Probability and statistics

october-25

Central - limit theorem:

Let $X_1, X_2, \ldots$ be a sequence of iid random variables with mean μ and variance σ^2.

Objective: study distribution of $S_n = \sum_{i=1}^{n} X_i$

$$\boxed{\frac{S_n - n\mu}{\sigma\sqrt{n}}} = Z_n$$

Notice: $E(S_n) = n\mu$.

$\operatorname{var}(S_n) = n\sigma^2$

Let X_1 have density f, then

$$f_{S_n}(x) = \sum_{y} f_{S_{n-1}}(y)\, f(x-y) \quad \text{if } X_i\text{'s are discrete}$$

$$f_{S_n}(x) = \int_{-\infty}^{\infty} f_{S_{n-1}}(y)\, f(x-y)\, dy \quad \text{if } X_i\text{'s are continuous.}$$

Central limit theorem: (CLT)

Let $x_1, x_2, \ldots$ be iid with mean μ and variance σ^2. Let $S_n = x_1 + x_2 + \cdots + x_n$

Then

$$\lim_{n\to\infty} P\left[\underbrace{\frac{S_n - n\mu}{\sigma\sqrt{n}}}_{S_n^*} \le x\right] = \Phi(x)$$

$-\infty < x < \infty$

where $\Phi(x)$ is the CDF of $N(0,1)$.

Note that $P\left[\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right] = F_{S_n^*}(x)$

$=$ CDF of $S_n^* = \frac{S_n - n\mu}{\sigma\sqrt{n}}$

$$\lim_{n\to\infty} F_{S_n^*}(x) = \Phi(x) \quad \text{for } -\infty < x < \infty$$

$$\Rightarrow \lim_{n\to\infty} F_{S_n^*}(x) = \Phi(x) \qquad -\infty < x < \infty$$

$$\Rightarrow \lim_{n\to\infty} \boxed{\frac{F_{S_n^*}(x)}{\Phi(x)}} = 1 \qquad -\infty < x < \infty$$

$\underbrace{X_1, \ldots, X_n, \ldots}$ a sequence of $\underline{\text{iid}}$ r.v.s. with mean μ and var σ^2.

$$S_n = \sum_{i=1}^{n} X_i \; ; \quad E(S_n) = n\mu \; ; \quad var(S_n) = n\sigma^2 \; ; \quad S_n^* = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$$E(S_n^*) = 0$$

$$var(S_n^*) = 1$$

$$\text{CLT} \Rightarrow \lim_{n\to\infty} F_{S_n^*}(x) = \Phi(x) \qquad -\infty < x < \infty$$

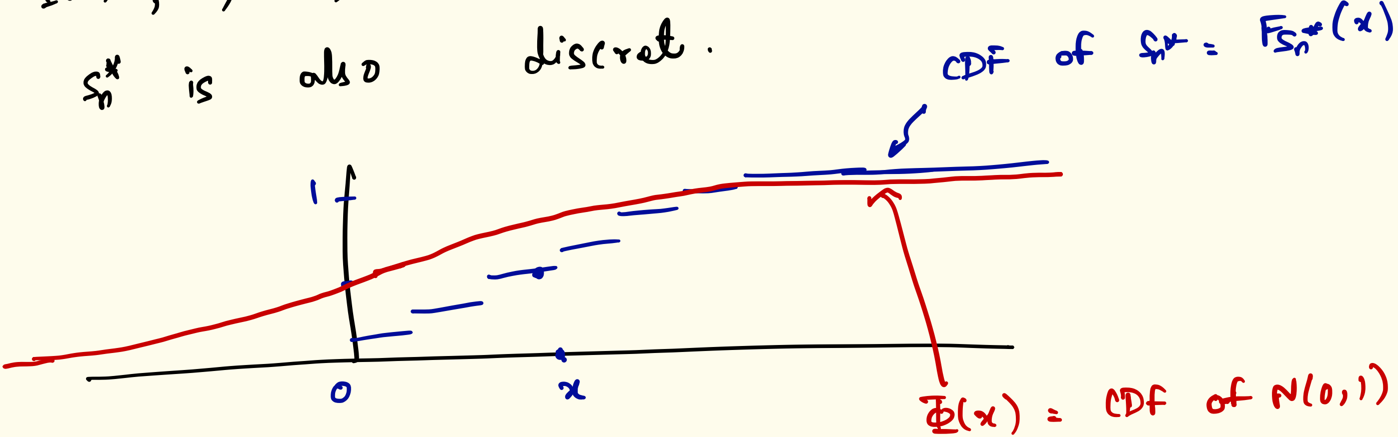
The CDF of S_n^* "behaves" like that of $N(0,1)$ as $n \to \infty$.

Notice:

i) the limit talks about CDFs and not pmfs/pdfs.

ii) Given random variables $X_1, X_2, \dots$ may be discrete or continuous.

If $X_1, \dots, X_n, \dots$ is a seq. of discrete r.v.s. S_n^* is also discret.



Exercise: