The Journal of the Astronautical Sciences, Vol. 48, Nos. 2 and 3, April–September 2000, pp. 423–433

A Scrapbook of Beautiful Equations and Great Ideas

Richard H. Battin[1]

The concept of a beautiful equation, the very idea that a mathematical equation could ever be regarded as attractive, would be foreign to anyone who lacks an appreciation of mathematics. Unfortunately, there are those for whom math is a drudge, to be avoided at all cost. This paper is not for them. But, for those who do not actually hate math, those who use math as a tool, those who may never have thought that an equation could have any aesthetic value, they are my audience. Those who like math should enjoy much of this paper. Those who love math and astrodynamics should find it a pleasant diversion.

Leonard Euler found great beauty and utility in mathematics. Indeed, modern mathematics began with Euler. His book *Introduction to Analysis of the Infinite* would be highly suitable even today as a textbook for the college student. It was translated into English in 1988 by John D. Blanton and published by Springer-Verlag New York Inc.

Much of our modern mathematic notation began with Euler. After he used the Greek letter π to represent the ratio of the circumference of a circle to its diameter, it became the standard. Before then, each author had his own symbol. Euler first used e as the base for the natural logarithms. He was the first to use i for the imaginary $\sqrt{-1}$. And it was Euler who is credited with $e^{i\phi} = \cos\phi + i\sin\phi$ from which follows the fantastic relationship among these three most important and seemingly unrelated mathematical concepts:

A Remarkable Equation

Leonard Euler

$$e^{i\pi} + 1 = 0$$

It would be most appropriate for this extraordinary and beautiful equation of Leonard Euler to be framed and given a prominent place in the Louvre—perhaps somewhere near the Mona Lisa.

In England, during the early part of the seventeenth century, mathematics was regarded as devilry. John Wallis, who was born in 1616, said "Mathematics at that

[1]Senior Lecturer, Massachusetts Institute of Technology, Department of Aeronautics and Astronautics. E-mail: battin@alum.mit.edu. Fellow AAS.

time was scarce looked on as academical but rather mechanical—as the business of tradesmen." He prepared in Cambridge to be a professor of mathematics, mainly by independent study, but left for Oxford because "no career was open to a teacher of that subject." In 1649 he was appointed to the Savilian chair of geometry at Oxford and remained there for the rest of his life. One of the notable contributions from his book *Arithmetic of the Infinite* is known today as Wallis' Theorem—expressing π as an infinite product. It qualifies as a truly beautiful equation.

Euler supplied a much simpler proof of Wallis' theorem in his book cited earlier and developed a representation of the constant e as a beautiful continued fraction. The infinite product and the continued fraction use each of the positive integers exactly twice. [Euler chose $1/(e-1)$ rather than simply e so that the pattern would start with the desired symmetry. Euler had an eye for beauty.]

Two Beautiful Equations

John Wallis (1616–1703)

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdots}$$

Leonard Euler (1707–1783)

$$\frac{1}{e-1} = \cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\ddots}}}}}$$

One of the twelve greatest theorems of all time, according to the author William Dunham in his book *Journey through Genius* published in 1990 by John Wiley & Sons, Inc., was the formula for the area of a triangle involving only the lengths of the three sides which was discovered by either Archimedes or, a century later, by Heron. No matter, it qualifies as a beautiful formula.

Another is Viéte's formula for the ratio of the area of a square to the area of the circumscribing circle. Imagine π expressed using only $\frac{1}{2}$ and $\sqrt{\ }$. François Viète lived in the second half of the sixteenth century and was one of the major mathematical stars of the Renaissance even though he was trained as a lawyer.

