

# The Realizability Bridge: Algebraic Closure in the Q-RAIF Framework

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## Abstract

This addendum provides a minimal mathematical bridge between the two foundational papers of the **Q-RAIF** (**Quantum Reference Algebra for Information Flow**) framework. Paper A [1] establishes that the observable algebra of a holographically consistent universe must contain  $Cl(1, 3)$  as its minimal Clifford-compatible structure. Paper B [2] establishes that the control algebra of a persistent subsystem must be Cliffordian  $Cl(V, q)$  to ensure Lyapunov stability under entropic constraints.

Here we prove the **Closure Theorem**: any *physically realizable* control algebra must embed into the environmental algebra as a subalgebra. We formalize the required feedback synchrony via a *Same-Clock* co-indexing lemma, ensuring the feedback loop is thermodynamically potent.

This note does not modify Papers A or B; it supplies only the realizability bridge needed for algebraic closure.

**Keywords:** Q-RAIF, realizability, representation, operator algebra, Clifford algebra, open quantum systems, Lyapunov stability, algebraic closure

## 1 Introduction

### 1.1 Context: The Q-RAIF Program

The Quantum Reference Algebra for Information Flow (Q-RAIF) framework investigates what algebraic structures are *necessary*—as opposed to merely convenient—for the self-consistent description of physical reality and persistence within it. The program builds on the Holographic Alaya-Field Framework (HAFF) [3, 4], which establishes that geometry emerges from coarse-graining of observable algebras.

Paper	Question	Analogy	Result
HAFF [3]	How does geometry emerge?	Ocean	$\text{Algebra} \rightarrow \text{Geometry}$
Q-RAIF A [1]	What algebra does geometry need?	Water	$\text{Cl}(1, 3)$
Q-RAIF B [2]	What algebra does survival need?	Fish	$\text{Cl}(V, q)$
This work	Must the fish fit the water?	Bridge	$\text{Cl}(V, q) \hookrightarrow \text{Cl}(1, 3)$

## 1.2 The Logical Gap

Papers A and B independently arrive at Clifford algebra from opposite directions. Both papers explicitly note that this convergence is *heuristic rather than deductive* [1, 2]. The present note closes the gap by proving a realizability constraint: the internal control algebra of any persistent subsystem must be representable within the external observable algebra.

## 1.3 Scope

This addendum introduces no new physical assumptions. It uses only the objects and results already established in Papers A and B, and derives their mutual constraint. Papers A and B remain unmodified.

## 2 Setup and Prerequisites

Let  $\mathcal{U}$  be a universe described by the Q-RAIF framework.

- **Environment (“water”).** Let  $\mathcal{A}_{\text{ext}}$  denote the algebra of observables accessible at the holographic boundary. Paper A [1] argues that  $\mathcal{A}_{\text{ext}}$  must contain  $\text{Cl}(1, 3)$  as its minimal Clifford-compatible subalgebra (Theorem 1 of Paper A, “Clifford Compatibility”).
- **Subsystem (“fish”).** Let  $\mathcal{O}_{\text{int}}$  denote the internal control algebra of a persistent subsystem  $R \subset \mathcal{U}$ . Paper B [2] argues that thermodynamic persistence requires  $\mathcal{O}_{\text{int}} \cong \text{Cl}(V, q)$  for some  $(V, q)$  (Theorem 1 of Paper B, “Persistence Compatibility”).

The remaining logical gap is the relationship between  $\mathcal{O}_{\text{int}}$  and  $\mathcal{A}_{\text{ext}}$ : can a stable Clifford control algebra exist while being structurally disjoint from the available environmental observables?

## 3 Realizability and Same-Clock Co-Indexing

**Definition 1** (Algebraic Realizability). *A control algebra  $\mathcal{O}_{\text{int}}$  is **physically realizable** within an environment  $\mathcal{A}_{\text{ext}}$  if there exists a homomorphism*

$$\phi : \mathcal{O}_{\text{int}} \rightarrow \mathcal{A}_{\text{ext}} \tag{1}$$

*such that  $\text{Im}(\phi)$  has non-zero action on the interaction Hamiltonian  $H_{\text{int}}$ , i.e.,  $[\text{Im}(\phi), H_{\text{int}}] \neq 0$ . This ensures that the controller can physically influence the system-environment boundary.*

Let  $I$  be an operational/causal index set (e.g., proper-time frames or discretized event slices). For a subset  $J \subseteq I$ , write  $\mathcal{A}|_J$  for the restriction of an algebra  $\mathcal{A}$  to the index set  $J$ .

