Probability and Statistics

(September 13)

Lecture

- Transformations in the case of continuous r.v.s.

- "scaling"

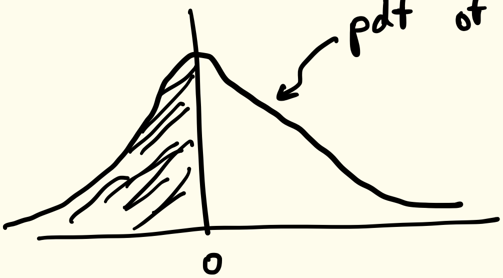
$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad -\infty < x < \infty$$

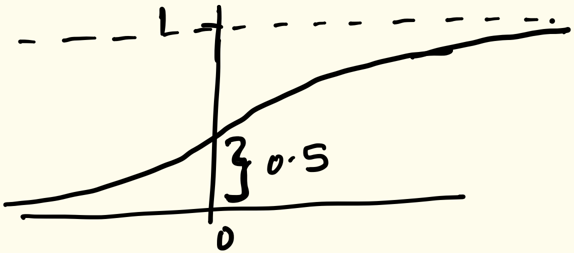
This pdf is called as standard normal density.

$X \sim N(0,1)$ (std. normal density)

$\Phi(x) =$ CDF of std. normal.

pdf of $N(0,1)$
$\varphi(x)$



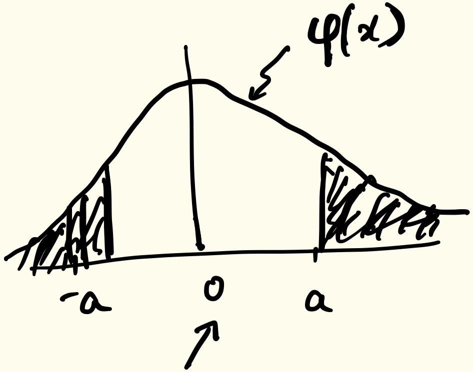


Probability computation:

$$P(a \le X \le b) = \Phi(b) - \Phi(a)$$

and $\Phi(x)$ for any real number $x \in \mathbb{R}$ can be "read" from std. normal table.

$$\Phi(a) = \int_{-\infty}^{a} \varphi(x)\,dx = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$



For $a>0$, $\Phi(a)$

For $a>0$, $\Phi(-a) = 1 - \Phi(a)$

True for any symmetric density.

Let X be standard normal r.v.

Define $\boxed{Y = \mu + \sigma X}$ where $\left.\begin{array}{l}\sigma > 0 \\ \mu\end{array}\right\} \in \mathbb{R}$

Q: Compute density of Y ??

let $g(y)$ be the density Y.

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$g(y) = \frac{1}{\sigma} \varphi\left(\frac{y-\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$$

$$\boxed{g(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \qquad -\infty < y < \infty}$$

Y is said to follow normal density with two parameters μ & σ^2.

$Y \sim N(\mu, \sigma^2)$

$$\boxed{g(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \;; \quad -\infty < y < \infty}$$

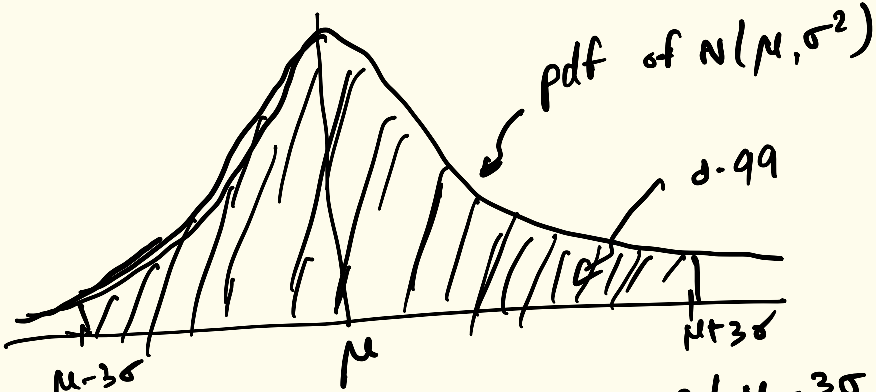
<u>Probability computation:</u>

$$\begin{aligned} P(a < Y < b) &= P(a < \mu + \sigma X < b) \\ &= P(a - \mu < \sigma X < b - \mu) \\ &= P\left(\frac{a-\mu}{\sigma} < X < \frac{b-\mu}{\sigma}\right) \\ &= \underbrace{\Phi\left(\frac{b-\mu}{\sigma}\right)}_{**} - \underbrace{\Phi\left(\frac{a-\mu}{\sigma}\right)}_{**} \end{aligned}$$

can be read from std. normal table.

$$Y \sim N(\mu, \sigma^2)$$

$$\frac{Y-\mu}{\sigma}$$



$$P(\mu - 3\sigma < Y < \mu + 3\sigma) = P(\mu - 3\sigma - \mu < Y - \mu < \mu + 3\sigma - \mu)$$

$$= P\left(\frac{-3\sigma}{\sigma} < \frac{Y-\mu}{\sigma} < \frac{3\sigma}{\sigma}\right)$$

$$= P(-3 < Z < 3)$$

$$\boxed{\text{where } Z \sim N(0,1)}$$

$$= \Phi(3) - \Phi(-3)$$

$$=$$

Gamma densities:

Let $X \sim N(0, \sigma^2)$

Consider the transformation

$$Y = X^2$$

$X \sim N(0, \sigma^2)$; $\boxed{f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \qquad -\infty < x < \infty}$

Let $g(y)$ be the density of $Y = X^2$.

$$g(y) = \frac{1}{2\sqrt{y}} \left(f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right) \qquad y > 0$$

$$\boxed{g(y) = \frac{1}{\sigma\sqrt{2\pi y}} e^{-y/2\sigma^2} \qquad y > 0}$$

Gamma density.

