Relations

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Cartesian Product

Definition: Cartesian Product or Cross Product of two sets, $\mathcal{A}$ and $\mathcal{B}$, denoted as $\mathcal{A} \times \mathcal{B}$, is defined by, $\mathcal{A} \times \mathcal{B} = \{(a, b) \mid a \in \mathcal{A}, b \in \mathcal{B}\}$

Generically, $\mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k = \{(x_1, x_2, \ldots, x_k) \mid \forall i,\ x_i \in \mathcal{A}_i\}$

Ordered Pairs: The elements of $(\mathcal{A} \times \mathcal{B})$ are called ordered pairs.

Generically, the elements, $(x_1, x_2, \ldots, x_k) \in \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k$ (k-fold Cartesian product), are called ordered k-tuples.

Cardinality: Let, $|\mathcal{A}_1| = n_1, |\mathcal{A}_2| = n_2, \ldots, |\mathcal{A}_k| = n_k$. Then, $|\mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k| = |\mathcal{A}_1||\mathcal{A}_2| \cdots |\mathcal{A}_k| = n_1 n_2 \cdots n_k$.

Properties: For $(a, b), (c, d) \in \mathcal{A} \times \mathcal{B}$, we have $(a, b) = (c, d)$ if and only if $a = b$ and $c = d$.

Note that, $\mathcal{A} \times \mathcal{B} \neq \mathcal{B} \times \mathcal{A}$, but $|\mathcal{A} \times \mathcal{B}| = |\mathcal{A}||\mathcal{B}| = |\mathcal{B} \times \mathcal{A}|$.

Other Properties: Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{U}$

(i) $\mathcal{A} \times \phi = \phi \times \mathcal{A} = \phi$

(ii) $\mathcal{A} \times (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \times \mathcal{B}) \cap (\mathcal{A} \times \mathcal{C})$

(iii) $\mathcal{A} \times (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \times \mathcal{B}) \cup (\mathcal{A} \times \mathcal{C})$

(iv) $(\mathcal{A} \cap \mathcal{B}) \times \mathcal{C} = (\mathcal{A} \times \mathcal{C}) \cap (\mathcal{B} \times \mathcal{C})$

(v) $(\mathcal{A} \cup \mathcal{B}) \times \mathcal{C} = (\mathcal{A} \times \mathcal{C}) \cup (\mathcal{B} \times \mathcal{C})$

(vi) $(\mathcal{A} - \mathcal{B}) \times \mathcal{C} = (\mathcal{A} \times \mathcal{C}) - (\mathcal{B} \times \mathcal{C})$

(vii) $\mathcal{A} \times (\mathcal{B} - \mathcal{C}) = (\mathcal{A} \times \mathcal{B}) - (\mathcal{A} \times \mathcal{C})$

Relations and Examples

(Binary) Relation

Definition: A (binary) relation, ρ, between two sets, $\mathcal{A}$ and $\mathcal{B}$, is defined as, $\rho \subseteq \mathcal{A} \times \mathcal{B}$. If an ordered pair, $(a, b) \in \rho$ (or $a \; \rho \; b$), then the element, $a \in \mathcal{A}$, is said to be *related* to the element, $b \in \mathcal{B}$.

- Any subset of $(\mathcal{A} \times \mathcal{A})$ (or $\mathcal{A}^2$) is called a relation on $\mathcal{A}$.
- The relation, $\rho = \mathcal{A} \times \mathcal{B}$, is called the *universal relation*.

Count: Total number of (binary) relations between two sets, $\mathcal{A}$ and $\mathcal{B}$ (where, $|\mathcal{A}| = m$ and $|\mathcal{B}| = n$), is the number of possible subsets of $(\mathcal{A} \times \mathcal{B})$, i.e. 2^{mn}.

Example

Let $\mathcal{A} = \{1, 2, 3\}$ and $\mathcal{B} = \{a, b\}$. So, the Cartesian products are defined as,

$\mathcal{A} \times \mathcal{B} = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$ and

$\mathcal{B} \times \mathcal{A} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

Clearly, $\mathcal{A} \times \mathcal{B} \neq \mathcal{B} \times \mathcal{A}$, however $|\mathcal{A} \times \mathcal{B}| = 6 = |\mathcal{B} \times \mathcal{A}|$.

There can be a total of $2^6 = 64$ different (binary) relations possible. Some are:

$\rho_1 = \{(1, a), (1, b), (1, c)\}$ or $\rho_2 = \{(2, a), (3, a), (1, b), (3, b)\}$.

