\}$ means $\phi \neq \pi/4$ since $\phi \in [0, \pi/4)$.

$$w_p{}'(\phi) = \frac{a_T r_0^2 \left(\sin^3(2k) - 3\right) \sin(2k) - 2\mu \cos(2k) \left(1 - \frac{a_T r_0^2}{\mu}(3\phi + 2)\right)}{2\mu r_0 \sin^3(2k)} = 0 \tag{37}$$

$$\implies a_T r_0^2 \left(\sin^3(2k) - 3\right) \sin(2k) - 2\mu \cos(2k) \left(1 - \frac{a_T r_0^2}{\mu}(3\phi + 2)\right) = 0 \tag{38}$$

$$\frac{\sin(2k)}{2\mu \cos(2k)} = \underbrace{\frac{\tan(2k)}{2\mu}}_{d_1(\phi)} = \underbrace{\frac{1 - \frac{a_T r_0^2}{\mu}(3\phi + 2)}{a_T r_0^2(\sin^3(2k) - 3)}}_{d_2(\phi)} \tag{39}$$

Numerically finding the intersection of $d_1(\phi)$ and $d_2(\phi)$ yields the value of ϕ at $\min(w_p(\phi))$, which can then be used to find r for $\min(w(r))$.

Numerical minimization of $w_p(\phi)$

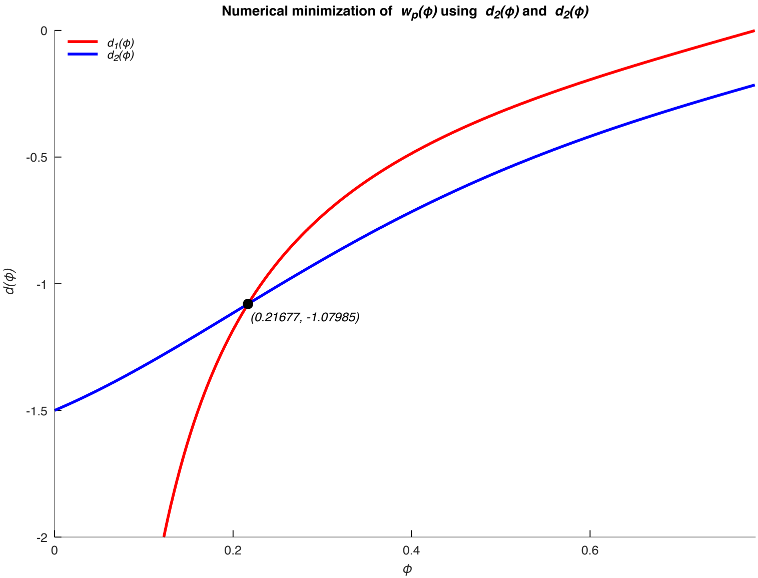
Code:

```
d1 = tan(2*k)/(2*mu);
d2 = (1-a_T*r_0^2/mu * (3*phi+2))./(a_T*r_0^2*(sin(2*k).^2-3));

% minimization
intersect = find(abs(d1 - d2) <= min(abs(d1 - d2)));

ix = phi(intersect)
iy = mean([d1(intersect) d2(intersect)])
r_min = r(intersect)
```

Output:

```
ix = 0.2168
iy = -1.0799
r_min = 1.4201
```

Intersection of $d_1(\phi)$ and $d_2(\phi)$



Visual of $\min(w_p(\phi))$