Lecture 04

How to Implement a Dictionary?

- Sequences
  - ordered
  - unordered
- Binary Search Trees
- **Hash tables**

Hashing

- Another important and widely useful technique for implementing dictionaries
- Constant time per operation (on the average)
- Worst case time proportional to the size of the set for each operation (just like array and chain implementation)

Hashing - Basic Idea

- Use *hash function* to map keys into positions in a *hash table*

<u>Ideally</u>

- If element *e* has key *k* and *h* is hash function, then *e* is stored in position *h(k)* of table
- To search for *e*, compute *h(k)* to locate position. If no element, dictionary does not contain *e*.

Hash function example

- elements = Integers
- *h(i) = i % 10*
- insert 41, 34, 7, and 18
- constant-time lookup:
  - just look at *i % 10* again later

- Hash tables have no ordering information!
  - Expensive to do following:
    - getMin, getMax, removeMin, removeMax,
    - the various ordered traversals
    - printing items in sorted order

0	
1	41
2	
3	
4	34
5	
6	
7	7
8	18
9	

Hashing Operations

- **Search**
  - looks for key *k*
- **Insert**
  - first searches for a slot, then inserts
- **Delete**
  - Cannot just turn the slot containing the key we want to delete to contain NIL. Why?

Hashing Analysis

- Analysis
  - O(b) time to initialize hash table (b number of positions or buckets in hash table)
  - O(1) time to perform *insert, remove, search*
- Reality
  - Works for implementing dictionaries, but many applications have key ranges that are too large to have 1-1 mapping between buckets and keys!
  - Example:
    - Suppose key can take on values from 0 .. 65,535 (2 byte unsigned int)
    - Expect $\approx$ 1,000 records at any given time
    - Impractical to use hash table with 65,536 slots!

Hash Collisions

- **Collision:** the event that two hash table elements map into the same slot in the array
  - example: insert 41, 34, 7, 18, then 21
  - 21 hashes into the same slot as 41!

- **Resolution**:
  - How can we choose the hash function to minimize collisions?
  - What do we do about collisions when they occur?

0	
1	21
2	
3	
4	34
5	
6	
7	7
8	18
9	

Collision Resolution Policies

- Two classes:
  1. Closed hashing / open addressing
  2. Open hashing / separate chaining

- Difference has to do with whether collisions are stored *outside the table* (open hashing) or whether collisions result in storing one of the records at *another slot in the table* (closed hashing)

