Non-Markovian Memory and the Thermodynamic Necessity of Temporal Accumulation

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Abstract

We investigate the thermodynamic constraints on open quantum systems that must persist far from equilibrium in stochastic environments. Working within the framework of stochastic thermodynamics and information thermodynamics (Sagawa–Ueda), we define a *survival functional* $\mathcal{S} := \Delta F - W$ measuring the difference between the non-equilibrium free energy gained and the work invested by an agent.

We prove a **Markovian Ceiling**: for any open-loop Markovian (GKSL) dynamics with no measurement or feedback, $\mathcal{S} \leq 0$—the agent cannot thermodynamically "profit." We then derive an exact identity—valid for *arbitrary* (possibly correlated) initial states under autonomous evolution in the weak-coupling limit—expressing the survival functional in terms of the change in system–environment mutual information and bath displacement: $\beta\,\mathcal{S} = -\Delta I(S{:}E) - \Delta D_{\mathrm{KL}}(\rho_E \| \rho_E^{\mathrm{th}})$. Pre-existing correlations $I(S{:}E;\,0) > 0$, built during prior interaction epochs, serve as a consumable thermodynamic resource; their consumption during non-Markovian backflow intervals yields $\mathcal{S} > 0$, bounded by the initial correlation budget.

This establishes **memory as a thermodynamic necessity** for sustained far-from-equilibrium persistence. The memory kernel induces a causal partial order on system trajectories that, when restricted to the classical sector selected by decoherence (quantum Darwinism), is consistent with the accessibility ordering of the Holographic Alaya-Field Framework (HAFF). A worked example—a spin-boson model with Lorentz–Drude spectral density—illustrates how non-Markovian backflow enables free-energy extraction unavailable to memoryless systems.

Finally, using the entropy rate and predictive information from computational mechanics, we quantify the intrinsic cost of memory and identify the **Memory Catastrophe**: unbounded memory under finite energy leads to thermodynamic collapse, motivating the symmetry-breaking mechanism of Paper II.

Keywords: non-Markovian dynamics, open quantum systems, Nakajima–Zwanzig equation, memory kernel, thermodynamic arrow of time, information backflow, entropy production, stochastic thermodynamics

1 Introduction

1.1 Context: The Problem of Persistence

A quantum system coupled to a thermal environment generically relaxes toward equilibrium. This is the content of the *zeroth crisis*: absent special structure, every open subsystem is eventually erased by thermal noise [1].

Yet the physical world contains persistent far-from-equilibrium structures—from molecular machines to living organisms—that maintain themselves against the entropic tide for timescales vastly exceeding their intrinsic relaxation times. What structural feature of their dynamics makes this possible?

The standard answer invokes free-energy input: a persistent system is one that continuously imports low-entropy energy and exports high-entropy waste [2]. This is correct but incomplete. Two systems receiving *identical* free-energy flux from *identical* environments may exhibit vastly different persistence characteristics. The distinguishing factor, we argue, is *memory*—the capacity to condition present dynamics on past environmental states.

1.2 Position within the Series

This paper is the first of three constituting the **T-DOME** (Thermodynamic Dynamics of Observer-Memory Entanglement) framework, the third pillar of a three-paper program.

Framework	**Question**		**Result**	**Status**
HAFF [17, 18]	How does geometry emerge?	Ocean	Algebra $\to$ Geometry	Complete
Q-RAIF [21, 22]	What algebra must an observer have?	Fish	$Cl(V, q) \hookrightarrow Cl(1, 3)$	Complete
T-DOME I (this work)	Why must agents carry memory?	Seed	Markovian ceiling; memory as necessity	**This paper**
T-DOME II	Why must agents break symmetry?	Ego	Reference-frame selection	Planned
T-DOME III	How does self-calibration arise?	Loop	Fisher self-referential bound	Planned

The three T-DOME papers form an irreversible logical chain. Each resolves a survival crisis created by its predecessor:

1. **Paper I (The Seed):** Without memory, a system is trapped in the *Markovian present*—no accumulation, no temporal arrow, inevitable thermal death. Memory breaks this trap but floods the system with unbounded historical data.

2. **Paper II (The Ego):** Unbounded memory under finite computational resources causes processing collapse. Spontaneous symmetry breaking of the reference frame (establishing a "self") resolves the overload but introduces systematic bias.

3. **Paper III (The Loop):** Uncorrected bias diverges from a changing environment. A self-referential calibration loop (monitoring one's own prediction error) resolves the bias but requires the system to "observe its own observation"—closing the self-calibration loop.

The present paper addresses only the first link in this chain.

1.3 Relation to HAFF Paper F

HAFF Paper F [20] establishes the arrow of time as the direction of *accessibility propagation*: informational redundancy $\mathcal{R}(\hat{O})$ generically expands, inducing a partial order $\prec$ on observable algebras. That analysis is purely algebraic—it characterizes temporal asymmetry without invoking dynamics.

The present paper complements Paper F by identifying the *dynamical* origin of temporal asymmetry: the non-Markovian memory kernel $\mathcal{K}(t, s)$. We show (Section 5) that the partial order induced by the kernel's temporal support embeds into the HAFF accessibility ordering as a sub-structure. The two descriptions are dual faces of the same phenomenon: Paper F provides the algebraic skeleton; Paper I provides the dynamical muscle.

1.4 Scope and Disclaimers

1. This work does *not* claim that non-Markovian dynamics is sufficient for persistence. Memory is identified as *necessary* under the conditions specified; sufficiency requires additional structure (Papers II and III).

2. We do *not* claim that all non-Markovian systems outperform all Markovian systems. The theorem establishes that the supremum of survival efficiency over non-Markovian dynamics strictly exceeds the Markovian supremum.

3. We do *not* derive the specific form of the memory kernel from first principles. The kernel is treated as a structural feature of the system-environment coupling.

4. The term "agent" is used in the control-theoretic sense (a subsystem that acts on its environment to maintain a target state) and carries no implication of consciousness, intention, or subjective experience.

5. A broader structural analogy with classical philosophical concepts of temporal persistence exists but is outside the scope of this paper.

2 Mathematical Preliminaries

2.1 Open Quantum Systems: The Markovian Baseline

Consider a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_R \otimes \mathcal{H}_E$, where R denotes the "agent" (reduced system) and E the environment. The total Hamiltonian is

$$H = H_R \otimes \mathbb{1}_E + \mathbb{1}_R \otimes H_E + \lambda\, H_{\text{int}}, \tag{1}$$

where λ parametrizes the coupling strength.

Under the Born–Markov and secular approximations, the reduced dynamics of $\rho_R(t) = \mathrm{Tr}_E[\rho(t)]$ is governed by the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) master equation [3, 4]:

$$\dot{\rho}_R(t) = -i[H_{\text{eff}},\, \rho_R(t)] + \sum_k \gamma_k \left(L_k \rho_R(t) L_k^\dagger - \tfrac{1}{2}\{L_k^\dagger L_k,\, \rho_R(t)\} \right), \tag{2}$$

with $\gamma_k \geq 0$ and Lindblad operators $\{L_k\}$.

Remark 1 (Markovian = Memoryless)**.** *The GKSL equation is* time-local*: $\dot{\rho}_R(t)$ depends only on $\rho_R(t)$, never on $\rho_R(s)$ for $s < t$. Physically, this corresponds to an environment with vanishing correlation time ($\tau_E \to 0$): the bath "forgets" its interaction with the system instantaneously. The semigroup property $\Lambda(t+s) = \Lambda(t)\Lambda(s)$ ensures complete positivity at all times but precludes any information backflow from environment to system [9].*

2.2 Beyond Markov: The Nakajima–Zwanzig Equation

When the environmental correlation time τ_E is non-negligible, the Born–Markov approximation fails. The exact reduced dynamics is captured by the Nakajima–Zwanzig (NZ) integro-differential equation [5, 6]:

$$\dot{\rho}_R(t) = -i[H_{\text{eff}},\, \rho_R(t)] + \int_0^t ds\, \mathcal{K}(t,s)\, \rho_R(s), \tag{3}$$

where $\mathcal{K}(t,s)$ is the **memory kernel**—a superoperator encoding the influence of the system's entire history on its present dynamics.

Definition 2 (Memory Kernel)**.** *The memory kernel $\mathcal{K} : [0,\infty)^2 \to \mathcal{L}(\mathcal{B}(\mathcal{H}_R))$ is the superoperator satisfying* (3)*. It encodes two types of information:*

1. ***Environmental structure:*** *the spectral density, correlation functions, and non-equilibrium features of the bath;*
2. ***Temporal reach:*** *the effective support $\tau_{\text{mem}} := \inf\{\tau : \|\mathcal{K}(t,s)\| < \epsilon \;\forall\; t-s > \tau\}$, the "memory depth."*

The Markovian limit corresponds to $\mathcal{K}(t,s) \to \mathcal{K}_0\, \delta(t-s)$, recovering the GKSL generator.

Remark 3 (Information Backflow)**.** *Non-Markovian dynamics admits* information backflow*: the distinguishability of two initial states, as measured by trace distance $D(\rho_1(t), \rho_2(t))$, can temporarily increase [11]. This is the operational signature of memory—the environment returns previously absorbed information to the system.*

2.3 Thermodynamic Framework

We adopt the framework of stochastic thermodynamics for open quantum systems [12]. The following conventions are fixed throughout.

Definition 4 (Thermodynamic Setup)**.**

1. ***Hamiltonian decomposition.*** *The system Hamiltonian is* $H_S(t) = H_R + H_{\mathrm{ctrl}}(t)$*, where* H_R *is the* fixed *bare Hamiltonian and* $H_{\mathrm{ctrl}}(t)$ *is the agent's time-dependent control protocol. The bath Hamiltonian* H_E *and coupling* H_{int} *are as in* (1)*.*

2. ***Reference state.*** *The thermal equilibrium state of the bare Hamiltonian is*

$$\rho_{\mathrm{eq}} := \frac{e^{-\beta H_R}}{Z_R}, \qquad Z_R := \mathrm{tr}\big(e^{-\beta H_R}\big), \qquad \beta := (k_B T)^{-1}. \tag{4}$$

   Since H_R *is time-independent,* ρ_{eq} *is a well-defined, fixed reference throughout the protocol.*

3. ***Non-equilibrium free energy.*** *For any state* ρ *of the reduced system,*

$$F(\rho) := \mathrm{tr}(\rho\, H_R) + \beta^{-1}\,\mathrm{tr}(\rho \ln \rho) = \langle H_R \rangle_\rho - \beta^{-1} S(\rho), \tag{5}$$

   where $S(\rho) = -\,\mathrm{tr}(\rho \ln \rho)$ *is the von Neumann entropy. The equilibrium value is* $F_{\mathrm{eq}} = -\beta^{-1} \ln Z_R$*.*

4. ***Free energy–relative entropy identity.***

$$D_{\mathrm{KL}}(\rho \| \rho_{\mathrm{eq}}) = \beta\big(F(\rho) - F_{\mathrm{eq}}\big) \geq 0. \tag{6}$$

