Probability & statistics

August 31

Lecture 6

$\mathbb{R}, (\Omega, \mathcal{F}, P)$, $X \longleftrightarrow$ pmf

$\downarrow$

discrete r.v.

$\{a_1, \cdots, a_N\}$

Ex:

x	a_1	a_2	$\cdots$	a_N
$f(x)$	$f(a_1)$	$f(a_2)$		$f(a_N)$

Given that $f(a_i) > 0 \quad \forall\ i = 1, 2, \cdots, N$

If $\sum_{i=1}^{N} f(a_i) = 1$, then it is a pmf.

$$g(a_i) = \frac{f(a_i)}{\sum_{i=1}^{N} f(a_i)} \qquad \text{if} \quad \sum_{i=1}^{N} f(a_i) \neq 1.$$

Then g becomes a pmf.

Ex: Let f_i be a seq. of real numbers s.t.

$f_i > 0$ for $i = 1, 2, \cdots$ $\qquad R_X = \{1, 2, \cdots\}$

Assume that $\sum_{i=1}^{\infty} f_i$ is convergent.

Let $\sum_{i=1}^{\infty} f_i = k$

pmf??

i	1	2	$\cdots$
f_i	f_1/k	f_2/k	$\cdots$

Ex: $f_i = \frac{\lambda^i}{i!}$ $\quad i = 0, 1, 2, \cdots$ $\quad$ (for $\lambda > 0$)

$\sum_{i=0}^{\infty} f_i = e^{\lambda} = k$

$$\boxed{P(\underset{\sim}{X} = i) = \frac{e^{-\lambda} \lambda^i}{i!} \qquad \underline{i = 0, 1, 2, \cdots}}$$

Poission pmf $\qquad X \sim \text{Poission}(\lambda)$

Another computation with pmf.

$$\underbrace{X \le t}_{t \in \mathbb{R}}$$

Event: $\underline{X \le t}$

$= \underline{\{w : X(w) \le t\}}$

$P(X \le t)$

i) $X \sim \text{uniform}(10)$

$f(x) = \frac{1}{10}$ for $x = \{0, 1, 2, \cdots, 9\}$

$= 0$ o.w.

