Probability Questions

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Problem

You roll a fair die until you get 2. What is the expected number of rolls (including the roll given 2) performed conditioned on the event that all rolls show even numbers?

Solution

Let ω_i denote the i^{th} toss and let us toss the dice n times. Then the sequence of our toss would be $\omega = \omega_1 \dots \omega_{n-1}\omega_n$.

We already know that the n^{th} toss results in 2 i.e. $\omega_n = 2$. And we also know that each of $\omega_i \in \{2,4,6\}, i \neq n$. It is easy to see that $\mathbf{P}[\omega_i \in \{2,4,6\}, i \neq n] = \frac{1}{2}$ And $\mathbf{P}[\omega_n = 2] = \frac{1}{6}$. This gives us the probability of our sequence $\omega = \omega_1 \dots \omega_{n-1}2$. Let X be the event such that in n die rolls, the first $n-1$ rolls are even and n^{th} roll is 2.

$$\begin{aligned}
\mathbf{P}[X] &= \mathbf{P}[\omega_1 \dots \omega_{n-1}2 | \omega_i \in \{2,4,6\}, i = 1,\dots,n-1] \\
&= \mathbf{P}[\omega_1 \in \{2,4,6\}] \dots \mathbf{P}[\omega_{n-1} \in \{2,4,6\}] \cdot \mathbf{P}[\omega_n = 2] \\
&= \underbrace{\frac{1}{2} \cdots \frac{1}{2}}_{n-1} \frac{1}{6} \\
&= \frac{1}{6}\left(\frac{1}{2}\right)^{n-1} \qquad (1)
\end{aligned}$$

We can see that X is similar to *geometric random variable* i.e. n is the number of trials that have passed until we get the first success (including the trial when we got the first success). So, a better way to write the 1 is:

$$\mathbf{P}[X = n] = \frac{1}{6}\left(\frac{1}{2}\right)^{n-1} \qquad (2)$$

Now, we just need to find the expectation value. By definition, we have:

$$
\begin{aligned}
\mathbb{E}[X] &= \sum_{n=1}^{\infty} n \cdot \mathbf{P}[X = n] \\
&= \sum_{n=1}^{\infty} n \cdot \frac{1}{6} \left(\frac{1}{2}\right)^{n-1} \\
&= \frac{1}{6} \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n-1} \\
&= \frac{1}{6} \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n-1} \frac{\frac{1}{2}}{\frac{1}{2}} \\
&= \frac{1}{3} \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n} \\
&= \frac{1}{3} \frac{1}{1 - \frac{1}{2}} \\
&= \frac{2}{3}
\end{aligned} \tag{3}
$$