And Two More Classics

Archimedes (287 B.C.–212 B.C.)

$$\text{Area of a Triangle} = \sqrt{s(s-a)(s-b)(s-c)}$$
$$s = \tfrac{1}{2}(a+b+c)$$

François Viète (1540–1603)

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}} \times \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}} \times \cdots$$

The semiperimeter of the triangle was featured in Archimedes' formula for the triangle's area. Almost two millennia later Leonard Euler discovered that the time to travel an arc of a parabolic orbit between two radii depends on the semiperimeter of the triangle whose sides are the two radii and the chord connecting the initial and terminal points:

Euler's Time Equation for a Parabolic Orbit

$$\Delta t \propto s^{3/2} \pm (s-c)^{3/2}$$

Isaac Newton said "If I have seen a little farther than others it is because I have stood on the shoulders of giants." Certainly, one of those giants was Kepler who was the founder of celestial mechanics. His most famous great ideas were his three laws and his three anomalies: the *true anomaly,* the *eccentric anomaly* and the *mean anomaly.* His relation between the mean and eccentric anomalies is the most famous of all transcendental equations and must have an honored place in this scrapbook:

Kepler's Equation
Johanness Kepler (1571–1630)

$$M = E - e \sin E$$

The elements of an ellipse (the semimajor axis a, the semiminor axis b and the parameter p) are each related to the pericenter and apocenter radii r_p and r_a in such a wonderful way that they too must have special recognition:

Elliptic Orbital Elements

Arithmetic Mean	$a = \frac{1}{2}(r_a + r_p)$
Geometric Mean	$b = \sqrt{r_a r_p}$
Harmonic Mean	$\frac{1}{p} = \left(\frac{1}{r_a} + \frac{1}{r_p}\right)$

One of the famous conjectures in celestial mechanics, known as Lambert's Theorem, has to do with the time to traverse an elliptic arc: *The orbital transfer time depends only upon the semimajor axis, the sum of the distances of the initial and final points of the arc from the center of force, and the length of the chord joining these points.* Lagrange was the first to provide a proof. He derived three equations which are as beautiful as they are important for the boundary-value problem.

Lagrange's Equations
Joseph-Louis Lagrange (1736–1783)

$$\tfrac{1}{2}\sqrt{\mu}\,(t_2 - t_1) = a^{\frac{3}{2}}(\psi - \sin\psi\cos\phi)$$

$$\tfrac{1}{2}(r_1 + r_2) = a(1 - \cos\psi\cos\phi)$$

$$\tfrac{1}{2}c = a\sin\psi\sin\phi$$

The secret lay in his choice of notation:

$$\psi = \tfrac{1}{2}(E_2 - E_1) \qquad \text{and} \qquad \cos\phi = e\cos\tfrac{1}{2}(E_1 + E_2)$$

To obtain his delightful time equation, Lagrange cleverly defined two other quantities $\alpha = \phi + \psi$ and $\beta = \phi - \psi$. After a bit of manipulation: Voilà:

The Lagrange Time Equation

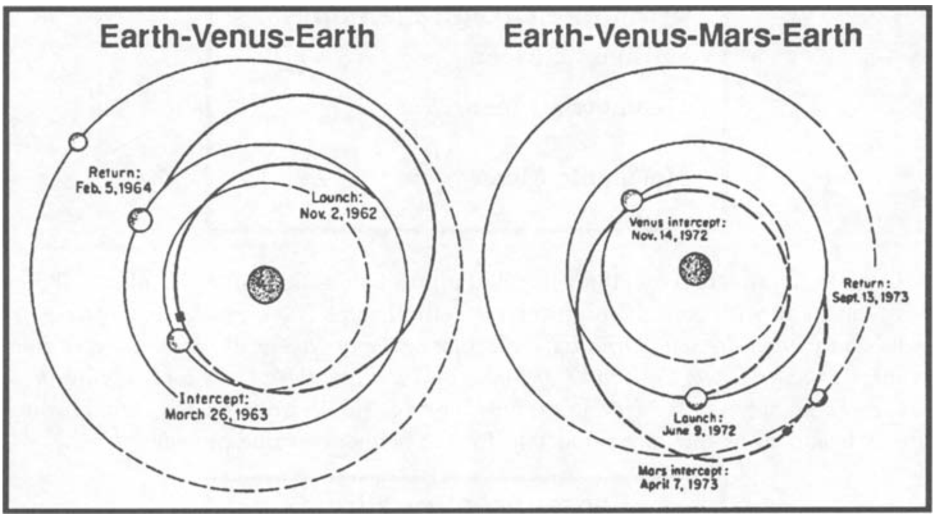
$$\sqrt{\frac{\mu}{a^3}}\,(t_2 - t_1) = (\alpha - \sin\alpha) - (\beta - \sin\beta)$$

$$\sin^2 \tfrac{1}{2}\alpha = \frac{s}{2a} \qquad \sin^2 \tfrac{1}{2}\beta = \frac{s-c}{2a}$$

The flight time $t_2 - t_1$ is now a function only of the single variable a. (Again, the semiperimeter s of the triangle has a prominent role.)