$t = 2.7$; $X \le t = \{w : X(w) \le t\}$

$P(X \le t) = P(X \le 2.7) = P(X = 0) + P(X = 1) + P(X = 2)$

$= f(0) + f(1) + f(2) = \frac{3}{10}$

$t = 0$

$P(X \le 0) = \frac{1}{10}$

$t = -0.09$

$P(X \le -0.09) = 0$

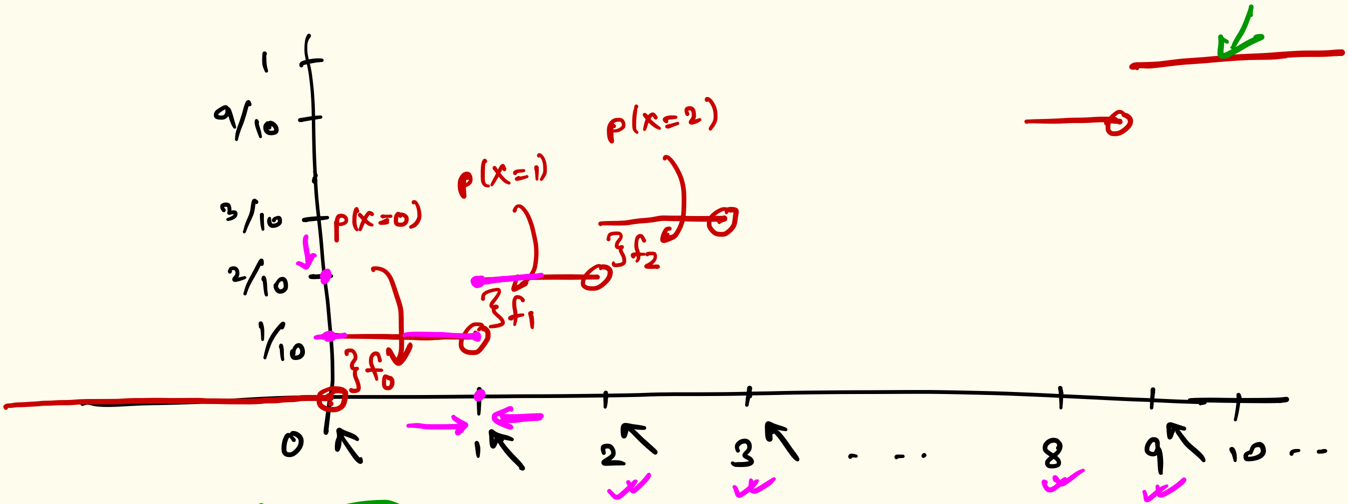
$t = 9$

$P(X \le 9) = 1$

Cumulative Distribution f^{n}: [CDF] $\to F(t)$

for $t \in \mathbb{R}$

$t \longmapsto \boxed{\mathrm{Prob}(X \le t)} = P(X \le t)$



$R_X = \{0, 1, 2, \ldots, 9\}$

Ex: Geometric(p)

pmf

$$f(x) = p(1-p)^x \qquad x = 0, 1, 2, \ldots$$

$$= 0 \qquad \text{o.w.}$$

For $t \in \mathbb{R}$, $t > 0$

CDF

$$F(t) = P(X \le t)$$

$$= \sum_{x=0}^{[t]} p(1-p)^x$$

$$= p \sum_{x=0}^{[t]} (1-p)^x$$

$$= p \, \frac{\left(1 - (1-p)^{[t]+1}\right)}{1-(1-p)}$$

$[t]$: largest integer $\le t$

If $t < 0$

$$F(t) = P(X \le t)$$

$$= 0$$

$$\boxed{F(t) = 1 - (1-p)^{[t]+1}}$$

$t \geq 0$

$[0] = 0$

$R_x = \{0, 1, 2, 3, \ldots\}$

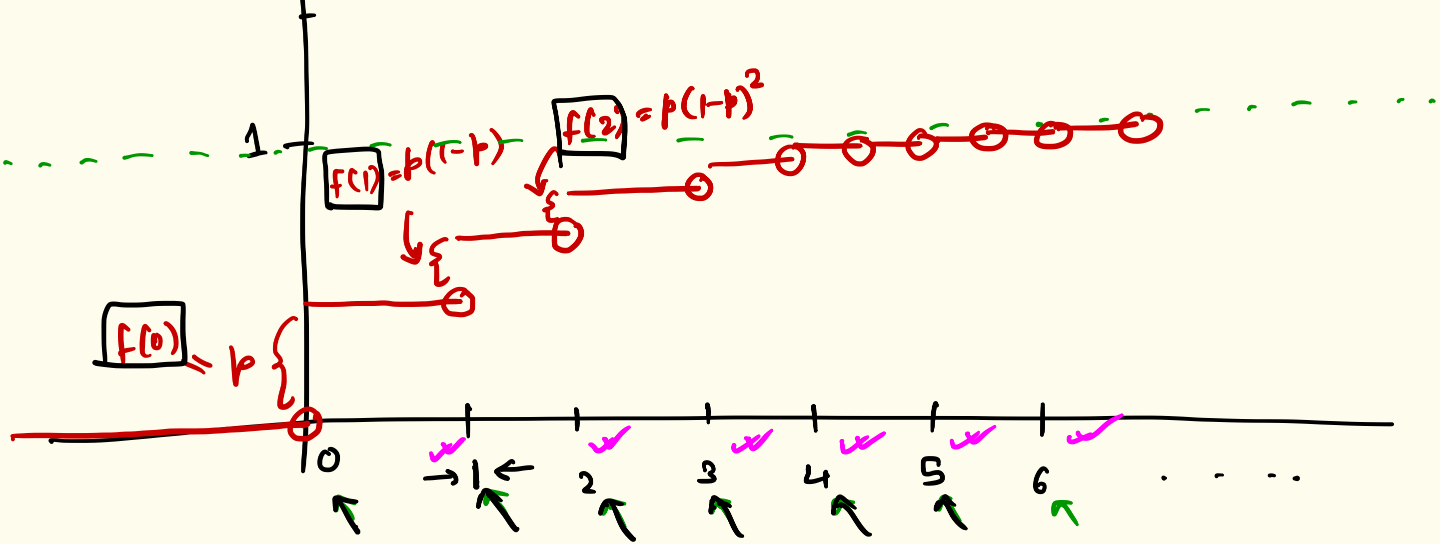
$F(1) = p + (1-p)p$

$F(2) = p + (1-p)p + (1-p)^2 p$

At every pt in $x \in R_x$

$F(t)$ has a jump.

