Prob-Stat/QUIZ/3/A

Fill in the blanks (Numerical)

Date of Exam : 16th Nov, 2021

Time : 12:05 pm to 12:55 am

Duration : 45min

No of questions: 10 out of 20 questions

Type: Random-sequential (navigation NOT allowed)

Each question carries 4 marks

A NOTE: $\Phi(1.25) = 0.8943502$, $\Phi(0.6667) = 0.7475181$, $\Phi(0.8889) = 0.8129716$, $\Phi(1.58113883) = 0.9430769$, $\Phi(1/\sqrt{2}) = 0.7602499$, $\Phi(0.25) = 0.5987063$

November 19, 2021

A:Q11. Let $(X_1, X_2 \dots X_9), (Y_1, Y_2, \dots, Y_9)$ and $(Z_1, Z_2, \dots, Z_9)$ be independent random samples from $N(-1, 9), N(1, 16)$ and $N(0, 25)$ populations respectively. Then find the expectation of the random variable

$$W = \frac{1}{25} \sum_{i=1}^{9} \{(X_i + Y_i)^2 + Z_i^2\}$$

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 18

ERROR RANGE: 0.005

Soln : $W \sim \chi^2_{18}$

A:Q12. Let $(X_1, X_2, \dots X_9), (Y_1, Y_2, \dots, Y_9)$ and $(Z_1, Z_2, \dots, Z_9)$ be independent random samples from $N(-1, 9), N(1, 16)$ and $N(0, 25)$ respectively. Then find the variance of the random variable

$$W = \frac{6\sqrt{2}(\bar{X} + \bar{Y})}{\sqrt{\sum_{i=1}^{9}(Z_i - \bar{Z})^2}}$$

where, $\bar{X} = \frac{1}{9}\sum_{i=1}^{9} X_i, \bar{Y} = \frac{1}{9}\sum_{i=1}^{9} Y_i, \bar{Z} = \frac{1}{9}\sum_{i=1}^{9} Z_i$.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 1.333333

ERROR RANGE: 0.005

ANS: $W \sim T_8$ so $Var(W) = 8/6$

A:Q13. Let $(X_1, X_2, \ldots X_9), (Y_1, Y_2, \ldots, Y_9)$ and $(Z_1, Z_2, \ldots, Z_9)$ be independent random samples from $N(-1, 9), N(1, 16)$ and $N(0, 25)$ respectively. Denote $\bar{X} = \frac{1}{9}\sum_{i=1}^{9} X_i, \bar{Y} = \frac{1}{9}\sum_{i=1}^{9} Y_i, \bar{Z} = \frac{1}{9}\sum_{i=1}^{9} Z_i$. If

$$F = \frac{\sum_{i=1}^{9}(X_i + Y_i - \bar{X} - \bar{Y})^2}{\sqrt{\sum_{i=1}^{9}(Z_i - \bar{Z})^2}}$$

has degrees of freedom (m, n) then find the value of $\frac{m}{m+n}$

ANSWER : 0.5

ERROR RANGE: 0.005

Soln: $F \sim F_{8,8}$

A:Q14. Let $X_1, \ldots X_{10}$ be independent and identically distributed Poisson random variables with mean 0.2. Let $Y = \sum_{i=1}^{10} X_i$. Find $P(Y \leq 1)$.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.4060

ERROR RANGE: 0.005

Sol. Y follows Poisson distribution with mean 2. So $P(Y \leq 2) = 3e^{-2} = 0.4060$.

A:Q15. Let (X, Y) follow a bivariate normal distribution with $(\mu_x = 2, \mu_y = 3, \sigma_x^2 = 4, \sigma_y^2 = 9, \rho = 1/3)$. Find $P(|3X - 2Y| \leq \sqrt{3})$.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.1974127

ERROR RANGE: 0.005

ANS: $3X-2Y \sim N(0,48)$. Hence $P(|3X-2Y| \leq \sqrt{3}) = P(|Z| \leq 0.25) = 2\Phi(0.25)-1 = 0.1974127$

NOTE $\Phi(0.25) = 0.5987063$

A:Q16. Let (X,Y) follow a bivariate normal distribution with $(\mu_x = 2, \mu_y = 3, \sigma_x^2 = 4, \sigma_y^2 = 9, \rho = 1/3)$. Find $P(Y \leq 6|X = 4)$.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER :0.7611

ERROR RANGE: 0.005

Sol. $Y|X = 4$ follows $N(4,8)$ distribution. So $P(Y \leq 6|X = 4) = P(Y \leq 1/\sqrt{2}) = 0.7602499$

NOTE: $\Phi(1/\sqrt{2}) = 0.7602499$

A:Q21. Let X and Y have the joint density function

$$f(x,y) = \begin{cases} 2x & \text{for } 0 < x < 1;\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability of $P(X+Y \leq 1|X \leq \frac{1}{2})$?

