Propositional Logic

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Introduction to Logic

History and Genesis

Indic: Geometry and Calculations, Nyaya and Vaisisekha, Argumentation Theory, Sanskrit and Binary Arguments, Chatustoki (Logical Argumentation), Philosophers, Vedanta, Formal Systems

China: Confucious, Mozi, Master Mo (Mohist School), Basic Formal Systems, Buddhist Systems from India

Greek: Thales and Pythagoras (Postulates, Geometry), Heraclitus and Permenides (Logos), Plato (Logic beyond Geometry), Aristotle (Syllogism, Syntax), Stoics

Middle-East: Egyptian logic, Arabic (Avisennian logic), Inductive logic

Medieval-Europe: Post Aristotle, Precursor to First-Order logic

Today: Propositional, Predicate, Higher-Order, Psychology, Philosophy

Applications

- Problem Solving using Logical Arguments
- Automated Reasoning and Artificial Intelligence
- Automated Learning and Deduction / Derivation
- Circuit Behaviour and Program Verification
- Cognition Models and Neural Network Analysis

Some Example Arguments

Example-1

a. If I am the VP of Gymkhana, then I am well-known in IIT. I am the VP of Gymkhana. Therefore, I am well-known in IIT.
b. If Ninaad is the VP of Gymkhana, then Ninaad is well-known in IIT. Ninaad is the VP of Gymkhana. Therefore, Ninaad is well-known in IIT.
c. If Neha is the VP of Gymkhana, then Neha is well-known in IIT. Neha is NOT the VP of Gymkhana. Therefore, Neha is NOT well-known in IIT.

Example-2

a. If Ninaad is elected as the VP of Gymkhana, then Ayushi is chosen as a G-Sec AND Devang is chosen as a Treasurer. Ayushi is NOT chosen as a G-Sec. Therefore, Ninaad is NOT elected as VP of Gymkhana.
b. If Ninaad is elected as the VP of Gymkhana, then Ayushi is chosen as a G-Sec AND Devang is chosen as a Treasurer. Devang is chosen as a Treasurer. Therefore, Ninaad is elected as VP of Gymkhana.

Representation and Deduction using Propositional Logic

Formal Representation

Propositions: Choice of Boolean variables with true or false values.

Connectors: Well-defined connectors, such as, $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\leftrightarrow$ (if and only if) etc.

The meaning (semantics) is given by their Truth-tables.

Codification: Boolean Formulas constructed from the statements in arguments.

Deduction Process

- Obtaining truth of the *combined formula* expressing complete argument.
  - Proving or Disproving the argument using truth-tables or formal deduction rules.
  - Checking the Validity and Satisfiability of the formula analyzing its truth or falsification over various Interpretations.

Deduction using Propositional Logic: *Example-1a*

Example

If I am the VP of Gymkhana, then I am well-known in IIT. I am the VP of Gymkhana. Therefore, I am well-known in IIT.

Argument Representation

Propositions: $\mathtt{v}$: I am the VP of Gymkhana, $\mathtt{w}$: I am well-known in IIT.

Codification: $\mathtt{F}_1 : \mathtt{v} \rightarrow \mathtt{w} \equiv (\neg\mathtt{v} \vee \mathtt{w})$, $\mathtt{F}_2 : \mathtt{v}$, $\mathtt{G} : \mathtt{w}$.

Complete Formula for Deduction: $(\mathtt{F}_1 \wedge \mathtt{F}_2) \rightarrow \mathtt{G} \equiv ((\mathtt{v} \rightarrow \mathtt{w}) \wedge \mathtt{v}) \rightarrow \mathtt{w}$

Deduction Steps: Using Truth-tables

$\mathtt{v}$	$\mathtt{w}$	$\mathtt{v} \rightarrow \mathtt{w}$	$(\mathtt{v} \rightarrow \mathtt{w}) \wedge \mathtt{v}$	$((\mathtt{v} \rightarrow \mathtt{w}) \wedge \mathtt{v}) \rightarrow \mathtt{w}$
True	True	True	True	True
True	False	False	False	True
False	True	True	False	True
False	False	True	False	True

Tautology: Formula, $(\mathtt{F}_1 \wedge \mathtt{F}_2) \rightarrow \mathtt{G}$, is Valid, i.e. True under *all* Interpretations.