   Thus D_{KL} *measures the free-energy surplus in units of* $k_B T$*.*

5. ***Work.*** *The work performed on the system by the control protocol over* $[0, \tau]$ *is*

$$W[0,\tau] := \int_0^\tau \mathrm{tr}\left(\rho(t)\, \frac{\partial H_{\mathrm{ctrl}}}{\partial t}\right) dt. \tag{7}$$

6. ***Entropy-production functional.*** *The* generalised entropy production *over* $[0, \tau]$ *is*

$$\Sigma[0,\tau] := \beta\big(W[0,\tau] - \Delta F\big), \tag{8}$$

   where $\Delta F = F(\rho(\tau)) - F(\rho(0))$*. For uncorrelated (product) initial states,* $\Sigma \geq 0$ *recovers the standard second-law bound. For initially correlated states,* Σ *can be* negative*, reflecting the consumption of pre-existing correlations (see Remark 26).*

Remark 5 (Why H_R is fixed)**.** *The bare Hamiltonian* H_R *defines the system's energy scale and hence the reference state* ρ_{eq}*. The agent acts on the world through* $H_{\mathrm{ctrl}}(t)$*, which may be time-dependent. This separation ensures that* ρ_{eq} *is well-defined and time-independent, avoiding the ambiguity that arises when the full* $H_S(t)$ *is used to define the thermal reference.*

Definition 6 (Standing Assumptions)**.** *The following minimal assumptions are in force throughout Sections 4–6 unless stated otherwise. Every main result (Lemma 20, Theorem 22, Corollary 25) relies* only *on items* (A1)–(A5) *below.*

(A1) ***Finite-dimensional bipartite system.*** $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$, *with total Hamiltonian* (1) *and global unitary evolution* $U(t) = \mathcal{T} \exp\Big(-i \int_0^t H(s)\, ds\Big)$.

(A2) ***Weak coupling.*** *The system–environment interaction satisfies* $\lambda \ll 1$ *in* (1), *so that* $\Delta\langle H_{\text{int}}\rangle = O(\lambda)$ *[1]. Energy conservation is then* $\Delta\langle H_R\rangle + \Delta\langle H_{\text{ctrl}}\rangle + \Delta\langle H_E\rangle \approx 0$ *up to controlled* $O(\lambda)$ *corrections.*

(A3) ***Fixed environmental reference.*** $\rho_E^{\text{th}} := e^{-\beta H_E}/Z_E$ *is a fixed* bookkeeping *Gibbs state at inverse temperature* β. *The* actual *initial bath state* $\rho_E(0)$ *need not coincide with* ρ_E^{th}; *when* $\rho_E(0) \neq \rho_E^{\text{th}}$, *the quantity* $D_{\text{KL}}(\rho_E(t)\|\rho_E^{\text{th}})$ *tracks the nonequilibrium free energy stored in the bath relative to this reference. The bath Hamiltonian* H_E *is time-independent.*

(A4) ***Arbitrary initial state.*** *The total initial state* $\rho_{SE}(0)$ *is* not *required to be a product state. In particular, initial system–environment correlations* $I(S{:}E;\, 0) > 0$ *and initial bath displacement* $D_{\text{KL}}(\rho_E(0)\|\rho_E^{\text{th}}) > 0$ *are both permitted.*

(A5) ***Regularity.*** *All quantum states appearing in the thermodynamic identities are assumed to have full rank (or are restricted to their support), so that all relative entropies* $D_{\text{KL}}(\rho\|\sigma)$ *are finite.*

Remark 7 (Bookkeeping conventions)**.** *The heat absorbed by the environment is* $Q := \Delta\langle H_E\rangle = \mathrm{Tr}[\rho_E(\tau)H_E] - \mathrm{Tr}[\rho_E(0)H_E]$ *(matching Esposito* et al. *[12]). We define* $\Sigma := \beta(W - \Delta F)$ *as a* generalised entropy-balance functional*; for correlated initial conditions* Σ *need not be nonnegative (see Remark 26).*

2.4 The Survival Functional

We now define the central quantity of this paper.

Definition 8 (Survival Functional)**.** *For a reduced system* R *evolving under dynamics* Λ *over* $[0,\tau]$, *the* ***survival functional*** *is*

$$\mathcal{S}[\Lambda,\tau] := \Delta F - W[0,\tau] = \big[F(\rho(\tau)) - F(\rho(0))\big] - W[0,\tau]. \tag{9}$$

Equivalently, using (8),

$$\beta\, \mathcal{S}[\Lambda,\tau] = -\Sigma[0,\tau]. \tag{10}$$

Note on nomenclature. We retain the term "survival functional" to emphasize the biological interpretation of persistence far from equilibrium; mathematically, $\mathcal{S}$ is strictly a *generalized entropy-balance functional* derived from the first and second laws.

Remark 9 (Interpretation)**.** *The survival functional has a transparent physical meaning:*

- $\mathcal{S} > 0$*: the system gained more free energy than was invested by the external protocol— a* thermodynamic profit*. The agent has extracted usable work from environmental correlations.*
- $\mathcal{S} = 0$*: the agent breaks even (reversible limit,* $\Sigma = 0$*).*
- $\mathcal{S} < 0$*: the agent paid more than it gained (the generic irreversible case).*

Under the standard second law ($\Sigma \geq 0$*),* $\mathcal{S} \leq 0$ *always. As we show in Sections 3 and 4, achieving* $\mathcal{S} > 0$ *requires* information*—and the memory kernel provides exactly this.*

Remark 10 (Connection to Information Thermodynamics)**.** *In the Sagawa–Ueda framework [7, 8], a system under feedback control satisfies the generalized second law*

$$\Sigma \geq -I_{\text{feedback}}, \tag{11}$$

where $I_{\text{feedback}} \geq 0$ *is the mutual information gained through measurement of the system. This permits* $\Sigma < 0$ *(and hence* $\mathcal{S} > 0$*) at the expense of information. The core thesis of this paper is that a non-Markovian memory kernel provides* implicit *feedback: the system's history encodes correlations with the environment that play the same thermodynamic role as explicit measurement outcomes.*

3 The Markovian Ceiling

We now establish the fundamental thermodynamic limitation of memoryless agents. The result is elementary given the framework of Section 2.3, but its consequences are far-reaching: under *open-loop* control—where the agent's protocol $H_{\text{ctrl}}(t)$ is fixed in advance and receives no information from the bath—the survival functional can never be positive.

3.1 Spohn's Inequality

Throughout this section we assume that the GKSL generator $\mathcal{L}$ is a *thermal Lindbladian*: it is obtained from the weak-coupling (Davies) limit of a system coupled to a single thermal bath at inverse temperature β, and satisfies **quantum detailed balance** (the KMS condition) [10, 1]. Under this assumption, the unique stationary state is the Gibbs state $\rho_{\text{ss}} = \rho_{\text{eq}}$ of (4), and the generator is self-adjoint with respect to the KMS inner product. This ensures that the entropy production rate below is well-defined and non-negative.

Definition 11 (Markovian Semigroup)**.** *Throughout this paper, "Markovian" dynamics refers strictly to a* ***dynamical semigroup*** *generated by a time-independent GKSL generator* $\mathcal{L}$ *with non-negative rates. While time-dependent CP-divisible maps [9] are often called Markovian in broader contexts, the ceiling theorem (Theorem 14) targets the semigroup case* $\Lambda(t) = e^{\mathcal{L}t}$*, where no memory effects or temporal correlations can be exploited.*

Lemma 12 (Spohn [10])**.** *For any GKSL dynamical semigroup* $\Lambda_t = e^{\mathcal{L}t}$ *satisfying quantum detailed balance with unique invariant state* ρ_{eq}*, the entropy production rate*

$$\sigma(t) := -\operatorname{tr}\big(\mathcal{L}[\rho(t)]\,(\ln\rho(t) - \ln\rho_{\text{eq}})\big) \tag{12}$$

satisfies $\sigma(t) \geq 0$*, with equality if and only if* $\rho(t) = \rho_{\text{eq}}$*.*

Proof. This follows from the contractivity of CPTP maps under quantum relative entropy [10, 1]: $D_{\mathrm{KL}}(\Lambda_t\rho\|\Lambda_t\rho_{\mathrm{eq}}) \leq D_{\mathrm{KL}}(\rho\|\rho_{\mathrm{eq}})$ for all $t \geq 0$. Differentiating at $t = 0$ yields $\sigma(t) \geq 0$. □

3.2 The Markovian Ceiling Theorem

Definition 13 (Open-loop Markovian control class $\mathcal{C}_{\mathrm{M}}$). *A protocol $H_{\mathrm{ctrl}}(t)$ belongs to the* ***open-loop Markovian control class*** $\mathcal{C}_{\mathrm{M}}$ *if and only if:*

(C1) $H_{\mathrm{ctrl}}(t)$ *is a* predetermined *function of t alone, fixed before the protocol begins.*

(C2) *No measurement of the system or environment is performed during $[0,\tau]$, and $H_{\mathrm{ctrl}}(t)$ receives no feedback from measurement outcomes.*

(C3) $H_{\mathrm{ctrl}}(t)$ *is statistically independent of the bath realization $\{\xi_E(s) : s \in [0,\tau]\}$.*

Protocols involving adaptive measurement-based feedback (Sagawa–Ueda [7]) are excluded *from $\mathcal{C}_{\mathrm{M}}$.*

Theorem 14 (Markovian Ceiling). *Let Λ^{M} denote GKSL dynamics* (2) *satisfying quantum detailed balance (Lemma 12), coupled to a stationary thermal bath at inverse temperature β, under a control protocol $H_{\mathrm{ctrl}}(t) \in \mathcal{C}_{\mathrm{M}}$ (Definition 13). Then the survival functional satisfies*

$$\mathcal{S}[\Lambda^{\mathrm{M}},\tau] \leq 0 \qquad \text{for all } \tau \geq 0. \tag{13}$$

Equality holds in the quasi-static limit ($\Sigma \to 0$), where the protocol varies slowly enough that the state remains close to the instantaneous Gibbs state $\rho_{\mathrm{eq}}(t)$ at all times.

Proof. The proof proceeds in two steps.
Step 1: Free-energy balance. Differentiating (6), the relative entropy evolves as

$$\frac{d}{dt}\, D_{\mathrm{KL}}(\rho(t)\|\rho_{\mathrm{eq}}) = \beta\, \dot{W}(t) - \sigma(t), \tag{14}$$

where $\dot{W}(t) = \mathrm{tr}(\rho(t)\,\partial_t H_{\mathrm{ctrl}})$ is the instantaneous power and $\sigma(t)$ is Spohn's entropy production rate (12). Integrating over $[0,\tau]$:

$$\Delta D_{\mathrm{KL}} = \beta\, W[0,\tau] - \underbrace{\int_0^\tau \sigma(t)\, dt}_{=\,\Sigma\ \geq\ 0}. \tag{15}$$

Step 2: Applying Spohn. By Lemma 12, $\sigma(t) \geq 0$ for all t, so $\Sigma \geq 0$. From (15):

$$\Delta D_{\mathrm{KL}} \leq \beta\, W[0,\tau]. \tag{16}$$

Converting via (6): $\Delta F \leq W[0,\tau]$, whence $\mathcal{S} = \Delta F - W \leq 0$.

