# Thrust Normal to Velocity Vector

July 22, 2021

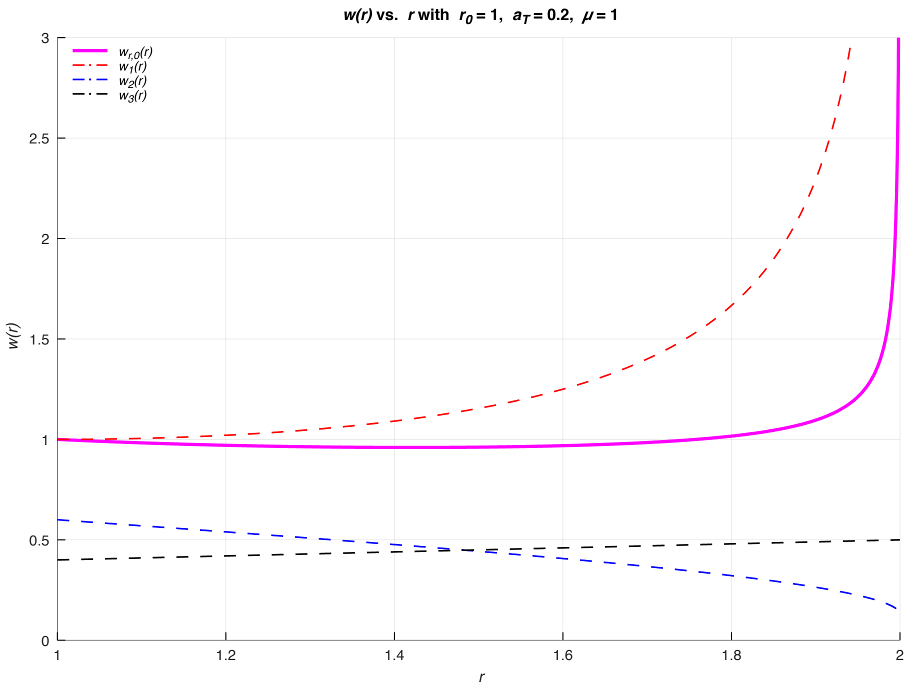
## Expressing $w(r)$ in terms of $\phi$ and $\beta$

Approach: reexpress components of $w(r)$ and substitute $r(\phi)$

$$w(r) = \underbrace{\frac{r_0}{\sqrt{r\,(2r_0 - r)}}}_{w_1(r)} \underbrace{\left[1 - \frac{a_T r_0^2}{\mu}\left(3\sin^{-1}\left(\sqrt{\frac{r}{2r_0}}\right) - \frac{3\pi}{4} + 2\right)\right]}_{w_2(r)} + \underbrace{\frac{a_T r_0}{2\mu}(r + 3r_0)}_{w_3(r)}$$

For plots: $r_0 = 1, \mu = 1, a_T = 0.2$
Definitions: $k(\phi) = \phi + \pi/4$, $r(\phi) = 2r_0 \sin^2(k)$, $\beta = \sqrt{\frac{4\mu}{(3\pi+8)a_T r_0^2}}$
Subscript $p$ denotes function of $\phi$.

## Plotting $w_1(r), w_2(r), w_3(r)$




## Reexpressing $w_1(r)$

$$
\begin{aligned}
w_{1p}(\phi) &= \frac{r_0}{\sqrt{r\,(2r_0 - r)}}\Big|_{r(\phi)} && (1)\\
&= \frac{r_0}{\sqrt{2r_0 \sin^2(k)(2r_0 - 2r_0 \sin^2(k))}} && (2)\\
&= \frac{r_0}{\sqrt{4r_0^2 \sin^2(k)\cos^2(k)}} && (3)\\
&= \frac{1}{\sin(2k)} && (4)
\end{aligned}
$$

Obtain 3 using $\cos^2(k) = 1 - \sin^2(k)$ and 4 by $\sin(2k) = 2\sin(k)\cos(k)$.

## Reexpressing $w_2(r)$

$$w_{2p}(\phi) = 1 - \frac{a_T r_0^2}{\mu}\left(3\arcsin(\sqrt{\frac{r}{2r_0}}) - \frac{3\pi}{4} + 2\right)\Big|_{r(\phi)} \tag{5}$$

$$= 1 - \frac{a_T r_0^2}{\mu}\underbrace{\left(3k - \frac{3\pi}{4} + 2\right)}_{3\phi+2} \tag{6}$$

$$= 1 - \frac{a_T r_0^2}{\mu}(3\phi + 2) \tag{7}$$

## Reexpressing $w_3(r)$

$$w_{3p}(\phi) = \frac{a_T r_0}{2\mu}(r + 3r_0)\Big|_{r(\phi)} \tag{8}$$

$$= \frac{a_T r_0}{2\mu}(r_0(2\sin^2(k) + 3)) \tag{9}$$

$$= \frac{a_T r_0^2}{2\mu}(2\sin^2(k) + 3) \tag{10}$$

$$= \frac{2}{\beta^2(3\pi + 8)}(2\sin^2(k) + 3) \tag{11}$$

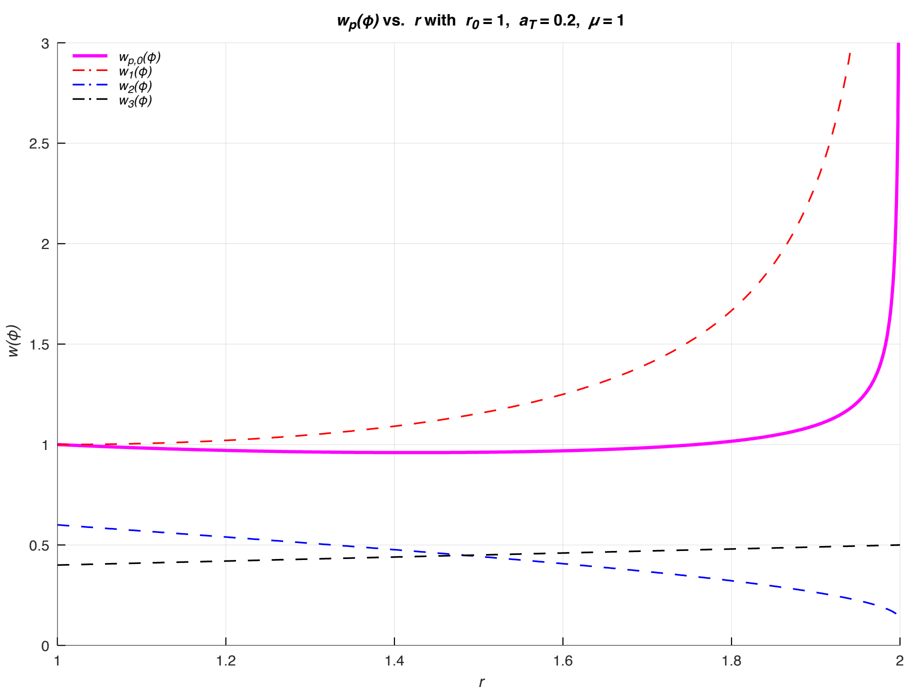
Obtain 11 by $\beta = \sqrt{\frac{4\mu}{(3\pi+8)a_T r_0^2}} \implies \frac{1}{\beta^2} = \frac{(3\pi+8)a_T r_0^2}{4\mu}$

## Expression for $w_p(\phi)$

$$w_p(\phi) = \frac{1}{\sin(2k)}\left[1 - \frac{a_T r_0^2}{\mu}(3\phi + 2)\right] + \frac{2}{\beta^2(3\pi + 8)}(2\sin^2(k) + 3) \tag{12}$$

$$= w_{1p}(\phi)w_{2p}(\phi) + w_{3p}(\phi) \tag{13}$$

## Checking answer with plots



## Expressing $y(r)$ in terms of $\phi$ and $\beta$

Approach similar to $w(r)$:

$$y(r) = \underbrace{\sqrt{\frac{\mu(2r_0 - r)}{rr_0}}}_{y_1(r)} \underbrace{\sqrt{1 - w(r)^2}}_{y_2(r)}$$