Deduction using Propositional Logic: *Example-1b*

Example

If Ninaad is the VP of Gymkhana, then Ninaad is well-known in IIT. Ninaad is the VP of Gymkhana. Therefore, Ninaad is well-known in IIT.

Argument Representation

Propositions: $\mathtt{v}$: Ninaad is the VP of Gymkhana, $\mathtt{w}$: Ninaad is well-known in IIT.

Codification: $\mathtt{F}_1 : \mathtt{v} \rightarrow \mathtt{w} \equiv (\neg\mathtt{v} \vee \mathtt{w})$, $\quad \mathtt{F}_2 : \mathtt{v}$, $\quad \mathtt{G} : \mathtt{w}$.

Complete Formula for Deduction: $(\mathtt{F}_1 \wedge \mathtt{F}_2) \rightarrow \mathtt{G} \equiv ((\mathtt{v} \rightarrow \mathtt{w}) \wedge \mathtt{v}) \rightarrow \mathtt{w}$

Deduction Steps: Using Truth-tables

$\mathtt{v}$	$\mathtt{w}$	$\mathtt{v} \rightarrow \mathtt{w}$	$(\mathtt{v} \rightarrow \mathtt{w}) \wedge \mathtt{v}$	$((\mathtt{v} \rightarrow \mathtt{w}) \wedge \mathtt{v}) \rightarrow \mathtt{w}$
True	True	True	True	True
True	False	False	False	True
False	True	True	False	True
False	False	True	False	True

Tautology: Formula, $(\mathtt{F}_1 \wedge \mathtt{F}_2) \rightarrow \mathtt{G}$, is Valid, i.e. True under *all* Interpretations.

Deduction using Propositional Logic: *Example-1c*

Example

If Neha is the VP of Gymkhana, then Neha is well-known in IIT. Neha is NOT the VP of Gymkhana. Therefore, Neha is NOT well-known in IIT.

Deduction Process: Using Truth-tables

Codification: $\mathtt{F}_1 : \mathtt{v} \rightarrow \mathtt{w}$, $\mathtt{F}_2 : \neg\mathtt{v}$, $\mathtt{G} : \neg\mathtt{w}$.

Formula: $(\mathtt{F}_1 \wedge \mathtt{F}_2) \rightarrow \mathtt{G} \equiv ((\mathtt{v} \rightarrow \mathtt{w}) \wedge \neg\mathtt{v}) \rightarrow \neg\mathtt{w}$

Truth-table:

v	w	$v \rightarrow w$	$(v \rightarrow w) \wedge \neg v$	$((v \rightarrow w) \wedge \neg v) \rightarrow \neg w$
True	True	True	False	True
True	False	False	False	True
False	True	True	True	False
False	False	True	True	True

Interpretations of a Complete Formula

- **Valid?** No! vs. **Satisfiable?** Yes!
- **Invalid?** Yes! vs. **Unsatisfiable?** No!

Fundamental Laws of Propositional Logic

Let, **Propositions**: $\mathtt{p}$, $\mathtt{q}$ and $\mathtt{r}$, **Tautology**: $\mathtt{T}$, **Contradiction**: $\mathtt{F}$.

Law	Explanation
If and Only If	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
Double Negation	$\neg\neg p \equiv p$
DeMorgan's Laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q, \quad \neg(p \vee q) \equiv \neg p \wedge \neg q$
Commutative Laws	$p \wedge q \equiv q \wedge p, \quad p \vee q \equiv q \vee p$
Associative Laws	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r, \quad p \vee (q \vee r) \equiv (p \vee q) \vee r$
Distributive Laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r), \quad p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Idempotent Laws	$p \wedge p \equiv p, \quad p \vee p \equiv p$
Identity Laws	$p \wedge T \equiv p, \quad p \vee F \equiv p$
Inverse Laws	$p \wedge \neg p \equiv F, \quad p \vee \neg p \equiv T$
Domination Laws	$p \wedge F \equiv F, \quad p \vee T \equiv T$
Absorption Laws	$p \wedge (p \vee q) \equiv p, \quad p \vee (p \wedge q) \equiv p$

Here, $\equiv$ abbreviates as 'equivalent to', and may also be denoted as $\Leftrightarrow$.