The ceiling $\mathcal{S} = 0$ is achieved in the reversible limit where the protocol is infinitely slow and $\sigma(t) \to 0$ pointwise. □

Remark 15 (The "Open-Loop" Qualifier)**.** *The restriction to the control class* $\mathcal{C}_{\mathrm{M}}$ *(Definition 13) is essential. If the agent can perform* measurements *on the bath and condition its protocol on the outcomes—i.e., violate condition* (C2)*—the Sagawa–Ueda generalized second law* (11) *permits* $\Sigma < 0$ *(and hence* $\mathcal{S} > 0$*) at the expense of mutual information. The Markovian ceiling is therefore not a universal bound on all Markovian agents, but on agents whose protocols satisfy* (C1)–(C3)*.*

This qualifier is precisely the point: the memory kernel of non-Markovian dynamics provides implicit access to bath correlations, playing the role of implicit measurement—the subject of Section 4.

Corollary 16 (Temporal Blindness)**.** *Under the Born–Markov approximation, the bath correlation function is replaced by its white-noise limit* $C(t,s) \to C_0\,\delta(t-s)$*, and the GKSL dissipator depends only on the spectral density* $J(\omega)$ *evaluated at the system's Bohr frequencies. The agent interacts with the environment's* power spectrum *but is structurally blind to its* temporal correlations*—the off-diagonal elements* $C(t,s)$ *for* $t \neq s$*.*

Consequently, the spectral gap $\lambda_{\min} \propto \sum_k J(\omega_k)$ *of the Liouvillian sets the rate of irreversible decay. Maintaining* $D_{\mathrm{KL}} > 0$ *requires continuous work at rate* $\dot{W} \geq \beta^{-1}\sigma(t) > 0$*, and the integrated cost always meets or exceeds the integrated gain.*

Remark 17 (Dissipative vs. Self-Nourishing Structures)**.** *The Markovian ceiling partitions far-from-equilibrium structures into two classes:*

- ***Dissipative structures*** *(*$\mathcal{S} \leq 0$*): sustained by continuous external free-energy input. Every unit of order is paid for in full. (Prigogine's sense [2].)*
- ***Self-nourishing structures*** *(*$\mathcal{S} > 0$*): extract structured advantage from environmental correlations, gaining more free energy than they consume. These require information flow, and hence memory.*

The ceiling is not a limitation of the agent's control strategy but a structural consequence *of temporal blindness: without memory, the environment's temporal correlations are thermodynamically invisible.*

4 The Non-Markovian Advantage

Having established that open-loop Markovian agents are thermodynamically capped at $\mathcal{S} \leq 0$, we now demonstrate how non-Markovian dynamics breaks this ceiling. The mechanism is grounded entirely in standard quantities: the quantum mutual information $I(S{:}E)$ between system and environment serves as a consumable thermodynamic resource. Non-Markovian backflow intervals are precisely those during which pre-existing correlations are consumed, enabling the system to extract free energy beyond what open-loop work provides.

4.1 System–Environment Mutual Information

We work with the total system–environment state $\rho_{SE}(t)$, evolving unitarily under the total Hamiltonian (1). The quantum mutual information

$$I(S{:}E;\,t) := S(\rho_S(t)) + S(\rho_E(t)) - S(\rho_{SE}(t)) = D_{\mathrm{KL}}\big(\rho_{SE}(t)\,\big\|\,\rho_S(t)\otimes\rho_E(t)\big) \geq 0 \tag{17}$$

quantifies the total correlations (classical and quantum) between the system S and the environment E at time t.

Remark 18 (Role of Initial Correlations)**.** *Under the Born approximation, the initial state is taken as a product* $\rho_{SE}(0) = \rho_S(0) \otimes \rho_E^{\mathrm{th}}$*, so* $I(S{:}E;\,0) = 0$*. For a system that has already been interacting with its environment (the physically generic situation for a "persistent agent"), the effective initial state at any restart time* $t_0 > 0$ *is* not *a product state: the preceding evolution has established correlations* $I(S{:}E;\,t_0) > 0$*. These pre-existing correlations—the system's "memory" of past interactions—are the thermodynamic resource that the memory kernel can exploit.*

4.2 The Information–Thermodynamic Identity

The following identity is the central technical tool of this section. It holds for **any** initial state—product or correlated—and relies only on unitarity and the definitions of mutual information and relative entropy.

Remark 19 (Relative-entropy chain rule)**.** *We repeatedly use the identity*

$$D_{\mathrm{KL}}\big(\rho_{SE} \,\big\|\, \rho_S \otimes \sigma_E\big) = I(S{:}E)_{\rho_{SE}} + D_{\mathrm{KL}}(\rho_E \| \sigma_E), \tag{18}$$

valid for arbitrary *(possibly correlated)* ρ_{SE} *and any full-rank reference state* σ_E*.*[1] *Importantly, this is a* pure algebraic identity *and does not assume product initial conditions.*

Lemma 20 (Information–Thermodynamic Identity)**.** *Let* $\rho_{SE}(t)$ *evolve unitarily under the total Hamiltonian. Then, for* ***any*** *initial state* $\rho_{SE}(0)$ *(product or correlated):*

$$\Delta I(S{:}E) + \Delta D_{\mathrm{KL}}\big(\rho_E \,\big\|\, \rho_E^{\mathrm{th}}\big) = \Delta S_S + \beta\,\Delta\langle H_E\rangle, \tag{19}$$

where $\Delta S_S = S(\rho_S(\tau)) - S(\rho_S(0))$ *is the change in the system's von Neumann entropy and* $\Delta\langle H_E\rangle = \mathrm{Tr}[\rho_E(\tau)\,H_E] - \mathrm{Tr}[\rho_E(0)\,H_E]$ *is the energy absorbed by the environment.*

Proof. Applying the chain rule (18) (Remark 19) with $\sigma_E = \rho_E^{\mathrm{th}}$:

$$D_{\mathrm{KL}}\big(\rho_{SE}(t) \,\big\|\, \rho_S(t) \otimes \rho_E^{\mathrm{th}}\big) = I(S{:}E;\,t) + D_{\mathrm{KL}}\big(\rho_E(t) \,\big\|\, \rho_E^{\mathrm{th}}\big). \tag{20}$$

Expanding the left side using $\ln \rho_E^{\mathrm{th}} = -\beta H_E - \ln Z_E$:

$$D_{\mathrm{KL}}\big(\rho_{SE}(t) \,\big\|\, \rho_S(t) \otimes \rho_E^{\mathrm{th}}\big) = -S(\rho_{SE}(t)) + S(\rho_S(t)) + \beta\langle H_E\rangle_t + \ln Z_E. \tag{21}$$

Since the total evolution is unitary, $S(\rho_{SE}(t)) = S(\rho_{SE}(0))$ for all t. Taking the difference between times τ and 0 cancels both $S(\rho_{SE})$ and $\ln Z_E$, yielding

$$\Delta\big[I(S{:}E) + D_{\mathrm{KL}}(\rho_E \| \rho_E^{\mathrm{th}})\big] = \Delta S_S + \beta\,\Delta\langle H_E\rangle. \tag{22}$$

□

[1] This follows from the definition of quantum relative entropy and $\ln(\rho_S \otimes \sigma_E) = \ln \rho_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes \ln \sigma_E$. See, e.g., M. M. Wilde, *Quantum Information Theory*, 2nd ed., Cambridge University Press (2017), Sec. 11; and M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2000), Ch. 11.

Remark 21 (No assumption on the initial state)**.** *The proof of Lemma 20 uses* only *unitarity* ($\Delta S(\rho_{SE}) = 0$) *and the algebraic structure of the KL divergence. No assumption is made about the initial state* $\rho_{SE}(0)$*, the coupling strength, or the character (Markovian or non-Markovian) of the reduced dynamics. When the initial state is a product state with the environment in thermal equilibrium, all initial-time terms vanish and the identity reduces to the Esposito decomposition [12]:* $\Sigma = I(S{:}E;\,\tau) + D_{\mathrm{KL}}(\rho_E(\tau)\|\rho_E^{\mathrm{th}})$*.*

4.3 The Survival Identity

We now connect the information–thermodynamic identity (19) to the survival functional $\mathcal{S}$ defined in Section 2.4.

Theorem 22 (Survival Functional: General Form)**.** *Under Assumptions* (A1)–(A5) *of Definition 6, let* $\rho_{SE}(t)$ *evolve unitarily from an* arbitrary *(possibly correlated) initial state* $\rho_{SE}(0)$*. The survival functional satisfies*

$$\boxed{\beta\,\mathcal{S}[\Lambda,\tau] = -\Delta I(S{:}E) - \Delta D_{\mathrm{KL}}\big(\rho_E \,\big\|\, \rho_E^{\mathrm{th}}\big) - \beta\,\Delta\langle H_{\mathrm{ctrl}}\rangle,} \tag{23}$$

where $\Delta\langle H_{\mathrm{ctrl}}\rangle = \mathrm{Tr}[\rho_S(\tau)\,H_{\mathrm{ctrl}}(\tau)] - \mathrm{Tr}[\rho_S(0)\,H_{\mathrm{ctrl}}(0)]$ *is the change in the control-field energy.*

For ***autonomous evolution*** *(*$H_{\mathrm{ctrl}} = 0$ *throughout* $[0,\tau]$*), the control term vanishes:*

$$\beta\,\mathcal{S}[\Lambda,\tau] = -\Delta I(S{:}E) - \Delta D_{\mathrm{KL}}\big(\rho_E \,\big\|\, \rho_E^{\mathrm{th}}\big). \tag{24}$$

Proof. The proof uses three ingredients: the definition of $\mathcal{S}$, the first law, and Lemma 20.

Step 1 (First law in weak coupling). Since $H_{\mathrm{ctrl}}(t)$ is the only time-dependent component of H, the work satisfies $W = \Delta\langle H\rangle \approx \Delta\langle H_R\rangle + \Delta\langle H_{\mathrm{ctrl}}\rangle + \Delta\langle H_E\rangle$ by Assumption (A2).