Size of jump $= f(x) = \text{Prob}(X = x)$



$$F(t) = 1 - (1-p)^{[t]+1}$$

$$F(0) = 1 - (1-p)^{0+1} = 1 - (1-p) = p$$

$t \in [0, 1)$

$F(t) = p$

Properties of $F(t)$

$F(t) = Prob(X \le t) \quad \forall t \in \mathbb{R}$

$\begin{cases} A \subseteq B \\ P(A) \le P(B) \end{cases}$ $\quad t_1 \le t_2$

$(X \le t_1) \subseteq (X \le t_2)$

$\Rightarrow P(X \le t_1) \le P(X \le t_2)$

i) $\underbrace{0 \le F(t)} \le 1 \quad \forall t \in \mathbb{R}$

ii) $\lim_{t \to \infty^+} F(t) = 1$

iii) $\lim_{t \to \infty^-} F(t) = 0$

iv) F is a non-decreasing function.

$\forall t_1, t_2 \in \mathbb{R}, \quad t_1 \le t_2 \Rightarrow \underbrace{F(t_1) \le F(t_2)}$

v) F is right continuous function.

for $\underbrace{\delta > 0}$, $\underbrace{\lim_{\delta \to 0} F(t+\delta) = F(t)}_{\text{RHL} \,=\, \text{value of the f}^{\text{n}}} \quad \forall t \in \mathbb{R}$

observe: The function is discontinuous only at $t \in R_X$.

The function $F: \mathbb{R} \to [0,1]$ is called a CDF (Cumulative Distribution Function) if it satisfies above five properties.

K. L. Chung
Intro. to prob. theory.

$\forall\, t_2 \geq t_1$

$F(t_2) - F(t_1)$

$\Omega, \mathcal{F}, P \xrightarrow{R} X \longleftrightarrow \text{pmf} \longleftrightarrow \text{CDF}$

discrete r.v.

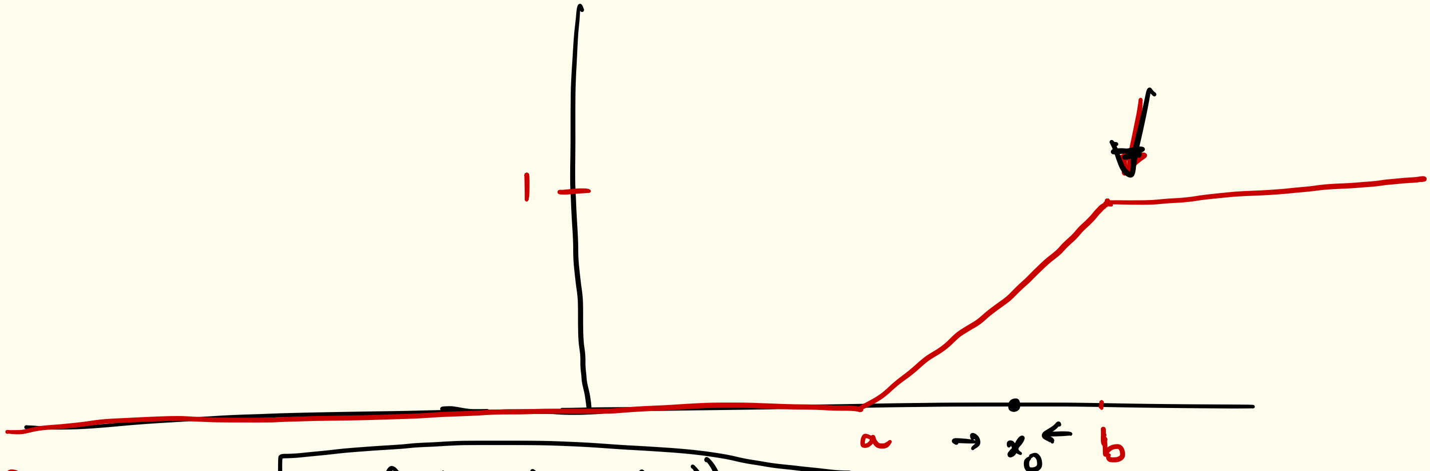
From CDF to pmf (for discrete r.v.)