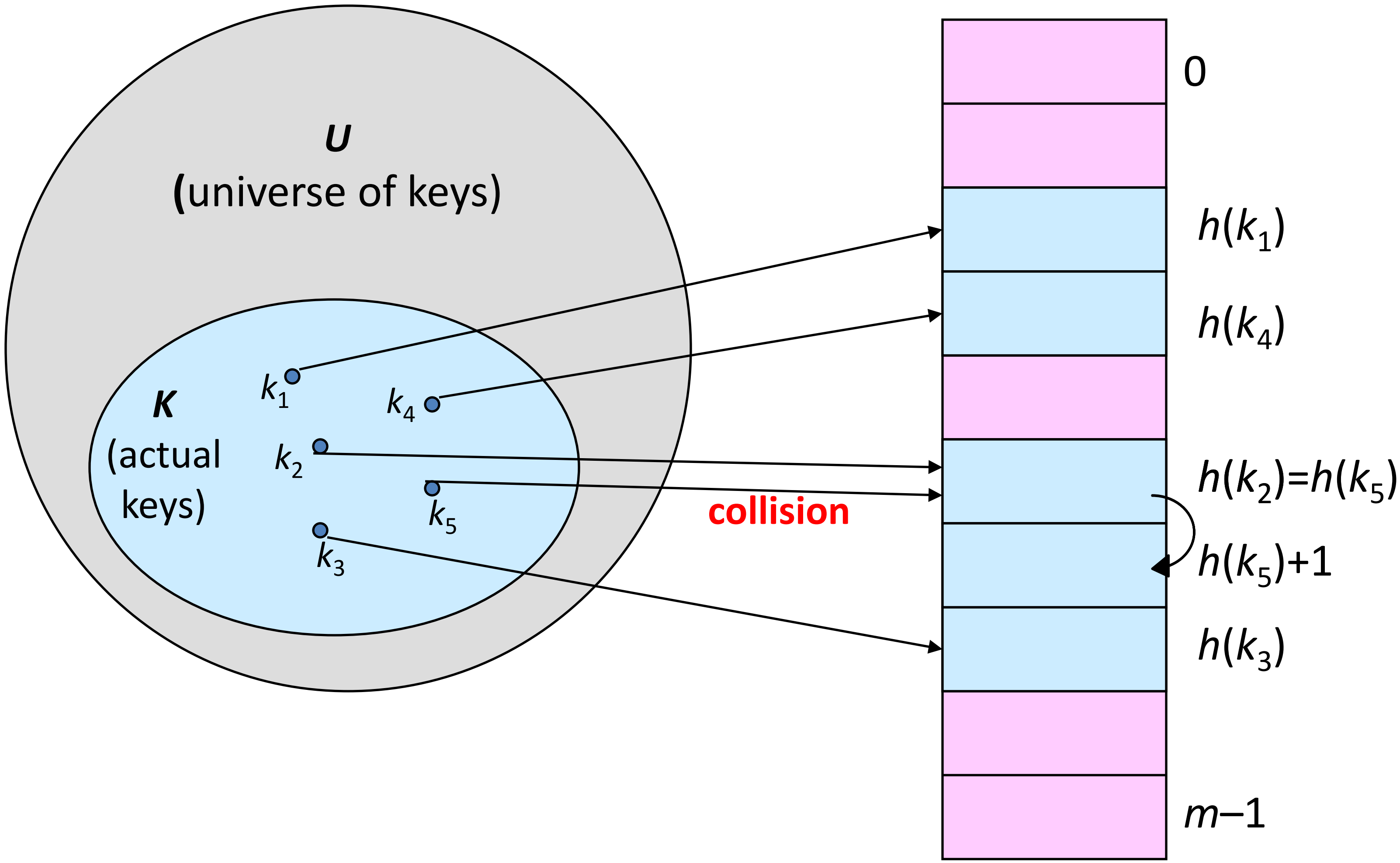
Open Addressing

- **Concept:**
  - Store all n keys in the m slots of the hash table itself.
  - Each slot contains either a key or NIL.
  - To ***search*** for key k:
    - Examine slot $h(k)$. Examining a slot is known as a probe.
    - If slot $h(k)$ contains key k, the search is successful. If the slot contains NIL, the search is unsuccessful.
    - There's a third possibility: slot $h(k)$ contains a key that is not k.
      - Compute the index of some other slot, based on k and which probe we are on.
      - Keep probing until we either find key k or we find a slot holding NIL.

Advantages: Avoids pointers; so less code, and we can dedicate the memory to the table.

What can you say about the load factor $\alpha = n/m$?

Open addressing - issue



Closed Hashing

- Associated with closed hashing is a *rehash strategy*:

  "If we try to place x in bucket $h(x)$ and find it occupied, find alternative location $h_1(x)$, $h_2(x)$, etc. Try each in order, if none empty table is full,"

- $h(x)$ is called *home bucket*

- Simplest rehash strategy is called *linear hashing*

$$h_i(x) = (h(x) + i) \% D$$

- In general, our collision resolution strategy is to generate a sequence of hash table slots (probe sequence) that can hold the record; test each slot until find empty one (probing)

Importance of Good Hash Functions

- Recall the assumption of *simple uniform hashing:*
  - Any key is equally likely to hash into any of the slots, independent of where any other key hashes to.
  - $O(1)$ time to compute $h(k)$.

- Hash values should be independent of any patterns that might exist in the data.
  - E.g. If each key is drawn independently from U according to a probability distribution P, we want

$$\text{for all } j \in [0 \ldots m-1], \sum_{k:h(k) = j} P(k) = 1/m$$

- Often use heuristics, based on the domain of the keys, to create a hash function that performs well.