Deduction using Propositional Logic: *Example-2a*

Example

If Ninaad is elected as the VP of Gymkhana, then Ayushi is chosen as a G-Sec AND Devang is chosen as a Treasurer. Ayushi is NOT chosen as a G-Sec. Therefore, Ninaad is NOT elected as VP of Gymkhana.

Deduction Process: Using Rule-based Logical Inferencing

Propositions: $\mathtt{v}$: Ninaad is elected as the VP, $\mathtt{s}$: Ayushi is chosen as a G-Sec, $\mathtt{t}$: Devang is chosen as a Treasurer.

Codification: $\mathtt{F}_1 : \mathtt{v} \rightarrow (\mathtt{s} \wedge \mathtt{t})$, $\mathtt{F}_2 : \neg\mathtt{s}$, $\mathtt{G} : \neg\mathtt{v}$.

Requirement: $\dfrac{\begin{matrix} F_1 \\ F_2 \end{matrix}}{\therefore\ \mathtt{G}} \equiv \dfrac{\begin{matrix} v \rightarrow (s \wedge t) \\ \neg s \end{matrix}}{\therefore\ \neg\mathtt{v}}$ **??**

Inferencing: $\dfrac{\mathtt{v} \rightarrow (\mathtt{s} \wedge \mathtt{t})}{\therefore\ (\mathtt{v} \rightarrow \mathtt{s})}$ and $\dfrac{\begin{matrix} v \rightarrow s \\ \neg s \end{matrix}}{\therefore\ \neg\mathtt{v}}$ **Yes! This is a Tautology!**

Deduction using Propositional Logic: *Example-2b*

Example

If Ninaad is elected as the VP of Gymkhana, then Ayushi is chosen as a G-Sec AND Devang is chosen as a Treasurer. Devang is chosen as a Treasurer. Therefore, Ninaad is elected as VP of Gymkhana.

Deduction Process: Using Rule-based Logical Inferencing

Propositions: $\mathtt{v}$: Ninaad is elected as the VP, $\mathtt{s}$: Ayushi is chosen as a G-Sec, $\mathtt{t}$: Devang is chosen as a Treasurer.

Codification: $\mathtt{F}_1 : \mathtt{v} \rightarrow (\mathtt{s} \wedge \mathtt{t})$, $\mathtt{F}_2 : \mathtt{t}$, $\mathtt{G} : \mathtt{v}$.

Requirement: $\dfrac{\begin{array}{c}F_1\\F_2\end{array}}{\therefore \mathtt{G}} \equiv \dfrac{\begin{array}{c}v \rightarrow (s \wedge t)\\t\end{array}}{\therefore \mathtt{v}}$ **??**

Inferencing: $\dfrac{\mathtt{v} \rightarrow (\mathtt{s} \wedge \mathtt{t})}{\therefore (\mathtt{v} \rightarrow \mathtt{t})}$ and $\dfrac{\begin{array}{c}v \rightarrow t\\t\end{array}}{\begin{array}{c}\therefore v \text{ or } \neg v\\(both\ possible)\end{array}}$ **No! This is Invalid!**