→ Check the points in $\mathbb{R}$ where CDF has jumps.

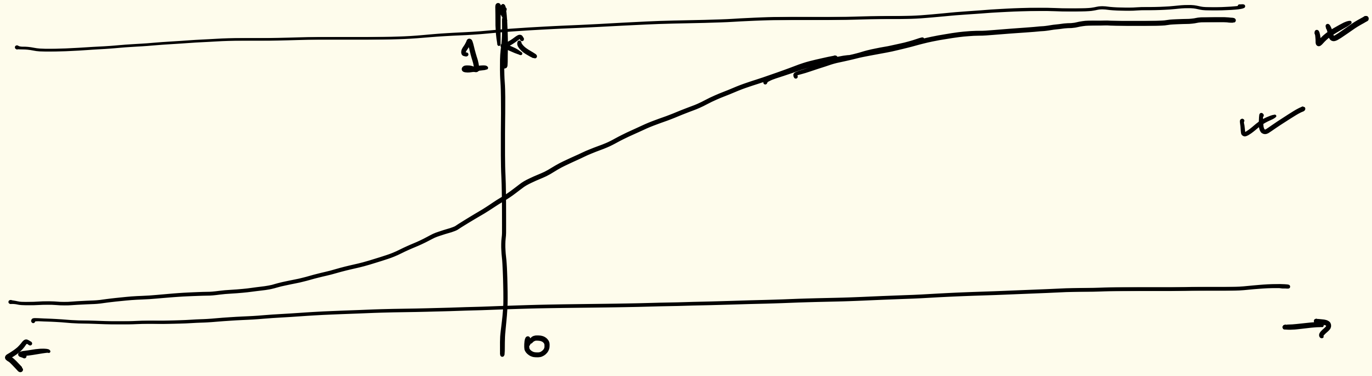
→ these pts. $\in R_X$

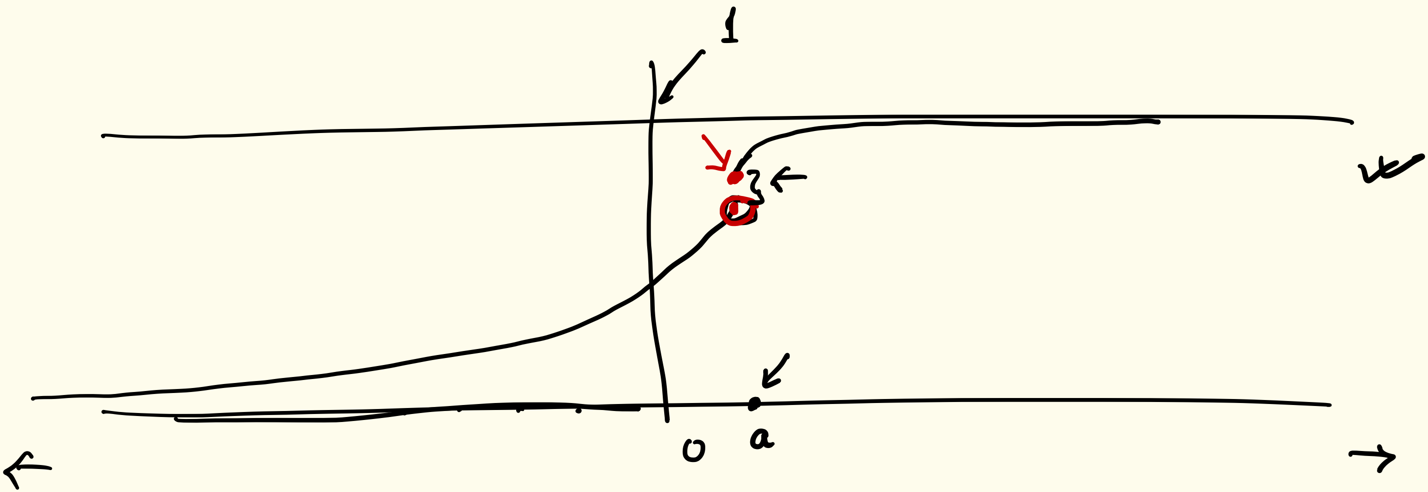
→ pmf at the points in R_X is the jump size.