Two examples only

- **Division method**
  - Map each key k into one of the m slots by taking the remainder of k divided by m.
    $$\boldsymbol{h(k) = k \bmod m}$$
  - Example: $m = 31$ and $k = 78 \Rightarrow h(k) = 16$.
  - Advantage: Fast, since requires just one division operation.
  - Disadvantage: For some values, such as $m=2^p$, the hash depends on just a subset of the bits of the key.
  - Note: Primes are good, if not too close to power of 2 (or 10).

- **Multiplication method**
  - Map each key k to one of the m slots indicated by the fractional part of k times a chosen real $0 < A < 1$.
    $$\boldsymbol{h(k) = \lfloor m\,(kA \bmod 1) \rfloor = \lfloor m\,(kA - \lfloor kA \rfloor) \rfloor}$$
  - Example: $m = 1000$, $k = 123$, $A \approx 0.6180339887\ldots$
    $$h(k) = \lfloor 1000(123 \cdot 0.6180339887 \bmod 1) \rfloor$$
    $$= \lfloor 1000 \cdot 0.0181\ldots \rfloor = 18.$$
  - Disadvantage: A bit slower than the division method.
  - Advantage: Value of m is not critical.

Homework

Implement these two techniques.

Example Linear (Closed) Hashing

- D=8, keys *a,b,c,d* have hash values h(a)=3, h(b)=0, h(c)=4, h(d)=3

- Where do we insert *d*? 3 already filled
- Probe sequence using linear hashing:

  $h_1(d) = (h(d)+1)\%8 = 4\%8 = 4$

  $h_2(d) = (h(d)+2)\%8 = 5\%8 = \mathbf{5}^*$

  $h_3(d) = (h(d)+3)\%8 = 6\%8 = 6$

  etc.

  7, 0, 1, 2
- Wraps around the beginning of the table!