Types and Properties of Relations

Let a relation, ρ, is defined over the set, $\mathcal{A}$ with $|\mathcal{A}| = n$, as $\rho \subseteq \mathcal{A} \times \mathcal{A}$. (Count: 2^{n^2})

Reflexive: ρ is reflexive if $\forall x \in \mathcal{A},\ (x, x) \in \rho$

Count: $2^{n^2 - n}$ (after choosing all n number of (x, x) pairs, any subset from $(n^2 - n)$ pairs can be taken as relation keeping reflexivity)

Symmetric: ρ is symmetric if $\forall x, y \in \mathcal{A},\ (x, y) \in \rho \Rightarrow (y, x) \in \rho$

Count: $2^{\frac{n^2+n}{2}}$ (selecting an $(x, y) + (x, x)$ pair in $\binom{n}{2} + n$ ways, any subset from $\binom{n}{2} + n$ pairs can be taken as relation keeping symmetry)

Transitive: ρ is transitive if $\forall x, y, z \in \mathcal{A},\ (x, y), (y, z) \in \rho \Rightarrow (x, z) \in \rho$

Count: *Unknown* (still an open-problem!)

Antisymmetric: ρ is antisymmetric if $\forall x, y \in \mathcal{A},\ (x, y), (y, x) \in \rho \Rightarrow (x = y)$

Count: $2^n 3^{\frac{n^2-n}{2}}$ (element (x, x) can either be included or excluded; element (x, y) have three options – (i) take only (x, y), (ii) take only (y, x), or (iii) take neither (x, y) nor (y, x). *What if take both?*)

Irreflexive: ρ is irreflexive if $\exists x \in \mathcal{A},\ (x, x) \notin \rho$

Asymmetric: ρ is asymmetric if $\exists x, y \in \mathcal{A},\ (x, y) \in \rho \wedge (y, x) \notin \rho$

Non-Transitive: ρ is non-transitive if $\exists x, y, z \in \mathcal{A},\ (x, y), (y, z) \in \rho \wedge (x, z) \notin \rho$

Not Antisymmetric: ρ is not antisymmetric if $\exists x, y \in \mathcal{A},\ (x, y), (y, x) \in \rho \wedge (x \neq y)$

Examples of Relations

1. *Reflexive and Symmetric, but NOT Transitive*:

   ρ is defined over $\mathbb{Z}$ as, $\rho = \{(x, y) \mid xy \geq 0 \text{ and } x, y \in \mathbb{Z}\}$

   (Reflexive as $x^2 \geq 0$, Symmetric as $xy = yx$, NOT Transitive for $x = 2, y = 0, z = -1$)

2. *Symmetric and Transitive, but NOT Reflexive*:

   ρ is defined over $\mathbb{R}$ as, $\rho = \{(x, y) \mid xy > 0 \text{ and } x, y \in \mathbb{R}\}$

   (NOT Reflexive for $x = 0$, Symmetric as $xy = yx$, Transitive as $xz = \frac{(xy).(yz)}{y^2} > 0$ since $xy > 0, yz > 0, y^2 > 0$)

3. *Reflexive and Transitive, but NOT Symmetric* (Antisymmetric):

   ρ is defined over $\mathbb{R}$ as, $\rho = \{(x, y) \mid x \leq y \text{ and } x, y \in \mathbb{R}\}$

   (Reflexive as $x \leq x$, NOT Symmetric for $x = 0.1, y = 1.0$, Transitive as $x \leq y \leq z$)

4. *NOT Reflexive, NOT Symmetric, NOT Transitive, BUT Antisymmetric*:

   ρ is defined over $\mathbb{Z}$ as, $\rho = \{(x, y) \mid y = x + 1 \text{ and } x, y \in \mathbb{Z}\}$

   (NOT Reflexive as $x \neq x + 1$, NOT Symmetric as $y = x + 1 \Rightarrow x = y - 1$, NOT Transitive as $z = y + 1 = x + 2$)

5. *Only Reflexive*: Relation $\rho = \{(A, B) \mid \text{Person-}A \text{ knows Person-}B\}$
6. *Only Symmetric*: Relation $\rho = \{(A, B) \mid A + B = 5 \text{ and } A, B \in \mathbb{Z}\}$
7. *Only Transitive*: Relation $\rho = \{(A, B) \mid A \subset B \text{ and } A, B \in \mathcal{U}\}$
8. *Only Antisymmetric*: *Left for You to find as an Exercise!*

Equivalence Relation and Equivalence Classes

Equivalence Relation: A relation $\rho \subseteq \mathcal{A} \times \mathcal{A}$ on set $\mathcal{A}$ is called an equivalence relation if it is reflexive, symmetric and transitive.