<u>Question</u> : Does every $f^{\underline{n}}$ which satisfy these five properties in the definition of CDF look like the graphs we have already drawn??



$$\boxed{\lim_{\delta \to 0} \left(F(x_0+\delta) - F(x_0) \right) = 0} \Rightarrow \underbrace{\text{Prob}}\,(X = x_0)$$





At a, $\lim_{\delta \to 0_-} F(a+\delta) - F(a) \neq 0$

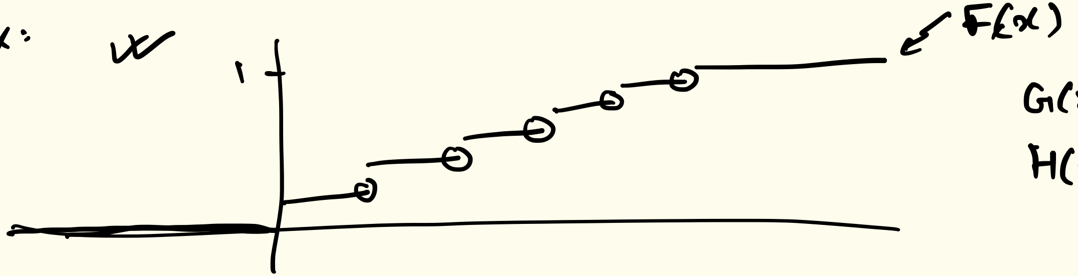
* **Lebesgue decomposition theorem.**

If a function $F(x)$ satisfies the properties (i) to (v) stated above, then $F(x)$ can be represented as sum of two functions, say $G(x)$ and $H(x)$ as

$$F(x) = G(x) + H(x)$$

where $G(x)$ is continuous and $H(x)$ is a right continuous step function with jumps coinciding with those of $F(x)$ and $H_x(-\infty) = 0$.

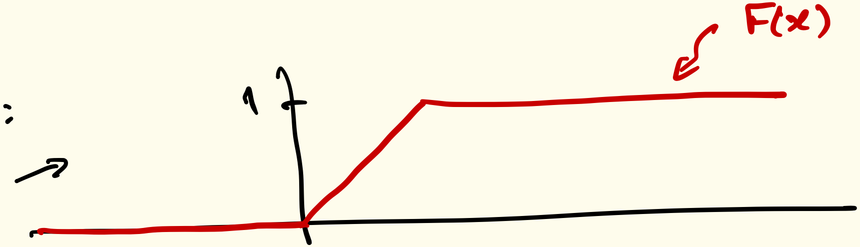
Ex:



$G(x) = 0$

$H(x) = F(x)$

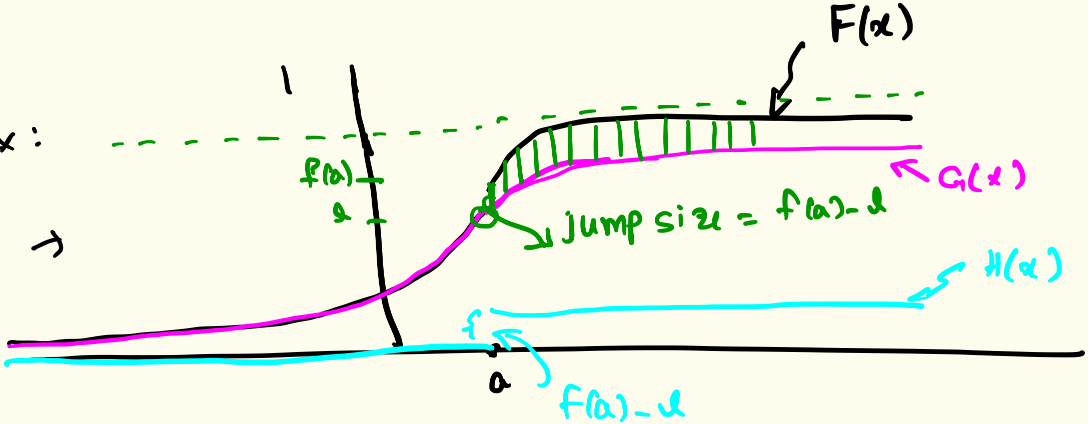
Ex:



$G(x) = F(x)$

$H(x) = 0$

Ex:



$F(x) = G(x) + H(x)$

In Lebesgue decomposition theorem if $G(x) \equiv 0$, then the random variable X is called discrete random variable.