**Lemma 2** (Same-Clock / Co-Indexing). *For a feedback loop to be causally closed and thermodynamically potent (capable of entropy export [8]), there must exist non-null index overlap between control and feedback windows: there exist  $J_{\text{ctrl}}, J_{\text{env}} \subseteq I$  such that*

1. **Non-null intersection:**  $J_{\text{ctrl}} \cap J_{\text{env}} \neq \emptyset$ .
2. **Window integrity:** on any critical lookback window  $W \subseteq J_{\text{ctrl}} \cap J_{\text{env}}$  used to define the controller,  $\mathcal{A}_{\text{ext}}|_W$  is well-defined (no holes on  $W$ ).

*Proof.* If  $J_{\text{ctrl}} \cap J_{\text{env}} = \emptyset$ , the control action is operationally decoupled from environmental feedback, so no entropy export channel exists; persistence (NESS [8]) fails. If window integrity fails on a critical lookback window  $W$ , the feedback map—and thus the Lyapunov descent condition (Eq. (4) of Paper B [2])—is not definable on the operational window. Therefore both conditions are necessary.  $\square$

### 3.1 Semisimplicity of Clifford Algebras

**Lemma 3** (Injectivity from Semisimplicity). *Let  $(V, q)$  be a finite-dimensional real vector space with non-degenerate quadratic form. Then  $Cl(V, q)$  is semisimple. If  $\dim V$  is even,  $Cl(V, q)$  is simple, and every non-zero algebra homomorphism  $\phi : Cl(V, q) \rightarrow \mathcal{A}$  is injective.*

*Proof.* By the periodicity theorem for real Clifford algebras [5, 7],  $Cl(V, q)$  with non-degenerate  $q$  is isomorphic to a matrix algebra  $M_n(K)$  or a direct sum  $M_n(K) \oplus M_n(K)$ , where  $K \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$  depends on the signature and dimension modulo 8. In either case the algebra is semisimple.

When  $\dim V$  is even,  $Cl(V, q)$  is simple (a single matrix block). The kernel of any algebra homomorphism is a two-sided ideal; a simple algebra admits no proper non-trivial ideals, so  $\ker \phi = \{0\}$  whenever  $\phi \neq 0$ .  $\square$

## 4 The Closure Theorem

**Theorem 4** (Q-RAIF Algebraic Closure). *Assume  $\mathcal{A}_{\text{ext}} \supseteq Cl(1, 3)$  (Paper A [1]). Let  $R$  be a persistent subsystem whose control algebra satisfies  $\mathcal{O}_{\text{int}} \cong Cl(V, q)$  (Paper B [2]). If  $\mathcal{O}_{\text{int}}$  is realizable in  $\mathcal{A}_{\text{ext}}$  (Definition 1) and the Same-Clock conditions of Lemma 2 hold, then the effective control algebra*

$$\mathcal{O}_{\text{eff}} := \text{Im}(\phi) \subseteq \mathcal{A}_{\text{ext}} \tag{2}$$

*is a Clifford subalgebra of the external geometry.*

*Proof.* By realizability, there exists a homomorphism  $\phi : \mathcal{O}_{\text{int}} \rightarrow \mathcal{A}_{\text{ext}}$  with non-trivial image. The operational content of the controller is its image  $\mathcal{O}_{\text{eff}} = \text{Im}(\phi)$ . Since  $\mathcal{O}_{\text{int}} \cong Cl(V, q)$  by the persistence requirement (Theorem 1 of Paper B), and  $\phi$  is structure-preserving,  $\mathcal{O}_{\text{eff}}$  inherits the Clifford relations  $v^2 = q(v)\mathbf{1}$  [6]. Moreover, by Lemma 3, if  $\dim V$  is even then  $Cl(V, q)$  is simple and  $\phi$  is necessarily injective; the image is therefore isomorphic to

