

Vel-Norm Problem Followup

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July 3, 2021

1 Simplifying expression for $w_p(\phi)$

The sine function is a phase shift of the cosine function, so

$$\sin(2k) = \sin(2\phi + \pi/2) = \cos(2\phi). \quad (1)$$

Further,

$$2 \sin^2(2k) = 1 - \cos(2k). \quad (2)$$

Because $\cos(2k) = -\sin(2\phi)$ by a similar argument,

$$2 \sin^2(2k) = 1 + \sin(2\phi). \quad (3)$$

Finally

$$w_p(\phi) = \frac{1}{\underbrace{\sin(2\phi + \pi/2)}_{\cos(2\phi)}} \left[1 - \underbrace{\frac{a_T r_0^2}{\mu}}_{-\frac{4}{\beta^2(3\pi+8)}} (3\phi + 2) \right] + \frac{2}{\beta^2(3\pi+8)} \underbrace{(2 \sin^2(\phi + \pi/4) + 3)}_{\sin(2\phi)+4} \quad (4)$$

$$= \frac{1}{\cos(2\phi)} \left[1 - \frac{4}{\beta^2(3\pi+8)} (3\phi + 2) \right] + \frac{2}{\beta^2(3\pi+8)} (\sin(2\phi) + 4) \quad (5)$$

2 Numerical evaluation of $y(\phi_m) = 0$

To evaluate $y_p(\phi) = 0$, we first define `phi = (1:1:1000)*pi/4/1000`. Using this vector to formulate `y_p(Beta)`, where `Beta` is a variable constant, we have a vector of length 1000 representing $y_p(\phi)$. Note that

$$y_p(\phi) = \frac{\sqrt{\varphi(\phi)}}{\beta^2(3\pi+8) \sin(2\phi + \pi/2)} \sqrt{\frac{\mu}{r_0}} \cot(\phi + \pi/4) \quad (6)$$

This implies that $\sin(2\phi + \pi/2) \neq 0$, and further that y_p is only defined for $\varphi(\phi) \geq 0$. Thus, the vector `y_p(Beta)` must be shortened to reflect this and exclude complex numbers. From calculations, we find that the real representation of y_p is `y_p(Beta)(1:548)` using $a_T = 0.2, r_0 = 1, \mu = 1$ with $\beta \approx 1.07$. To find the real portion of this vector, we have called a function `realBreakpoint(vector)` in Figure 1. Sorting this vector and determining the corresponding ϕ_m solves the problem, where we expect `y_minValue = 0` and `phi_m` to be the corresponding value of ϕ :

```

[y_values index_vector] = sort(y_p(Beta)(1:548));
y_minValue = y_values(1);
phi_m = index_vector(1);

```

A similar approach can be employed for a changing parameter β . Since $\delta = \frac{1}{\beta^2} \in (0.001, 1)$, $\beta = \sqrt{\frac{1}{\delta}} \in (1, \sqrt{1000})$. In Octave/MATLAB, we can implement it as

```

delta = (1:1:1000)/1000;
Beta_vec = sqrt(1./delta);

```

Mathematically, we say that

$$\vec{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \text{ and } \vec{\delta} = \begin{bmatrix} 1/\beta_1^2 \\ \vdots \\ 1/\beta_n^2 \end{bmatrix} \quad (7)$$

where we choose $n = 1000$. Thus, by iterating through the values of **Beta_vec**, we can generate a corresponding vector **phi_m1** to represent how ϕ_m changes with respect to it:

```

phi_m1 = zeros(1,n);
w_1 = zeros(1,n);
w_min = zeros(1,n);
y_0 = zeros(1,n);

for i = 1:n
    y_real = y_p(Beta_vec(i))(1:realBreakpoint(y_p(Beta_vec(i))));
    [yvals idx] = sort(y_real);
    if (length(idx) > 1)
        w_1(i) = w_p(Beta_vec(i))(idx(2));
        phi_m1(i) = phi(idx(2));
        y_0(i) = yvals(2);
    endif
end

