FIM 548 Monte Carlo Simulations and Methods

Final Research Project

Fixed Income Product Pricing - Black Derman Toy Model

Siddharth Thakur

INTRODUCTION

This report describes the final research project done for the FIM548 course. The project is aimed at pricing fixed income product (Swaption) using a short interest rate stochastic model. These models that describes the future evolution of interest rates by describing the future evolution of the short rate, usually written r_t.

This project aims to calibrate the Black Derman Toy model using current treasury yield data. The model has the following equation.

$$dlog(r_t) = \theta(t)dt \quad + \sigma dW_t$$

This is a log normal version of the Ho Lee model which is widely used in the industry.

This project first models this equation, calibrates its parameters by fitting the current zero coupon bond prices in the market. Once these are fitted/calibrated, this model can be used to project interest rates in the future. These rates can then be used to model the prices of Bonds and other interest rate derivatives.

CONTENTS OF THE REPORT

Introduction 2

Black Derman Toy Model 3

Model Calibration 3

SDE solution for Equation 3

Treasury Yields and Bootstrapping 5

Swaptions 5

Results 7

Bootstrapping results 7

Calibration results 7

Swaption Valuation 8

Appendix 9

References 9

BLACK DERMAN TOY MODEL

In 1990, Black, Derman, and Toy proposed a binomial-tree model for a lognormal short-rate process. . It can be shown that the stochastic process corresponding to the model is:

$$dlog(r_t) = (\theta(t) - a(t)\log(r_t))\ \text{dt} \quad + \sigma dW_t$$

$$a(t) = \frac{-\sigma'(t)}{\sigma(t)}$$

where $-\sigma'(t)$ is the derivative of σ with respect to t. This model has the advantage over Ho–Lee and Hull–White that the interest rate cannot become negative. The Wiener process dz can cause $\log r_t$ to be negative, but r itself is always positive. One disadvantage of the model is that there are no analytic properties. A more serious disadvantage is that the way the tree is constructed imposes a relationship between the volatility parameter σ(t) and the reversion rate parameter a(t). The reversion rate is positive only if the volatility of the short rate is a decreasing function of time.

In practice, the most useful version of the model is when σ is constant. The parameter a is then zero, so that there is no mean reversion, and the model reduces to

$$dlog(r_t) = \theta(t)dt \quad + \sigma dW_t$$

This can be characterized as a lognormal version of the Ho–Lee model.

This model is used to model the short rates and hence the future interest rates. However, due to its form, there are no analytical versions of the closed form equations that can be used to value bonds.

MODEL CALIBRATION

The Black Derman Toy model was originally introduced as a binomial tree model. In this report, the model has been implemented using the Monte Carlo method and discounted cashflow methodology. As part of this method, the model was implemented using the Euler discretization method of the closed form solution of equation.

SDE SOLUTION FOR EQUATION

Model equation

$$dlog(r_t) = \theta(t)dt \quad + \sigma dW_t$$

Assume:

$$dr_t = A(t)dt \quad + B(t)dW_t$$

Getting a solution using Ito's Lemma:

$$Let\ \ F(x,t) = Log(x)$$

Hence,

$$\frac{df(x,t)}{dt} = 0$$

$$\frac{df(x,t)}{dx} = \frac{1}{x}$$

$$\frac{d^2f(x,t)}{dx^2} = -\frac{1}{x^2}$$

Substituting these in Ito's Lemma equation:

$$df(x,t) = \frac{df(x,t)}{dx}dx + \frac{1}{2}\frac{d^2f(x,t)}{dx^2}dt + \frac{df(x,t)}{dt}dt$$

$$df(x,t) = \frac{1}{r_t^2}\left(A(t)r_t \ \ -\frac{1}{2}B(t)^2\right)dt + B(t)r_t \ \ dW_t$$

We can now compare it to the actual model equation to get the value of A(t) and B(t). We get

$$A(t) = \ r_t \ \ \left(\theta(t) + \frac{1}{2}\sigma^2\right)$$

$$B(t) = \ \sigma$$

Hence the final SDE is:

$$dr_t \quad = \quad r_t \ \ \left(\theta(t) + \frac{1}{2}\sigma^2\right)dt + \ r_t \ \ \sigma dW_t$$

The report assumes a simple linear model for $\theta(t)$

$$\theta(t) = \ \alpha \ + \ \beta * t$$

Therefore, the closed form solution for this equation would be:

$$r_t = r_0 \exp^{\left(\alpha * t + \ \beta * \frac{t^2}{2} + \frac{\sigma^2 t}{2} + \sigma r_t Z_t\right)}$$

We can use this equation and simulate short rates using Euler method of discretization.