## Reexpressing $y_1(r)$

$$y_{1p}(\phi) = \sqrt{\frac{\mu(2r_0 - r)}{rr_0}}\Bigg|_{r(\phi)} \tag{14}$$

$$= \sqrt{\frac{\mu 2r_0(1 - \cos^2(k))}{2r_0^2 \sin^2(k)}} \tag{15}$$

$$= \sqrt{\frac{\mu 2r_0 \cos^2(k)}{2r_0^2 \sin^2(k)}} \tag{16}$$

$$= \sqrt{\frac{\mu}{r_0}} \cot(k) \tag{17}$$

# Reexpressing $y_2(r)$

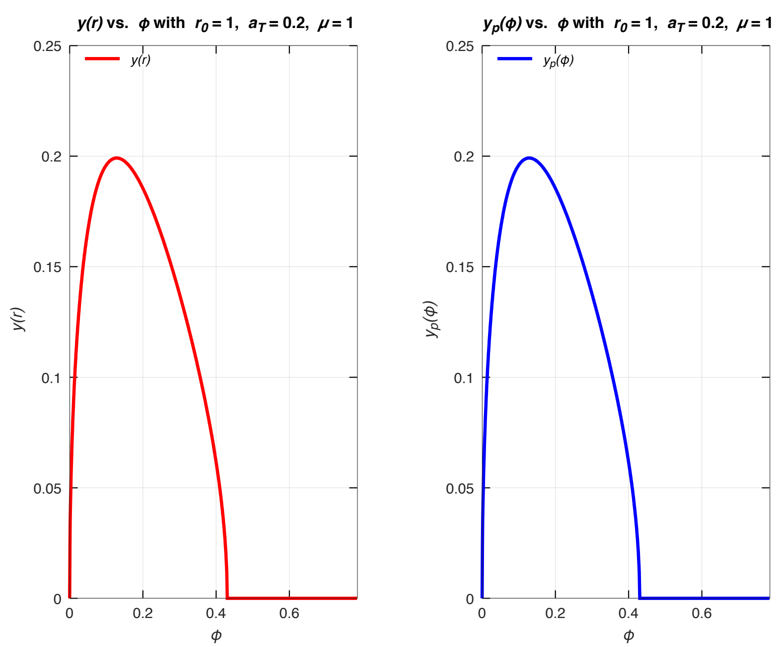
Given $y_{2p}(\phi) = \sqrt{1 - w_p(\phi)^2}$.
Define $\varphi(\phi)$ such that $\frac{\varphi(\phi)}{(3\pi+8)^2\beta^4 \sin^2(2k)} = 1 - w_p(\phi)^2$.
Then, by squaring 12,

$$\begin{aligned} 1 - w_p(\phi)^2 &= \frac{(3\pi+8)^2\beta^4\sin^2(2k) - \left[(3\pi+8)^2\beta^4 - 8(3\phi+2)(3\pi+8)\beta^2 + 16(3\phi+2)^2\right]}{(3\pi+8)^2\beta^4\sin^2(2k)} && (18) \\ &- \frac{2\sin(2k)(4\sin^2(k)+6)((3\pi+8)\beta^2 - 4(3\phi+2))}{(3\pi+8)^2\beta^4\sin^2(2k)} && (19) \\ &- \frac{\sin^2(2k)(16\sin^4(k) + 48\sin^2(k) + 36)}{(3\pi+8)^2\beta^4\sin^2(2k)} && (20) \end{aligned}$$

Extracting the denominator,

$$y_p(\phi) = \frac{\sqrt{\varphi(\phi)}}{\beta^2(3\pi+8)\sin(2k)}\sqrt{\frac{\mu}{r_0}}\cot(k) \qquad (21)$$

# Plots to verify $y_p(\phi)$



## Minimizing $w(r)$ or $w_p(\phi)$

- Began by differentiating $w(r)$ and simplifying
- Substituted $r(\phi)$ to find $w_p{}'(\phi)$
- Solving for $\phi$ in $w_p{}'(\phi)$ would not be feasible by hand since there were trigonometric and linear terms of $\phi$
    - Obtained a greatly simplified expression that was used for minimization
- Used Octave to minimize the expression numerically

## First derivative test on $w(r)$

$$w_1{}'(r) = \frac{d}{dr} r_0(2r_0 r - r^2)^{-1/2} \tag{22}$$

$$= -\frac{1}{2} r_0(2r_0 - 2r)(2r_0 r - r^2)^{-3/2} \tag{23}$$

$$= \frac{-r_0(r_0 - r)}{(2r_0 r - r^2)^{3/2}} \tag{24}$$

$$w_2{}'(r) = \frac{-3a_T r_0^2}{\mu} \frac{d}{dr} \arcsin\sqrt{\frac{r}{2r_0}} \tag{25}$$

$$= \frac{-3a_T r_0^2}{\mu} \frac{\frac{1}{4r_0}\left(\frac{r}{2r_0}\right)^{-\frac{1}{2}}}{\sqrt{1-\frac{r}{2r_0}}} \tag{26}$$

$$= \frac{-3a_T r_0}{\mu} \frac{1}{4\sqrt{\frac{r}{2r_0}\left(\frac{2r_0-r}{2r_0}\right)}} \tag{27}$$

$$= -\frac{3a_T r_0}{2\mu\sqrt{2r_0 r - r^2}} \tag{28}$$

$$w_3{}'(r) = \frac{d}{dr} \frac{a_T r_0}{2\mu}(r + 3r_0) \tag{29}$$

$$= \frac{a_T r_0}{2\mu} \tag{30}$$

## First derivative test on $w(r)$ (continued)

$$w'(r) = \frac{d}{dr}\left[w_1(r)w_2(r) + w_3(r)\right] \tag{31}$$
$$= w_1{}'(r)w_2(r) + w_1(r)w_2{}'(r) + w_3{}'(r) \tag{32}$$

Substituting expressions found prior,

$$w'(r) = \underbrace{\frac{-r_0(r_0 - r)}{(2r_0 r - r^2)^{3/2}}}_{w_1{}'(r)} \underbrace{\left[1 - \frac{a_T r_0^2}{\mu}\left(3\arcsin\left(\sqrt{\frac{r}{2r_0}}\right) - \frac{3\pi}{4} + 2\right)\right]}_{w_2(r)} \tag{33}$$
$$\underbrace{- \frac{3a_T r_0}{2\mu\,(2r_0 r - r^2)}}_{w_1(r)w_2{}'(r)} + \underbrace{\frac{a_T r_0}{2\mu}}_{w_3(r)} \tag{34}$$

## Reubstituting with $r(\phi)$ and simplifying

First, $2r_0 r - r^2 = 4r_0^2\sin^2(k) - 4r_0 sin^4(k) = 4r_0^2\sin^2(k)\cos^2(k)$.

$$w_{1p}{}'(\phi) = \frac{2r_0^2\sin^2(k) - r_0^2}{(2r_0\sin(k)\cos(k))^3} = \frac{2\sin^2(k) - 1}{r_0\sin^3(2k)} \tag{35}$$
$$w_{2p}(\phi) = 1 - \frac{a_T r_0^2}{\mu}(3\phi + 2) \tag{36}$$
$$w_1(r)w_2{}'(r) = -\frac{3a_T r_0}{2\mu(4r_0^2\sin^2(k)\cos^2(k))} = -\frac{3a_T r_0}{2\mu\sin^2(2k)} \tag{37}$$

Then,

$$w_p{}'(\phi) = \frac{2\sin^2(k) - 1}{r_0\sin^3(2k)}\left(1 - \frac{a_T r_0^2}{\mu}(3\phi + 2)\right) - \frac{3a_T r_0}{2\mu\sin^2(2k)} + \frac{a_T r_0}{2\mu} \tag{38}$$

