MA 20205 Probability and Statistics
Assignment No. 8

1. The life of a special type of battery is a random variable with mean 40 hrs and standard deviation 20 hrs. A battery is used until it fails, at which point it is replaced by a new one. Assuming a stockpile of 25 such batteries, whose lives are independent, use the Central Limit Theorem to approximate the probability that over 1100 hrs of use can be obtained.

2. Sylvania's 40-watt light bulbs will burn a random time X before failing. Let X have mean μ and standard deviation 100 hours. If n of these bulbs are placed on test till they burn out, resulting in observations $X_1, \dots, X_n$, how large should n be so that the probability that $\bar{X}$ differs by μ by less than 50 hours is at least 0.95?

3. A certain component is critical to the operation of an electrical system and must be replaced immediately upon failure. If the mean lifetime of this type of component is 100 hours and its standard distribution is 30 hours, how many of these components must be in stock so that the probability that the system is in continual operation is for the next 2000 hours is at least 0.95?

4. Let $X_1, \dots, X_n$ be i.i.d. $N(\mu, \sigma^2)$. Find $(|\bar{X} - \mu| \leq 1.028\, S)$.

5. Let $X_1, \dots, X_n, X_{n+1}$ be i.i.d. $N(\mu, \sigma^2)$ and let $\bar{X}$ and S^2 denote the sample mean and sample variance based on $X_1, \dots, X_n$. Find the distribution of $\sqrt{\frac{n}{n+1}}\left(\frac{X_{n+1}-\bar{X}}{S}\right)$.

6. Let $X_1, \dots, X_m$ be a random sample from $N(\mu_1, \sigma^2)$ population and $Y_1, \dots, Y_n$ be another independent random sample from $N(\mu_2, \sigma^2)$ population. Let $\bar{X}$, $\bar{Y}$, S_1^2, S_2^2 be the sample means and sample variances based on X and Y - samples respectively. Further, let

   $S^2 = \frac{(m-1)S_1^2+(n-1)S_2^2}{m+n-2}$. Determine the distribution of

   $$U = \frac{\alpha(\bar{X}-\mu_1)+\beta(\bar{Y}-\mu_2)}{S\sqrt{\left(\frac{\alpha^2}{m}+\frac{\beta^2}{n}\right)}} \ , \alpha \neq 0, \beta \neq 0.$$

7. Consider two independent samples- the first of size 10 from a normal population with variance 4 and the second of size 5 from a normal population with variance 2. Compute the probability that the sample variance from the second sample exceeds the one from the first.

8. The temperature at which certain thermostat are set to go on is normally distributed with variance σ^2. A random sample is to be drawn and the sample variance S^2 computed. How many observations are required to ensure that $P\left(\frac{S^2}{\sigma^2} \leq 1.8\right) \geq 0.95$?

9. Let X_1 and X_2 be independent $N(0, \sigma^2)$ random variables. Find $P(X_1^2 + X_2^2 \leq \sigma^2)$.

10. Let $X_1, \dots, X_n$ be a random sample from $N(\mu, \sigma^2)$ population. For $1 < k < n$, define

$$U = \frac{1}{k}\sum_{i=1}^{k} X_i \ , V = \frac{1}{n-k}\sum_{i=k+1}^{n} X_i \ , S^2 = \frac{1}{k-1}\sum_{i=1}^{k}(X_i - U)^2,$$

$T^2 = \frac{1}{n-k-1}\sum_{i=k+1}^{n}(X_i - V)^2$. Find the distributions of $W_1 = \frac{U+V}{2}$,

$W_2 = \frac{(k-1)S^2+(n-k-1)T^2}{\sigma^2}$, $W_3 = \frac{S^2}{T^2}$, $W_4 = \frac{\sqrt{k}(U-\mu)}{T}$ and $W_5 = \frac{\sqrt{(n-k)}(V-\mu)}{T}$.

11. Let $X_1, X_2, \dots, X_{25}$ be a random sample from a $N(2,4)$ population and let $\bar{X}$ and S^2 denote the sample mean and the sample variance respectively. Find $P(2 - 0.3422\, S < \bar{X} < 2 + 0.4984\, S)$.

12. Let $(X_1, \dots, X_9), (Y_1, Y_2, \dots, Y_9)$ and $(Z_1, Z_2, \dots, Z_{25})$ be independent random samples from $N(-1, 9), N(1, 16)$ and $N(0, 9)$ populations respectively. Find the distribution of $W = \frac{\sum_{i=1}^{9}(X_i+Y_i)^2}{\sum_{j=1}^{25} Z_j^2}$.

13. Let $(X_1, \dots, X_7$ be a random sample from a normal population with variance 2 and let $Y_1, Y_2, \dots, Y_9$ be an independent random sample from a normal population with variance 4. Let S_1^2 and S_2^2 denote the variances of X and Y samples respectively. Find $P\left(0.12057 < \frac{S_1^2}{S_2^2} < 1.7903\right)$.

14. Let $(X_1, \dots, X_{25})$ be independent and identically distributed $N(3,5)$ random variables. Let $\bar{X}$ and S^2 denote the sample mean and the sample variance respectively. Find distributions of $W = 5(\bar{X} - 3)^2$, $T = \frac{5(\bar{X}-3)}{S}$ and $U = \frac{25(\bar{X}-3)^2}{S^2}$.