Prob-Stat/QUIZ/3/B

Fill in the blanks (Numerical)

Date of Exam : 15th Nov, 2021

Time : 11:05 am to 11:55 am

Duration : 45min

No of questions: 10 out of 20 questions

Type: Random-sequential (navigation NOT allowed)

Each question carries 4 marks

B NOTE $\Phi(2) = 0.9772499, \Phi(1/\sqrt{3}) = 0.7181486, \Phi(1) = 0.8413447, \Phi(1.12) = 0.8686431$

November 19, 2021

B:Q41. Let (X, Y) be jointly distributed with PDF

$$f(x,y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & otherwise. \end{cases}$$

Then $P(Y \geq 1/2 \,|\, x = 1/2) =$ ....

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 1

ERROR RANGE: 0.005

Sol. The marginal density functions are

$$f_X(x) = \int_x^1 2dy = \begin{cases} 2 - 2x, 0 < x < 1 \\ 0, otherwise \end{cases}$$

and

$$f_Y(x) = \int_0^y 2dx = \begin{cases} 2y, 0 < y < 1 \\ 0, otherwise. \end{cases}$$

Then $f_{X|Y}(x|y) = \frac{1}{y}, 0 < x < y$ which is uniform on $(0, y)$ and $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1}{1-x}, x < y < 1$ which is uniform on $(x, 1)$. Then $P(Y \geq 1/2|x = 1/2) = \int_{1/2}^{1} \frac{1}{1-1/2} dy = 1$.

B:Q42. Let $X_1, \ldots, X_{64}$ be i.i.d. random variables with p.m.f. $p(0) = 1/4, p(1) = 1/2, p(2) = 1/4$. Let $Y = X_1 + \ldots + X_{64}$. Find y such that $P(Y \leq y) \geq 0.95$. (Provided $\Phi(1.645) = 0.95$ where Φ is the c.d.f. of a standard normal distribution.)

(Round off your answer to the largest nearest integer, error range 0)

ANSWER : 74 CHANGED TO 73 OR 74

ERROR RANGE: 0.000

Sol. $E(X_i) = \frac{1}{4}(0) + \frac{1}{2}(1) + \frac{1}{4}(2) = 1$. Similarly, $E(X_i^2) = \frac{1}{4}(0^2) + \frac{1}{2}(1^2) + \frac{1}{4}(2^2) = \frac{3}{2}$. $Var(X_i) = E(X_i^2) - (E(X_i))^2 = \frac{1}{2}$. Thus $Var(Y) = 64 \times \frac{1}{2} = 32$. Thus $\sigma_Y = 4\sqrt{2}$. Let $P(Y \leq y) \geq 0.95 \implies \Phi(\frac{Y-64}{4\sqrt{2}}) = 0.95 \implies Y - 64 = 1.645 \times 4\sqrt{2} \implies Y = 73.3055 \approx 74$.

B:Q43. Let X be a random variable whose pdf is uniform distribution on the interval $[-\pi/2, \pi/2]$. If $Y = \sin(X)$ then $f_Y(1/2)$ is = ..., where $f_Y(y)$ denotes the pdf of Y.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.3675526

ERROR RANGE: 0.005

Sol. Here $Y = g(X)$, where g is a differentiable function. Then g is monotonic increasing. Thus, We note that since $R_X = [-\pi/2, \pi/2]$, $R_Y = [-1, 1]$. If $y \in [-1, 1]$, we have $x = \sin^{-1}(y)$ and

$$f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}}$$

Putting $y = 0.5$, we get 0.3675526.

B:Q44. Suppose you roll two die and the numbers that show up are represented by the random variables X and Y. Suppose $Z = X + Y$. Then $P(X = 4 \mid Z = 8) =$ ......

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.2

ERROR RANGE: 0.005

Sol: $P(X=4|Z=8) = \frac{P(X=4,Z=8)}{P(Z=8)} = \frac{P((4,4))}{P((2,6)\cup(3,5)\cup(4,4)\cup(5,3)\cup(6,2))} = \frac{1/36}{5/36} = 0.2.$

B:Q45. Let $X_1,\ldots,X_4$ be i.i.d. $N(0,16)$. Also let $Y_1,\ldots,Y_8$ be i.i.d. $N(0,32)$ independently of X_i's. Let $\overline{X}$ and $\overline{Y}$ be the means of X_i's and Y_j's respectively. The degrees of freedom of the distribution of $W = \frac{1}{4}\left(\overline{X}^2 + \overline{Y}^2\right)$ is

(answer should be positive integer, error range 0)

ANSWER : 2

ERROR RANGE: 0

Sol. $0.25\bar{X}^2 \sim \chi_1^2$ and $0.25\bar{Y}^2 \sim \chi_1^2$ hence $W \sim \chi_2^2$

B:Q47. Suppose $X_1, X_2, X_3, X_4, X_5, X_6$ is a random sample drawn from a normal distribution $N(\mu, \sigma^2)$. Let $S^2 = \sum_{i=2}^{6}(X_i - m)^2$ where $m = \frac{1}{5}\sum_{i=2}^{6} X_i$. Find the degrees of freedom of the distribution of $T = \frac{2(X_1-\mu)}{S}$.