## Reubstituting with $r(\phi)$ and simplifying (continued)

$$w_p{}'(\phi) = \underbrace{\frac{2\sin^2(k)-1}{r_0\sin^3(2k)}\left(1-\frac{a_T r_0^2}{\mu}(3\phi+2)\right)}_{\frac{-2\mu\cos(2k)\left(1-\frac{a_T r_0^2}{\mu}(3\phi+2)\right)}{2\mu r_0\sin^3(2k)}}\underbrace{-\frac{3a_T r_0}{2\mu\sin^2(2k)}+\frac{a_T r_0}{2\mu}}_{\frac{a_T r_0^2\sin^3(2k)-3a_T r_0^2\sin(2k)}{2\mu r_0\sin^3(2k)}} \tag{39}$$

$$= \frac{a_T r_0^2\left(\sin^2(2k)-3\right)\sin(2k)-2\mu\cos(2k)\left(1-\frac{a_T r_0^2}{\mu}(3\phi+2)\right)}{2\mu r_0\sin^3(2k)} \tag{40}$$

## Minimization of $w_p(\phi)$

Note that $2\mu r_0\sin^3(2k)\neq 0 \implies \sin(2k)\neq 0$. So $2(\phi+\pi/4)\notin\{0,\pi\}$ means $\phi\neq\pi/4$ since $\phi\in[0,\pi/4)$.

$$w_p{}'(\phi) = \frac{a_T r_0^2\left(\sin^2(2k)-3\right)\sin(2k)-2\mu\cos(2k)\left(1-\frac{a_T r_0^2}{\mu}(3\phi+2)\right)}{2\mu r_0\sin^3(2k)} = 0 \tag{41}$$

$$\implies a_T r_0^2\left(\sin^2(2k)-3\right)\sin(2k)-2\mu\cos(2k)\left(1-\frac{a_T r_0^2}{\mu}(3\phi+2)\right) = 0 \tag{42}$$

$$\frac{\sin(2k)}{2\mu\cos(2k)} = \underbrace{\frac{\tan(2k)}{2\mu}}_{d_1(\phi)} = \underbrace{\frac{1-\frac{a_T r_0^2}{\mu}(3\phi+2)}{a_T r_0^2(\sin^3(2k)-3)}}_{d_2(\phi)} \tag{43}$$

Numerically finding the intersection of $d_1(\phi)$ and $d_2(\phi)$ yields the value of $\phi$ at $\min(w_p(\phi))$, which can then be used to find $r$ for $\min(w(r))$.

## Numerical minimization of $w_p(\phi)$

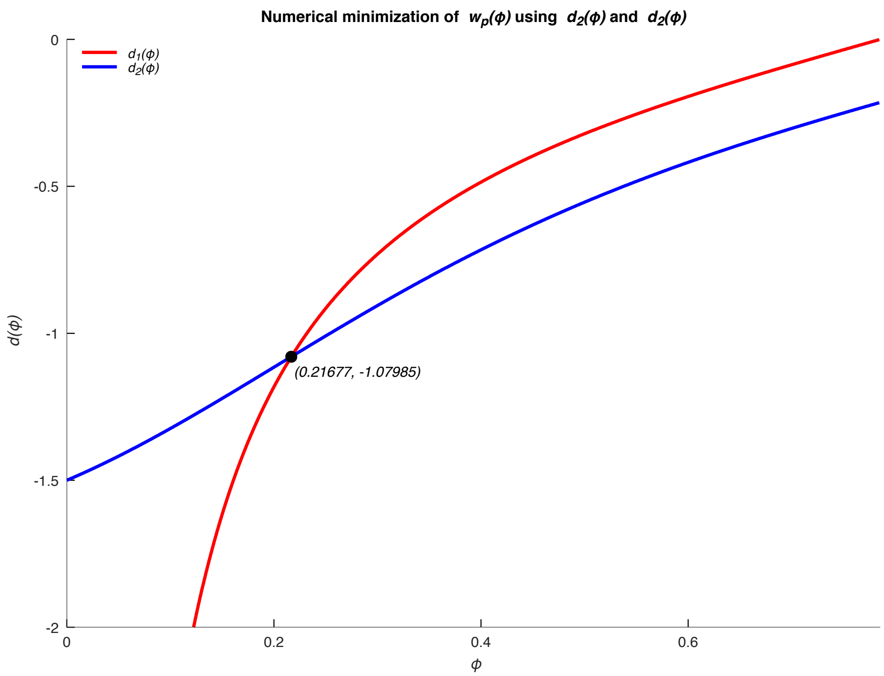
Code:

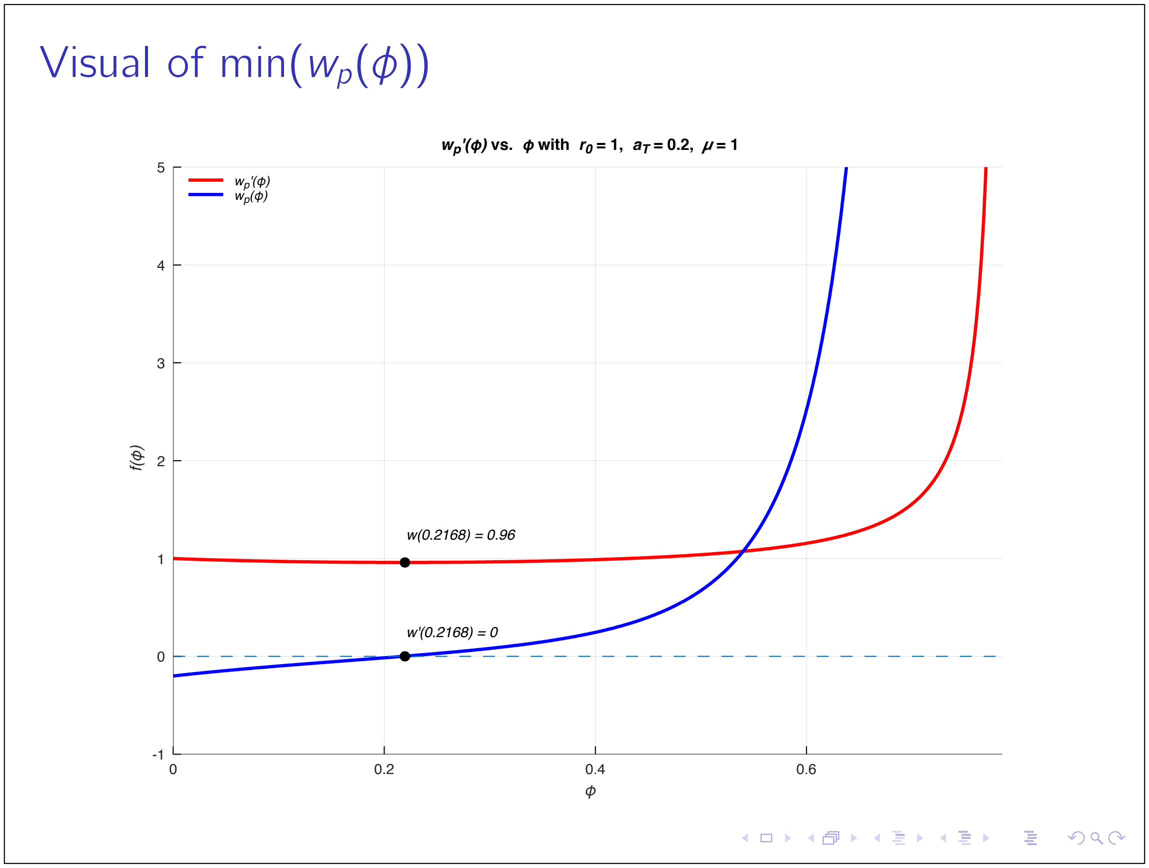
```
d1 = tan(2*k)/(2*mu);
d2 = (1-a_T*r_0^2/mu * (3*phi+2))./(a_T*r_0^2*(sin(2*k).^2-3));

% minimization
intersect = find(abs(d1 - d2) <= min(abs(d1 - d2)));

ix = phi(intersect)
iy = mean([d1(intersect) d2(intersect)])
r_min = r(intersect)
```