Gamma functions:

$$\int_0^\infty x^{\alpha-1} e^{-\lambda x}\, dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}$$

Gamma pdf:

$$\underbrace{\Gamma(x;\alpha,\lambda)}_{\text{pdf}} = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x>0 \\ 0 & \text{otherwise} \end{cases}$$

parameters of this density are α and λ.

(when $\alpha = 1$; $\Gamma(1) = 0! = 1$

$\Gamma(x; 1, \lambda) = \lambda e^{-\lambda x}$ $x > 0$: exponential density.

$\alpha = \frac{1}{2}$; $\lambda = \frac{1}{2\sigma^2}$

then one gets $g(y) = \frac{1}{\sigma\sqrt{2\pi y}} e^{-y/2\sigma^2}$ $y > 0$

$$Y = X^2 \qquad (X \sim N(0, \sigma^2))$$

H.W. $B(x; \alpha, \beta)$: densities.

r.v.s.

discrete

R_x
countable
finite/infinite

p.m.f.

$p(X=i) = f(i)$

i) Bernoulli (p)
ii) Binomial (n, p)
iii) geometric (p)
iv) NB
v) Poission
vi) Hypergeometric
vii) Discrete uniform

continuous

R_x : continuous interval

p.d.f.

$f(x) \neq P(X=x)$

$P(a<x<b) = F(b) - F(a)$

$= \int_a^b f(x)\,dx$

i) continuous uniform
ii) exp
iii) Cauchy
iv) Normal
v) Gamma
vi) Beta

only defined expectation, variance, median

Defn: Let X be discrete r.v. with R_X as its range.

$$E(X) = \sum_{x \in R_X} x\, P(X = x)$$

provided that the sum is convergent.

Ex: Let X be discrete uniform r.v. on

$R_X = \{x_1, x_2, \dots, x_n\}$

$f(x) = \frac{1}{n}$ for $x \in R_X$

$= 0$ o.w.

$$E(X) = \sum_{x \in R_X} x f(x) = \sum_{i=1}^{n} x_i\, P(x_i) = \frac{x_1 + \dots + x_n}{n}$$

Ex: Let X be a discrete r.v. with the pmf

x	x_1	x_2	$\cdots$	x_n
$f(x)$	p_1	p_2	$\cdots$	p_n

$$E(X) = \sum_{x \in R_X} x f(x) = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n$$

(In general, we can understand $E(X)$ as weighted average).

Ex: $X \sim$ Bernouli (p)

x	0	1
$f(x)$	$1-p$	p

$$E(X) = 0(1-p) + 1.(p)$$

$$\boxed{E(X) = p}$$

$0 \leq p \leq 1$

Ex: $X \sim$ Binomial (n, p)

$x = 0, 1, 2, \ldots, n$

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

0.w

$= 0$

$$E(x) = \sum_{x \in R_x} x f(x)$$

$$= \sum_{j=0}^{n} j \binom{n}{j} p^{j} (1-p)^{n-j}$$

$$n \binom{n-1}{j-1}$$

Notice: $j \binom{n}{j} = \dfrac{j\; n!}{(n-j)\; j!} = n \binom{n-1}{j-1}$

$$\boxed{E(x) = np}$$

$X \sim \text{Binomial}(n, p) \;;\; E(x) = np$

Ex: $X \sim \text{Poisson}(\lambda)$

$$f(x) = \frac{e^{-\lambda} \lambda^x}{x!} \qquad x = 0, 1, 2, \ldots$$

$$= 0 \qquad \text{o.w.}$$

$$E(X) = \sum_{j=0}^{\infty} j \, \frac{e^{-\lambda} \lambda^j}{j!}$$

$$\boxed{E(X) = \lambda}$$

Ex: Let X be a discrete r.v. with pmf

$$f(x) = \begin{cases} \frac{1}{x(x+1)} & x = 1, 2, 3, \ldots \\ 0 & \text{o.w.} \end{cases}$$

To check whether $f(x)$ is a pmf.

i) $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

ii) $$\sum_{x \in R_X} f(x) = \sum_{x=1}^{\infty} \frac{1}{x(x+1)} = \sum_{x=1}^{\infty} \left[\frac{1}{x} - \frac{1}{x+1}\right]$$

$$= 1$$

$$E(X) = \sum_{x=1}^{\infty} x f(x) = \sum_{x=1}^{\infty} x \cdot \frac{1}{x(x+1)} = \sum_{x=1}^{\infty} \frac{1}{x+1}$$

does NOT exist.

Ex: $X \sim$ geometric(p)

$$E(X) = \frac{1-p}{p}$$

} H.W.

Prepare a chart.

c.r.v.s	d.r.v.s.
1) name / pdf / cdf / $E(X)$ / var(x) / median(x) / mgf	