Step 2 (Connecting Σ to the identity). From Definition 8 and (8), using $\Delta F = \Delta\langle H_R\rangle - \beta^{-1}\Delta S_S$:

$$\begin{aligned}\Sigma = \beta(W - \Delta F) &= \beta\big(W - \Delta\langle H_R\rangle\big) + \Delta S_S \\ &= \beta\big(\Delta\langle H_{\mathrm{ctrl}}\rangle + \Delta\langle H_E\rangle\big) + \Delta S_S \\ &= \big(\Delta S_S + \beta\,\Delta\langle H_E\rangle\big) + \beta\,\Delta\langle H_{\mathrm{ctrl}}\rangle.\end{aligned} \tag{25}$$

By Lemma 20, the parenthesized term equals $\Delta I(S{:}E) + \Delta D_{\mathrm{KL}}(\rho_E\|\rho_E^{\mathrm{th}})$. Hence

$$\Sigma = \Delta I(S{:}E) + \Delta D_{\mathrm{KL}}\big(\rho_E \,\big\|\, \rho_E^{\mathrm{th}}\big) + \beta\,\Delta\langle H_{\mathrm{ctrl}}\rangle. \tag{26}$$

Step 3 (Survival functional). $\beta\,\mathcal{S} = -\Sigma$ by (10), yielding (23). For $H_{\mathrm{ctrl}} = 0$: $\Delta\langle H_{\mathrm{ctrl}}\rangle = 0$, recovering (24). □

Remark 23 (Nature of the result)**.** *Equation* (23) *is an exact* accounting identity*, not an inequality or optimality bound. It establishes that any thermodynamic profit (*$\mathcal{S} > 0$*) in the autonomous regime must be perfectly balanced by the consumption of system–environment correlations (*$\Delta I < 0$*) or the relaxation of the bath (*$\Delta D_{\mathrm{KL}} < 0$*). The "non-Markovian advantage" arises because memory kernels allow access to regimes where* $\Delta I(S{:}E)$ *is negative and dominant—a channel that memoryless (Born–Markov) dynamics resets to zero at every time step (Remark 28).*

Remark 24 (Scope of the theorem)**.** *Theorem 22 holds for* any *initial state* $\rho_{SE}(0)$*—product or correlated. The proof requires only Assumptions* (A1)–(A5) *of Definition 6 and the definitions of* $\mathcal{S}$*,* $I(S{:}E)$*, and* D_{KL}*. No assumption about the reduced dynamics (Markovian, non-Markovian, or otherwise) is needed. This generality is essential: a persistent agent that has already been interacting with its environment necessarily carries correlations (*$I(S{:}E;\,0) > 0$*), and it is precisely these correlations that constitute the thermodynamic resource for survival.*

4.4 Three Regimes of Survival

We specialize to the autonomous case ($H_{\mathrm{ctrl}} = 0$), which is the natural setting for the "memory as a resource" argument: the agent benefits from pre-existing correlations without external driving.

Corollary 25 (Three Regimes)**.** *Under autonomous evolution, identity* (24) *identifies three regimes:*

1. ***Product initial state*** *(*$I(S{:}E;\,0) = 0$*,* $D_{\mathrm{KL}}(\rho_E(0)\|\rho_E^{\mathrm{th}}) = 0$*): Both* ΔI *and* ΔD_{KL} *are increases from zero to non-negative final values, so*

   $$\beta\,\mathcal{S} = -\big(I(S{:}E;\,\tau) + D_{\mathrm{KL}}(\rho_E(\tau)\|\rho_E^{\mathrm{th}})\big) \leq 0.$$

   This recovers the Markovian ceiling (Theorem 14), now with a precise accounting of where *the entropy goes: into system–environment correlations and bath displacement.*

2. ***Correlated initial state*** *(*$I(S{:}E;\,0) > 0$*): If the dynamics* consumes *pre-existing correlations (*$\Delta I < 0$*, i.e.,* $I(S{:}E;\,\tau) < I(S{:}E;\,0)$*), the first term contributes* positively *to* $\mathcal{S}$*. Provided*

   $$|\Delta I(S{:}E)| > \Delta D_{\mathrm{KL}}\big(\rho_E \,\big\|\, \rho_E^{\mathrm{th}}\big), \tag{27}$$

   the survival functional is strictly positive: $\mathcal{S} > 0$*. The agent has converted pre-existing correlations into usable free energy.*

3. ***Upper bound:*** *Since* $I(S{:}E;\,\tau) \geq 0$ *and* $D_{\mathrm{KL}}(\rho_E(\tau)\|\rho_E^{\mathrm{th}}) \geq 0$*, the maximum survival gain is bounded by*

   $$\beta\,\mathcal{S} \leq I(S{:}E;\,0) + D_{\mathrm{KL}}\big(\rho_E(0) \,\big\|\, \rho_E^{\mathrm{th}}\big). \tag{28}$$

   The thermodynamic profit cannot exceed the total initial "resource budget"—the pre-existing correlations plus the initial displacement of the bath from equilibrium.

4.5 The Correlation Battery

The three regimes of Corollary 25 raise a natural question: *where do the initial correlations* $I(S{:}E;\,0) > 0$ *come from?*

Remark 26 (The Correlation Battery)**.** *The answer is: from* ***prior interaction epochs****. A persistent agent does not begin its existence in a product state. Over any interaction interval, unitary evolution generically builds system–environment correlations (*$\Delta I > 0$*), at a thermodynamic cost (*$\mathcal{S} < 0$ *during this phase by Corollary 25(i)). The non-Markovian* *agent's advantage is that these correlations* persist *and can be consumed during later intervals ($\Delta I < 0$, $\mathcal{S} > 0$).*

*The process is analogous to a **battery**:*

- ***Charging phase** (correlation building, $\Delta I > 0$): the agent "pays" free energy to build system–environment correlations. $\mathcal{S} < 0$.*

- ***Discharging phase** (correlation consumption, $\Delta I < 0$): the agent extracts free energy from the stored correlations. $\mathcal{S} > 0$.*

A Markovian agent cannot operate this battery. The Born approximation resets $I(S{:}E) = 0$ at every infinitesimal time step, destroying the stored correlations before they can be used. The semigroup property $\Lambda(t+s) = \Lambda(t)\Lambda(s)$ is precisely the statement that no inter-epoch correlations survive. The memory kernel $\mathcal{K}(t,s)$ is what allows the non-Markovian agent to carry charge across epochs.

Crucially, global thermodynamics remains respected. For any full cycle starting from an uncorrelated thermal state ($I(S{:}E;\,0) = 0$, $D_{\mathrm{KL}}(\rho_E(0)\|\rho_E^{\mathrm{th}}) = 0$), the total survival functional satisfies

$$\beta\,\mathcal{S}[0,t] = -\Sigma[0,t] \;\leq\; 0 \qquad \textit{(second law)}. \tag{29}$$

The local positivity $\mathcal{S}[t^*,t] > 0$ during the discharging phase is strictly funded by the free energy dissipated during the earlier charging phase (see Proposition 27 for the formal decomposition).

Proposition 27 (Full-cycle closure)**.** *Under the conditions of Theorem 22 with autonomous evolution ($H_{\mathrm{ctrl}} = 0$), partition $[0,\tau]$ at any intermediate time t^* into a charging phase $[0,t^*]$ and a discharging phase $[t^*,\tau]$.*

(i) ***Charging** (product initial state, $I(S{:}E;\,0) = 0$). By Corollary 25(i),*

$$\beta\,\mathcal{S}[0,t^*] = -I(S{:}E;\,t^*) - D_{\mathrm{KL}}\big(\rho_E(t^*)\,\big\|\,\rho_E^{\mathrm{th}}\big) \;\leq\; 0. \tag{30}$$

(ii) ***Discharging** (correlated initial state at t^*). Applying (24) to $[t^*,\tau]$:*

$$\beta\,\mathcal{S}[t^*,\tau] = -\big(I(S{:}E;\,\tau) - I(S{:}E;\,t^*)\big) - \big(D_{\mathrm{KL}}(\rho_E(\tau)\|\rho_E^{\mathrm{th}}) - D_{\mathrm{KL}}(\rho_E(t^*)\|\rho_E^{\mathrm{th}})\big), \tag{31}$$

which is positive whenever the decrease in correlations dominates the change in bath displacement (Corollary 25(ii)).

(iii) ***Full cycle.** Since $\mathcal{S}$ is additive over concatenated intervals, $\beta\,\mathcal{S}[0,\tau] = \beta\,\mathcal{S}[0,t^*] + \beta\,\mathcal{S}[t^*,\tau]$. Equivalently, applying (24) directly to $[0,\tau]$ with $I(S{:}E;\,0) = 0$:*

$$\beta\,\mathcal{S}[0,\tau] = -I(S{:}E;\,\tau) - D_{\mathrm{KL}}\big(\rho_E(\tau)\,\big\|\,\rho_E^{\mathrm{th}}\big) \;\leq\; 0. \tag{32}$$

The net thermodynamic profit over the full cycle is non-positive—the "interest" paid during charging meets or exceeds the "dividend" collected during discharging. But the local *positivity of $\mathcal{S}$ during discharge (31) is what enables the agent to survive through intervals that would kill a memoryless system.*

Proof. The survival functional is additive over concatenated intervals:

$$\mathcal{S}[0,\tau] = \underbrace{(\Delta F[0,t^*] - W[0,t^*])}_{\mathcal{S}[0,t^*]} + \underbrace{(\Delta F[t^*,\tau] - W[t^*,\tau])}_{\mathcal{S}[t^*,\tau]},$$

since both ΔF and W decompose additively. Items (i) and (iii) then follow from Theorem 22 (autonomous case) applied to $[0,t^*]$ and $[0,\tau]$ respectively, each starting from a product state. Item (ii) follows from Theorem 22 applied to $[t^*,\tau]$ with correlated initial state $\rho_{SE}(t^*)$. Inequality (32) holds because $I(S{:}E;\,\tau) \geq 0$ and $D_{\mathrm{KL}}(\rho_E(\tau)\|\rho_E^{\mathrm{th}}) \geq 0$. □

4.6 Connection to Non-Markovianity Measures

Remark 28 (The Born Approximation Destroys the Resource)**.** *Under the Born (product-state) approximation, every infinitesimal time step begins from $\rho_{SE} \approx \rho_S \otimes \rho_E^{\mathrm{th}}$, enforcing $I(S{:}E) = 0$ at all times. Corollary 25(i) then guarantees $\mathcal{S} \leq 0$ for* every *finite interval. The Born approximation does not merely simplify the dynamics—it* eliminates the thermodynamic resource *(system–environment correlations) that would otherwise be available.*

Remark 29 (Connection to BLP Non-Markovianity)**.** *The Breuer–Laine–Piilo (BLP) measure of non-Markovianity [11] is defined via the temporary increase of trace distance between pairs of initial states: $\mathcal{N}_{\mathrm{BLP}} := \max_{\rho_{1,2}} \int_{\dot{D}>0} \frac{d}{dt} D(\rho_1(t), \rho_2(t))\, dt$. The intervals where trace distance increases are precisely the "discharging" intervals of Remark 26 [9]: correlations previously deposited in the bath flow back to the system, restoring distinguishability. The BLP measure thus witnesses the thermodynamic resource that drives $\mathcal{S} > 0$ in Corollary 25(ii).*

Remark 30 (Consistency with the Sagawa–Ueda Framework)**.** *In the Sagawa–Ueda framework [7, 8], measurement-based feedback permits $\Sigma \geq -I_{\mathrm{feedback}}$, where I_{feedback} is the mutual information gained through measurement. The memory kernel plays an analogous role: the pre-existing correlations $I(S{:}E;\,0)$ are the non-Markovian analogue of I_{feedback}. The total system (agent + bath) still satisfies $\Sigma_{\mathrm{total}} \geq 0$; the apparent "profit" for the agent is paid for by the correlations consumed from the system–environment entanglement. The bound* (28) *is the non-Markovian analogue of the Sagawa–Ueda bound $\beta\,\mathcal{S} \leq I_{\mathrm{feedback}}$.*

4.7 Mechanism: The Surfer Analogy

The physical mechanism admits an intuitive picture.