Output:

```
ix = 0.2168
iy = -1.0799
r_min = 1.4201
```

## Intersection of $d_1(\phi)$ and $d_2(\phi)$






## 1 Simplifying expression for $w_p(\phi)$

The sine function is a phase shift of the cosine function, so

$$\sin(2k) = \sin(2\phi + \pi/2) = \cos(2\phi). \tag{1}$$

It is also given that

$$2\sin^2(2k) = 1 - \cos(2k). \tag{2}$$

Because $\cos(2k) = -\sin(2\phi)$ by a similar argument,

$$2\sin^2(2k) = 1 + \sin(2\phi). \tag{3}$$

Finally

$$w_p(\phi) = \underbrace{\frac{1}{\sin(2\phi + \pi/2)}}_{\cos(2\phi)}[1 \underbrace{-\frac{a_T r_0^2}{\mu}}_{-\frac{4}{\beta^2(3\pi+8)}}(3\phi + 2)] + \frac{2}{\beta^2(3\pi + 8)}\underbrace{(2\sin^2(\phi + \pi/4) + 3)}_{\sin(2\phi)+4} \tag{4}$$

$$= \frac{1}{\cos(2\phi)}\left[1 - \frac{4}{\beta^2(3\pi + 8)}(3\phi + 2)\right] + \frac{2}{\beta^2(3\pi + 8)}(\sin(2\phi) + 4) \tag{5}$$

## 2 Numerical evaluation of $y(\phi_0) = 0$

***Notation:*** **Let $\phi_0$ satisfy $y(\phi_0) = 0, w_p(\phi_0) = 1$ and $\phi_{\min}$ minimize $w_p(\phi)$.**

To evaluate $y_p(\phi) = 0$, we first define `phi = (1:1:1000)*pi/4/1000`. Using this vector to formulate `y_p(Beta)`, where `Beta` is a variable constant, we have a vector of length 1000 representing $y_p(\phi)$. Note that

$$y_p(\phi) = \frac{\sqrt{\varphi(\phi)}}{\beta^2(3\pi + 8)\sin(2\phi + pi/2)}\sqrt{\frac{\mu}{r_0}}\cot(\phi + \pi/4) \tag{6}$$

$\varphi(\phi)$ was defined in Eq. 18-20 of the presentation, and the above expression comes from Eq. 21 of the presentation. This implies that $\sin(2\phi + \pi/2) \neq 0$, and further that $y_p$ is only defined for $\varphi(\phi) \geq 0$. Thus, the vector `y_p(Beta)` must be shortened to reflect this and exclude complex numbers. From calculations, we find that the real representation of $y_p$ is `y_p(Beta)(1:548)` using $a_T = 0.2, r_0 = 1, \mu = 1$ with $\beta \approx 1.07$. To find the section of this vector that is $\in \mathbb{R}$, we have called a function `realBreakpoint(vector)` in Figure 1. Sorting this vector and determining the corresponding $\phi_0$ solves the problem, where we expect `y_minValue = 0` and `phi_0` to be the corresponding value of $\phi$:

```
[y_values index_vector] = sort(y_p(Beta)(1:548));
y_minValue = y_values(1);
phi_0 = index_vector(1);
```

This approach can be extended for a changing parameter $\beta$. Since $\delta = \frac{1}{\beta^2} \in (0.001, 1)$, $\beta = \sqrt{\frac{1}{\delta}} \in (1, \sqrt{1000})$. In Octave/MATLAB, we can implement it as

```
delta = (1:1:1000)/1000;
Beta_vec = sqrt(1./delta);
```

Mathematically, we say that

$$\vec{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \text{ and } \vec{\delta} = \begin{bmatrix} 1/\beta_1^2 \\ \vdots \\ 1/\beta_n^2 \end{bmatrix} \tag{7}$$

where we choose $n = 1000$. Thus, by iterating through the values of `Beta_vec`, we can generate a corresponding vector `phi_m1` to represent how $\phi_0$ changes with respect to $\delta$:

```
phi_m1 = zeros(1,n);
w_1 = zeros(1,n);
w_min = zeros(1,n);
y_0 = zeros(1,n);

for i = 1:n
    y_real = y_p(Beta_vec(i))(1:realBreakpoint(y_p(Beta_vec(i))));
    [yvals idx] = sort(y_real);
    if (length(idx) > 1)
        w_1(i) = w_p(Beta_vec(i))(idx(2));
        phi_m1(i) = phi(idx(2));
        y_0(i) = yvals(2);
    endif
end
```

Figure 1: Numerically finding $\vec{\phi}_0$ over $\vec{\delta}$

In Figure 1 above, the vector `y_0` is updated with the value $y_p(\phi_{0,i})$ in each iteration for each value of $\beta$, and is expected to be 0. The vector `w_1` is updated with $w_p(\phi_{m,i})$, which is expected to be unity. As was done with `y_p(Beta)`, we have defined a vector `w_p(Beta)` to represent $w_p(\phi)$ where `Beta` is a constant that can be varied. Thus, we should expect $w_p(\vec{\phi}_{\min}) = \vec{1}$, $y_p(\vec{\phi}_0) = \vec{0} \in \mathbb{R}^n$, and the existence of $\vec{\phi}_m = \begin{bmatrix} \phi_{m,1} \\ \vdots \\ \phi_{m,n} \end{bmatrix} \in \mathbb{R}^n$ at the end of the loop. The second index of `y_p(Beta_vec(i))` is accessed to find $\phi_{0,i}$ within the loop because $\phi = 0$ always satisfies $y_p(\phi) = 0$, and we want to find the second such value. The conditional check is to account for cases where the domain of $y_p(\phi) \in \mathbb{R}$ is very small (i.e. `length(y_p(Beta_vec(i))` is 1), so we are only able to find $\phi_{0,i} = 0 \implies y(\phi_0) = 0$.

The results of $\phi_0$ vs. $\delta$ are expressed in Figure 2 below. $y_p(\phi_0) \approx 0$ and $w_p(\phi_{\min}) \approx 1$ within numerical error, which verifies the validity of these results.

