Probability and Statistics

(September - 20)

Random variable:

pmf / pdf

cdf

R - random expt.

$(\Omega, \mathcal{F}, P)$ -

X r.v. $\quad X : \Omega \longrightarrow \mathbb{R}$

$$X^{-1}(x) = \{ \omega : X(\omega) = x \} \in \mathcal{F}$$

Random vector:

$(\Omega, \mathcal{F}, P)$

let $X_1, X_2, \ldots, X_r$ be r discrete random variables on $(\Omega, \mathcal{F}, P)$.

Define $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \end{bmatrix}$ is an r-dimensional vector.

For an $\omega \in \Omega$

$$\underline{X}(\omega) = \begin{bmatrix} X_1(\omega) \\ X_2(\omega) \\ \vdots \\ X_r(\omega) \end{bmatrix} \in \mathbb{R}^r$$

Suppose $x_1 = X_1(\omega)$; $x_2 = X_2(\omega), \dots, x_r = X_r(\omega)$

$$\underline{X}(\omega) = \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} \in \mathbb{R}^r$$

Definition:

Let $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}$ be a vector where x_i is a random variable on $(\Omega, \mathcal{F}, P)$.

For $\underline{x} \in \mathbb{R}^r$ the set

$$\{\omega \in \Omega \mid \underline{X}(\omega) = \underline{x}\} \in \mathcal{F}.$$

Then $X : \Omega \to \mathbb{R}^r$ is called as an r-dimensional random vector.

We are interested in $\text{Prob}(\underline{X} = \underline{x})$

Let $\underline{X} = \begin{pmatrix} X_1(\omega) \\ \vdots \\ X_r(\omega) \end{pmatrix}$ be a random vector.

Let $\underline{x}$ denote the value assumed by the random vector $\underline{X}$, then if

$\{\underline{x} \mid \text{Prob}(\underline{X}(\omega) = \underline{x}) > 0\}$ is finite or countably infinite.

Then $\underline{X}$ is a discrete random vector.

<u>Definition</u>: The discrete joint p.m.f. of the random vector $\underline{X}$ is defined as:

$$f(x_1, \cdots, x_r) = \text{Prob}(X_1 = x_1, X_2 = x_2, \cdots, X_r = x_r)$$

where $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix}$ and $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_r \end{bmatrix}$.

In the vector notation

$$f(\underline{x}) = P(\underline{X} = \underline{x}) \quad \forall \ \underline{x} \in \mathbb{R}^r$$

For a subset $A \subseteq \mathbb{R}^r$

$$P(\underline{X} \in A) = \sum_{\underline{x} \in A} f(\underline{x})$$

Definition:

A function f is called as discrete joint p.m.f. if

i) $f(\underline{x}) \geq 0 \quad \forall \; \underline{x} \in \mathbb{R}^r$

$R_{\underline{X}}$: range of $\underline{X}$.

ii) $\{\underline{x} \mid f(\underline{x}) > 0\}$ is finite or countably infinite. This set $\underline{x}_1, \underline{x}_2, \underline{x}_3, \ldots$

iii) $$\sum_i f(\underline{x}_i) = 1$$

Ex: A discrete r.v. X_1

x_i	1	2
$f(x_i)$	$1/2$	$1/2$

$R_{X_1} = \{1, 2\}$

Ex:

$X_1 \backslash X_2$	1	2	3	4	
1	$1/4$	$1/8$	$1/16$	$1/16$	$1/2$
2	$1/16$	$1/16$	$1/4$	$1/8$	$1/2$
	$5/16$	$3/16$	$5/16$	$3/16$	$\boxed{1}$ ✓

(All the three conditions are satisfied)

Is this function a joint pmf??

$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \underline{X}$; $R_{\underline{X}} = \left\{ \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \boxed{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}, \boxed{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$

$$\text{Prob}(x_1 \geq x_2) = \text{Prob}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$$

$$= \frac{1}{4} + \frac{1}{16} + \frac{1}{16}$$

$$= \frac{3}{8}$$

$$\text{Prob}(x_1 = 1) = \text{Prob}(x_1 = 1, x_2 = 1) + \text{Prob}(x_1 = 1, x_2 = 3)$$
$$+ \text{Prob}(x_1 = 1, x_2 = 2)$$
$$+ \text{Prob}(x_1 = 1, x_2 = 4)$$

$$= \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16}$$

$$= \frac{1}{2}$$

$$\left.\begin{aligned} \text{Prob}(X_1 = 1) &= \sum_{x_2} f(1, x_2) = \frac{1}{2} \\ \text{Prob}(X_1 = 2) &= \sum_{x_2} f(2, x_2) = \frac{1}{2} \end{aligned}\right\} \text{ pmf of } X_1$$

$$\boxed{f_{X_1}(x_1) = \sum_{x_2} f(x_1, x_2) \quad \text{for } x_1 \in R_{X_1}}$$

