Partial Order and Hasse Diagram

Partial Order: A relation $\rho \subseteq \mathcal{A} \times \mathcal{A}$ on set $\mathcal{A}$ is called a partial ordering relation (or partial order) if it is reflexive, antisymmetric and transitive.

We call $(\mathcal{A}, \rho)$ as a Poset (Partial Ordered Set).

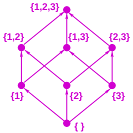
Example: Let $\mathcal{S} = \{1, 2, 3\}$ and $\rho = \{(\mathcal{A}, \mathcal{B}) \mid \mathcal{A} \subseteq \mathcal{B} \text{ and } \mathcal{A}, \mathcal{B} \in \mathcal{P}(\mathcal{S})\}$, therefore $(\mathcal{P}(\mathcal{S}), \rho)$ or $(\mathcal{P}(\mathcal{S}), \subseteq)$ is a poset.

Also, $(\mathcal{P}(\mathcal{S}), \supseteq)$ is a poset and called dual of the poset $(\mathcal{P}(\mathcal{S}), \subseteq)$.

Covering Relation: Let $(\mathcal{A}, \rho)$ is a poset and $p, q, r \in \mathcal{A}$. We call q as the cover for p (denoted as $p \prec q$) when $(p, q) \in \rho$, and no element $r \in \mathcal{A}$ exists such that $p \prec r \prec q$, that is $(p, r) \in \rho$ and $(r, q) \in \rho$.

Hasse Diagram: A directed acyclic graph (DAG) with elements of set $\mathcal{A}$ as nodes and (p, q) as directed edges from p to q $(p, q \in \mathcal{A})$ iff $p \prec q$ (q covers p).

Example: Note that, $(\{2\}, \{1, 3\}) \notin \rho$ and $\{1, 2\} \prec \{1, 2, 3\}$ (forming the cover), but $\{1\} \not\prec \{1, 2, 3\}$ as $\{1\} \prec \{1, 3\} \prec \{1, 2, 3\}$.



Total Order: If $(\mathcal{A}, \rho)$ is a Poset, we call $\mathcal{A}$ is totally ordered (or linearly ordered) if for all $x, y \in \mathcal{A}$ either $(x, y) \in \rho$ or $(y, x) \in \rho$. In this case, ρ is also called a total order (or linear order).

Properties of Partial Orders

Maximal Element: In the poset $(\mathcal{A}, \rho)$, an element $x \in \mathcal{A}$ is called a maximal element of $\mathcal{A}$ if $\forall a \in \mathcal{A}\ [(a \neq x) \Rightarrow (x, a) \notin \rho]$ $(\equiv \nexists a \in \mathcal{A}\ [(x, a) \in \rho \Rightarrow (a = x)])$.

Minimal Element: In the poset $(\mathcal{A}, \rho)$, an element $y \in \mathcal{A}$ is called a minimal element of $\mathcal{A}$ if $\forall b \in \mathcal{A}\ [(b \neq y) \Rightarrow (b, y) \notin \rho]$ $(\equiv \nexists b \in \mathcal{A}\ [(b, y) \in \rho \Rightarrow (b = y)])$.

Example: In the poset $(\mathcal{P}(\mathcal{S}), \subseteq)$ where $\mathcal{S} = \{1, 2, 3\}$, we have $\{1, 2, 3\}$ and $\{\}$ as the maximal and minimal elements, respectively.

If $(\mathcal{A}, \rho)$ is a poset and $\mathcal{A}$ is finite, then $\mathcal{A}$ has both a maximal and a minimal element.

Least Element: Let $(\mathcal{A}, \rho)$ is a poset. An element $x \in \mathcal{A}$ is called the least element if $\forall a \in \mathcal{A},\ (x, a) \in \rho$.

Greatest Element: Let $(\mathcal{A}, \rho)$ is a poset. An element $y \in \mathcal{A}$ is called the greatest element if $\forall a \in \mathcal{A},\ (a, y) \in \rho$.

Example: In the poset $(\mathcal{P}(\mathcal{S}), \subseteq)$ where $\mathcal{S} = \{1, 2, 3\}$, we have $\{\}$ and $\{1, 2, 3\}$ as the least and greatest elements, respectively.

If $(\mathcal{A}, \rho)$ is a poset has a least (greatest) element, then that element is unique.

Properties of Partial Orders

Lower Bound: Let $(\mathcal{A}, \rho)$ is a poset and $\mathcal{B} \subseteq \mathcal{A}$. An element $x \in \mathcal{A}$ is called a lower bound of $\mathcal{B}$ if $\forall b \in \mathcal{B},\ (x, b) \in \rho$.

Upper Bound: Let $(\mathcal{A}, \rho)$ is a poset and $\mathcal{B} \subseteq \mathcal{A}$. An element $y \in \mathcal{A}$ is called a upper bound of $\mathcal{B}$ if $\forall b \in \mathcal{B},\ (b, y) \in \rho$.

Greatest Lower Bound: Let $(\mathcal{A}, \rho)$ is a poset. An element $x' \in \mathcal{A}$ is called the greatest lower bound (glb) of $\mathcal{B}$ if it is a lower bound of $\mathcal{B}$ and $(x'', x') \in \rho$ for all other lower bounds x'' of $\mathcal{B}$.

Least Upper Bound: Let $(\mathcal{A}, \rho)$ is a poset. An element $y' \in \mathcal{A}$ is called the least upper bound (lub) of $\mathcal{B}$ if it is an upper bound of $\mathcal{B}$ and $(y', y'') \in \rho$ for all other upper bounds y'' of $\mathcal{B}$.

Example: In the poset $(\mathcal{P}(\mathcal{S}), \subseteq)$ where $\mathcal{S} = \{1, 2, 3\}$ and let $\mathcal{B} = \{\{1\}, \{2\}, \{1, 2\}\} \subseteq \mathcal{P}(\mathcal{S})$. Then, $\{1, 2\}$ and $\{1, 2, 3\}$ both are the upper bounds for $\mathcal{B}$ in $(\mathcal{P}(\mathcal{S}), \rho)$; whereas $\{1, 2\}$ is the lub (and is in $\mathcal{B}$). However, the glb for $\mathcal{B}$ is $\{\}$, i.e. ϕ, which does not belong to $\mathcal{B}$.

If $(\mathcal{A}, \rho)$ is a poset and $\mathcal{B} \subseteq \mathcal{A}$, then $\mathcal{B}$ has at most one lub (glb).

Lattice

Definition

A lattice is a poset, $(\mathcal{A}, \rho)$, in which for every pair of elements $a, b \in \mathcal{A}$, the $lub\{a, b\}$ and $glb\{a, b\}$ both exists in $\mathcal{A}$.
A lattice is complete in which every subset of elements has a lub and glb.

Examples: *All the following posets are lattice.*

1. Poset $(\mathbb{N}, \rho)$, where $\rho = \{(x, y) \mid x \leq y \text{ and } x, y \in \mathbb{N}\}$ is a lattice.
   Here, for any $x, y \in \mathbb{N}$, $lub\{x, y\} = max\{x, y\}$ and $glb\{x, y\} = min\{x, y\}$.
2. Poset $(\mathcal{P}(\mathcal{S}), \rho)$, where $\rho = \{(\mathcal{A}, \mathcal{B}) \mid \mathcal{A} \subseteq \mathcal{B} \text{ and } \mathcal{A}, \mathcal{B} \in \mathcal{P}(\mathcal{S})\}$ is a lattice.
   Here, for any $\mathcal{A}, \mathcal{B} \in \mathcal{P}(\mathcal{S})$, $lub\{\mathcal{A}, \mathcal{B}\} = \mathcal{A} \cup \mathcal{B}$ and $glb\{\mathcal{A}, \mathcal{B}\} = \mathcal{A} \cap \mathcal{B}$.
3. Poset $(\mathbb{Z}^+, \rho)$, where $\rho = \{(x, y) \mid x \text{ divides } y \text{ and } x, y \in \mathbb{Z}^+\}$ is a lattice.
   Here, for any $x, y \in \mathbb{Z}^+$, $lub\{x, y\} = LCM\{x, y\}$ and $glb\{x, y\} = GCD\{x, y\}$.

Example: *The following poset is NOT a lattice.*

Let $\mathcal{S} = \{1, 2, 3\}$ and $\mathcal{Q} \subset \mathcal{P}(\mathcal{S})$ (all proper subsets) where $\phi \notin \mathcal{Q}$. Poset $(\mathcal{Q}, \rho)$, where $\rho = \{(\mathcal{A}, \mathcal{B}) \mid \mathcal{A} \subseteq \mathcal{B} \text{ and } x, y \in \mathcal{Q}\}$ is NOT a lattice.
Here, the pair of elements $\{1, 2\}$ and $\{1, 3\}$ in $\mathcal{Q}$ do not have a *lub*, whereas the pair of elements $\{1\}$ and $\{2\}$ in $\mathcal{Q}$ do not have a *glb*.

Thank You!