```

Figure 1: Numerically finding $\vec{\phi}_m$ over $\vec{\delta}$

In Figure 1 above, the vector **y_0** is updated with the value $y_p(\phi_{m,i})$ in each iteration for each value of β , and is expected to be 0. The vector **w_1** is updated with $w_p(\phi_{m,i})$, which is expected to be unity. As was done with **y_p(Beta)**, we have defined a vector **w_p(Beta)** to represent $w_p(\phi)$ where **Beta** is a constant that can be varied. Thus, we should expect $\vec{w}_p(\vec{\phi}_m) = \vec{1}$, $\vec{y}_p(\vec{\phi}_m) = \vec{0}$, and the

existence of $\vec{\phi}_m = \begin{bmatrix} \phi_{m,1} \\ \vdots \\ \phi_{m,n} \end{bmatrix}$ at the end of the loop. The conditional check is to account for cases where the domain of $y_p(\phi) \in \mathbb{R}$ is very small, so we are only able to find $\phi_m = 0 \implies y(\phi_m) = 0$

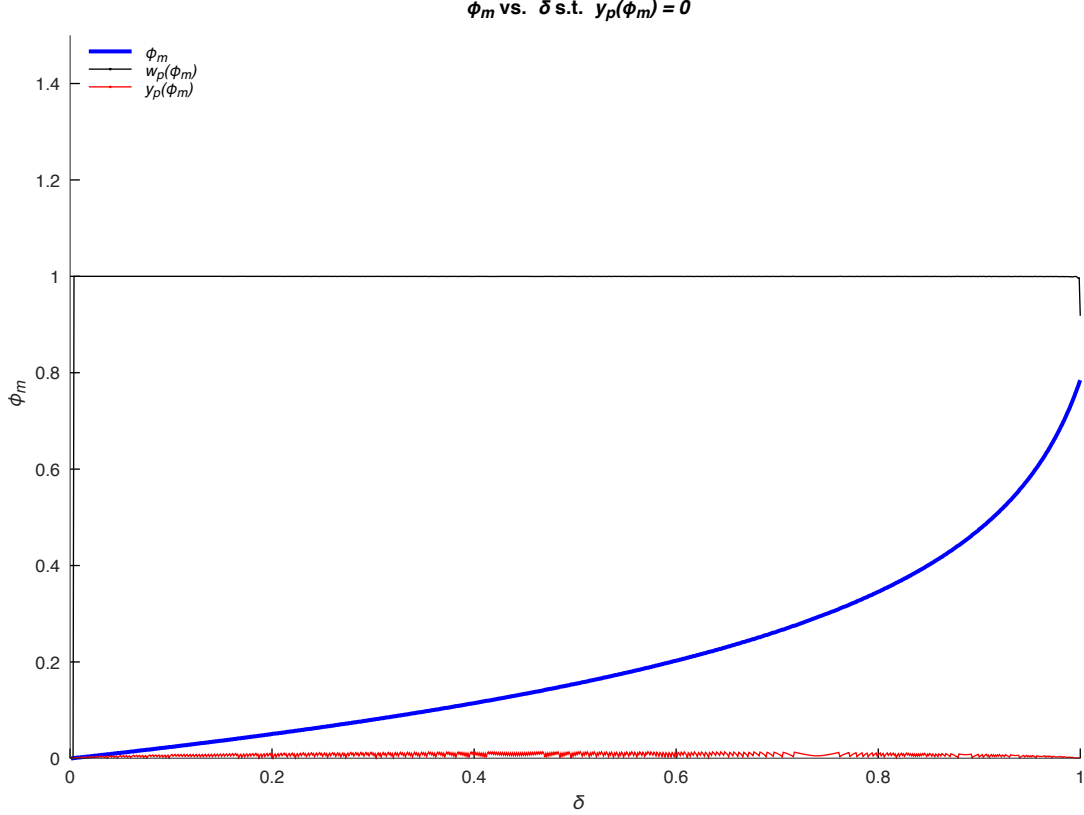


Figure 2: A plot of these results

We see that $y_p(\phi_m) \approx 0$, $w_p(\phi_m) \approx 1 \forall \phi_m$, which is consistent with the expectations.