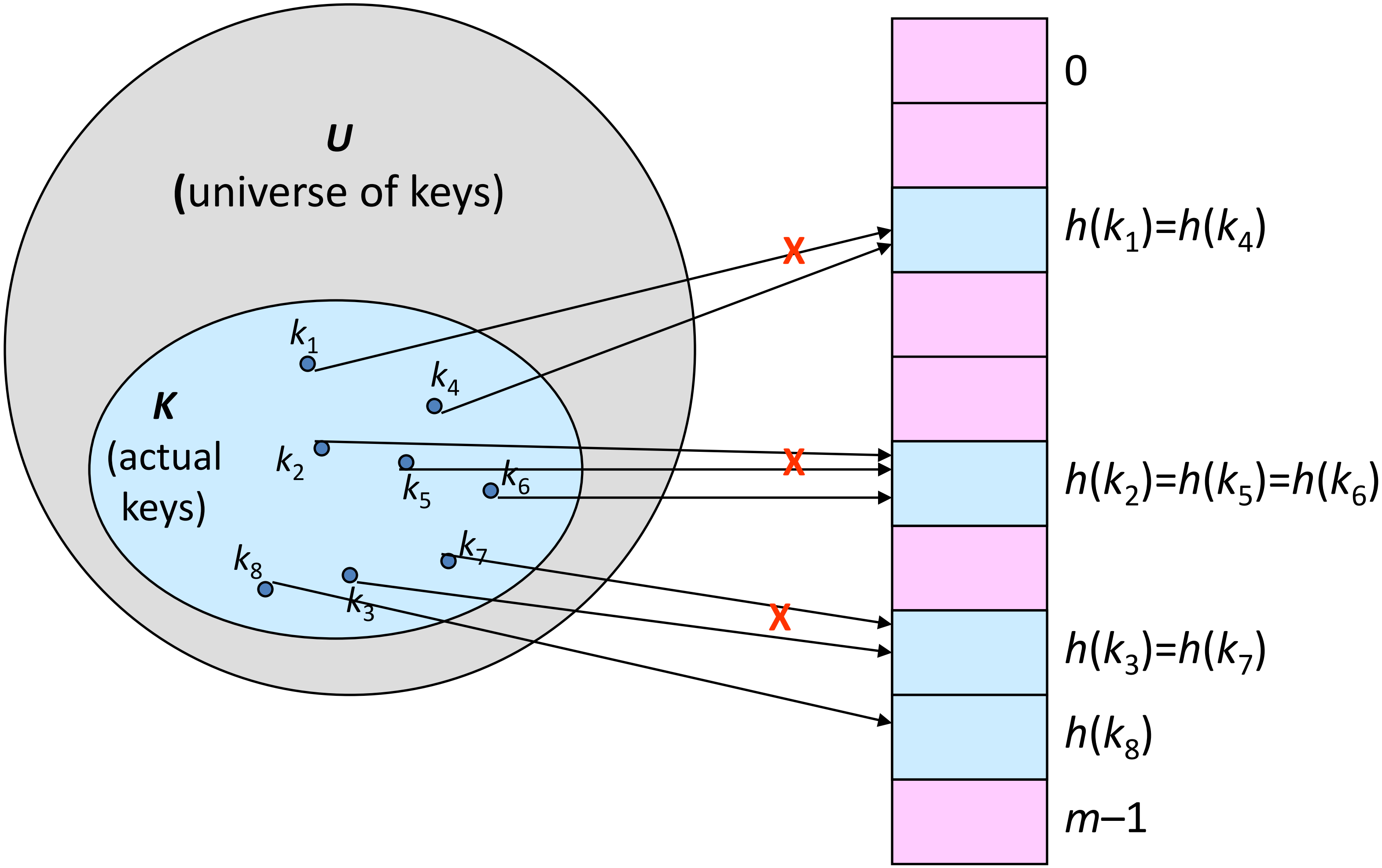
0	b
1	
2	
3	a
4	c
5	**d**
6	
7	

Performance Analysis

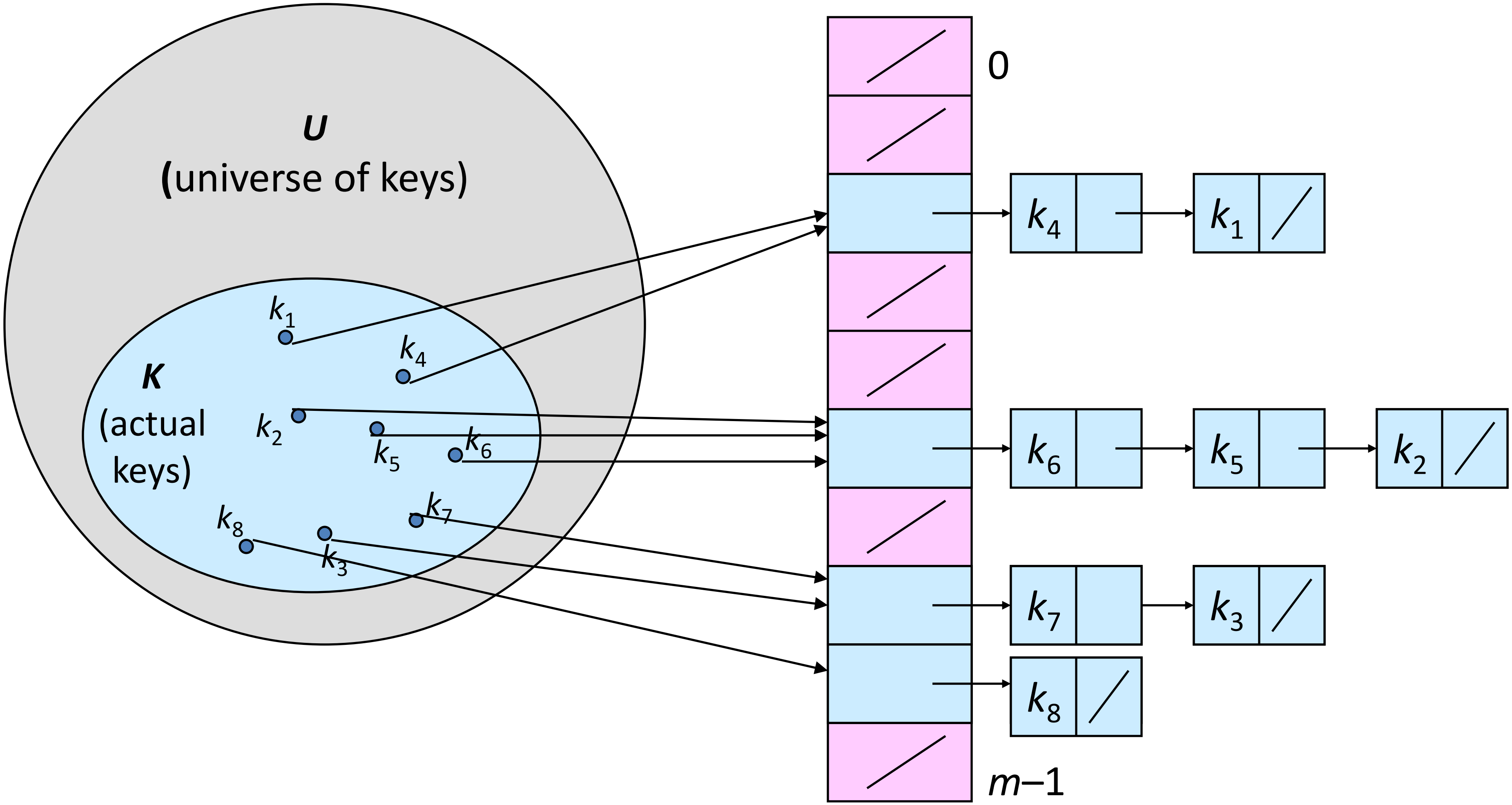
Computational complexity for initialization?