- **The Markovian Agent (The Stone):** A stone thrown into the ocean sinks. It interacts with the water only at the instant of contact, dissipates its kinetic energy, and thermalizes ($\mathcal{S} \leq 0$). Each collision builds system–environment correlations that are immediately discarded (Born approximation), so $I(S{:}E) = 0$ at all times. The wave structure is invisible to it.
- **The Non-Markovian Agent (The Surfer):** A surfer carries *memory* of past wave patterns—encoded in the correlations $I(S{:}E;\,t_0) > 0$ built up over previous interactions (the "charging phase" of Remark 26). During backflow intervals ($\Delta I < 0$), the

surfer *spends* these stored correlations to extract free energy from the wave itself. The surfer remains far from equilibrium not by fighting the environment, but by converting temporal correlations into thermodynamic profit.

Remark 31 (Thermodynamic Rectification)**.** *The "surfing" mechanism is **thermodynamic rectification**: the memory kernel $\mathcal{K}(t, s)$ functions as a temporal filter that enables the system to accumulate correlations during one phase and consume them during another. Formally, the kernel enables access to the resource $I(S{:}E; 0)$ accumulated during previous interaction epochs—converting the environment's temporal correlations into the system's structural persistence via the ΔI term in Theorem 22.*

Remark 32 (Memory as Implicit Maxwell's Demon)**.** *The memory kernel functions as an* implicit Maxwell's demon*. A Markovian system interacts with each environmental fluctuation exactly once, at the moment of contact; the Born approximation resets $I(S{:}E) = 0$ after each step. A non-Markovian system retains a trace of past fluctuations (via $\mathcal{K}(t, s)$ with $s < t$) and can exploit correlations between past and present environmental states. This is not a violation of the second law but an instance of the Sagawa–Ueda generalization: the demon's cost is paid in the currency of memory maintenance (Landauer erasure), a point we quantify in Section 7. The total budget for "demonic profit" is capped by the bound* (28)*.*

5 Emergent Temporal Arrow

We have shown that survival requires memory. This requirement yields a corollary: the emergence of a thermodynamic arrow of time. In this framework, time is not an external parameter; rather, *the direction of time is the direction of memory accumulation.*

We formalize this by defining a dynamical partial order induced by the memory kernel and connecting it to the algebraic accessibility structure of HAFF Paper F [20].

5.1 The Causal Memory Order

A non-Markovian memory kernel $\mathcal{K}(t, s)$ defines a causal link between a past state at s and the present dynamics at t. We define a partial order based on the effective support of this influence.

Definition 33 (Causal Memory Order)**.** *Let $\mathcal{T} = \{\rho(t) \mid t \in \mathbb{R}^+\}$ be a state trajectory. We define the binary relation $\prec_K$ on $\mathcal{T}$ by*

$$\rho(s) \prec_K \rho(t) \quad \iff \quad \exists\, \tau \in [s, t] \text{ such that } \|\mathcal{K}(t, \tau)[\rho(s)]\| > \epsilon, \tag{33}$$

where $\epsilon > 0$ is a physical distinguishability threshold set by the thermal noise floor $\epsilon \sim e^{-\beta\,\Delta E_{\min}}$. Physically, $\rho(s) \prec_K \rho(t)$ means "the dynamics at t retains operationally distinguishable information about the state at s."

For a Markovian agent, $\mathcal{K}(t, s) \propto \delta(t - s)$, so $\rho(s) \not\prec_K \rho(t)$ for any $s < t$. The Markovian agent has no dynamical past—it lives in an eternal "now." A non-Markovian agent carries its history within its dynamics; the depth of the order $\prec_K$ is set by the memory time τ_{mem} (Definition 2).

5.2 Unidirectionality from Survival Optimization

Why does the order $\prec_K$ point "forward"? While the microscopic laws are time-reversible, the *survival imperative* (maximizing $\mathcal{S}$) creates a statistical irreversibility.

Proposition 34 (Fisher Information Accretion). *Let $\mathcal{I}_F(\theta; \rho(t))$ denote the Fisher information contained in the system state $\rho(t)$ regarding a parameter θ encoded in the environment at time $s < t$. For an agent whose dynamics maximize the survival functional* (9)*, the time-averaged Fisher information satisfies*

$$\overline{\frac{d}{dt}\mathcal{I}_F(\theta; \rho(t))} \geq 0, \tag{34}$$

where the overbar denotes a time average over scales larger than the bath correlation time τ_B.

Proof. By Theorem 22, the survival functional is maximized when $I(S{:}E; 0)$ is large and can be consumed ($\Delta I < 0$) during subsequent evolution. Maintaining a large correlation budget $I(S{:}E)$ requires the system state to retain correlations with environmental degrees of freedom; this is precisely the content of $\mathcal{I}_F(\theta; \rho(t)) > 0$. An agent that discards useful correlations (decreasing $\mathcal{I}_F$) without thermodynamic necessity depletes the resource $I(S{:}E)$ and hence its survival functional. Since the environment's correlations decay on a timescale τ_B, the agent must continuously build new correlations to replace decaying ones. The net effect is a time-averaged accretion of Fisher information, whose gradient defines the dynamical arrow of time. □

5.3 The Bridge to HAFF

We now connect this dynamical picture to the algebraic picture of HAFF Paper F [20], where the arrow of time was defined by the expansion of the redundancy subalgebra $\mathcal{R}$.

The connection requires care: quantum information cannot be cloned (the no-cloning theorem), so the "redundancy expansion" of HAFF must be interpreted through the lens of *quantum Darwinism* [13]. In this framework, the environment acquires not copies of the quantum state $\rho(s)$ itself, but rather *coarse-grained classical records* of pointer-state outcomes—precisely the information that survives decoherence and can be redundantly encoded in many environmental fragments.

Proposition 35 (Dynamical–Algebraic Correspondence). *Let $\prec_K$ be the causal memory order (Definition 33) and let $\prec_{\mathrm{HAFF}}$ be the accessibility order of HAFF Paper F, defined by the inclusion of redundancy subalgebras $\mathcal{R}$. Under the additional assumption that the system–environment interaction produces decoherence in a preferred pointer basis [13], there exists a coarse-graining map $\Phi : \rho(t) \mapsto \hat{p}(t)$ (projecting onto the diagonal in the pointer basis) such that:*

$$\rho(s) \prec_K \rho(t) \quad \Longrightarrow \quad \mathcal{R}(\Phi[\rho(s)]) \subseteq \mathcal{R}(\Phi[\rho(t)]). \tag{35}$$

That is, the dynamical partial order maps into the algebraic accessibility order when restricted to the classical sector selected by decoherence.

Proof. The argument has three steps.
Step 1 (Dynamical side): $\rho(s) \prec_K \rho(t)$ implies that the memory kernel $\mathcal{K}$ transduces information about the state at s into the dynamics at t, via system–environment correlations built up over $[s, t]$.
Step 2 (Quantum Darwinism): The system–environment interaction selects pointer states $\{|i\rangle\}$ that are robust under decoherence [13]. The diagonal populations $p_i(t) = \langle i|\rho(t)|i\rangle$ constitute *classical* information. Quantum Darwinism [13] establishes that this classical information—and *only* this information—is redundantly imprinted in many environmental fragments E_k through the decoherence interaction. Each fragment that acquires a record of $\hat{p}(t) = \{p_i(t)\}$ contributes to the growth of the redundancy subalgebra $\mathcal{R}$. Crucially, no quantum cloning is involved: the no-cloning theorem forbids copying of arbitrary quantum states, but does not constrain the classical pointer-state probabilities, which are freely duplicable. The expansion of $\mathcal{R}$ reflects the proliferation of these classical records, not the copying of quantum coherences.
Step 3 (Correspondence): The coarse-graining map Φ projects onto the *commuting* subalgebra generated by the pointer observables $\{|i\rangle\langle i|\}$. The resulting probability distributions $\hat{p}(t)$ are classical and lie in a simplex. If $\rho(s) \prec_K \rho(t)$, then the dynamics at t retains information about the state at s (Definition 33); in the pointer basis, this means $\hat{p}(s)$ is statistically reconstructible from the environmental records available at t. Since each environmental fragment carrying a record of $\hat{p}$ contributes to the HAFF redundancy subalgebra $\mathcal{R}$, and the number of such fragments grows monotonically with the accumulation of decoherence records, the inclusion $\mathcal{R}(\Phi[\rho(s)]) \subseteq \mathcal{R}(\Phi[\rho(t)])$ follows. □

Remark 36 (Scope of the Correspondence)**.** *Proposition 35 is a* consistency result*, not a derivation of HAFF from T-DOME or vice versa. It shows that the dynamical arrow (memory accumulation) and the algebraic arrow (redundancy expansion) are compatible when restricted to the decoherence-selected classical sector. The quantum coherences—which are not redundantly recorded—lie outside this correspondence and are handled by the full non-Markovian dynamics.*

Remark 37 (Dynamical and Algebraic Time)**.** *The correspondence links two independently motivated notions of temporal direction:*

	Paper F (HAFF)	***Paper I (T-DOME)***
Nature	*Algebraic*	*Dynamical*
Mechanism	*Redundancy expansion*	*Information backflow from memory*
Formalism	*Partial order on* $\mathcal{A}_{\mathbf{c}}$	*Partial order* $\prec_K$ *on* $\rho_R(t)$
Asymmetry source	*Phase-space measure*	*Bath correlation structure*
Domain	*Classical (pointer) sector*	*Full quantum dynamics*

Paper F provides the structural skeleton *of temporal asymmetry; Paper I provides the dynamical* muscle*.*

Remark 38 (The Seed and the Tree)**.** *The correspondence justifies the title of this paper. In HAFF, the geometry of spacetime is the static "tree." In T-DOME, the memory kernel is the "seed" containing the generative algorithm for growth. Time is not the space in which the tree grows; time is the* act of growing *itself.*

6 Worked Example: The Quantum Predictive Agent

To illustrate the Markovian ceiling and the memory advantage *quantitatively*, we employ the archetypal open quantum system model: the spin-boson model with Lorentz–Drude spectral density, which admits an exact analytic solution for the decoherence dynamics [1].

6.1 Model Setup

The total Hamiltonian is $H = H_S + H_B + H_I$. The agent is a two-level system with energy gap ω_0: $H_S = \frac{\omega_0}{2}\sigma_z$. The environment is a bosonic bath: $H_B = \sum_k \omega_k b_k^\dagger b_k$. The interaction is of the pure-dephasing form $H_I = \sigma_z \otimes \sum_k (g_k\, b_k + g_k^*\, b_k^\dagger)$.