When I arrive at this point in my Astrodynamics lectures, I try to engage my students with the sheer beauty of what is before them. "If you have just a ho-hum attitude when you look at these beautiful equations, you probably shouldn't be in this course." or "Maybe you should rethink your career." This usually gets their attention.

The Lagrange equations were used to calculate round-trip trajectories to Mars and to Venus as well as for calculating multiple planetary fly-by orbits.[2] The round-trip Venus trajectory shows dramatically the increase in velocity from the energy exchange between the spacecraft and Venus. The return to Earth portion of the trajectory carries the vehicle to a distance of 1.35 a.u. from the Sun—almost to the orbit of Mars. Wouldn't it be great if Mars were there? Then you could visit two planets



[2]Lecture Notes on "The Trajectory Problem As It Relates To The Mission For Interplanetary Flight" by Richard H. Battin and J. Halcombe Laning, Jr., August 1961, for MIT Class 16.46 Astronautical Guidance. Copyright © 1961 by Richard H. Battin and J. Halcombe Laning, Jr., published by the Instrumentation Laboratory of Massachusetts Institute of Technology. Printed in the United States of America.

on one mission and return to Earth in just a little more than one year from launch. An example of such a mission is shown beside the round-trip Venus trajectory. A nice idea but not too practical. The launch windows are narrow and far between.

One of the truly great ideas that belongs in this scrapbook is the "George" Algebraic Compiler which was conceived and developed by Hal Laning. After about six months of experience as a user of the MIT Whirlwind computer in 1952, Hal decided that computers should be programmed using a straight-forward mathematical notation rather than an awkward machine language. His concept was completely original at the time. It is proper to recognize this achievement by including in these pages the very first program executed by "George."

The First Algebraic Compiler "George"

J. Halcombe Laning, Jr. 1952

x = 1,

Print x.

The key element in the "Q-Guidance" system is a differential equation for the velocity-to-be-gained vector $\mathbf{v}_g$. Consider a missile or spacecraft propelled, in a gravity field, by a thrust acceleration $\mathbf{a}_T$ whose direction, but not magnitude, can be controlled. At any point in space $\mathbf{r}$ there is a velocity vector $\mathbf{v}_c$ (correlated with the position vector) which would carry the vehicle to the target if the thrust acceleration were terminated. The velocity-to-be-gained is, therefore, the difference between the correlated velocity and the actual vehicle velocity. A major virtue of the $\mathbf{v}_g$ differential equation is the absence of the gravity vector. Gravity calculation in the onboard computer is not required!

Q-Guidance

J. Halcombe Laning, Jr.

$$\frac{d\mathbf{v}_g}{dt} = -\mathbf{Q}\mathbf{v}_g + \mathbf{a}_T \quad \text{where} \quad \mathbf{Q} = \frac{\partial \mathbf{v}_c}{\partial \mathbf{r}}$$

This system was feasible in the early 1950's for ballistic missiles (when the on-board computers were analog devices) because the elements of the Q matrix could be approximated by constants and/or linear functions of time.

The elegant solution to the control problem was the realization that if you wanted to drive the velocity-to-be-gained vector to zero efficiently, you had only to align it with its derivative. The vector would remain fixed in space (not rotating) and shrink to zero. Another great idea from Hal Laning.