↑ marginal pmf of X_1

x_1	$f(x_1)$
1	1/2
2	1/2

: marginal pmf of x_1

x_2	1	2	3	4
$f(x_2)$	5/16	3/16	5/16	3/16

: marginal pmf x_2

$$\boxed{f_{X_2}(x_2) = \sum_{x_1} f(x_1, x_2) \quad \forall x_2 \in R_{X_2}}$$

marginal pmf of X_2

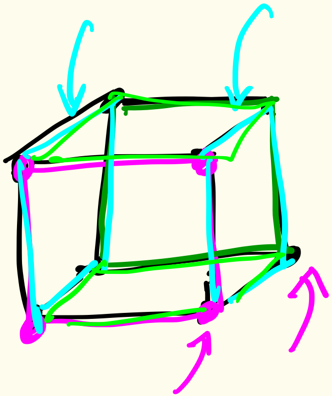
Ex: Toss three coins simultaneously.

Heads – 1

Tails – 0

$$\left\{ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$$

joint pmf.



$$f_{X_1}(x_1) = \sum_{\boxed{x_2, x_3}} f(x_1, \boxed{x_2, x_3})$$

$\forall x_1 \in R_{X_1}$

$\boxed{\text{tensor}}$

Independent random variables:

Let $X_1, X_2, \ldots, X_r$ be r discrete random variables with p.m.fs $f_1, f_2, \ldots, f_r$ respectively.

Then the random variables $X_1, \ldots, X_r$ are called as mutually independent if their joint p.m.f. $f(x_1, \ldots, x_r)$ is given as:

$$f(x_1, \ldots, x_r) = f_1(x_1) \cdot f_2(x_2) \cdot \ldots f_r(x_r)$$

[joint pmf can be written as the product of marginal pmfs.]

Notation:

$$f(x_1, \ldots, x_r) = \text{Prob}(X_1 = x_1, X_2 = x_2, \ldots, X_r = x_r)$$

Define $A_i \subseteq \Omega$ s.t. $A_i = [X_i = x_i]$

$$= \{\omega \in \Omega \mid X_i(\omega) = x_i\}$$

for $i = 1, 2, \ldots, r$

$$f(x_1, \ldots, x_r) = \text{Prob}(X_1 = x_1, X_2 = x_2, \ldots, X_r = x_r)$$

$$= \text{Prob}(A_1 \cap A_2 \cap \cdots \cap A_r)$$

independence of events

$$= \text{Prob}(A_1) \cdot \text{Prob}(A_2) \cdots \text{Prob}(A_r)$$

$$= \text{Prob}(X_1 = x_1) \cdot \text{Prob}(X_2 = x_2) \cdots \text{Prob}(X_r = x_r)$$

$$\underbrace{f(x_1, \ldots, x_r)} = \underbrace{f_1(x_1) \cdot f_2(x_2) \cdots f_r(x_r)}$$

Ex:

$X_1 \backslash X_2$	0	1	
0	1/4	1/4	1/2
1	1/4	1/4	1/2
	1/2	1/2	

Are X_1 and X_2 indep.??

$$f(x_1, x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2)$$

Ex:

$X_1 \backslash X_2$	0	1
0	$(1-p_1)(1-p_2)$	$(1-p_1)p_2$
1	$(1-p_2)p_1$	$p_1 p_2$

$\leftarrow$ independent X_1 & X_2 ??

$$p_1 = p_2 = \frac{1}{2}$$

$f_{X_1}(0) = 1-p_1$ $\qquad$ $f_{X_2}(0) = 1-p_2$

$f_{X_1}(1) = p_1$ $\qquad$ $f_{X_2}(1) = p_2$

Ex: Let x_1 and x_2 be two independent random variable each with geometric distribution with parameter p.

Find the distribution of $\min(x_1, x_2) = Z$.

$R_Z = \{0, 1, 2, \ldots\}$

$$\text{Prob}(\min(x_1, x_2) \geq z) = \text{Prob}(x_1 \geq z,\ x_2 \geq z)$$

$$= \text{Prob}(x_1 \geq z) \cdot \text{Prob}(x_2 \geq z)$$

$$= (1-p)^z \ (1-p)^z$$

$$= ((1-p)^2)^z$$

$\blacksquare$