Lecture 22-23

Monte Carlo Simulations

Monte Carlo Methods

- Monte Carlo is another name for statistical sampling methods of great importance to physics and computer science
- Applications of Monte Carlo Method
  - Evaluating integrals of arbitrary functions of 6+ dimensions
  - Predicting future values of stocks
  - Solving partial differential equations
  - Sharpening satellite images
  - Modeling cell populations
  - Finding approximate solutions to NP-hard problems
  - ……

What is Monte Carlo (MC) method ?

The Monte Carlo method is a numerical method for statistical simulation which utilizes sequences of random numbers to perform the simulation.

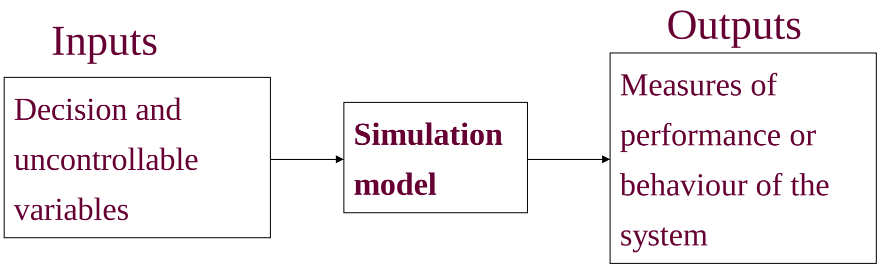
What the meaning of MC simulation?

MC simulation is a versatile tool to analyze and evaluate complex measurements

Constructing a *model* of a *system*.

Experimenting with the model to draw inferences of the system's behavior

A simulation model



A simulation model cont..

- Model inputs capture the environment of the problem
- The simulation model
  - Conceptual model: set of assumptions that define the system
  - Computer code: the implementation of the conceptual model
- Outputs describe the aspects of system behaviour that we are interested in

Random numbers

Uniform Random numbers or pseudo-random numbers (PRN) are essentially independent random variables uniformly Distributed over the unit interval (0,1).

The PRNs are good if they are uniformly distributed, statistically independent and reproducible.

Classic Example

Find the value of π

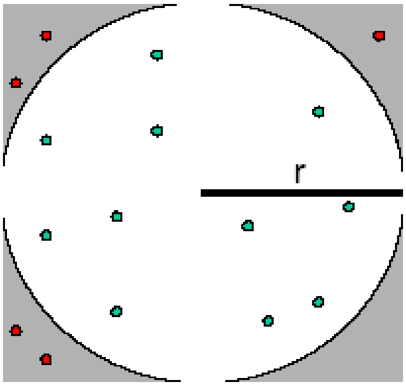
Use the reject and accept method
Or hit and miss method

The area of square=$(2r)^2$

The area of circle = $r^2\pi$

$$\frac{area \cdot of \cdot square}{area \cdot of \cdot circle} = \frac{4r^2}{\pi r^2} = \frac{4}{\pi}$$

$$\pi = 4 * \frac{area \cdot of \cdot circle}{area \cdot of \cdot square}$$

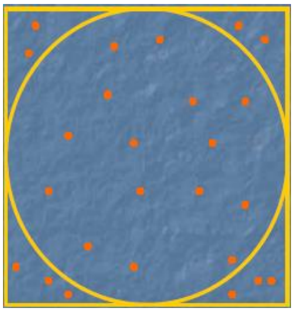


Cont....

$$\frac{area.of\,.circle}{area.of\,.square} = \frac{\#.of\,.dots.inside.circle}{total.number.of\,.dots}$$

Hit and miss algorithm

- Generate two sequences of N of PRN :: R_i,,R_j
- X_i=-1+2R_i
- Y_j=-1+2R_j
- Start from s=zero
- If (X^2+Y^2<1) s=s+1
- # of dots inside circle=s
- total number of dots=N



$$\pi = 4 * S / N$$

Random versus Pseudo-random

- Virtually all computers have "random number" generators
- Their operation is deterministic
- Sequences are predictable
- More accurately called "pseudo-random number" generators
- In this chapter "random" is shorthand for "pseudo-random"
- "RNG" means "random number generator"

Properties of an Ideal RNG

- Uniformly distributed
- Uncorrelated
- Never cycles
- Satisfies any statistical test for randomness
- Reproducible
- Machine-independent
- Changing "seed" value changes sequence
- Easily split into independent subsequences
- Fast
- Limited memory requirements

No RNG Is Ideal

- Finite precision arithmetic $\Rightarrow$ finite number of states $\Rightarrow$ cycles
  - Period = length of cycle
  - If period > number of values needed, effectively acyclic
- Reproducible $\Rightarrow$ correlations
- Often speed versus quality trade-offs

Linear Congruential RNGs

$$X_i = (a \times X_{i-1} + c) \bmod M$$

Modulus

Additive constant

Multiplier

Sequence depends on choice of seed, X_0

Period of Linear Congruential RNG

- Maximum period is M
- For 32-bit integers maximum period is 2^{32}, or about 4 billion
- This is too small for modern computers
- Use a generator with at least 48 bits of precision

Producing Floating-Point Numbers

- X_i, a, c, and M are all integers
- X_is range in value from 0 to M-1
- To produce floating-point numbers in range [0, 1), divide X_i by M

Defects of Linear Congruential RNGs

- Least significant bits correlated
  - Especially when M is a power of 2
- k-tuples of random numbers form a lattice
  - Points tend to lie on hyperplanes
  - Especially pronounced when k is large

Lagged Fibonacci RNGs

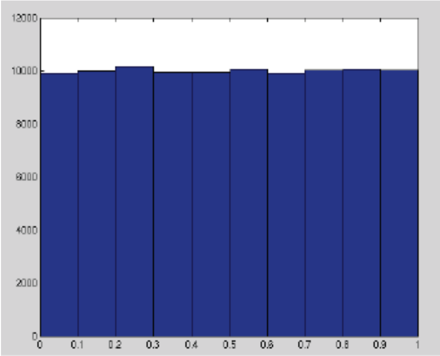
$$X_i = X_{i-p} * X_{i-q}$$

1. p and q are lags, $0 < p < q$
2. * is any binary arithmetic operation
   a. Addition modulo M
   b. Subtraction modulo M
   c. Multiplication modulo M
   d. Bitwise exclusive or
3. M is usually a power of 2

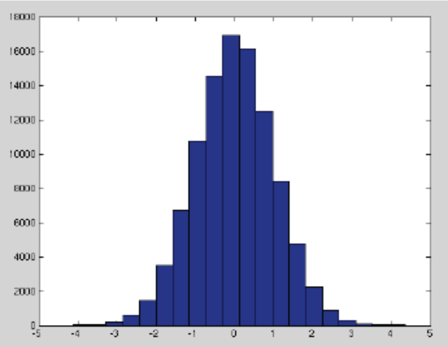
Properties of Lagged Fibonacci RNGs

- Require p seed values
- Careful selection of seed values, p, and q can result in very long periods and good randomness
- For example, suppose M has b bits
- Maximum period for additive lagged Fibonacci RNG is $(2^p - 1)2^{b-1}$

Types of distribution



Uniform distribution



Gaussian or normal distribution

Monte Carlo Integration