$Cl(V, q)$  itself, giving a genuine embedding  $Cl(V, q) \hookrightarrow \mathcal{A}_{\text{ext}}$ . In the physically relevant case ( $\dim V = 4$ , signature  $(1, 3)$  or compatible sub-signature), the even-dimensionality condition is satisfied. Since  $\mathcal{O}_{\text{eff}} \subseteq \mathcal{A}_{\text{ext}}$ , the internal geometry  $(V, q)$  is induced by a restriction of the ambient algebraic structure.  $\square$

**Corollary 5** (No Ghost Algebra). *A control algebra that is mathematically stable (Cliffordian) but not representable in  $\mathcal{A}_{\text{ext}}$  is not physically realizable. In particular, a control structure with signature incompatible with  $(1, 3)$  cannot underwrite persistent feedback in a universe whose observable algebra contains  $Cl(1, 3)$ .*

## 5 Discussion

### 5.1 What This Result Does and Does Not Show

**Does show:** Realizability forces the internal control algebra of a persistent subsystem to embed into the external observable algebra. Combined with Papers A and B, this converts the previously heuristic convergence ( $Cl(V, q)$  from stability,  $Cl(1, 3)$  from geometry) into a constrained embedding:  $Cl(V, q) \hookrightarrow Cl(1, 3)$ .

**Does not show:** That  $\phi$  must be injective in general—however, by Lemma 3, injectivity is guaranteed when  $\dim V$  is even (which includes the physically relevant case  $\dim V = 4$ ). For odd  $\dim V$ , the image  $\text{Im}(\phi)$  is isomorphic to a simple factor of  $Cl(V, q)$  and still carries the Clifford structure. That the specific signature  $(V, q)$  is uniquely determined—only that it must be compatible with  $(1, 3)$ . That this constitutes a derivation of physics from first principles—it is a consistency constraint within the Q-RAIF framework.

### 5.2 The Bridge Statement

**Remark 6** (Closing the Loop). *Paper A fixes the realizable operator content of the world ( $\mathcal{A}_{\text{ext}}$ ). Paper B fixes the algebraic form required for persistence ( $\mathcal{O}_{\text{int}}$ ). Theorem 4 locks them together: realizable persistence forces the agent’s control algebra to be built from the same algebraic atoms as its environment. The fish’s gills must be made of water’s molecules.*

### 5.3 Connection to HAFF

Within the HAFF program [3, 4], geometry emerges from coarse-graining of observable algebras. The Closure Theorem adds a further structural consequence: not only does the world’s geometry emerge from its algebra, but any persistent subsystem’s internal geometry is constrained to be a restriction of that emergent geometry. This is algebraic natural selection operating at the level of geometric structure.

## References

- [1] S. Liu, *Algebraic Constraints on the Emergence of Lorentzian Metrics in Entropic Gravity Frameworks*, Zenodo (2026), DOI: 10.5281/zenodo.18525876.

- [2] S. Liu, *Thermodynamic Stability Constraints on the Operator Algebra of Persistent Open Quantum Subsystems*, Zenodo (2026), DOI: 10.5281/zenodo.18525890.
- [3] S. Liu, *Emergent Geometry from Coarse-Grained Observable Algebras: The Holographic Alaya-Field Framework*, Zenodo (2026), DOI: 10.5281/zenodo.18361706.
- [4] S. Liu, *Accessibility, Stability, and Emergent Geometry: Conceptual Clarifications on the Holographic Alaya-Field Framework*, Zenodo (2026), DOI: 10.5281/zenodo.18367060.
- [5] M. F. Atiyah, R. Bott and A. Shapiro, *Clifford modules*, Topology **3**, Suppl. 1, 3–38 (1964).
- [6] D. Hestenes, *Space-Time Algebra*, Gordon and Breach (1966).
- [7] H. B. Lawson and M.-L. Michelsohn, *Spin Geometry*, Princeton University Press (1989).
- [8] U. Seifert, *Stochastic thermodynamics, fluctuation theorems and molecular machines*, Rep. Prog. Phys. **75**, 126001 (2012).