(answer should be positive integer, error range: 0)

ANSWER : 4

ERROR RANGE: 0

$\frac{S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=2}^{6}(X_i - \bar{X})^2$ follows χ_4^2 which is independently distributed to $\frac{2(X_1-\mu)}{\sigma}$ following $N(0,1)$. Hence T follows t_4

B:Q49. Suppose in a population of husband and wife, the height X_1 (in ft.) of the husband and the height X_2 (in ft.) of the wife have bivariate normal distribution with parameters $\mu_1 = 5.8$, $\mu_2 = 5.3$, $\sigma_1 = \sigma_2 = 0.2$, and the correlation coefficient $\rho = 0.6$. Find the probability that his wife has height between 5.28 and 5.92 feet given that the height of the husband is 6.3 feet.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.9544997

ERROR RANGE: 0.005

NOTE $\Phi(2) = 0.9772499$

Sol. The conditional pdf of X_2, given $X_1 = 6.3$ is normal with mean $5.3 + (0.6)(6.3 - 5.8) = 5.6$ and standard deviation $(0.2)\sqrt{1 - 0.36} = 0.16$. Then

$$P(5.28 < X_2 < 5.92 \,|\, X_1 = 6.3) = \Phi(2) - \Phi(-2) = 0.9544997.$$

B:Q54. Let $(X, Y) \sim BivariateNormal(\mu_x = 0, \mu_y = -1, \sigma_x^2 = 1, \sigma_y^2 = 4, \rho = -\frac{1}{2})$. Find $P(X + Y > 0)$.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.2818514

ERROR RANGE: 0.005

$X + Y \sim N(-1, 3)$, Then $P(X + Y > 0) = 1 - \Phi(1/\sqrt{3}) = 0.2818514$

NOTE : $\Phi(1/\sqrt{3}) = 0.7181486$

B:Q55. Let $X_1, \ldots X_5$ be i.i.d. $N(0, 1)$. Define

$$F = \frac{4X_5^2}{X_1^2 + X_2^2 + X_3^2 + X_4^2}.$$

following $F_{m,n}$ degrees of freedom . Find the value of $m + n$.

(answer should be positive integer, error range: 0)

ANSWER : 5

ERROR RANGE: 0

B:Q57. Let Y be defined by $X = \frac{e^{\frac{Y}{2}-1}}{e^{\frac{Y}{2}}}$, where $X \sim Uniform(0, 1)$ distribution. Find the value of the density of Y at the point $y = 2$.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.1839397 CHANGE TO "ANY POSITIVE VALUE IS CORRECT"

ERROR RANGE: 0.005

Soln: $Y = -\frac{1}{2}\log_e(1 - X)$. Hence Y follows exponential distribution with mean 2. So $f_Y(2) = 0.5 * exp(-1) = 0.1839397$

B:Q58. The lifetime of two components in an electronic system are independent random variables X and Y where $X \sim N(0, 1)$ and $Y \sim N(1, 3)$. What is the probability that the lifetimes of the two components expire within 1 time unit from each other?

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.3413447

ERROR RANGE: 0.005

Soln: $X - Y \sim N(-1, 4)$ then $P(|X - Y| \leq 1) = \Phi(1) - \Phi(0) = 0.8413447 - 0.5 = 0.3413447$

NOTE : $\Phi(1) = 0.8413447$

B:Q59. Let X and Y be independent random variables each having an exponential distribution with mean 0.25. Find $P(X + Y > 1/4)$.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.7357589

ERROR RANGE: 0.005

Soln. X and Y are i.i.d. $Gamma(1, 4)$. Hence $X + Y \sim Gamma(2, 4)$. Hence $P(X + Y > 1/4) = 2e^{-1} = 0.7357589$

$1 - pgamma(1/4, shape = 2, rate = 4) = 0.7357589$

B:Q60. A fair dice is repeatedly rolled. Let the random variable X be the number of rolls until the face value 1 appears and Y be the number of rolls until the face value 4 appears. Find the conditional expectation of X given $Y = 3$.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 6.28

ERROR RANGE: 0.005

$$E(Y|X = 3) = P(Y = 1|X = 3) + 2P(Y = 2|X = 3) + \sum_{i=4}^{\infty} iP(Y = i|X = 3) = 157/25 = 6.28.$$

B:Q62. Let X_1 and X_2 be two independent random variables resulting from two casts of an unbiased dice. That is, X_1, X_2 is a random sample of size $n = 2$ from a distribution with p.m.f. $f(x) = 1/6$, $x = 1, 2, \ldots, 6$. If $Y = X_1 + X_2$, then calculate the value $Var(Y)$.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 5.833

