Indian Institute of Technology Kharagpur
Mid-Semester Examination: Autumn 2022

Date of Examination: 27/09/2022 (FN) **Duration: 2 Hrs**
Subject. No: AI61003 **Subject Name: Linear Algebra for AI and ML**
Department: CoEAI **TOTAL MARKS: 40**
Specific Chart, graph paper log book etc. required: None
Special Instruction: None

ANSWER ALL THE QUESTIONS

1. State whether the following statements are *TRUE* or *FALSE*. Justify your answer with a proof or a counter example. No marks will be awarded without justification. [10 marks]

   ✓(a) Whenever a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is such that $\mathbf{A}^2 = 0$, then the matrix $\mathbf{A}$ is a zero matrix.

   ✓(b) For $\mathrm{Q} \in \mathbb{R}^{n \times n}$ orthogonal and $\mathbf{x} \in \mathbb{R}^n$, $\|Qx\|_1 = \|x\|_1$.

   ✓(c) If rows of a matrix $\mathbf{A}$ are linearly independent, then $\mathbf{A}$ is right invertible.

   (d) For a square matrix, eigenvectors corresponding to distinct eigenvalues are linearly independent.

   (e) Let $\det(\mathbf{A})$ denote the determinant of a square matrix $\mathbf{A}$. If $\det(\mathbf{A})$ is close to zero, then $\mathbf{A}$ is close to singularity.

✓ 2. A matrix vector product $\mathbf{Ax}$ takes $2n^2$ flops in general when $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{x} \in \mathbb{R}^n$. Device a computationally more efficient algorithm to multiply matrix $\mathbf{A}$ with a vector $\mathbf{x}$ when $\mathbf{A}$ is of the form $\mathbf{A} = \mathbf{I}_n + \mathbf{a}\mathbf{b}^\top$ where $\mathbf{I}_n$ is the $n \times n$ identity matrix and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ are some given vectors. [3 marks]

3. Let $\mathbf{x} \in \mathbb{R}^n$ and let $\mathbf{y} \in \mathbb{R}^n$ be a vector with non-negative entries such that $\mathbf{y}$ is closest to $\mathbf{x}$. (Note that the closeness is measured using $\|\cdot\|_2$ norm on $\mathbb{R}^n$.) Determine the expression of $\mathbf{y}$. Further, show that $\mathbf{y}^\top(\mathbf{y} - \mathbf{x}) = 0$. [3 marks]

✓ 4. Let $\mathbf{x}_1 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^\top \in \mathbb{R}^3$ and $\mathbf{x}_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^\top \in \mathbb{R}^3$ be two vectors. Does there exist a common left inverse for these two vectors? If yes, compute. If no, justify. (A common left inverse is a matrix which is simultaneously a left inverse for $\mathbf{x}_1$ and $\mathbf{x}_2$.) [3 marks]

✓ 5. For a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, define the maximum magnification of $\mathbf{A}$ (denoted as maxmag($\mathbf{A}$)) and the minimum magnification of $\mathbf{A}$ (denoted as minmag($\mathbf{A}$)). Further, for an invertible matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, prove that

$$\kappa(\mathbf{A}) = \frac{\mathrm{maxmag}(\mathbf{A})}{\mathrm{minmag}(\mathbf{A})}$$

where $\kappa(\mathbf{A})$ denotes the condition number of $\mathbf{A}$. [4 marks]

6. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and the columns of $\mathbf{A}$ are linearly independent. Let $\widehat{x}$ denote the least squares solution to the problem $\mathbf{Ax} = \mathbf{b}$. Prove that the least squares solution is unique. Discuss when the least squares solution is not unique. [4 marks]

7. Suppose vectors $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$ are such that they are approximately linearly related as $\mathbf{y} \approx \mathbf{Ax}$. Here we do not know the matrix $\mathbf{A}$; however, we have observed data vectors

$$\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(N)}, \quad \mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \ldots, \mathbf{y}^{(N)}.$$

Formulate this problem as the least squares problem to estimate the matrix $\mathbf{A}$. Write down the least squares solution to this problem in terms of pseudo inverse. [6 marks]

P.T.O.

8. For a given invertible matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and a given vector $\mathbf{b} \in \mathbb{R}^n$, let $\mathbf{c} = \mathbf{A}\mathbf{b}$. Further, let $\delta\mathbf{b} \in \mathbb{R}^n$ and $\delta\mathbf{c} \in \mathbb{R}^n$ be such that

$$\mathbf{A}\left(\mathbf{b} + \delta\mathbf{b}\right) = \mathbf{c} + \delta\mathbf{c}$$

✓ (a) Prove the following. [3 marks]

$$\frac{\|\delta\mathbf{c}\|_2}{\|\mathbf{c}\|_2} \leqslant \kappa_2(\mathbf{A})\,\frac{\|\delta\mathbf{b}\|_2}{\|\mathbf{b}\|_2}$$

where $\kappa_2(\mathbf{A})$ is the condition number of $\mathbf{A}$.

(b) Determine the direction of $\delta\mathbf{b}$ such that $\mathbf{c} + \delta\mathbf{c}$ is possibly farthest from $\mathbf{c}$. [2 marks]

(c) From the inequality in item (8a), discuss why orthogonal matrices are preferred in numerical linear algebra. [2 marks]

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