Computational complexity for insertion / search?

Collision Resolution by Chaining



Collision Resolution by Chaining



Hashing with Chaining

Dictionary Operations:

- Chained-Hash-Insert (T, x)
  - Insert x at the head of list $T[h(key[x])]$.
  - Worst-case complexity: $O(1)$.

- Chained-Hash-Search (T, k)
  - Search an element with key k in list $T[h(k)]$.
  - Worst-case complexity: proportional to length of list.

- Chained-Hash-Delete (T, x)
  - Delete x from the list $T[h(key[x])]$.
  - Worst-case complexity: search time + $O(1)$.
    - Need pointer to preceding element, or a doubly-linked list.

Analysis of Chained-Hash-Search

- ✓ Worst-case search time: time to compute $h(k) + \Theta(n)$.
- ✓ Average time: depends on how h distributes keys among slots.
  - ✓ **Assumptions:**
    - *Simple uniform hashing*: Any key is equally likely to hash into any of the slots, independent of where any other key hashes to.
    - $O(1)$ time to compute $h(k)$.
  - ✓ **Define** Load factor $\alpha = n/m$ = average # of keys per slot.
    - n – number of keys stored in the hash table.
    - m – number of slots = # linked lists.

Implications for separate chaining

- If $n = O(m)$, then load factor $\alpha = n/m = O(m)/m = O(1)$.

- Deletion takes $O(1)$ worst-case time if you have a pointer to the preceding element in the list.

- Hence, for hash tables with chaining, all dictionary operations take $O(1)$ time on average, given the assumptions of simple uniform hashing and O(1) time hash function evaluation.

- Extra memory needed for linked list pointers.
- Can we satisfy the simple uniform hashing assumption?

Probe Sequence

- Sequence of slots examined during a key search constitutes a *probe sequence*.
- Probe sequence must be a permutation of the slot numbers.
  - We examine every slot in the table, if we have to.
  - We don't examine any slot more than once.