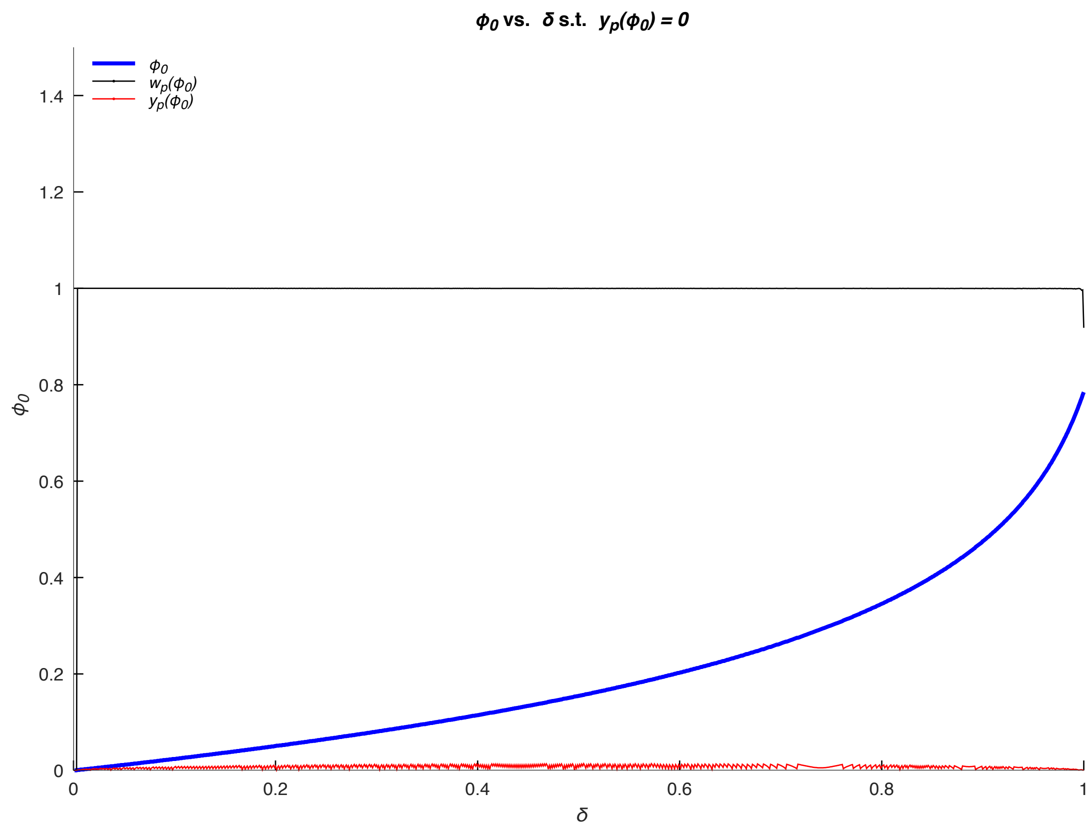



Figure 2: A plot of these results

# 3 Numerical minimization of $w_p(\phi)$

We use a similar approach as before to minimize $w_p(\phi)$ numerically. In Figure 1, `w_1(i)` was updated to reflect the value of $w_p(\phi_{\min,i})$ where $\phi = \phi_{\min,i}$ minimized $y_p(\phi)$ for corresponding values of $\beta_i, \delta_i$ from Eq. 7. The following code defines a vector `phi_m2` of length $n = 1000$ and populates it with the value of $\phi_{\min,i}$ that minimizes $w_p(\phi)$ for $\beta_i$. The corresponding minimum values are stored in another vector `w_min`. Note that the first index of the sorted vector `w_vals` is accessed, since we are looking for the absolute minimum.

```
phi_m2 = zeros(1,n);
w_min = zeros(1,n);
for i = 1:n
    [w_vals idx] = sort(w_p(Beta_vec(i)));
    phi_m2(i) = phi(idx(1));
    w_min(i) = w_vals(1);
end
```

Figure 3: Iteratively finding $\vec{\phi}_{\min}$ for various $\beta$

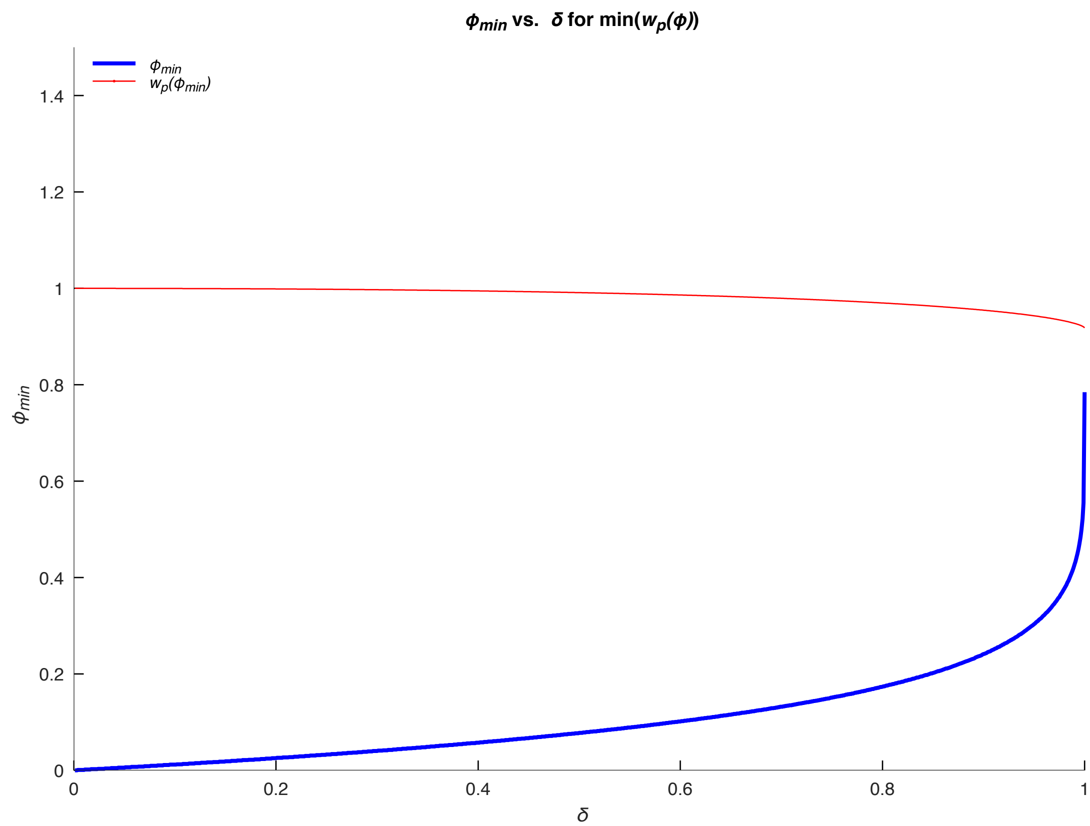



Figure 4: Values of $\phi_{\min}$ that minimize $w_p(\phi)$ for various $\beta$

Figure 4, showing $\phi_{\min}$ (representing `phi_m2`) vs. $\delta$, summarizes these results.

# 4 Bisection and Newton-Raphson method for finding roots

## 4.1 Bisection method

We will use the bisection method to find when $y_p(\phi) = 0$ for a given tolerance $\epsilon$. The requirement for the bisection method is that the function in question is continuous on the closed interval $[a, b]$[1], so the roots of $y_p(\phi)$ can be found. The roots of $w'(\phi)$ will be found using the Newton's method later.

## 4.2 Newton-Raphson method

In Eq. 40 from the presentation, ${w_p}'(\phi)$ was not expressed in terms of $\beta$, so the following expression will be used:

$$ {w_p}'(\phi) = \frac{2(\sin^2(2\phi + \pi/2) - 3)\sin(2\phi + \pi/2)}{(3\pi + 8)\beta^2 \sin^3(2\phi + \pi/2)} - \frac{\cos(2\phi + \pi/2)\left(1 - \frac{4}{(3\pi+8)\beta^2}(3\phi + 2)\right)}{r_0 \sin^3(2\phi + \pi/2)} \tag{8} $$

[1] Bisection method - Wolfram MathWorld

This method cannot be used to solve for the root of $y_p(\phi)$ because although the function exists in the domain $\phi \in [0, \pi/4)$, $\lim_{\phi\to 0^+} {y_p}'(\phi)$ and $\lim_{\phi\to \pi/4^-} {y_p}'(\phi)$ do not exist, so the function is not differentiable on the entire interval[2].

From observing the graph of $y_p(\phi)$ on Page 6, this is confirmed visually. Since the endpoints of $y_p(\phi)$ on this interval are its two roots, they cannot be found through this method. The bisection method works, however, since the only requirement is continuity of $y_p(\phi)$ on the interval.