3 Numerical minimization of $w_p(\phi)$

We use a similar approach as before to minimize $w_p(\phi)$ numerically. In Figure 1, `w_1(i)` was updated to reflect the value of $w_p(\phi_{m,i})$ where $\phi = \phi_{m,i}$ minimized $y_p(\phi)$ for corresponding values of β_i, δ_i from Eq. 7. The following code defines a vector `phi_m2` of length $n = 1000$ and populates it with the value of $\phi_{m,i}$ that minimizes $w_p(\phi)$ for β_i . The minimum value is stored in another vector `w_min`.

```
phi_m2 = zeros(1,n);
w_min = zeros(1,n);
for i = 1:n
    [w_vals idx] = sort(w_p(Beta_vec(i)));
    phi_m2(i) = phi(idx(1));
    w_min(i) = w_vals(1);
end
```

Figure 3: Iteratively finding $\vec{\phi}_m$ for $\min(w_p(\phi))$ for various β

Figure 4 below, showing ϕ_m (representing phi_m2) vs. δ , summarizes these results.

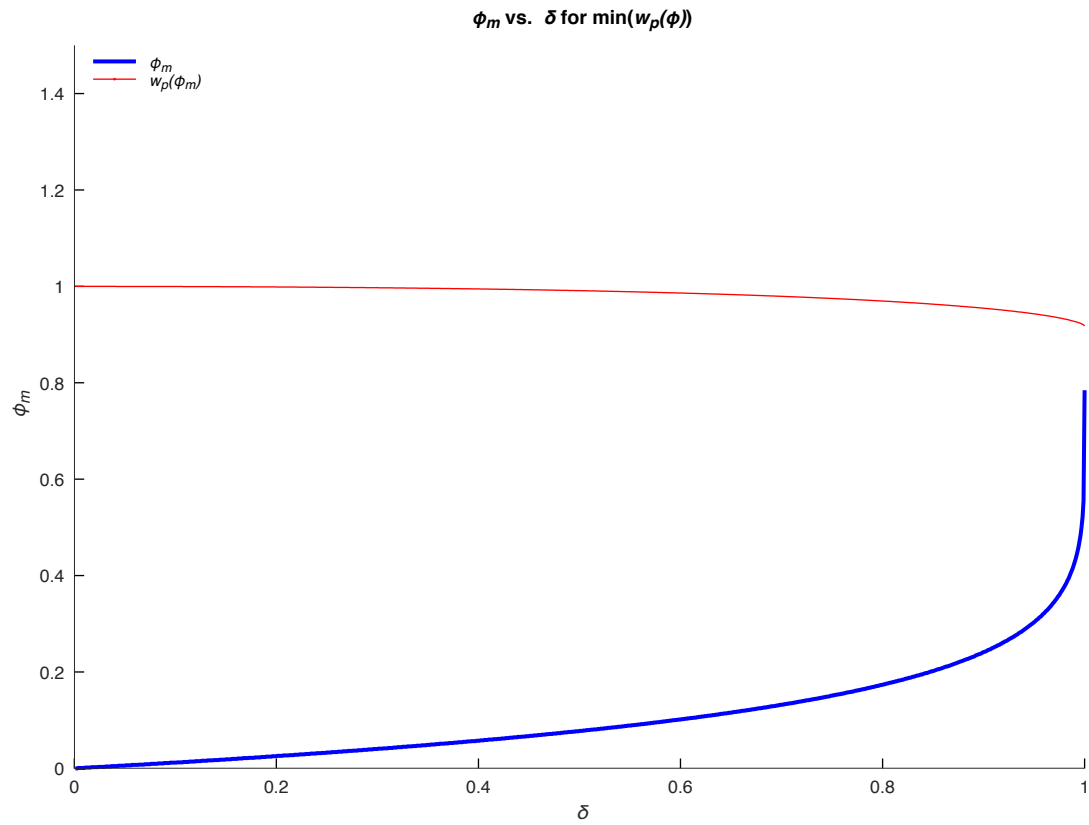


Figure 4: Values of ϕ_m that minimize $w_p(\phi)$ for various β