The models can be used to generate short rates into the future which can be used to calculate the price of bonds using the discounted cashflow method.

Treasury Yields and Bootstrapping

This model is now calibrated using the zero-coupon bond prices. In order to get the market prices of ZCBs, we use the current treasury par curves. These rates are released daily by the US Fed department. They can be found here[1].

Since these are the par rates for US Treasuries, they were needed to be converted to spot rates/ZCB rates. The following method was followed for this calculation.

1. Download the Par Rate curve from the Official Website [1].
2. Interpolate the missing tenure rates from the given Par curve.
3. Use Bootstrapping to convert par rates to Spot rates/ZCB prices.

The zero-coupon yield curve can be constructed from the observed prices of coupon bearing bonds by the technique of bootstrapping – that is, we build up the pattern of zero-coupon yields that is consistent with the market prices of the conventional bonds, using linear interpolation between observed values, where necessary. Bootstrapping involves finding the zero-coupon yields recursively. Having first determined the zero-coupon yield for the bond with the shortest outstanding term, we then use that information to determine the spot yield for the next shortest bond and so on. The technique, which implicitly assumes that bond markets are arbitrage-free, is best explained by means of an example. It follows the following formula

$$\boxed{\frac{C}{(1+r)^1} + \frac{C}{(1+r)^2} + \ldots + \frac{1+C}{(1+r)^n} = \$100}$$

Where c is the par rate for term n. This is done recursively starting with term 1 and going till the desired term.

The entire calculation for the yield curve can be found in the Excel sheets present in the supplement material.

Swaptions

A swaption, also known as a swap option, refers to an option to enter an interest rate swap or some other type of swap. In exchange for an options premium, the buyer gains the right but not the obligation to enter into a specified swap agreement with the issuer on a specified future date.

A swap can be considered as a short position in the fixed-rate bond and long position in floating-rate bond (The same value will be negative for a receiver swap). This can be incorporated in the following formula:

$$Value_{swap} = Notional * \left(B(t,To) - B(t,Tn) - \left(k * Payment_{freq} * \sum_{i=1}^{n} B(t,T_i) \right) \right)$$

Where:

B(t,T): Price of bond at time t maturing at time T.

K: Swap Rate

Notional: The amount underlying the swap payments

Payment_freq: Frequency of payment exchange in the swap

Since a swaption is an option to get to this cashflow, we can consider the payoff of the swaption as :

$$Payoff\ = Notional * \left(1 - B(To,Tn) - \left(\bar{R} * Payment_{freq} * \sum_{i=1}^{n} B(To,T_i) \right) \right)^{+}$$

Where $\bar{R}$ is the forwards swap rate /strike rate. To is the term of the option and Tn is the term of the swap.

The value of the swaption can then be shown to be:

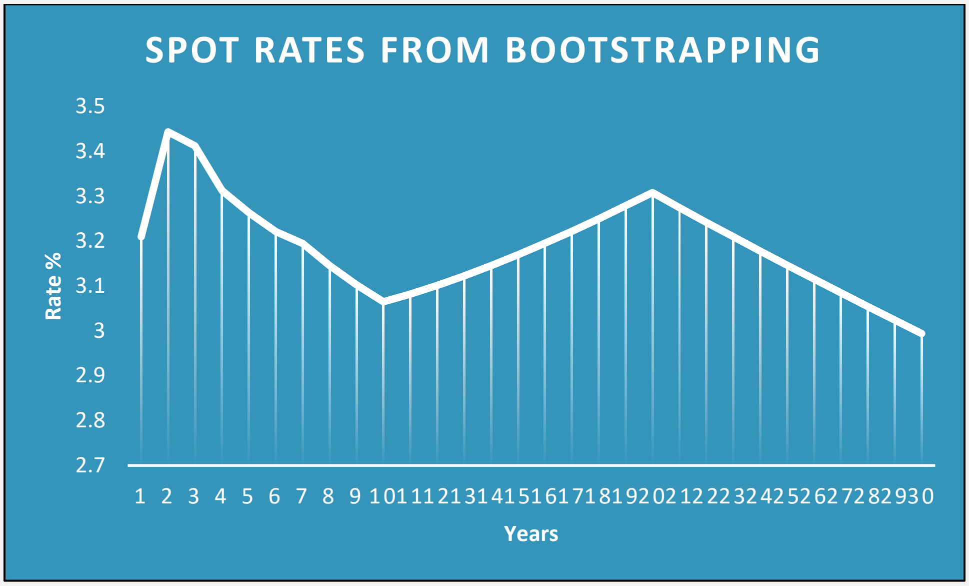
$$Price_{Swaption}\ = E\left[e^{\int_o^{To} r_t\, dt}\, Notional * \left(1 - B(To,Tn) - \left(\bar{R} * Payment_{freq} * \sum_{i=1}^{n} B(To,T_i) \right) \right)^{+} \right]$$

This can be calculated using the Monte Carlo methods learnt during the course.