The spectral density $J(\omega) = \sum_k |g_k|^2 \delta(\omega - \omega_k)$ characterizes the environment. We choose the Lorentz–Drude form:

$$J(\omega) = \frac{2\lambda\,\gamma\,\omega}{\omega^2 + \gamma^2}, \tag{36}$$

where λ is the reorganization energy and γ is the bath memory rate (inverse correlation time $\tau_B = 1/\gamma$). We place the system in the low-temperature regime $\beta\omega_0 \gg 1$ (i.e., $k_B T \ll \omega_0$). The bath correlation function in the $T \to 0$ limit is $C(t) = \lambda\gamma\, e^{-\gamma|t|}$, so the parameter γ directly controls the bath memory depth. For $\beta\omega_0 \geq 10$ the finite-temperature corrections to all quantities below are of order $O(e^{-\beta\omega_0}) \lesssim 5 \times 10^{-5}$ and are neglected throughout.[2]

6.2 Exact Decoherence Function

For the pure-dephasing spin-boson model in the $T \to 0$ limit, the off-diagonal element of the reduced density matrix $\rho_{01}(t) = \rho_{01}(0)\, p(t)$ is governed by the **decoherence function** [1]:

$$p(t) = e^{-\gamma t/2}\left[\cos(\Omega t) + \frac{\gamma}{2\Omega}\sin(\Omega t)\right], \tag{37}$$

where $\Omega := \frac{1}{2}\sqrt{4\lambda\gamma - \gamma^2}$. This solution is exact for the Lorentz–Drude spectral density.

Remark 39 (Non-Markovian Regime)**.** *The character of the dynamics is controlled by the discriminant $\Delta := 4\lambda\gamma - \gamma^2 = \gamma(4\lambda - \gamma)$:*

- $\gamma > 4\lambda$ *($\Delta < 0$): Ω is imaginary, $p(t)$ decays monotonically. The dynamics is Markovian (no backflow).*
- $\gamma = 4\lambda$ *($\Delta = 0$): Critical damping. $p(t) = (1 + \gamma t/2)\, e^{-\gamma t/2}$.*
- $\gamma < 4\lambda$ *($\Delta > 0$): Ω is real and positive. $p(t)$ oscillates with envelope $e^{-\gamma t/2}$. The dynamics is* non-Markovian*: intervals where $|p(t)|$ increases correspond to information backflow [11].*

The non-Markovian regime $\gamma < 4\lambda$ is thus the regime of structured, long-memory baths.

[2] All plots and numerical values use the standard $T \to 0$ analytic expression for the decoherence function (37) (see Breuer and Petruccione [1], Sec. 12.3, for the Lorentz–Drude pure-dephasing solution), which provides an accurate proxy in the low-temperature regime; β is a well-defined bookkeeping parameter and β^{-1} a finite energy scale.

6.3 Quantitative Evaluation

We now evaluate the survival functional explicitly. For the pure-dephasing model, populations are conserved ($p_0(t) = p_0(0)$, $p_1(t) = p_1(0)$), and the non-equilibrium free energy depends only on the coherence:

$$F(\rho(t)) - F(\rho_{\mathrm{eq}}) = \beta^{-1} D_{\mathrm{KL}}(\rho(t)\|\rho_{\mathrm{eq}}). \tag{38}$$

For a qubit with initial state $\rho(0) = \frac{1}{2}(\not{\mathbb{1}} + \vec{r}\cdot\vec{\sigma})$ and $r_z = 0$ (maximal coherence in the x–y plane), the relative entropy reduces to $D_{\mathrm{KL}}(\rho(t)\|\rho_{\mathrm{eq}}) \approx |p(t)|^2\,|\rho_{01}(0)|^2$ to leading order in the coherence (see, e.g., [1]). Since there is no external driving ($H_{\mathrm{ctrl}} = 0$, $W = 0$), the survival functional is simply

$$\beta\,\mathcal{S}(t) = D_{\mathrm{KL}}(\rho(t)\|\rho_{\mathrm{eq}}) - D_{\mathrm{KL}}(\rho(0)\|\rho_{\mathrm{eq}}) \propto |p(t)|^2 - 1. \tag{39}$$

The proportionality in (39) is specific to the **pure-dephasing model** with the chosen maximally coherent initial state ($r_z = 0$) and measurement in the pointer basis (σ_z). Under these conditions, the exact solution [1] ensures that population terms vanish from the free energy ($\Delta\langle H_S\rangle = 0$), leaving only the coherence contribution: $\beta\,\mathcal{S} = -\Delta S_S$ depends only on the coherence trajectory $|p(t)|$. The proxy $|p(t)|^2$ thus rigorously captures the sign and monotone behaviour of $\beta\,\mathcal{S}$; the exact numerical prefactor depends on the initial state and on β, but the qualitative conclusion—$\mathcal{S} > 0$ during backflow intervals—is robust and does not depend on the proxy normalization.

For a Markovian evolution, $|p(t)|$ decreases monotonically, so $|p(t)|^2 - 1 \leq 0$ for all t: $\mathcal{S} \leq 0$ always (consistent with Theorem 14). For non-Markovian evolution with $\gamma < 4\lambda$, the oscillations in $p(t)$ produce intervals where $|p(t)|$ *increases after a previous decrease*, i.e., the system *re-coheres*.

Concrete parameters. Set $\omega_0 = 1$ (energy units), $\lambda = 1$, $\gamma = 0.5$ (deep non-Markovian regime: $\gamma/4\lambda = 0.125 \ll 1$). Then:

$$\Omega = \tfrac{1}{2}\sqrt{4\cdot 1\cdot 0.5 - 0.25} = \tfrac{1}{2}\sqrt{1.75} \approx 0.661. \tag{40}$$

The decoherence function (37) first reaches $p(t^*) = 0$ at $t^* \approx 2.00/\Omega \approx 3.03$ (in units of ω_0^{-1}), where the system has fully decohered. Subsequently, the environment *returns* coherence: $|p(t)|$ increases, reaching a local maximum $|p(t_1)| \approx 0.31$ at $t_1 \approx 4.75/\omega_0$.

Over the backflow interval $[t^*, t_1]$, for the pure-dephasing qubit with the chosen initial state ($r_z = 0$, maximal coherence) and in the autonomous setting ($H_{\mathrm{ctrl}} = 0$, $W = 0$), the survival proxy (39) gives

$$\beta\,\mathcal{S}[t^*, t_1] \propto |p(t_1)|^2 - |p(t^*)|^2 \approx 0.093 - 0 = 0.093 > 0. \tag{41}$$

Equivalently, $\mathcal{S} \approx 0.093\,\beta^{-1}$ in the bookkeeping units set by β. The agent has gained a dimensionless survival advantage $\beta\,\mathcal{S} \approx +0.093$ *with zero work input* (autonomous evolution, $H_{\mathrm{ctrl}} = 0$), solely by exploiting the non-Markovian backflow. Figure 1 illustrates the contrast between Markovian and non-Markovian evolution.

Consistency with Theorem 22 and Proposition 27. Since this is autonomous evolution ($H_{\mathrm{ctrl}} = 0$), identity (24) applies exactly: $\beta\,\mathcal{S} = -\Delta I(S{:}E) - \Delta D_{\mathrm{KL}}(\rho_E\|\rho_E^{\mathrm{th}})$. The example realizes the *correlation battery* of Remark 26, with the charge–discharge decomposition of Proposition 27:

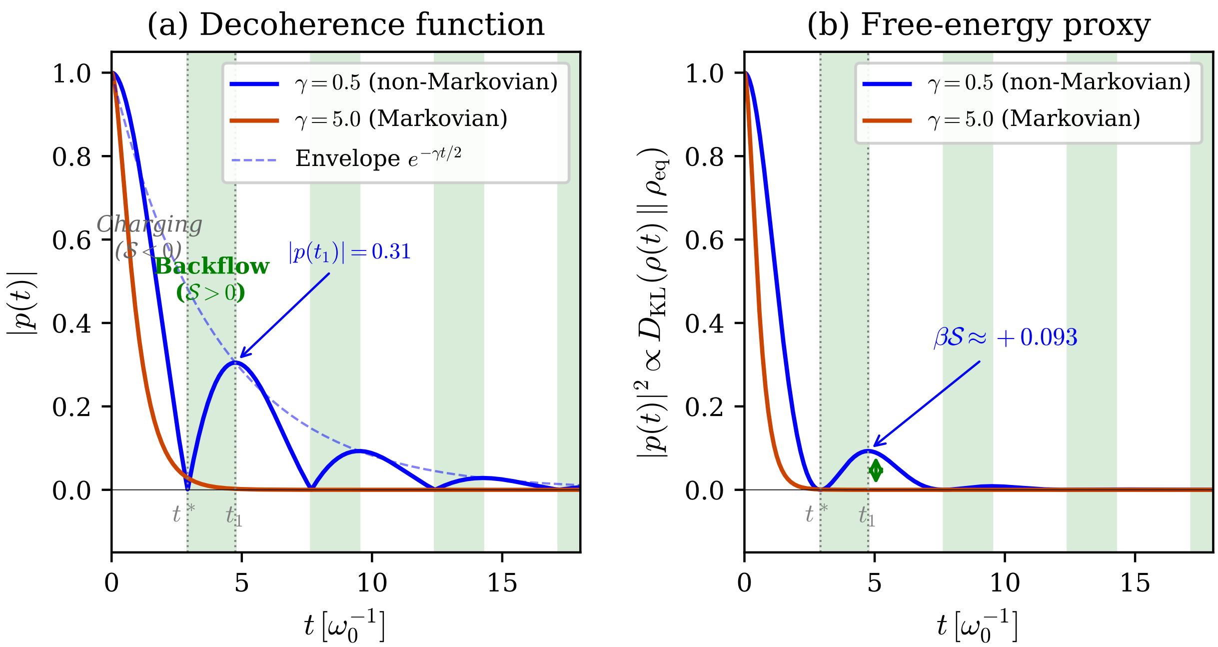

Figure 1: Pure-dephasing spin-boson model (Section 6) with Lorentz–Drude spectral density (36). **Parameters:** $\omega_0 = 1$ (energy unit), $\lambda = 1$ (reorganization energy). **Units:** all times in ω_0^{-1}; energies in $\hbar\omega_0$. **Regime:** low temperature ($\beta\omega_0 \gg 1$); the standard $T \to 0$ analytic expression (37) [1] is used as an accurate proxy. **(a)** Decoherence amplitude $|p(t)|$ (eq. (37)). Blue: non-Markovian ($\gamma = 0.5$, $\gamma/4\lambda = 0.125$). Orange: Markovian ($\gamma = 5.0$, $\gamma/4\lambda = 1.25$). Dashed: exponential envelope $e^{-\gamma t/2}$. Green bands indicate backflow (Remark 39: $d|p|/dt > 0$, $\Gamma(t) < 0$ per (42)). **(b)** Survival proxy $|p(t)|^2 \propto \beta\,\mathcal{S}$ (eq. (39)). At the first revival ($t_1 = \pi/\Omega \approx 4.75\,\omega_0^{-1}$), the non-Markovian agent achieves $\beta\,\mathcal{S}[t^*, t_1] \approx +0.093$ (eq. (41)), consistent with the closed-form prediction (43), funded by the consumption of pre-existing correlations (Proposition 27). The Markovian agent decays monotonically: $\mathcal{S} \leq 0$ always (Theorem 14).