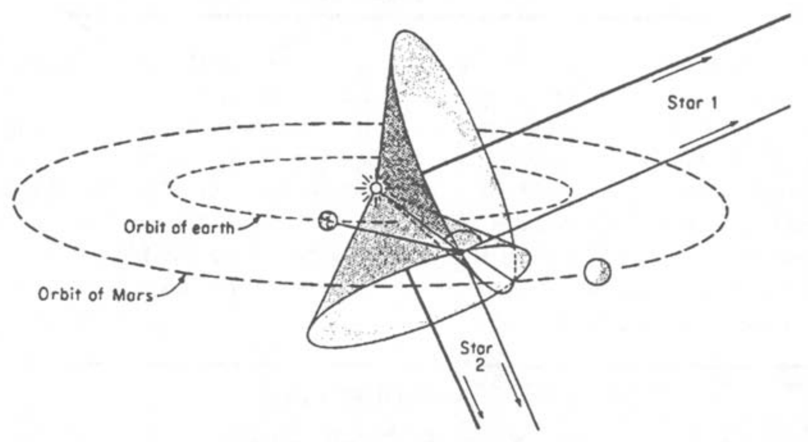
Cross-Product Steering

$$\mathbf{v}_g \times \frac{d\mathbf{v}_g}{dt} = 0$$

An important measurement, which can be made onboard a spacecraft, to determine a position fix is to measure the angle between the direction of a near body (like a planet or the Sun) and the direction to a star. Such a measurement places the spacecraft somewhere on the surface of a cone. Using a different star will determine another cone of position. The intersection of these two cones provides two possible lines of position. A third measurement, using a different near body, will complete the navigation fix.

I've been intrigued by this concept and the neat illustration which it spawned. The color version of this figure was used for both the frontispiece and the dust jacket of my book *Astronautical Guidance*. It appears as a logo on the cover of my astrodynamics text. But of greater importance, measurements of this type were used by Astronaut Jim Lovell to navigate the Apollo 8 spacecraft to the Moon at Christmas time in 1968.

Geometry of the Navigation Fix



The recursive navigation algorithm which was implemented in the Apollo Guidance Computer would not have been practical had the inversion of matrices of large order been required. (The Apollo state vector had nine dimensions.) The "lifesaver" was a matrix inversion formula which became known, affectionately, as the "Magic Lemma." In its simplest form, consider two rectangular matrices X_{nm} and Y_{mn} which are compatible for either $X_{nm}Y_{mn}$ or $X_{mn}X_{nm}$. If I_{mm} and I_{nn} are the identity matrices of the indicated dimension, then the Magic Lemma takes the form where

The Magic Lemma

$$(\mathrm{I}+\mathrm{XY})^{-1}=\mathrm{I}-\mathrm{X}(\mathrm{I}+\mathrm{YX})^{-1}\mathrm{Y}$$

the appropriate subscripts are implied. I'm not certain who was responsible for this great idea but we would have been in serious trouble without it.

Another critical problem in the Apollo navigation algorithm was solved by a great idea from James E. Potter. The covariance matrix of estimation errors had the nasty habit of becoming non-positive definite after several dozen measurements. Disaster was inevitable as the estimator rapidly went unstable—certainly not safe or reliable for navigating to the Moon. Jim Potter's idea of the square root matrix saved the day.

Square Root Estimator

$$\delta\widehat{\mathbf{r}}^{\star} = \delta\widehat{\mathbf{r}} + \mathbf{w}(\delta\widetilde{q} - \delta\widehat{q})$$

Recursive Estimator	Jim Potter's Estimator
$a = \sigma^2 + \mathbf{h}^{\mathrm{T}}\mathbf{P}\mathbf{h}$	$a = \sigma^2 + \mathbf{z}^{\mathrm{T}}\mathbf{z}$
$\mathbf{w} = \frac{1}{a}\mathbf{P}\mathbf{h}$	$\mathbf{w} = \frac{1}{a}\mathbf{W}\mathbf{z} \quad (\mathbf{z} = \mathbf{W}^{\mathrm{T}}\mathbf{h})$
$\mathbf{P}^{\star} = (\mathbf{I} - \mathbf{w}\mathbf{h}^{\mathrm{T}})\mathbf{P}$	$\mathbf{W}^{\star} = \mathbf{W}\left(\mathbf{I} - \frac{\mathbf{z}\mathbf{z}^{\mathrm{T}}}{a + \sqrt{a\sigma^2}}\right)$
$\mathbf{P}_n = \mathbf{\Phi}_{n,n-1}\mathbf{P}_{n-1}\mathbf{\Phi}^{\mathrm{T}}_{n,n-1}$	$\mathbf{W}_n = \mathbf{\Phi}_{n,n-1}\mathbf{W}_{n-1}$