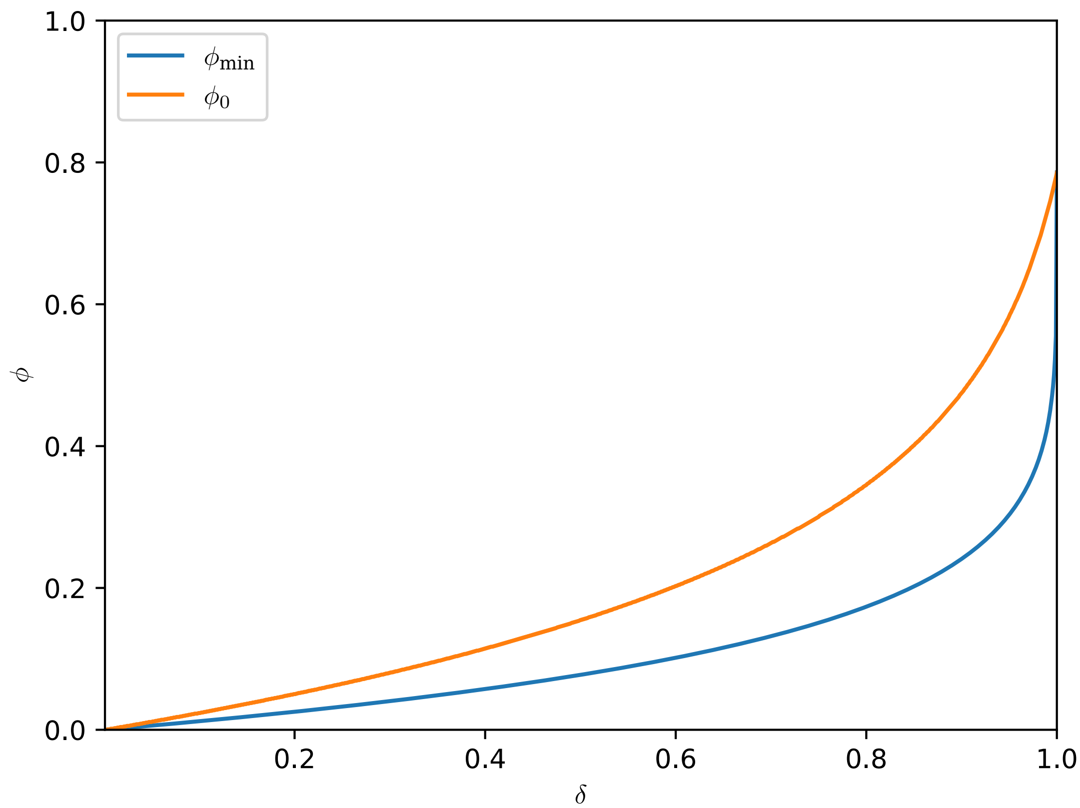



Figure 5: A plot of $\phi_{\min}$ and $\phi_0$ vs $\delta$

The code used to implement both algorithms is located here. Figure 5 demonstrates the bisection method for finding $\phi_0$ and Newton's method for $\phi_{\min}$.

# 5 Curve fitting

## 5.1 Objective

The least-squared method of curve fitting[3] was used to generate the closed-form equations $\phi_0(\delta)$ and $\phi_{\min}(\delta)$.

The approach is to define an error function $E(\phi) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\phi_i - \phi(\delta_i))^2}$ that represents the RMS error between the fitted curve and original data. $n$ is the number of data points in the set

[2] Newton's method
[3] Curve fitting: least squares methods

$\{(\delta_1, \phi_1), \ldots, (\delta_n, \phi_n)\}$ and $\phi(\delta)$ represents the fitted curve. Because a polynomial curve is desired, the coefficients of the terms in $\delta(\phi)$ belong to $\vec{c} \in \mathbb{R}^{k+1}$, so there are $k+1$ coefficients in the fitted polynomial. Since $\phi(\delta; \vec{c})$, $E : \mathbb{R}^{k+1} \to \mathbb{R}; \vec{c}$. The objective is then to find a $\vec{c}$ which minimizes $E(\vec{c})$. Similarly, each $\phi(\delta_i)$ becomes a function of $\vec{c}$.

Minimizing $E(\vec{c}) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\phi_i - \phi(\delta_i))^2}$ is the same as minimizing $E_1(\vec{c}) = \sum_{i=1}^{n}(\phi_i - \phi(\delta_i))^2$, so the next step is to solve

$$\frac{\partial E_1}{\partial c_j} = 2\sum_{i=1}^{n}(\phi_i - \phi(\delta_i))\frac{\partial \phi(\delta_i)}{\partial c_j} = 0 \tag{9}$$

$$\sum_{i=1}^{n}\frac{\partial \phi(\delta_i)}{\partial c_j}\phi_i = \sum_{i=1}^{n}\frac{\partial \phi(\delta_i)}{\partial c_j}\phi(\delta_i) \tag{10}$$

$$\sum_{i=1}^{n}\delta_i^{j-1}\phi_i = \sum_{i=1}^{n}c_j\delta_i^{2j-1} + \ldots + c_1\delta_i^{j-1} = \sum_{\ell=1}^{j}\sum_{i=1}^{n}c_{j-\ell+1}\delta_i^{2j-\ell-1} \tag{11}$$

where $j = 1, \ldots, k+1$ and the polynomial is of degree $k$.

A vectorized implementation yields

$$\begin{bmatrix} \sum_{i=1}^{n}\delta_i^k\phi_i \\ \vdots \\ \sum_{i=1}^{n}\delta_i^1\phi_i \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n}\delta_i^{2k} & \cdots & \sum_{i=1}^{n}\delta_i^k \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n}\delta_i^k & \cdots & \sum_{i=1}^{n} \end{bmatrix} \begin{bmatrix} c_{k+1} \\ \vdots \\ c_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n}c_{k+1}\delta_i^{2k} & \cdots & \sum_{i=1}^{n}c_1\delta_i^k \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n}c_{k+1}\delta_i^k & \cdots & \sum_{i=1}^{n}c_1 \end{bmatrix} \tag{12}$$

$$\vec{b} = A\vec{c} \tag{13}$$

The matrix-vector equation $A\vec{c} = \vec{b}$ can then be solved for $\vec{c}$ using an optimized linear algebra library, and the resulting coefficients used to generate a function $\phi(\delta)$.

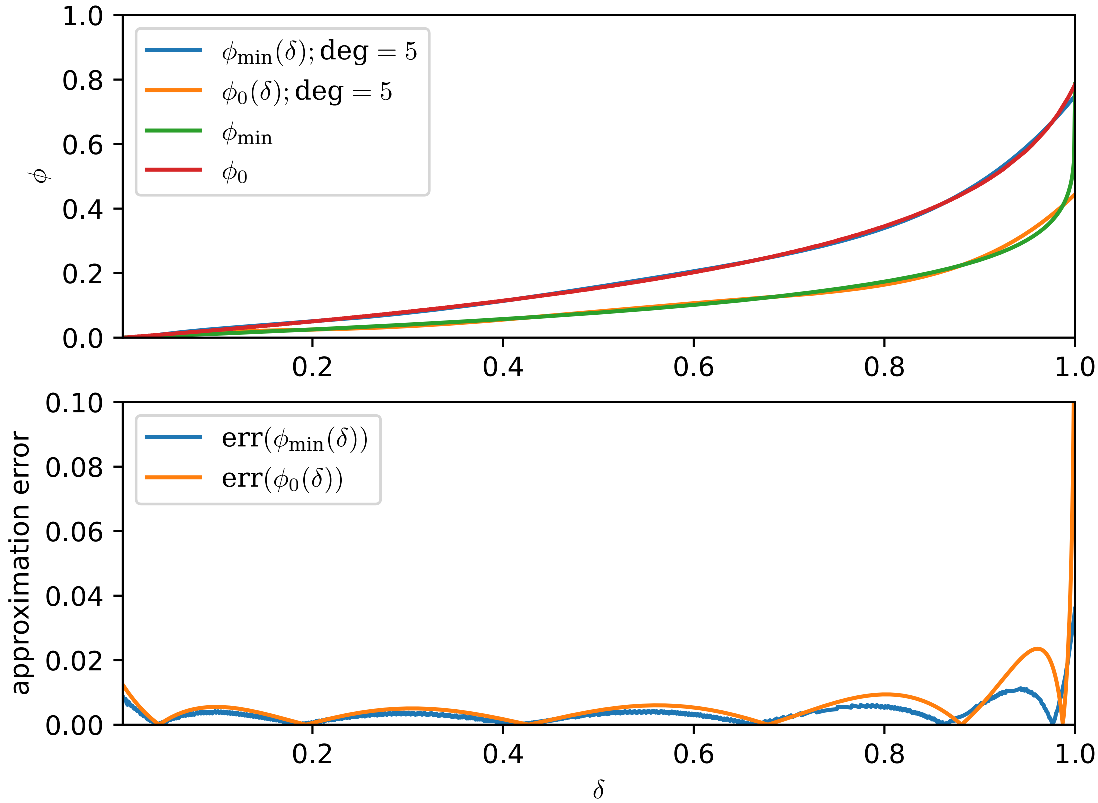



Figure 6: Using degree 5 polynomials to approximate $\phi_0(\delta)$ and $\phi_{\min}(\delta)$

A better approximation is desired since even with degree 5, a polynomial approximation cannot fit well to the smooth curves of $\phi_0$ and $\phi_{\min}$. This is evidenced by the Runge effect visible in the error plot of Figure 6.