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.6667

ERROR RANGE: 0.005

ANS : $P(X+Y \leq 1|X \leq \frac{1}{2}) = \frac{1/3}{1/2} = 2/3$

A:Q23. Let X and Y have the joint density function

$$f(x,y) = \begin{cases} \frac{6}{5}(x^2+y) & \text{for } 0 \leq x \leq 1;\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $Cov(X,Y)$.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : -0.01

ERROR RANGE: 0.005

Sol. $E(XY) = \frac{7}{20}, E(X) = \frac{3}{5} = E(Y)$. So $Cov(X, Y) = -\frac{1}{100}$.

A:Q25. Let X_1, X_2 be i.i.d. random variables from a distribution with the density function

$$f(x) = \begin{cases} e^{-x} & \text{for } 0 \leq x < \infty \\ 0 & \text{otherwise} \end{cases}$$

What is the value the density function of Y = $\min\{X_1, X_2\}$ at $y = 1$?

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.2706706

ERROR RANGE: 0.005

ANS: Y follows exponential distribution with mean 0.5. Hence $f_y(1) = 2 * exp(-2) = 0.2706706$

A:Q46. Let $X_1, \ldots, X_{100}$ be a random sample from a population that has mean 72 and variance 64. Then the value of $P(71 \leq \overline{X} \leq 73)$ is ...., where $\overline{X}$ denotes the sample mean.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.7887005

ERROR RANGE: 0.005

NOTE: $\Phi(1.25) = 0.8943502$

A:Q48. Let

$$f(x, y) = \begin{cases} x + y, \text{ if } 0 < x < 1,\ 0 < y < 1 \\ 0, \text{otherwise.} \end{cases}$$

be the joint pdf of a pair of random variables (X, Y). $P(X + Y \leq 1)$ is ....

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.3333

ERROR RANGE: 0.005

Sol. $P(X+Y\leq 1)=\int_0^1\int_0^{1-y}(x+y)dxdy=1/3$

A:Q61. For discrete random variables X and Y with the joint probability distribution provided as $P(X=0,Y=0)=0.2$, $P(X=1,Y=1)=0.1$, $P(X=1,Y=2)=0.1$, $P(X=2,Y=1)=0.1$, $P(X=2,Y=2)=0.1$, $P(X=3,Y=3)=0.4$. Determine the value of the correlation ρ_{XY}.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.926

ERROR RANGE: 0.005

Solution: $E(X)=1.8=E(Y)$, $E(XY)=4.5$, $Var(X)=1.36=Var(Y)$

$\sigma_{XY}=E(XY)-E(X)E(Y)=4.5-1.8*1.8=1.26$; $\sigma_X=\sqrt{1.36}$, $\sigma_Y=\sqrt{1.36}$

$\rho_{XY}=\sigma_{XY}/\sigma_X\sigma_Y=0.926$

A:Q63. Let X and Y have the joint pmf

$$f(x,y)=\frac{x+y}{21},\ x=1,2,3\ y=1,2.$$

Calculate the probability $P(X\leq 2|Y=2)$.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.5833333

ERROR RANGE: 0.005

Solution: The conditional pmf $g(x|y)$ of X, given $Y=y$ is equal to $\frac{f(x,y)}{f_2(y)}$, where $f_2(y)$ marginal pmf for Y.

$f_2(y)=\frac{3y+6}{21}$, $y=1,2$.

So $g(x|y)=\frac{x+y}{3y+6}$, $x=1,2,3$, when $y=1$ or $y=2$.