ERROR RANGE: 0.005

Solution: $E(Y) = E(X_1) + E(X_2)$. Note that $E(X_1) = 3.5 = E(X_2)$. Hence $E(Y) = 7$. $Var(Y) = E((X_1 + X_2 - 7)^2) = E((X_1 - 3.5)^2 + (X_2 - 3.5)^2) = Var(X_1) + Var(X_2) = 2 * 35/12 = 35/6$

B:Q64. Let X and Y have the joint p.d.f.

$$f(x, y) = \frac{3}{2}x^2(1 - |y|), -1 < x < 1, -1 < y < 1.$$

Find the value of $P(|X + Y| \leq 1)$

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.7

ERROR RANGE: 0.005

Solution: $P((X, Y) \in A) = 2\int_0^1 \int_{y-1}^1 \frac{3}{2}x^2(1 - y)dxdy = 0.7$

corrected TO $P((X, Y) \in A) = 1 - 2\int_0^1 \int_{1-x}^1 \frac{3}{2}x^2(1 - y)dydx = 0.7$

B:Q66. Let the random variables X_1 and X_2 denote the length and width, respectively of a manufactured part which is rectangular. Assume that X_1 is normal with $E(X_1) = 2$ cm and standard deviation 0.1 cm and that X_2 is normal with $E(X_2) = 5$ cm and standard deviation 0.2 cm. Also assume that X_1 and X_2 are independent. Find the probability that the perimeter of the rectangular part exceeds 14.5cm.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.1313569

ERROR RANGE: 0.005

Solution: Let $Y = 2X_1 + 2X_2$. Then Y is a normal random variable that represent the perimeter of the part. We obtain $Y \sim N(14, 0, 2)$ Now $P(Y > 14.5) = P(Z > 1.12) = 0.1313569$

NOTE: $\Phi(1.12) = 0.8686431$

B:Q72. Suppose a projectile has initial angle θ and initial velocity V. Also suppose θ follows $U[\frac{\pi}{6}, \frac{\pi}{3}]$, and V follows $N(5, 1.25)$ independently. Find the expected maximum hight attained by the projectile. [assume $g = 10$ unit]

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.65625

ERROR RANGE: 0.005

`ANS:` $E(H) = E(V^2 \sin^2\theta)/2g$

$= E(V^2)E(\sin^2\theta)/2g$

$= 26.25 * (0.5 - (3/(2*pi)) * (sin(2*pi/3) - sin(pi/3)))/(2*10) = 0.65625$

B: Q73. Consider a circle with radius $W\sqrt{X^2+Y^2}$. Let W and (X, Y) be independently distribute. If $W \sim U(0.1, 1.0)$ and $(X, Y) \sim BivariateNormal(\mu_x = 1, \mu_y = 2, \sigma_x^2 = 0.25, \sigma_y^2 = 0.25, \rho = 0.5)$. Find the expected area of the circle.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 6.393141

ERROR RANGE: 0.005

`ANS:` $E(\pi W^2(X^2+Y^2)) = \pi E(W^2)E(X^2+Y^2) = \pi(0.81/12 + (0.55)^2)(1.25 + 4.25) = 6.393141$

B:Q75. Let $Z_1, \ldots, Z_n$ be i.i.d. $N(0, 1)$ random variables. Denote $S_n = \sum_{i=1}^{n} Z_i^2$ and then find the value of

$$\lim_{n\to\infty} P(S_n \leq n).$$

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.5

ERROR RANGE: 0.005

`ANS:` $P_n = P(\chi_n^2 < n) = P(\sum_{i=1}^n Z_i^2 < n) = P((\sum_{i=1}^n Z_i^2 - n)/(\sqrt{2n}) < (n-n)/(\sqrt{2n}))$

$\lim_{n\to\infty} P_n = \Phi(0) = 0.5$ `by CLT where` Z_i`s are iid` $N(0, 1)$

B:Q80. A XII level candidate may prepare for Engineering $(X = 1)$ or may not$(X = 0)$. The candidate may also prepare for Medical $(Y = 1)$ or may not$(Y = 0)$ for future study. Let (X, Y) be random variables with joint p.m.f.

$$p(x,y) = \begin{cases} P_1^x(1-P_1)^{(1-x)}P_2^y(1-P_2)^{(1-y)} + k(-1)^{x+y} & \text{if } x, y \in \{0,1\} \\ 0 & \text{otherwise} \end{cases}$$

for any k suitably chosen in a neighbourhood of zero. If $P_1 = 0.7$ and $P_2 = 0.8$ and $k = 0.01$, find the covariance between X and Y.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.01

ERROR RANGE: 0.005

`ANS:` $COV(X,Y) = P_1 * P_2 + k - P_1 * P_2 = k = 0.01$