## 5.2 Attempted approximation using Chebyshev polynomial interpolation

The Chebyshev polynomial approximations resulted in a more pronounced Runge effect, as depicted below.

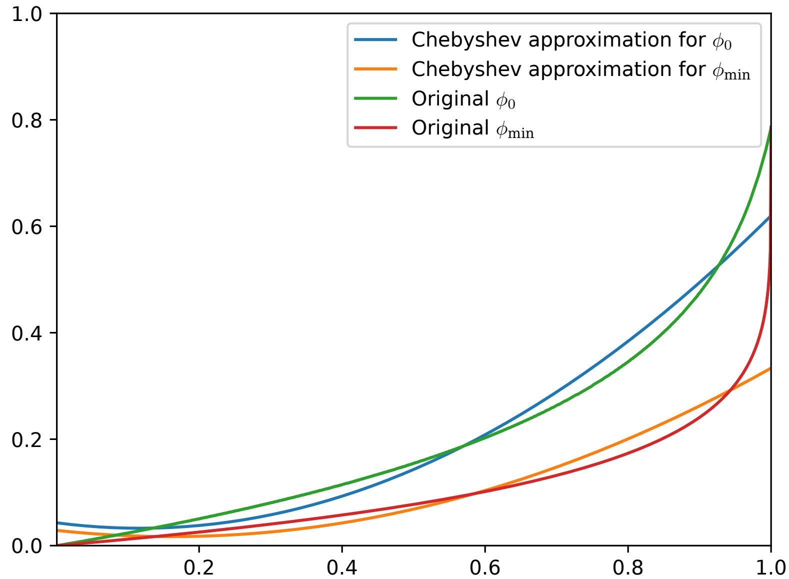



Figure 7: Chebyshev polynomial approximations

## 5.3 Attempted approximation using exponential function

By far the most accurate fit before the original polynomial approximation, an exponential function of the form $\phi = a\exp(b(\delta - c)) + d$ was used. Significant Runge effect is still evident in the plot below.

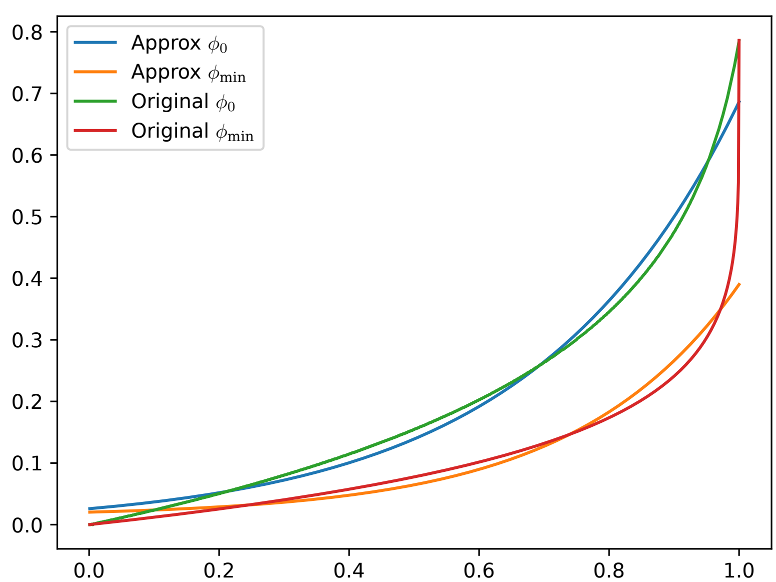



Figure 8: Exponential approximations

### 5.4 Attempted elliptical integral of first kind approximation

Through parameterizing the first-order elliptic integral $K(k;n)$ for some $n \in \mathbb{R}$ and $k \in [0,1]$ for both translation and dilation control, curve fitting was attempted:

$$K(k;n) = -\frac{\pi}{2n} + \int_0^1 \frac{dx}{n\sqrt{(1-x^2)(1-k^2x^2)}} \tag{14}$$

The optimization routine returned a value of $n = 1$ for both $\phi_0$ and $\phi_{\min}$, indicating that the first-order approximation is not effective. The plot is shown below.

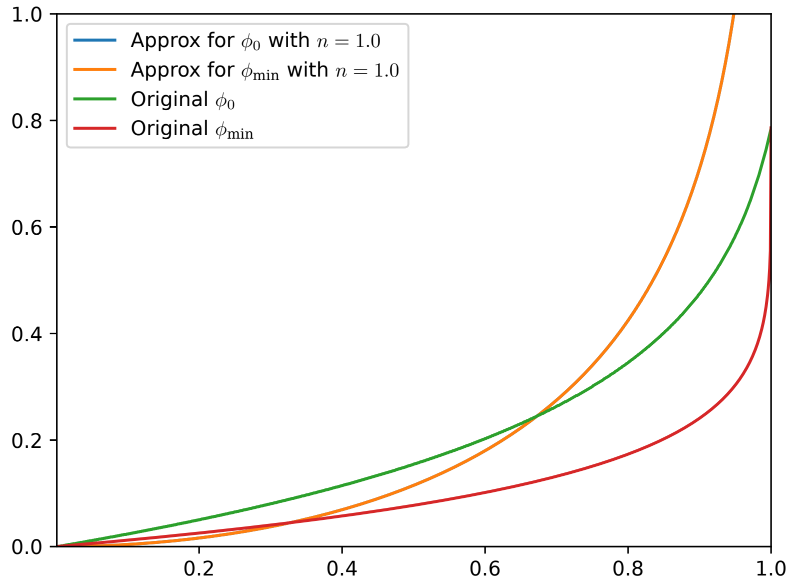



Figure 9: Elliptical approximations

Further parameterization attempts of $K(k)$ did not yield better results, as the curvature of the elliptic integral function cannot be manipulated easily to fit the data.

### 5.5 Univariate spline approximation

This method yielded the best results. A univariate spline of degree $k = 3$ with smoothness factor $s = 0.001$ was used to approximate $\phi_0$ and $\phi_{\min}$. Figure 10 shows that the method was successful, with minimized Runge effect and typical error magnitudes lower than that of the polynomial approximation.

