Question No. 63

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Given:

$N(t)$ is the population density observed at time t

K is the rate of reproduction per unit time

The differential equation in terms of population density at time t and the time of calculation is given by,

$$\frac{dN(t)}{dt} = KN(t)$$

Assumption:

Consider the value of K is independent wih respect to time t

Method - 1:

From the given condition,

$$\frac{dN(t)}{dt} = KN(t)$$
$$L\left(\frac{dN(t)}{dt}\right) = L(K.N(t))$$

Applying Laplace Transform on both sides,

$$sN(s) - N(0) = KN(s)$$
$$sN(s) - KN(s) = N_0$$
$$N(s)(s-K) = N_0$$
$$N(s) = \frac{N_0}{(s-K)}$$

Applying Inverse Laplace Transform on both sides,

$$N(t) = N_0.e^{Kt} \tag{a}$$

The time taken to double the population density w.r.t. initial population density is,

$$\begin{aligned} N(t) &= 2N_0 \\ \frac{N(t)}{N_0} &= 2 \end{aligned}$$

From (a),

$$\begin{aligned} e^{Kt} &= 2 \\ Kt &= \log_e 2 \\ t &= \frac{\log_e 2}{K} \end{aligned} \tag{b}$$

Method - 2:

From the given condition,

$$\begin{aligned} \frac{dN(t)}{dt} &= KN(t) \\ \frac{dN(t)}{N(t)} &= Kdt \end{aligned}$$

Integrating on both the sides,

$$\int_{N_0}^{N(t)} \frac{dN(t)}{N(t)} = \int_0^t K dt$$
$$\log_e N(t)\Big|_{N_0}^{N(t)} = K.(t-0)$$
$$\log_e N(t) - \log_e N_0 = K.t$$
$$\log_e \left(\frac{N(t)}{N_0}\right) = K.t$$
$$\left(\frac{N(t)}{N_0}\right) = e^{Kt}$$
$$N(t) = N_0.e^{Kt} \tag{a}$$

The time taken to double the population density w.r.t. initial population density is,

$$N(t) = 2N(0)$$
$$\frac{N(t)}{N(0)} = 2$$

From (a),

$$e^{Kt} = 2$$
$$Kt = \log_e 2$$
$$t = \frac{\log_e 2}{K} \tag{b}$$

Therefore the answers for (a) and (b) of the question are,

$$N(t) = N_0.e^{Kt} \tag{a}$$
$$t = \frac{\log_e 2}{K} \tag{b}$$