- One way to think of it: extend hash function to:
  - $\boldsymbol{h : U \times \underbrace{\{0, 1, \ldots, m-1\}}_{\text{probe number}} \rightarrow \underbrace{\{0, 1, \ldots, m-1\}}_{\text{slot number}}}$

Universe of Keys

Computing Probe Sequences

- The ideal situation is *uniform hashing*:
  - Generalization of simple uniform hashing.
  - Each key is equally likely to have any of the $m!$ permutations of $\langle 0, 1, \ldots, m-1 \rangle$ as its probe sequence.
  - It is hard to implement true uniform hashing.

- Approximate with techniques that guarantee to probe a permutation of $[0 \ldots m-1]$, even if they don't produce all $m!$ probe sequences
  - Linear Probing.
  - Quadratic Probing.
  - Double Hashing.

Linear Probing

- $\boldsymbol{h(k, i) = (h(k,0)+i) \bmod m}$

key Probe number Original hash function

- The initial probe determines the entire probe sequence.
- Suffers from *primary clustering*:
  - Long runs of occupied sequences build up.
  - Long runs tend to get longer, since an empty slot preceded by i full slots gets filled next with probability $(i+1)/m$.

Clustering problem

- Clustering: nodes being placed close together by probing, which degrades hash table's performance
  - add 89, 18, 49, 58, 9
  - now searching for the value 28 will have to check half the hash table! no longer constant time...

0	49
1	58
2	9
3	
4	
5	
6	
7	
8	18
9	89

Quadratic Probing

- $\boldsymbol{h(k,i) = (h'(k) + c_1 i + c_2 i^2) \bmod m}$ $\quad c_1 \neq c_2$
- Can suffer from *secondary clustering*
- Example: resolving collisions on slot *i* by putting the colliding element into slot *i*+1, *i*+4, *i*+9, *i*+16, ...
  - add 89, 18, 49, 58, 9
    - 49 collides (89 is already there), so we search ahead by +1 to empty slot 0
    - 58 collides (18 is already there), so we search ahead by +1 to occupied slot 9, then +4 to empty slot 2
    - 9 collides (89 is already there), so we search ahead by +1 to occupied slot 0, then +4 to empty slot 3
  - clustering is reduced
  - what is the lookup algorithm?