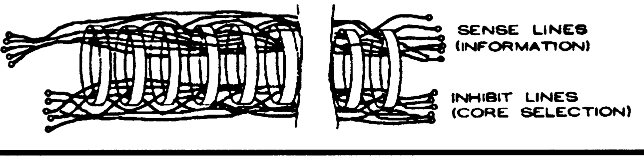
The fixed memory of the Apollo Guidance Computer was called a "core rope memory" since early models resembled lengths of rope—a great idea of Ray Alonzo and Hal Laning and originally intended for the onboard computer of a round-trip photo reconnaissance mission to Mars in the early 1960's.

The high density of this memory was achieved by "storing" a large number of bits in *each* magnetic core. A stored bit is a "one" whenever a sense wire threads a core and a "zero" when it fails to thread a core. To read a word, a core is switched to induce a voltage drop in every sense line that threads that core. The word length was 16 bits so that 16 sense wires, connected to sense amplifiers, could detect which had voltage drops. Thus, *one* core could store a complete word. With additional networking, each core could hold several words.

The Apollo computer read-only memory of 76 kilobytes was compact and failure proof; but NASA was never reconciled to its inflexibility. After manufacturing not a single bit could be changed, either intentionally or unintentionally.

Core Rope Memory

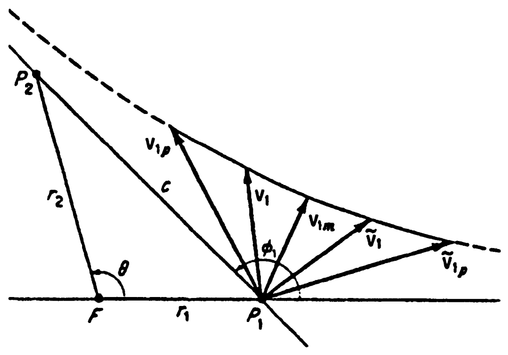
Apollo Guidance Computer



Some of the classical developments of the two-point boundary-value problem have appeared in earlier pages of this scrapbook, especially Lagrange's beautiful time equations. A recent revival of interest in this classic problem has resulted in several modern additions of beautiful equations and great ideas that significantly augment the classical ones. It will be worthwhile to include some of these in our collection.

One of these great ideas in orbital mechanics was presented in Stockholm, Sweden in 1960 by a Norwegian Naval Academy student named Thore Godal. He

Godal's Hyperbolic Locus



showed, for the two-point two-body boundary-value problem, that the velocity vectors at the initial and final points terminate on a hyperbolic locus.

The minimum-energy velocity $\mathbf{v}_m$ is apparent in the diagram. Velocity vectors for elliptic orbits occur in pairs $\mathbf{v}$ and $\tilde{\mathbf{v}}$ having the same energy (or semimajor axis) and are contained in the sector bounded by the parabolic velocity vectors $\mathbf{v}_p$ and $\tilde{\mathbf{v}}_p$. Those to the left of the minimum-energy orbit are for fast orbits while those to the right (with the tilde) are for the high-arcing slow orbits.

Godal's representation follows from using a skewed-axis coordinate system with velocity components v_c and v_ρ projected on the chord and the initial radius extended. The product of these components results in the hyperbolic locus and their ratio yields the ratio of the parameters of the orbit p and the minimum-energy orbit p_m. There are some beautiful equations for the elements of the minimum-energy orbit in terms of the semiperimeter of the triangle as well as equations expressing p_m as a harmonic mean and a geometric mean.

All is right with the world!