If $H(x) \equiv 0$, then X is called continuous random variable.

In the situation, where neither G, nor H is identically zero, the random variable is called as mixed random variable.

Aim: To understand CDF $\longrightarrow$ pmf or (equivalent concept) $\longrightarrow$ X

pdf
prob. density fn
continuous r.v.

In case of discrete r.v. CDF $\longrightarrow$ pmf by taking differences $\underbrace{F(t_i) - F(t_j)}$ $t_i, t_j \in \mathbb{R}$

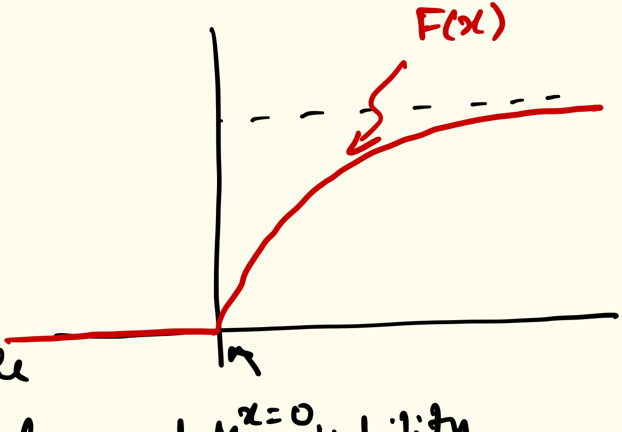
Case of continuous r.v.

$H(x) \equiv 0$ in Lebesgue decomposition.

$$\underbrace{P(X = x)} = \lim_{\delta \to 0} [F(x+\delta) - F(x)]$$

$$= 0$$

Define $f(x) = \frac{d}{dx} F(x)$



The function F is differentiable almost everywhere. The points of non-differentiability are "few".

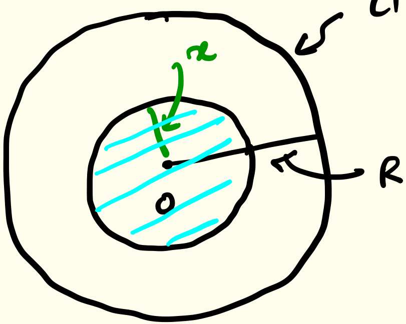
$$f(x) = \frac{d}{dx} F(x)$$

$\uparrow$ probability density function of a continuous r.v.

<u>Note of caution:</u> pdf does <u>NOT</u> give probability like pmf.

<u>Ex:</u>

circular board



<u>Expt:</u> Throw dart on this circular board.

$$(\Omega, \mathcal{F}, P)$$

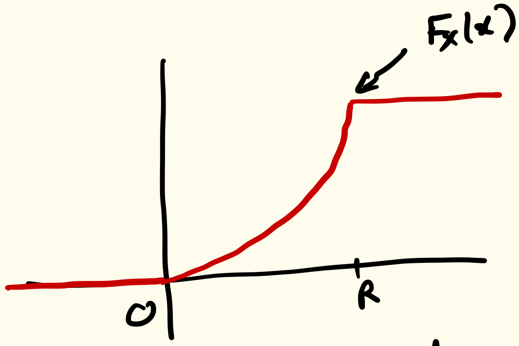
$\uparrow$ uniform probability space.

X : distance of the dart from the centre of the board.

Let $F(x)$ denote the CDF of X.

$$F(x) = \text{Prob}(X \le x)$$
$$= \frac{\pi x^2}{\pi R^2}$$
$$= \frac{x^2}{R^2}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2/R^2 & 0 \le x \le R \\ 1 & x > R \end{cases}$$



Note: CDF is a cts. f^n

$\Rightarrow X$ is a cts. r.v.