Results

Bootstrapping results

The following results were generated using excel. The following can be shown in the graphs below:



The calculations for these have been shown inside the bootstrapping excel sheet.

Calibration results

The model was calibrated using the zero-coupon prices which we got above. The model solution and discretization were used to calculate the theoretical value of zero-coupon bonds using the discounted cashflow method in the discretized time periods. The theoretical and actual prices for up to 10-year ZCBs were compared and the sum of the squared differences was taken to be a function. The calibration was done by changing the parameters σ, α and β in the model,

The following were the model results of the calibration.

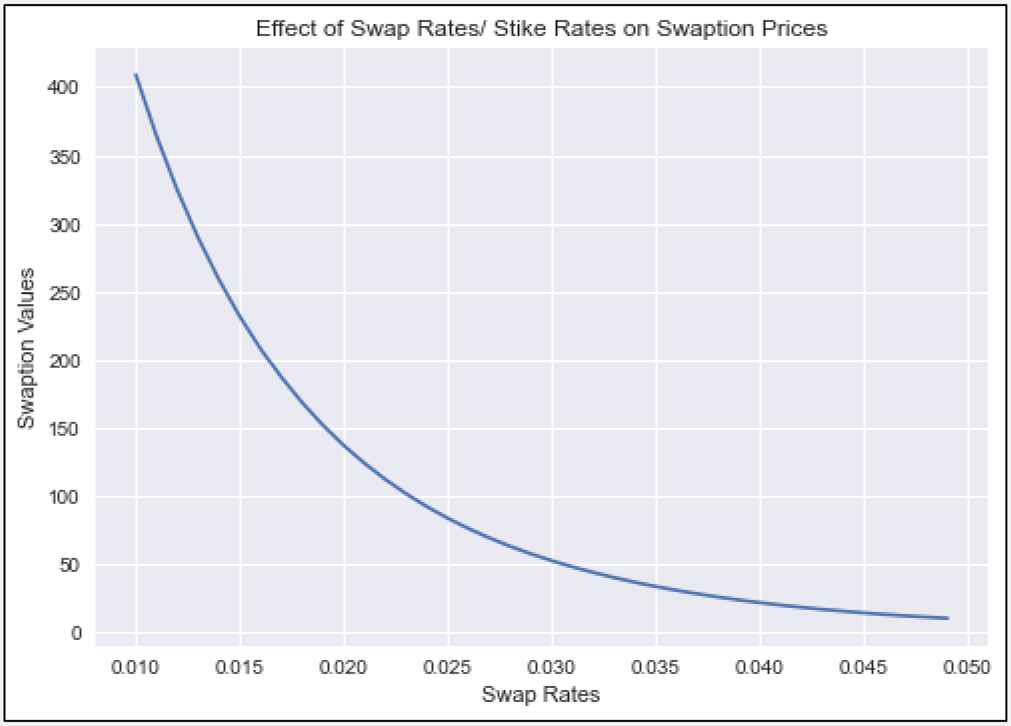
Calibrated Parameter Values	
Alpha α	0.30985707
Beta β	0.50549342
Sigma σ	0.46633875

SWAPTION VALUATION

Swaptions were valued using the Monte Carlo method discussed above. The following are the terms of a sample swaption taken for which the prices were calculated.

Sample Dummy Swaption	
Option Term	2 years
Swap Term	4 years
Swap Rate/Strike Rate	4%
Payment Freq	2 payments per year
Notional	100000
Calculated Value	$ 89.1526

Next, the prices were plotted as a function of swap rates.



The code for this calculation (Monte Carlo implementation) can be found in the code file provided separately.

APPENDIX

REFERENCES

[1] = Resource Center | U.S. Department of the Treasury

[2] - http://www-2.rotman.utoronto.ca/~hull/technicalnotes/TechnicalNote23.pdf

[3] - Swaption - Guide to Swap Options (investopedia.com)

[4] – Book : Options, Futures, and Other Derivatives 9th Edition by John C. Hull- Chapter 28-32

[5] – Book: Monte Carlo Methods In Financial Engineering – Section 3.5