Rule-based Deduction and Logical Inferencing

Name of Rule	Inferencing Rule	Logical Implications (Tautology)
Modus Ponens	$\dfrac{p \quad p\to q}{\therefore\ q}$	$[p \land (p \to q)] \to q$
Modus Tollens	$\dfrac{p\to q \quad \neg q}{\therefore\ \neg p}$	$[(p \to q) \land \neg q] \to \neg p$
Syllogism	$\dfrac{p\to q \quad q\to r}{\therefore\ p\to r}$	$[(p \to q) \land (q \to r)] \to (p \to r)$
Conjunction	$\dfrac{p \quad q}{\therefore\ (p\land q)}$	$[p \land q] \to (p \land q)$
Disjunctive Syllogism	$\dfrac{p\lor q \quad \neg p}{\therefore\ q}$	$[(p \lor q) \land \neg p] \to q$
Contradiction	$\dfrac{\neg p\to F}{\therefore\ p}$	$[\neg p \to F] \to p$
Conjunctive Simplification	$\dfrac{p\land q}{\therefore\ p}$	$[p \land q] \to p$
Disjunctive Amplification	$\dfrac{p}{\therefore\ (p\lor q)}$	$[p] \to (p \lor q)$
Conditional Proof	$\dfrac{p\land q \quad p\to(q\to r)}{\therefore\ r}$	$[(p \land q) \land (p \to (q \to r))] \to r$
Proof by Cases	$\dfrac{p\to r \quad q\to r}{\therefore\ (p\lor q)\to r}$, $\dfrac{p\to q \quad p\to r}{\therefore\ p\to(q\land r)}$	$[(p\to r)\land(q\to r)]\to((p\lor q)\to r)$, $[(p\to q)\land(p\to r)]\to(p\to(q\land r))$
Constructive Dilemma	$\dfrac{p\to q \quad r\to s \quad p\lor r}{\therefore\ (q\lor s)}$	$[(p\to q)\land(r\to s)\land(p\lor r)] \to(q\lor s)$
Destructive Dilemma	$\dfrac{p\to q \quad r\to s \quad \neg q\lor\neg s}{\therefore\ (\neg p\lor\neg r)}$	$[(p\to q)\land(r\to s)\land(\neg q\lor\neg s)] \to(\neg p\lor\neg r)$

Some More Example Arguements

Example

If Ninaad is elected as the VP, then Ayushi is chosen as a G-Sec OR Devang is chosen as a Treasurer. Ayushi is NOT chosen as a G-Sec. Therefore, if Ninaad is elected as VP then Devang is chosen as a Treasurer.

Rule-based Deduction Setup

Propositions: $\mathtt{v}$: Ninaad is elected as the VP, $\mathtt{s}$: Ayushi is chosen as a G-Sec, $\mathtt{t}$: Devang is chosen as a Treasurer.

Codification: $\mathtt{F}_1 : \mathtt{v} \rightarrow (\mathtt{s} \vee \mathtt{t})$, $\mathtt{F}_2 : \neg \mathtt{s}$, $\mathtt{F}_3 : \mathtt{v}$, $\mathtt{G} : \mathtt{t}$.

Requirement: $\dfrac{\begin{array}{c} F_1 \\ F_2 \\ F_3 \end{array}}{\therefore\ \mathtt{G}} \equiv \dfrac{\begin{array}{c} v \rightarrow (s \vee t) \\ \neg s \\ v \end{array}}{\therefore\ \mathtt{t}}$ **??**

Inferencing: $\dfrac{\begin{array}{c} v \rightarrow (s \vee t) \\ v \end{array}}{\therefore\ (\mathtt{s} \vee \mathtt{t})}$ (Modus Ponens) and $\dfrac{\begin{array}{c} s \vee t \\ \neg s \end{array}}{\therefore\ \mathtt{t}}$ (Disjunctive Syllogism)

Yes! This is a Tautology!

Some More Example Arguements

Example

If Ninaad is elected as the VP, then EITHER Ayushi is chosen as a G-Sec OR Devang is chosen as a Treasurer, *but not both*. Ayushi is NOT chosen as a G-Sec. Therefore, if Ninaad is elected as VP then Devang is chosen as a Treasurer.

Rule-based Deduction Setup

Propositions: $\mathtt{v}$: Ninaad is elected as the VP, $\mathtt{s}$: Ayushi is chosen as a G-Sec, $\mathtt{t}$: Devang is chosen as a Treasurer.

Codification: $\mathtt{F}_1 : \mathtt{v} \rightarrow ((\mathtt{s} \wedge \neg \mathtt{t}) \vee (\neg \mathtt{s} \wedge \mathtt{t}))$, $\mathtt{F}_2 : \neg \mathtt{s}$, $\mathtt{F}_3 : \mathtt{v}$, $\mathtt{G} : \mathtt{t}$.