Then the pdf of X is, f, computed as:

$$f(x) = \frac{d}{dx} F(x)$$

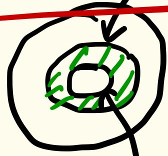
$$= \begin{cases} 0 & x < 0 \\ 2x/R^2 & 0 \le x \le R \\ 0 & x > R \end{cases}$$

$$\Rightarrow \boxed{f(x) = \begin{cases} 2x/R^2 & 0 \le x \le R \\ 0 & \text{o.w.} \end{cases}}$$

Remember

$f(a) = \frac{2a}{R^2}$

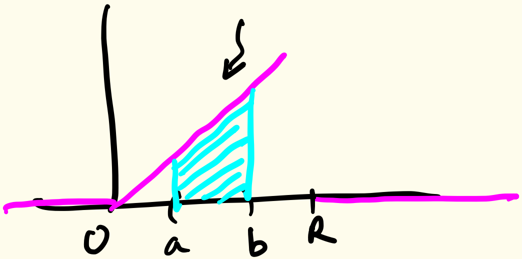
$\neq P(X=a) = 0$



$R_X = (0, R]$

$$P(a \le X \le b) = F(b) - F(a)$$

$$= \int_0^b f(x)\,dx - \int_0^a f(x)\,dx$$



$$\boxed{P(a \le X \le b) = \int_a^b f(x)\,dx}$$

Let X be a continuous random variable with CDF $F(x)$ and $f(x) = \frac{d}{dx} F(x)$ be its probability density function (pdf). Then the range of X is the set $R_X \subseteq \mathbb{R}$ s.t. $\forall x \in R_X$; $\underline{f(x) > 0}$.

<u>Defn</u>: A density function or pdf is a non-negative function such that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

<u>Check</u>: $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$$P(a \le X \le b) = F(b) - F(a) = \int_a^b f(x)\,dx$$

satisfies all the properties of CDF.

Ex: For what value of k,

$$f(x) = \begin{cases} kx^3 & 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

is a pdf ??

$k = 4$

Definitions :

$E(X) = \mu$

i) Expectation of a r.v.

discrete case: $E(X) = \sum_{x \in R_X} x f(x) = \sum_{x \in R_X} x P(X = x)$

X: r.v.

f : pmf

continuous case: $E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_{R_X} x f(x) dx$

X: r.v., f: pdf

ii) <u>variance</u>:

discrete case:

x: r.v.

f: p.m.f.

$\mu = E(x)$

$$E(x-\mu)^2 = var(x)$$

$$= \sum_{x \in R_x} (x_i - \mu)^2 \, Prob(x = x_i)$$

$$= \sum_{x \in R_x} (x_i - \mu)^2 f(x_i)$$

continuous case:

x: r.v.

f: p.d.f.

$\mu = E(x)$

$$var(x) = E(x-\mu)^2$$

$$= \int_{R_x} (x-\mu)^2 f(x)\, dx$$

iii) <u>Median</u> : M

$M \in \mathbb{R}$ is median s.t.

$$F(M) = 1/2$$

<u>Ex</u>: $X \sim$ uniform(10) discrete uniform.

$R_X = \{0, 1, 2, \dots, 9\}$

$$E(X) = \mu = \sum_{x \in R_X} x f(x)$$

$$= 0 \cdot f(0) + 1 \cdot f(1) + 2 \cdot f(2) + \dots + 9 \cdot f(9)$$

$$= \frac{1 + 2 + \dots + 9}{10}$$

$$= 4.5$$

$$\operatorname{var}(X) = \sum_{x_i \in R_X} (x_i - \mu)^2 f(x_i)$$

$$= (0-4.5)^2 f(0) + (1-4.5)^2 f(1) + \cdots + (9-4.5)^2 f(9)$$

$$= \frac{(0-4.5)^2 + (1-4.5)^2 + \cdots + (8-4.5)^2 + (9-4.5)^2}{10}$$

$$=$$

Ex: Median.

X is cts. r.v. with pdf.

$$f(x) = \begin{cases} \frac{2x}{R^2} & 0 \le x \le R \\ 0 & \text{o.w.} \end{cases}$$

Compute M: median ??

$$F(M) = \frac{1}{2}$$

$$\int_0^M \frac{2x}{R^2}\,dx = \frac{1}{2} \qquad \Rightarrow M = \_\_\_\_\_\_$$