0	49
1	
2	58
3	9
4	
5	
6	
7	
8	18
9	89

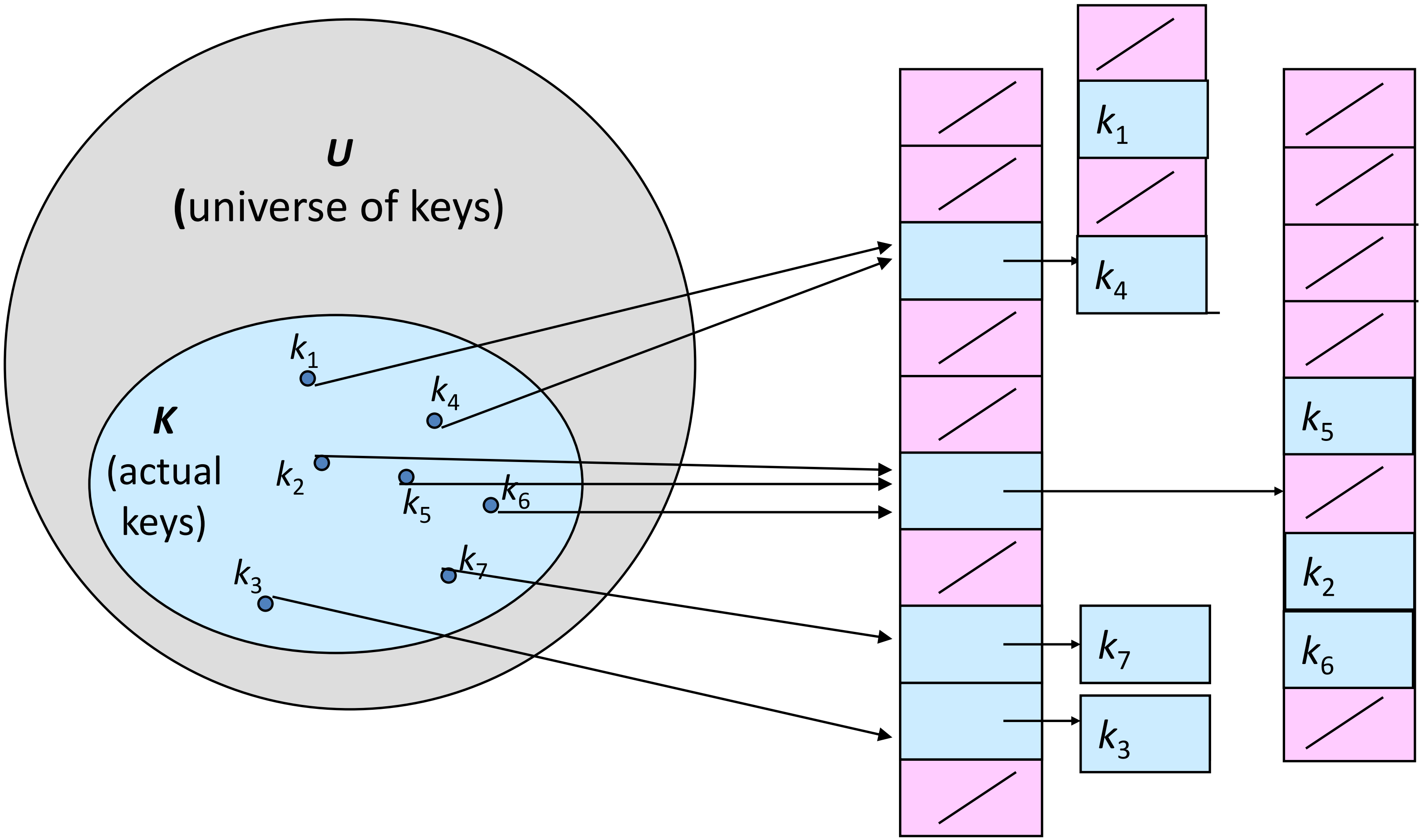
Double Hashing

- $h(k,i) = (h_1(k) + i\, h_2(k)) \bmod m$

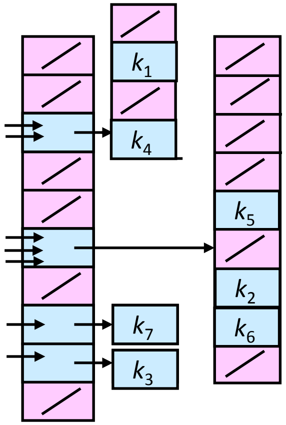
  key Probe number Auxiliary hash functions

- Two auxiliary hash functions.
  - h_1 gives the initial probe. h_2 gives the remaining probes.
- Must have $h_2(k)$ relatively prime to m, so that the probe sequence is a full permutation of $\langle 0, 1, \ldots, m-1 \rangle$.
  - Choose m to be a power of 2 and have $h_2(k)$ always return an odd number. Or,
  - Let m be prime, and have $1 < h_2(k) < m$.
- $\Theta(m^2)$ different probe sequences.
  - One for each possible combination of $h_1(k)$ and $h_2(k)$.
  - Close to the ideal uniform hashing.

Perfect Hashing



Perfect Hashing



- If you know the n keys in advance, makes a hash table with O(n) size, and worst-case O(1) lookup time.
- Just use two levels of hashing: A table of size n, then tables of size n_j^2.
- Dynamic versions have been created, but are usually less practical than other hash methods.
- Key idea: exploit both ends of space/#collisions tradeoff.

Analysis of hash tables

- Main operation: lookup of item in table
- What is worst-case cost of finding an item?
- Is the worst-case cost different for chaining, and the various open addressing schemes?
- Worst-case analysis doesn't make sense for hash tables, look at average case cost
- Cost highly depend on the **load factor** (no. of elements / array size)
- Which is better – hashing or tree based representation?