Requirement: $\dfrac{\begin{matrix} F_1 \\ F_2 \\ F_3 \end{matrix}}{\therefore\ \mathtt{G}} \equiv \dfrac{\begin{matrix} v \rightarrow ((s \wedge \neg t) \vee (\neg s \wedge t)) \\ \neg s \\ v \end{matrix}}{\therefore\ \mathtt{t}}$ **??**

Inferencing: **...** **Left for You as an Exercise!**

Problem Solving and Reasoning using Propositional Logic

Example

While walking in a labyrinth, you find yourself in front of three possible roads. The road on your left is paved with gold, the road in front of you is paved with marble, while the road on your right is made of small stones.
Each road is protected by a guard. You talk to the guards, and this is what they tell you.

- *The guard of the gold road: "This road will bring you straight to the center. Moreover, if the stones take you to the center, then also the marble takes you to the center."*
- *The guard of the marble road: "Neither the gold nor the stones will take you to the center."*
- *The guard of the stone road: "Follow the gold, and you will reach the center. Follow the marble, and you will be lost."*

You know that all the guards are liars. Your goal is to choose the correct road that will lead you to the center of the labyrinth.

Problem Solving and Reasoning using Propositional Logic

Solution

Let us introduce the following propositions:

- $\mathtt{GG}$: The guard of the gold road is telling the truth
- $\mathtt{GM}$: The guard of the marble road is telling the truth
- $\mathtt{GS}$: The guard of the stone road is telling the truth
- $\mathtt{G}$: The gold road leads to the center
- $\mathtt{M}$: The marble road leads to the center
- $\mathtt{S}$: The stone road leads to the center

The statements of the three guards can be logically encoded as follows:

$$\mathtt{GG} \leftrightarrow \big[\mathtt{G} \wedge (\mathtt{S} \rightarrow \mathtt{M})\big], \qquad \mathtt{GM} \leftrightarrow \big[\neg\mathtt{G} \wedge \neg\mathtt{S}\big], \qquad \mathtt{GS} \leftrightarrow \big[\mathtt{G} \wedge \neg\mathtt{M}\big]$$

Also, you also know that the following statement is true:

$$\begin{aligned} \mathtt{Z} \equiv \neg\mathtt{GG} \wedge \neg\mathtt{GM} \wedge \neg\mathtt{GS} \quad &\equiv \quad \neg\big[\mathtt{G} \wedge (\mathtt{S} \rightarrow \mathtt{M})\big] \wedge \neg\big[\neg\mathtt{G} \wedge \neg\mathtt{S}\big] \wedge \neg\big[\mathtt{G} \wedge \neg\mathtt{M}\big] \\ &\equiv \quad \big[\neg\mathtt{G} \vee (\mathtt{S} \wedge \neg\mathtt{M})\big] \wedge \big[\mathtt{G} \vee \mathtt{S}\big] \wedge \big[\neg\mathtt{G} \vee \mathtt{M}\big]. \end{aligned}$$

Problem Solving and Reasoning using Propositional Logic

Solution

The truth table of Z is given below:

G	M	S	$\neg G \lor (S \land \neg M)$	$G \lor S$	$\neg G \lor M$	Z
F	F	F	T	F	T	F
F	F	T	T	T	T	T
F	T	F	T	F	T	F
F	T	T	T	T	T	T
T	F	F	F	T	F	F
T	F	T	T	T	F	F
T	T	F	F	T	T	F
T	T	T	F	T	T	F

Conclusion: This truth table implies that $Z \equiv \left[\neg G \land S\right]$.
(This also implies that, the stone road will surely lead you to the center; the gold road will surely not lead you to the center; and the marble road may or may not lead you to the center.)

Insufficiency of Propositional Logic

Example

- Wherever Ankush goes, so does the pet dog. Ankush goes to school. So, the dog goes to school.
- No contractors are dependable. Some engineers are contractors. Therefore, some engineers are not dependable.
- All actresses are graceful. Anushka is a dancer. Anushka is an actress. Therefore, some dancers are graceful.
- Every passenger either travels in first class or second class. Each passenger is in second class if and only if he or she is not wealthy. Some passengers are wealthy. Not all passengers are wealthy. Therefore, some passengers travel in second class.

Limitations in Expressability

- **Quantifications:** 'some', 'none', 'all', 'every', 'wherever' etc.
- **Functionalities:** 'x goes to some place y', 'z travels in train' etc.

Thank You!