Velocity along Skewed Axes

Thore Godal 1960

$$\mathbf{v}_1 = v_c \mathbf{i}_c + v_\rho \mathbf{i}_{r_1} \qquad v_c v_\rho = \text{constant}$$

$$\mathbf{v}_2 = v_c \mathbf{i}_c - v_\rho \mathbf{i}_{r_2} \qquad \frac{v_c}{v_\rho} = \frac{p}{p_m}$$

Minimum-energy orbit $(v_c = v_\rho)$

$$p_m = \frac{2}{c}(s - r_1)(s - r_2) \qquad \frac{1}{p_m} = \frac{1}{2}\left(\frac{1}{P_1F_m^*} + \frac{1}{P_2F_m^*}\right)$$

$$a_m = \tfrac{1}{2}s \qquad p_m = \sqrt{p\tilde{p}}$$

Recently, the "Fundamental Ellipse" (I could not think of a better name) has begun to play an important role in the two-body boundary-value problem. In my book *Astronautical Guidance* I called it the "symmetric ellipse" which describes one of its properties but does not truly herald its significance. The major axis is parallel to the chord, which accounts for the symmetry, but it is also the ellipse having the smallest possible eccentricity. Furthermore, its orbital elements are nicely related to the sides of the triangle.

But that's not all. The straight lines connecting the focus to the extremities of the minor axis are the axes of the two parabolic orbits. And these axes also coincide with the locus of *mean points.* I chose to call the point in an orbit, where the velocity vector is parallel to the chord, the "mean point" because the eccentric anomaly of this point is the arithmetic mean: $E_0 = \frac{1}{2}(E_1 + E_2)$.

The Fundamental Ellipse

- **Orbit of Minimum Eccentricity**
- **Symmetric Ellipse**
- **Major Axis is parallel to the Chord**

$$e_F = \frac{r_2 - r_1}{c} \qquad a_F = \tfrac{1}{2}(r_1 + r_2) \qquad \frac{p_F}{p_m} = \frac{r_1 + r_2}{c}$$

- **Axes of Parabolas coincide with the Locus of Mean Points along lines connecting the focus and the extremities of the minor axis.**

The wonderful property of these mean points is that they all lie on a straight line. Indeed, there are three straight-line loci associated with the boundary-value problem. (But, who knows, there may be more.)

Three Straight Line Loci

- **Orbital Tangents and Transfer Angle Bisector meet at a point.**
- **Eccentricity Vectors terminate on line perpendicular to the Chord.**
- **Loci of Mean Points connect Focus and the extremities of the Minor Axis of the Fundamental Ellipse.**

There are also two hyperbolic loci: the one discovered by Thore Godal and the other being the locus of vacant foci. The foci of this hyperbolic locus are the end points P_1 and P_2. Note that the eccentricity of this locus is the reciprocal of the eccentricity of the fundamental ellipse. Will the wonders ever cease?

Two Hyperbolic Loci

- **Hyperbolic Locus of Vacant Foci**

$$2a^* = -(r_2 - r_1) \qquad e^* = \frac{c}{r_2 - r_1} \qquad e^* e_F = 1$$

- **Hyperbolic Locus of Velocity Vectors** $\quad v_c v_\rho = \text{constant}$

The locus of mean-points add some new and important additions to the original set of Lagrange's Equations, e.g., there are two expressions for the line segment FS measured along the locus from the focus to the point S of intersection with the chord. Also, the radius to the mean point of the orbit can be expressed in two different ways, one in terms of the line segment FP_{0p} which is the mean point radius of the parabola. As a reminder

$$\psi = \tfrac{1}{2}(E_2 - E_1) \qquad \text{and} \qquad \cos\phi = e\cos\tfrac{1}{2}(E_1 + E_2)$$

New Additions to Lagrange's Equations

$$\tfrac{1}{2}\sqrt{\mu}\,(t_2 - t_1) = a^{\frac{3}{2}}(\psi - \sin\psi\cos\phi)$$

$$a_F = \tfrac{1}{2}(r_1 + r_2) = a(1 - \cos\psi\cos\phi)$$

$$\tfrac{1}{2}c = a\sin\psi\sin\phi$$

$$FS = \sqrt{r_1 r_2}\cos\tfrac{1}{2}\theta = a(\cos\psi - \cos\phi)$$

$$FP_0 = r_0 = a(1 - \cos\phi) = r_{0p}(1 + \tan^2\tfrac{1}{2}\psi)$$

$$\text{where} \qquad FP_{0p} = r_{0p} = \tfrac{1}{2}(a_F + FS)$$

These new equations lead to a major improvement over the classic Gauss algorithm for solving the boundary value problem—specifically, the elimination of the singularity for the 180° transfer. The comparison of the two is both interesting and remarkable. It deserves a place in this scrapbook.

