Probability and statistics

Lecture #3

(August 23)

Model random experiments.

R: Random expt.

Ω: sample space

Countable
- finite
- infinite

Uncountable
- $[a,b]$
- $\mathbb{R}$
- $\mathbb{R}^2$

<u>Probability assignment</u>

- Uniform probability principle

Axiomatic definition of probability

$\mathcal{F}$: collection of all events to which probability is assigned.

In case finite countable sample space

$$\mathcal{F} = 2^{\Omega}$$

In case of $[0,1] = \Omega$

$\mathcal{F}$ = Borel-σ-algebra on $[0,1]$

Conditional probability :

R : Random expt.

Ω : sample space (set of all possible outcomes).

R: Rolling a "$\underline{\text{fair}}$" die

$\Omega = \{1, 2, 3, 4, 5, 6\}$; $P\{a\} = \frac{1}{6} \quad \forall a \in \Omega$

Now if I have additional information about the random expt.

$\left\{ \begin{array}{l} \text{An even number appears on the top face} \\ \text{of the die.} \end{array} \right\} = B$

$B = \{2, 4, 6\}$

$P\{1\} = \frac{1}{6}$

Under the assumption that B has occured

$P\{1\} = 0$

$A_1 = \{1\}$ = 1 appears on the top face of the die.

$A_2 = \{2\}$

$P(A_1) = 1/6$

$P(A_2) = \frac{1}{6}$

$P(A_1 | B) = 0$

$P(A_2 | B) = \frac{1}{3}$

$$P(A|B) = \frac{\#(A \cap B)}{\#(B)} = \frac{\#(A \cap B) / \#(\Omega)}{\#(B) / \#(\Omega)} = \frac{P(A \cap B)}{P(B)}$$

Definition (conditional probability)

Let $A, B \in \mathcal{F}$ s.t. $P(B) > 0$

conditional probability of A given B is defined as:

$$P(A \mid B) = \boxed{\frac{P(A \cap B)}{P(B)}}$$

$$P(A) = P(A \mid \Omega) = \frac{P(A \cap \Omega)}{P(\Omega)}$$

<u>Product rule:</u> $P(A \cap B) = P(A \mid B)\, P(B)$ — (1)

If $P(A) > 0$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(B \mid A)\, P(A) \quad \text{— (2)}$$

Given $P(A) > 0$ and $P(B) > 0$ and from (1) & (2)

$$P(A|B)\,P(B) = P(B|A)\,P(A)$$

$$\Rightarrow \boxed{P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} \qquad P(B|A) = \frac{P(A|B)\,P(B)}{P(A)}}$$

Baye's rule

Consider $\{B_1, \ldots, B_n\}$ to be a disjoint cover of Ω.

$B_i \cap B_j = \phi$ for $i \neq j$ ← disjoint

$\bigcup_{i=1}^{n} B_i = \Omega$ ← cover

Addition requirement:

$P(B_i) \neq 0$ $\forall i$

For any $A \in \mathcal{F}$

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$$ ← disjoint union

$$P(A) = P(A \cap B_1) + \cdots + P(A \cap B_n)$$

$$= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots$$

$$\cdots + P(A|B_n)P(B_n)$$

$$\boxed{P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)}$$ ← Total law of probability.

$$\boxed{P(B_j|A) = \frac{P(A|B_j)\,P(B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}}$$

A = You score grade 'A' in Prob/stats

B_i = Attendance record

$B_1 = [0,10]$; $B_2 = [10,20]$

$\vdots$

Independence of events.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A \mid B)\, P(B)$$

Definition: A & $B \in \mathcal{F}$ are indep.

if $P(A \cap B) = P(A)\, P(B)$