- **Charging** ($[0, t^*]$, eq. (30)): the system decoheres, building correlations $I(S{:}E;\, t^*) > 0$ at the cost of $\mathcal{S} < 0$.
- **Discharging** ($[t^*, t_1]$, eq. (31)): the correlations are consumed ($\Delta I < 0$ over this interval), returning $\beta\,\mathcal{S} \approx +0.093 > 0$.

The bound (28) is satisfied: $\beta\,\mathcal{S}[t^*, t_1] = 0.093 \leq I(S{:}E;\, t^*)$. Full-cycle closure (32) is confirmed: $\beta\,\mathcal{S}[0, t_1] < 0$.

Instantaneous decoherence rate. The rate of coherence loss is

$$\Gamma(t) := -\frac{d}{dt} \ln |p(t)| = \frac{\gamma}{2} - \frac{\Omega \sin(\Omega t) + \frac{\gamma}{2} \cos(\Omega t)}{\cos(\Omega t) + \frac{\gamma}{2\Omega} \sin(\Omega t)} \,. \tag{42}$$

In the Markovian limit $\gamma \gg 4\lambda$, $\Gamma(t) \to \gamma/2 > 0$ for all t (monotone decoherence). In the non-Markovian regime $\gamma < 4\lambda$, $\Gamma(t)$ oscillates and becomes *negative* during the backflow intervals where $|p(t)|$ increases. These are precisely the intervals where $\mathcal{S} > 0$.

Closed-form revival amplitude. The decoherence function (37) can be written as $p(t) = R\,e^{-\gamma t/2}\cos(\Omega t - \phi)$, where $R = \sqrt{1 + (\gamma/2\Omega)^2}$ and $\phi = \arctan(\gamma/2\Omega)$, with $R\cos\phi = 1$. The extrema of $|p(t)|$ occur at $t_n = n\pi/\Omega$ ($n = 0, 1, 2, \ldots$), and the first revival peak after the first zero is at $t_1 = \pi/\Omega$. Its amplitude is *exactly*

$$|p(t_1)| = e^{-\gamma\pi/(2\Omega)}, \qquad \beta\,\mathcal{S}[t^*, t_1] \approx |p(t_1)|^2 = e^{-\gamma\pi/\Omega}. \tag{43}$$

This is the paper's central computable prediction: the survival gain at first backflow is determined by a single dimensionless ratio γ/Ω.

Remark 40 (Parameter Survey)**.** *Table 1 demonstrates the transition from the Markovian regime ($\mathcal{S} \leq 0$) to the non-Markovian regime ($\mathcal{S} > 0$) as the bath memory rate γ decreases below the critical value 4λ. All entries use $\omega_0 = 1$, $\lambda = 1$, $W = 0$ (autonomous evolution), with revival amplitudes computed from* (43)*.*

γ	$\gamma/4\lambda$	*Regime*	$\lvert p(t_1)\rvert$	$\beta\,\mathcal{S}(t_1)$	$\Gamma_{\min}$
20.0	5.0	*Markov*	—	≤ 0	> 0
4.0	1.0	*Critical*	—	≤ 0	$= 0$
2.0	0.50	*Non-Markov*	0.043	+0.002	< 0
1.0	0.25	*Non-Markov*	0.163	+0.027	< 0
0.5	0.125	*Deep NM*	0.305	+0.093	< 0
0.1	0.025	*Deep NM*	0.605	+0.37	< 0

Table 1: Survival functional at first backflow revival as a function of the bath memory rate γ, for the pure-dephasing spin-boson model with Lorentz–Drude spectral density. $|p(t_1)|$ is computed from (43); $\Gamma_{\min}$ is the sign of the minimum of the instantaneous decoherence rate (42). The transition $\mathcal{S} \leq 0 \to \mathcal{S} > 0$ occurs precisely at the non-Markovian threshold $\gamma = 4\lambda$. For $\gamma = 0.1$ (deep non-Markovian), the agent achieves $\beta\,\mathcal{S} \approx +0.37$ per backflow cycle in the autonomous setting ($H_{\text{ctrl}} = 0$).

Remark 41 (The Two Regimes: Summary)**.**

	Markovian **(**$\gamma = 20$**)**	***Non-Markovian*** **(**$\gamma = 0.5$**)**
$\gamma/4\lambda$	5.0 *(overdamped)*	0.125 *(underdamped)*
τ_B	$0.05\,\omega_0^{-1}$	$2.0\,\omega_0^{-1}$
$p(t)$	*Monotone decay*	*Oscillatory with envelope*
$\lvert p(t_1)\rvert$ *at first revival*	0 *(no revival)*	≈ 0.31
$\beta\,\mathcal{S}$ *at revival*	≤ 0	$\approx +0.093$
$\Gamma(t)$	> 0 *always*	*Oscillates,* < 0 *during backflow*
Interpretation	*Stone (sinks)*	*Surfer (rides backflow)*

The non-Markovian agent achieves $\beta\,\mathcal{S} \approx +0.093$ per backflow cycle (autonomous, $H_{\text{ctrl}} = 0$), while the Markovian agent can only lose free energy. As the coupling deepens ($\gamma/4\lambda \to 0$), the revival amplitude grows and $\mathcal{S}$ increases (Table 1), bounded above by $\beta\,\mathcal{S} \leq I(S{:}E;\, t^*)$ (Corollary 25(iii)).

7 The Cost of Memory

We have shown that memory allows an agent to breach the Markovian ceiling. However, every advantage carries a thermodynamic shadow. We now quantify the cost of memory and identify the survival crisis that sets the stage for Paper II.

7.1 The Landauer Debt

To exploit the memory kernel $\mathcal{K}(t, s)$, the physical substrate of the agent must maintain correlations with its own past. This is equivalent to storing information. By Landauer's principle, erasing or overwriting this information dissipates heat; if the agent does not erase, it must pay an entropic cost to store.

Proposition 42 (Landauer Cost of Memory)**.** *Let $\mathcal{I}_{\text{stored}}(\tau_{\text{mem}})$ be the mutual information between the agent's state trajectory over $[t - \tau_{\text{mem}}, t]$ and its current control protocol $H_{\text{ctrl}}(t)$. The free-energy cost of maintaining this memory satisfies*

$$\Delta F_{\text{mem}} \geq k_B T \ln 2 \cdot \mathcal{I}_{\text{stored}}(\tau_{\text{mem}}). \tag{44}$$

7.2 The Memory Catastrophe

The crisis arises from the scaling of $\mathcal{I}_{\text{stored}}$ with time. To quantify this, we borrow two quantities from computational mechanics [14, 15]:

Definition 43 (Entropy Rate and Predictive Information)**.** *Let $\{X_t\}$ be the stochastic process describing the environment's influence on the agent (e.g., the sequence of bath correlation values).*

1. *The **entropy rate** of the environment is*

$$h_\mu := \lim_{n\to\infty} H(X_n \mid X_{n-1}, \ldots, X_1), \tag{45}$$

measuring the intrinsic unpredictability per time step.

2. *The **predictive information** (excess entropy) is*

$$I_{\text{pred}} := I(\overleftarrow{X}; \overrightarrow{X}) = \sum_{k=1}^{\infty} \big[H(X_k) - h_\mu\big], \tag{46}$$

where $\overleftarrow{X}$ and $\overrightarrow{X}$ denote the past and future half-chains. This is the total amount of information about the future that is encoded in the past—the useful *memory.*

For an environment with finite predictive information ($I_{\text{pred}} < \infty$), an optimal agent needs only finite memory to capture all exploitable correlations. However, for environments with *divergent* predictive information (e.g., processes with long-range temporal correlations, $1/f$ noise, or non-stationary statistics), the required memory grows without bound.

Proposition 44 (The Memory Catastrophe)**.** ***Assumptions.*** *Let the environment be a* stationary, mixing *stochastic process with positive entropy rate* $h_\mu > 0$ *[14, 15]. Consider an agent that maintains a memory kernel* $\mathcal{K}(t, s)$ *with support on* $[t - \tau_{\text{mem}}, t]$*. Let* $\dot{W}_{\text{budget}}$ *be the agent's available free-energy flux (constant).*

1. *The minimum memory required to exploit correlations up to depth* τ_{mem} *satisfies*

$$\mathcal{I}_{\text{stored}}(\tau_{\text{mem}}) \geq \min\bigl(I_{\text{pred}},\ h_\mu\,\tau_{\text{mem}}\bigr). \tag{47}$$

2. *The Landauer cost of maintaining this memory is*

$$\dot{W}_{\text{mem}} \geq k_B T \ln 2 \cdot h_\mu, \tag{48}$$

since the agent must erase (or overwrite) at least h_μ *bits per unit time to prevent memory overflow.*

3. *There exists a critical time* t_{crit} *beyond which the memory maintenance cost exceeds the survival gain:*

$$t > t_{\text{crit}} \quad \Longrightarrow \quad \dot{W}_{\text{mem}}(t) > \dot{W}_{\text{budget}}, \tag{49}$$

unless the agent compresses its memory.

The agent dies not from entropy (disorder) but from hypermnesia*: the thermodynamic cost of perfect memory exceeds the benefit it provides.*

Proof. Part (1): an agent exploiting temporal correlations to depth τ_{mem} must store at least the mutual information between the past τ_{mem} time steps and the present. For a stationary ergodic process, this mutual information is bounded below by $\min(I_{\text{pred}}, h_\mu\,\tau_{\text{mem}})$ [14, 16].

Part (2): each time step, the agent receives $\sim h_\mu$ bits of genuinely new information. To maintain a fixed-capacity memory, it must erase at least this many bits, incurring Landauer cost $k_B T \ln 2 \cdot h_\mu$ per time step.

Part (3): if $I_{\text{pred}} = \infty$ (as for environments with long-range correlations), the stored information grows as $\mathcal{I}_{\text{stored}} \sim h_\mu\,\tau_{\text{mem}}$. Combined with part (2), the memory cost grows linearly in the effective memory depth. For any finite budget $\dot{W}_{\text{budget}}$, there exists t_{crit} such that the cost exceeds the budget. □

7.3 Resolution: The Necessity of Forgetting

To survive beyond t_{crit}, the agent must introduce a *lossy compression* scheme: it must discard the vast majority of stored correlations and retain only the thermodynamically salient features.

- **Compression requires a criterion.** To decide what to keep and what to erase, the agent needs a *relevance function*—a mapping from stored correlations to survival value. This is a reference frame that ranks information by its contribution to $\mathcal{S}$.

