CS21201 Discrete Structures, Autumn 2021–2022

Second Test

Date: Oct 05, 2021 Time: 10:15am–11:30am Maximum marks: 40

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1. Prove that the following C function terminates for all non-negative integer inputs a, b, c. Here, the divisions by 2 are to be considered as divisions of `int` variables. **(10)**

```
void wow ( int a, int b, int c )
{
   int r, s, t;

   while (1) {
      if ((a == b) || (b == c) || (c == a)) break;
      r = (a + b) / 2; s = (b + c) / 2; t = (c + a) / 2;
      a = r; b = s; c = t;
   }
}
```

2. 65 distinct integers are chosen in the range $1, 2, 3, \ldots, 2021$. Prove that there must exist four of the chosen integers (call them a, b, c, d) such that $a - b + c - d$ is a multiple of 2021. **(10)**

3. Let ρ and σ be two binary relations over the set $\mathscr{A}$. A *composite relation* $\rho \circ \sigma$ over $\mathscr{A}$ is defined as

$$\rho \circ \sigma = \{(p, r) \mid \text{ there exists some } q \in \mathscr{A} \text{ such that } (p, q) \in \rho \text{ and } (q, r) \in \sigma\}.$$

Prove the following assertions with precise formal justifications.

(a) If ρ and σ are equivalence relations, then $\rho \circ \sigma$ is an equivalence relation if and only if $\rho \circ \sigma = \sigma \circ \rho$. **(6)**

(b) The *inverse* of a relation τ over $\mathscr{A}$ is defined as $\tau^{-1} = \{(q, p) \mid (p, q) \in \tau\}$ $(p, q \in \mathscr{A})$. Prove that $(\rho \circ \sigma)^{-1} = (\sigma^{-1} \circ \rho^{-1})$. **(4)**

4. Let $\mathscr{P}(S)$ denote the power set of S. For a function $f : X \to Y$, define two functions $g : \mathscr{P}(A) \to \mathscr{P}(B)$ and $h : \mathscr{P}(B) \to \mathscr{P}(A)$ as

$$\begin{aligned} g(A) &= \{b \mid \exists\, a \in A, f(a) = b\}, \text{ and} \\ h(B) &= \{a \mid f(a) \in B\} \end{aligned}$$

for all $A \subseteq X$ and $B \subseteq Y$. Prove the following assertions with precise formal justifications.

(a) f is injective if and only if $h(g(A)) = A$ for all $A \subseteq X$. **(5)**

(b) f is surjective if and only if $g(h(B)) = B$ for all $B \subseteq Y$. **(5)**