For each of these algorithms there is a method for improving the convergence. Suffice it to say that the new method converges almost uniformly over the entire

Comparing Boundary Value Problem Algorithms

Gauss Method	Battin–Vaughan Method
$D \equiv FS = \sqrt{r_1 r_2}\cos\frac{1}{2}\theta$	$D \equiv FP_{0p} = \frac{1}{4}(r_1 + r_2 + 2\sqrt{r_1 r_2}\cos\frac{1}{2}\theta)$
$\ell = \dfrac{r_1 + r_2 - 2\sqrt{r_1 r_2}\cos\frac{1}{2}\theta}{4D}$	$\ell = \dfrac{r_1 + r_2 - 2\sqrt{r_1 r_2}\cos\frac{1}{2}\theta}{4D}$
$m = \dfrac{\mu(t_2 - t_1)^2}{8D^3}$	$m = \dfrac{\mu(t_2 - t_1)^2}{8D^3}$
$x \equiv \sin^2\frac{1}{2}\psi$	$x \equiv \tan^2\frac{1}{2}\psi$
$y^2 = \dfrac{m}{\ell + x}$	$y^2 = \dfrac{m}{(\ell + x)(1 + x)}$
$y^3 - y^2 = m\dfrac{2\psi - \sin 2\psi}{\sin^3\psi}$	$y^3 - y^2 = m\dfrac{\psi - \sin\psi}{4\tan^3\frac{1}{2}\psi}$
$\dfrac{p}{p_m} = \dfrac{cy^2}{4mD}$	$\dfrac{p}{p_m} = \dfrac{cy^2}{4mD}(1 + x)^2$
$\dfrac{1}{a} = \dfrac{2y^2 x}{mD}(1 - x)$	$\dfrac{1}{a} = \dfrac{2y^2 x}{mD}$

range of interest. Gauss' method converges fast for small transfer angles but must be reformulated for transfer angles greater than 180°. It is reasonable to assume that if

Gauss had been aware of the mean point relations, he would have used them in much the same way as they were for the new method.

There are two remarkable and beautiful equations for the orbital parameter which exploit the mean-point locus and the transfer angle bisector. Two ratios are involved: first, is the ratio FS/FP_0 of the two line segments each measured along the mean point locus from the focus to the chord and from the focus to the orbital mean point. The second is the ratio FR/FN of the line segments measured along the transfer-angle bisector from the focus to the chord and from the focus to the point of intersection with the orbital tangents. Two specific orbital parameters appear in the numerators: the parameter of the conjugate of the parabolic orbit and the parameter of the conjugate of the fundamental ellipse.

Two Remarkable Expressions for the Orbital Parameter

$$p = \frac{\tilde{p}_p}{1 - \dfrac{FS}{FP_0}} \qquad p = \frac{\tilde{p}_F}{1 - \dfrac{FR}{FN}}$$

As a bit of frosting on the cake we also have

$$\frac{FR}{FS} = \frac{\sqrt{r_1 r_2}}{\frac{1}{2}(r_1 + r_2)}$$

Finally, let us return to Archimedes and one of the discoveries of which he was most proud. Consider a sphere just fitting inside a cylinder with a top and bottom. Archimedes found that the ratio of the volumes of the cylinder and the sphere is 3/2 and the ratio of the surface areas of the cylinder and the sphere is also 3/2. He was so elated by this theorem that he wanted the figure of the sphere in the cylinder carved on his tombstone along with the fraction 3/2. And in accordance with his wishes it was so inscribed.

Whenever I discuss the beautiful equations for the orbital parameter with my class, I tell the Archimedes story. Then, facetiously,

"I would not object if these were on my tombstone."