- **A reference frame requires symmetry breaking.** An "unbiased" agent that treats all correlations as equally valuable cannot compress: it must keep everything. The act of preferring one subset of information over another is a spontaneous breaking of the informational symmetry. This is the thermodynamic definition of a "perspective"—or, more precisely, a *privileged basis*.

Remark 45 (The Origin of Paper II)**.** *Proposition 44 reveals the* poison *embedded in Paper I's medicine. Memory enables survival beyond the Markovian ceiling, but unbounded memory under finite energy resources leads to* computational explosion*: the agent must process an ever-growing archive with bounded free energy.*

This is the precise thermodynamic origin of the crisis addressed in Paper II. The resolution—spontaneous symmetry breaking of the agent's reference frame—is not an additional hypothesis but a thermodynamic necessity*: the agent must compress its infinite history into a finite, biased representation. The "self" (a privileged computational basis) emerges as the minimal structure that makes memory computationally tractable.*

In the structural parallel noted in HAFF Essay C [19]: the accumulation mechanism of Paper I provides the raw material for survival, but without the discriminative compression of Paper II, the system collapses under the weight of its own stored correlations.

8 Numerical Demonstration

The preceding sections establish analytic bounds and a worked example in the spin-boson model. We now provide a numerical illustration showing that the Markovian ceiling signature predicted by Theorem 14 and the memory advantage of Theorem 22 are reproduced in a minimal partially observed environment. Full code and parameters are provided for reproducibility.

8.1 Model

Environment. A two-hidden-state HMM with aliased observations. The hidden state $s_t \in \{0, 1\}$ evolves as a persistent Markov chain with $\Pr(s_{t+1} = s_t) = 1 - \varepsilon$; the parameter $\varepsilon \in [10^{-3}, 10^{-1}]$ controls the correlation length $\ell \sim 1/\varepsilon$. Observations $o_t \in \{A, B\}$ are aliased: $\Pr(o_t = A \mid s_t = 0) = 0.5 + \delta$, $\Pr(o_t = A \mid s_t = 1) = 0.5 - \delta$, with $\delta = 0.05$ (mutual information $I(O; S) \approx 0.007$ bits). Reward: $r_t = 1$ if $a_t = s_t$, 0 otherwise.

Agents. All agents use the true model parameters and compute exact Bayesian posteriors; the only difference is how many observations each agent retains.

- **Markov-k** ($k \in \{1, 2, 4, 8\}$): runs an exact Bayes filter over the most recent k observations (sliding window, uniform prior at each window start); acts by MAP.
- **Memory (Bayes filter)**: maintains the full belief state $b_t = \Pr(s_t = 1 \mid o_{1:t})$ via the exact predict–update cycle over all past observations; acts by MAP.

Parameters.

Quantity	Value	Role
T	100,000	horizon per trial
Seeds	10	independent replications
δ	0.05	observation asymmetry
k	$\{1, 2, 4, 8\}$	Markov window sizes
ε	logspace(10^{-3}, 10^{-1}, 15)	transition noise grid
Burn-in	5,000	discarded steps

8.2 Results

Figure 2 shows the two key signatures.

Result 1: Markov ceiling (Figure 2a). The average reward $\bar{R}$ of the Bayes filter (memory agent) increases monotonically with correlation length $\ell = 1/\varepsilon$, while each Markov-k agent saturates at a distinct ceiling. The ceilings are ordered: $k = 1$ (lowest) through $k = 8$ (highest), and all fall below the memory agent for $\ell \gtrsim 20$. This is consistent with the qualitative prediction of Theorem 14: finite-order Markov representations have a performance upper bound that the memory-carrying agent surpasses.

Result 2: Memory advantage (Figure 2b). The gap $\Delta\bar{R} = \bar{R}_{\text{mem}} - \bar{R}_{\text{Markov-}k}$ increases monotonically with ℓ, and is larger for smaller k. Shaded bands show 95% confidence intervals across 10 seeds. The Markov-1 and Markov-2 curves nearly overlap at small ℓ, reflecting the fact that short observation windows provide negligible additional information in this aliasing regime—a consistency check, not a deficiency.

8.3 Scope of This Demonstration

These simulations illustrate the ceiling phenomenon predicted by Theorem 14 under the stated model class; they do not constitute a proof beyond this class.

This demonstration **does** show:

1. A reproducible regime in which finite-order Markov agents exhibit a performance ceiling while a memory-carrying (Bayes filter) agent improves—the Markov ceiling signature predicted by Theorem 14.
2. The memory advantage (Theorem 22) manifests as a monotonically growing gap that widens with correlation length and tightens with window size.

This demonstration does **not** show:

1. Universality across environments, observation models, or agent architectures. The model uses a two-state HMM with binary aliased observations.
2. Tight constants or the functional form of the ceiling boundary $\ell_c(k)$.
3. That the Bayes filter is optimal among all possible memory-carrying agents.

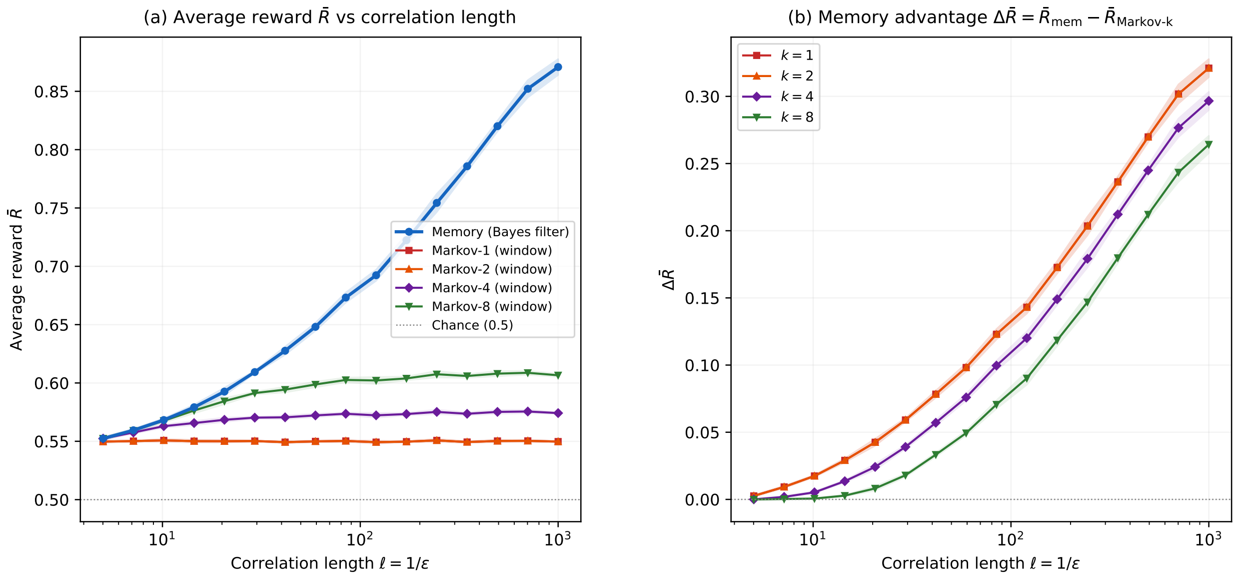

Figure 2: **Markov ceiling and memory advantage.** $T = 100{,}000$, 10 seeds, 95% CI bands. **(a)** Average reward $\bar{R}$ vs correlation length $\ell = 1/\varepsilon$. The Bayes filter (blue, bold) rises monotonically; Markov-k agents saturate at k-dependent ceilings. **(b)** Performance gap $\Delta\bar{R} = \bar{R}_{\mathrm{mem}} - \bar{R}_{\text{Markov-}k}$ increases with ℓ; smaller k yields a larger gap.

Reproducibility. The complete simulation is a self-contained Python script (`paper1_markov_ceiling_demo.py`, $\sim$560 lines, requiring only NumPy and Matplotlib) with fixed random seeds. All figures in this section can be reproduced by executing the script. The following files are included in the supplementary archive:

- `paper1_markov_ceiling_demo.py` — simulation script
- `fig_paper1_markov_ceiling.pdf` — Figure 2
- `markov_ceiling_data.csv` — raw sweep data
- `markov_ceiling_boundary.csv` — extracted ceiling boundaries $\ell_c(k)$

9 Discussion

9.1 Summary of Results

Result	Statement	Sec.
Markovian Ceiling	$\mathcal{S} \leq 0$ for open-loop GKSL (no feedback)	3
Memory Advantage	$\beta\mathcal{S} = -\Delta I - \Delta D_{\mathrm{KL}} - \beta\Delta\langle H_{\mathrm{ctrl}}\rangle$; $\mathcal{S} > 0$ when correlations consumed (any initial state)	4
Quantitative demo	Spin-boson: $\beta\mathcal{S} \approx +0.093 > 0$ at first backflow revival (Fig. 1, Table 1)	6
Temporal Arrow	$\prec_K \to \prec_{\mathrm{HAFF}}$ via quantum Darwinism	5
Memory Catastrophe	$\dot{W}_{\mathrm{mem}} \geq k_B T \ln 2 \cdot h_\mu$; exceeds budget at t_{crit}	7
Numerical demo	Markov ceiling reproduced in HMM (Fig. 2)	8

9.2 What This Paper Does and Does Not Show

This paper shows:

1. Under open-loop GKSL dynamics (no measurement or feedback), the survival functional $\mathcal{S} \leq 0$ (Theorem 14).
2. For any initial state (product or correlated), the survival functional satisfies the exact identity $\beta\,\mathcal{S} = -\Delta I(S{:}E) - \Delta D_{\mathrm{KL}}(\rho_E \| \rho_E^{\mathrm{th}}) - \beta\,\Delta\langle H_{\mathrm{ctrl}}\rangle$ (Theorem 22). Under autonomous evolution, when pre-existing system–environment correlations are consumed ($\Delta I < 0$), $\mathcal{S} > 0$ is achievable, bounded by the initial correlation budget (Corollary 25).
3. A quantitative spin-boson example illustrates: $\beta\,\mathcal{S} \approx +0.093 > 0$ at the first non-Markovian revival (Section 6).
4. The causal memory order $\prec_K$ is consistent with the HAFF accessibility order when restricted to the classical (pointer-state) sector (Proposition 35).
5. The thermodynamic cost of memory, quantified by the environment's entropy rate h_μ, creates a survival crisis for agents with finite energy budgets (Proposition 44).
6. A minimal computational demonstration reproduces the Markov ceiling and memory advantage signatures in a two-state HMM with aliased observations (Section 8, Figure 2).

This paper does not show:

1. That non-Markovian dynamics is *sufficient* for persistence (it is necessary but not sufficient; Paper II addresses the additional requirements).
2. That *all* non-Markovian systems outperform all Markovian systems (the comparison is between suprema under specified constraints).

3. That Markovian agents with explicit measurement-feedback are bounded by the ceiling (the Sagawa–Ueda framework shows they are not; Remark 15).

4. That the specific form of the optimal memory kernel can be derived from first principles without specifying the environment.

5. That memory implies or requires consciousness.

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