$P(X=1|Y=2)+P(X=2|Y=2)=g(1|2)+g(2|2)=7/12=0.5833333$

A:Q67. An automated filling machine fills soft drink cans. The mean fill volume is 12.1 fluid ounces (oz), and the standard deviation is 0.22 oz. Assume that the fill volumes of the cans are independent normal random variables. What is the probability that the average volume of 10 cans selected from this process is less that 12 oz?

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER :0.0571 CORRECTED TO 0.075 ERROR RANGE: 0.01

Solution: Let $X_1, X_2, \ldots, X_{10}$ denote the fill volumes of 10 cans. The average fill volume (denoted as $\overline{X}$) is a normal random variable with $E(\overline{X}) = 12.1$ and $Var(\overline{X}) = 0.22^2/10$.

Consequently, $P(\overline{X} < 12) = 0.075$

A:Q68. The weight of a population of workers have mean 167 and standard deviation 27. If a sample of 36 workers is chosen, find the approximate probability that the sample mean of their weights lies between 163 and 170.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.5604897
ERROR RANGE: 0.005

Solution: It follows from the central limit theorem that $\overline{X}$ is approximately normal with mean 167 and standard deviation $27/\sqrt{36} = 4.5$. Therefore, with Z being a standard normal random variable, $P(163 < \overline{X} < 170) = P(\frac{163-167}{4.5} < \frac{\overline{X}-167)}{4.5} < \frac{170-167}{4.5}) \approx P(-0.8889 < Z < 0.6667) = \Phi(0.6667) - \Phi(-0.8889) = 0.5604897$

NOTE: $\Phi(0.6667) = 0.7475181$ and $\Phi(0.8889) = 0.8129716$

A:Q70. Let X be a random variable with p.d.f.

$$f(x) = \begin{cases} \frac{1}{x^2} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

Define $Y = -\log_e(1 - \frac{1}{X})$, then find the value of the p.d.f of Y at $Y = 1$

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.3678794
ERROR RANGE: 0.005

Solution: c.d.f of X is $(1-\frac{1}{X})$ so $Y = -\log_e(1-\frac{1}{X}) \sim exp(1)$ hence $f_y(1) = 1/e = 0.3678794$

A:Q74. Let $(X, Y) \sim BivariateNormal(\mu_x = 0, \mu_y = 0, \sigma_x^2 = 1, \sigma_y^2 = 1, \rho = 0.5)$. Find $E(Y^2|X = 1)$.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 1.0

ERROR RANGE: 0.005

`ANS:` $Y|x \sim$ `Normal distribution with`

$E(Y|x) = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x) = 0.5$

$V(Y|x) = (1-\rho^2)\sigma_y^2 = 0.75$

$E(Y^2|x) = 1$

A:Q76. Let X_is be i.i.d. $Bernoulli(0.5)$ random variables for $i = 1, 2, 3 \ldots$. Define $Y_k = \sum_{i=1}^{k} X_i$. Find $Cov(Y_{100}, Y_{2021})$.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 25

ERROR RANGE: 0.005

`ANS:` $Cov(Y_{100}, Y_{2021}) = Cov(Y_{100}, Y_{100}) = Var(Y_{100}) = (100)0.25 = 25$

A:Q77. Let X_is be i.i.d. $Gamma(3, 2)$ for i=1,2,3,4 . Then find

$$E\left(\frac{X_1 + X_2}{X_3 + X_4}\right).$$

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 7.5 CORRECTED TO 1.2

ERROR RANGE: 0.005

`ANS` $E\left(\frac{X_1+X_2}{X_3+X_4}\right) = E(X_1+X_2)E\left(\frac{1}{X_3+X_4}\right) = 3 \times \frac{2}{5} = 1.2$,

`where` $X_1 + X_2$ `and` $X_3 + X_4$ `are i.i.d Gamma(6,2)`

A:Q79. Let X be a continuous random variables with c.d.f. $F(x) = \frac{e^x}{1+e^x}$. Let $Y = e^X$. Find the value of the density of Y at the point $y = 1$.

(answer should be correct up to three decimal places, error range: 0.005)

ANSWER : 0.25

ERROR RANGE: 0.005

`ANS`: $f_X(x) = \frac{e^X}{(1+e^X)^2}$ so $f_Y(y) = \frac{1}{(1+y)^2}$ when $Y > 0$ Hence $f_y(1) = 1/4$