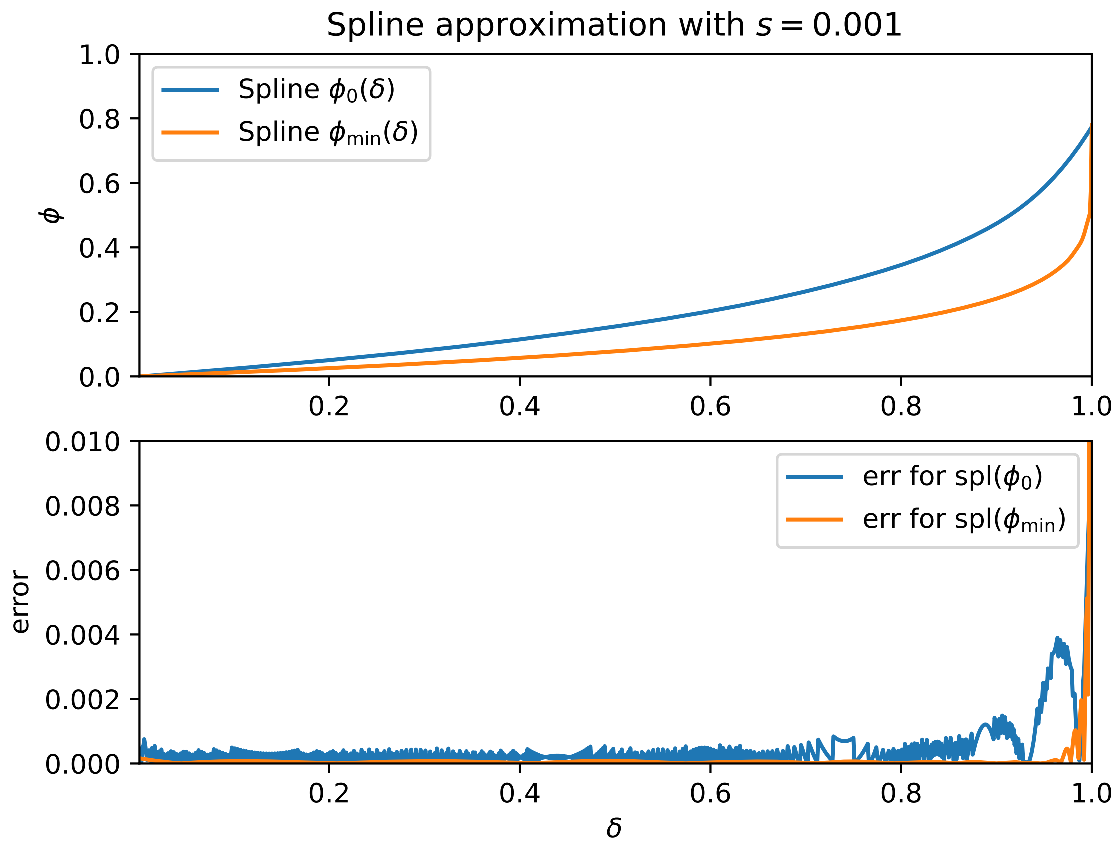



Figure 10: Spline approximation plot

The computer output below details the properties of each spline:

```
    ---------- phi_0 spline approx ----------
LSQ Error =          0.001000051241244917
Knots =          [0.001 0.501 0.751 0.876 1.   ]
Coeffs =         [-4.40529375e-04  3.92784349e-02  1.04654145e-01  2.56034241e-01
  4.15005836e-01  5.76096213e-01  7.73738936e-01]

---------- phi_min spline approx ----------
LSQ Error =          0.0009999540547108812
Knots =          [0.001 0.501 0.626 0.751 0.814 0.876 0.907 0.938 0.969 0.977 0.985 0.993
 0.997 1.   ]
Coeffs =         [-4.58365176e-05  1.99513345e-02  4.68542634e-02  1.06810792e-01
  1.41001801e-01  1.78919227e-01  2.10975405e-01  2.46198594e-01
  2.81539962e-01  3.21468750e-01  3.64172354e-01  3.99187610e-01
  4.27132615e-01  5.02562701e-01  4.92718655e-01  7.78749733e-01]
```

## 6 Simplifying the expression $1 - w_p(\phi)^2$ from $y_p(\phi)$

*Objective:* obtain the simplest expression for $1 - w_p(\phi)^2$ in terms of $\phi, \beta$.

The original expression for $1-w_p(\phi)^2$ is below, with the denominator removed for simplicity.

$$
\begin{aligned}
(3\pi+8)^2\beta^4\sin^2(2k)(1-w_p(\phi)^2) &= \underbrace{(3\pi+8)^2\beta^4\sin^2(2k) - \left[(3\pi+8)^2\beta^4 - 8(3\phi+2)(3\pi+8)\beta^2 + 16(3\phi+2)^2\right]}_{\gamma_1(\phi)} && (15)\\
&- \underbrace{2\sin(2k)(4\sin^2(k)+6)((3\pi+8)\beta^2 - 4(3\phi+2))}_{\gamma_2(\phi)} && (16)\\
&- \underbrace{\sin^2(2k)(16\sin^4(k)+48\sin^2(k)+36)}_{\gamma_3(\phi)} && (17)\\
&= \gamma_1(\phi) - \gamma_2(\phi) - \gamma_3(\phi) && (18)
\end{aligned}
$$

Applying Equations 1-3 to the denoted sub-functions $\gamma_1(\phi), \gamma_2(\phi), \gamma_3(\phi)$, we get

$$
\begin{aligned}
\gamma_1(\phi) &= (3\pi+8)^2\beta^4\left(\cos^2(2\phi)-1\right) + 8(3\phi+2)\left((3\pi+8)\beta^2 - 2(3\phi+2)\right) && (19)\\
\gamma_2(\phi) &= -2(8\cos(2\phi) + 2\sin(2\phi)\cos(2\phi))\left((3\pi+8)\beta^2 - 4(3\phi+2)\right) && (20)\\
&= -4\cos(2\phi)(\sin(2\phi)+2)\left((3\pi+8)\beta^2 - 4(3\phi+2)\right) && (21)\\
\gamma_3(\phi) &= -\cos^2(2\phi)\left(16\sin\left(\phi+\frac{\pi}{4}\right)^4 + 48\sin^2\left(\phi+\frac{\pi}{4}\right) + 36\right) && (22)\\
&= -\cos^2(2\phi)\left(4\sin^2(2\phi)+6\right)^2 && (23)
\end{aligned}
$$

Putting this together, a simplified expression is

$$
\begin{aligned}
1-w_p(\phi)^2 &= \frac{\gamma_1}{(3\pi+8)^2\beta^4\cos^2(2\phi)} + \frac{\gamma_2}{(3\pi+8)^2\beta^4\cos^2(2\phi)} + \frac{\gamma_3}{(3\pi+8)^2\beta^4\cos^2(2\phi)} && (24)\\
&= \underbrace{\frac{\cos^2(2\phi)-1}{\cos^2(2\phi)}}_{\frac{-(1-\cos^2(2\phi))}{\cos^2(2\phi)} = \frac{-\sin^2(2\phi)}{\cos^2(2\phi)} = \boxed{-\tan^2(2\phi)}} + \frac{8(3\phi+2)((3\pi+8)\beta^2 - 2(3\phi+2))}{(3\pi+8)^2\beta^4\cos^2(2\phi)} && (25)\\
&- \frac{4(\sin(2\phi)+2)((3\pi+8)\beta^2 - 4(3\phi+2))}{(3\pi+8)^2\beta^4\cos(2\phi)} - \frac{\left(4\sin^2(2\phi)+6\right)^2}{(3\pi+8)^2\beta^4} && (26)
\end{aligned}
$$

Because $y_p(\phi) = \sqrt{1-w_p(\phi)^2}\sqrt{\frac{\mu}{r_0}}\cot(\phi+\pi/4)$, by substitution

$$
\boxed{y_p(\phi) = \sqrt{-\tan^2(2\phi) + \frac{8(3\phi+2)((3\pi+8)\beta^2 - 2(3\phi+2))}{(3\pi+8)^2\beta^4\cos^2(2\phi)} - \frac{4(\sin(2\phi)+2)((3\pi+8)\beta^2 - 4(3\phi+2))}{(3\pi+8)^2\beta^4\cos(2\phi)} - \frac{(4\sin^2(2\phi)+6)^2}{(3\pi+8)^2\beta^4}}\sqrt{\frac{\mu}{r_0}}\cot(\phi+\pi/4)} \tag{27}
$$