Example: $\rho = \{(x, y) \mid (x - y)$ is divisible by 5 and $x, y \in \mathbb{Z}\}$

- Reflexive since $(x - x) = 0$ is divisible by 5.
- Symmetric since $(y - x) = -(x - y)$ is divisible by 5.
- Transitive since $(x - z) = (x - y) + (y - z)$ is divisible by 5.

Fallacy: Does *Symmetric* + *Transitive* $\Rightarrow$ *Reflexive*? Why define Reflexivity?

[from $(x, y) \in \rho \Rightarrow (y, x) \in \rho$ and $(x, y), (y, x) \in \rho \Rightarrow (x, x) \in \rho$]

Reason: NO, since for all x, an y may not be found/associated!

Equivalence Class: Let ρ be an equivalence relation on $\mathcal{A}$. For each $y \in \mathcal{A}$, the equivalence class is denoted by $[y] = \{x \mid (x, y) \in \rho$ and $x \in \mathcal{A}\}$.

Example: In the relation, $\rho = \{(x, y) \mid (x - y)$ is divisible by 3 and $x, y \in \mathbb{Z}\}$, the four equivalence classes are defined as:

- $[0] = \{\ldots, -6, -3, 0, +3, +6, \ldots\} = \{3k \mid k \in \mathbb{Z}\}$
- $[1] = \{\ldots, -5, -2, 1, +4, +7, \ldots\} = \{3k + 1 \mid k \in \mathbb{Z}\}$
- $[2] = \{\ldots, -4, -1, 2, +5, +8, \ldots\} = \{3k + 2 \mid k \in \mathbb{Z}\}$

Note: $[0] = [-3] = [+3] = [-6] = [+6] = \cdots$ (from definition)

$[0] \neq [1] \neq [2]$ and $\mathbb{Z} = [0] \cup [1] \cup [2]$ (details in next slide)

Equivalence Classes and Partitions

Theorem: If ρ is an equivalence relation on $\mathcal{A}$ and $x, y \in \mathcal{A}$, then
(i) $x \in [x]$; **(ii)** $(x, y) \in \rho$ iff $[x] = [y]$; and **(iii)** $[x] = [y]$ or $[x] \cap [y] = \phi$
Proof:

1. From Reflexive property, $(x, x) \in \rho$.
2. *[If]* Let $a \in [x] \Rightarrow (a, x) \in \rho$. As $(x, y) \in \rho$, so using transitivity, we get $(a, y) \in \rho \Rightarrow a \in [y]$. Hence, $[x] \subseteq [y]$. Again, let $b \in [y] \Rightarrow (b, y) \in \rho$. By symmetry, $(x, y) \in \rho \Rightarrow (y, x) \in \rho$. So, using transitivity, $(b, x) \in \rho \Rightarrow b \in [x]$. Hence, $[y] \subseteq [x]$.
   [Only-If] $x \in [x]$ and $[x] = [y]$ implies $x \in [y] \Rightarrow (x, y) \in \rho$.
3. Assume $[x] \neq [y]$, then $[x] \cap [y] = \phi$ must hold. If otherwise $[x] \cap [y] \neq \phi$, then let $u \in [x]$ and $u \in [y]$. Thus, $(u, x) \in \rho$ and by symmetry, $(x, u) \in \rho$. With $(u, y) \in \rho$, applying transitivity we get, $(x, y) \in \rho \Rightarrow [x] = [y]$, which contradicts the assumption!

Partitions of a Set (Revisited)

Given set $\mathcal{A}$ and index set $\mathcal{I}$, let $\forall i,\ \phi \neq \mathcal{A}_i \subseteq \mathcal{A}$. Then $\{\mathcal{A}_i\}_{i \in \mathcal{I}}$ induces a partition on $\mathcal{A}$ if:
(i) $\mathcal{A} = \bigcup_{i \in \mathcal{I}} \mathcal{A}_i$, and (ii) $\mathcal{A}_i \cap \mathcal{A}_j = \phi,\ \forall i, j \in \mathcal{I}\ (i \neq j)$.

Results: (i) Any equivalence relation ρ on set $\mathcal{A}$ induces a partition of $\mathcal{A}$.

Proof: Follows from the above theorem.

(ii) Any partition of $\mathcal{A}$ gives rise to an equivalence relation ρ on $\mathcal{A}$.

Proof: Left for You as an Exercise!