CS21201 Discrete Structures, Autumn 2021–2022

Third Test

Date: Nov 16, 2021 | Time: 10:15am–11:30am | Maximum marks: 40

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1. Consider the following three sets.

$$\begin{aligned} S &= \text{The set of all infinite bit sequences,} \\ A &= \text{The set of all infinite bit sequences containing two consecutive 0's (at least once),} \\ B &= \text{The set of all infinite bit sequences not containing two consecutive 0's.} \end{aligned}$$

In the class, we have seen that S is uncountable. This exercise deals with the countability/uncountability of A and B.

(a) Propose an *injective* map $f : S \to A$, and argue about the countability/uncountability of A. **(5)**

(b) Prove whether B is countable or uncountable. **(5)**

2. Consider the sequence $a_0, a_1, a_2, \ldots$ defined recursively as follows.

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 1, \\ a_2 &= 2, \\ a_n &= 2a_{n-2} + a_{n-3} + 2 \text{ for all } n \geqslant 3. \end{aligned}$$

(a) Derive a closed-form expression for the (ordinary) generating function $A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ of the sequence. **(5)**

(b) From the closed-form expression of $A(x)$ derived in Part (a), establish that $a_n = F_{n+2} - 1$ for all $n \geqslant 0$, where $F_0, F_1, F_2, \ldots$ is the Fibonacci sequence. <u>Use no other method.</u> **(5)**

3. Solve the following recurrence, and obtain the closed-form expression for a_n.

$$a_n = 8a_{n-2} - 16a_{n-4} + 2^n \quad (\text{for } n \geqslant 4) \quad \text{with} \quad a_0 = 1,\ a_1 = \frac{17}{4},\ a_2 = 30,\ a_3 = 41.$$

Note: Use of generaing fuctions is **not** allowed in this exercise. **(10)**

4. **(a)** Let $A = \mathbb{Z} \times \mathbb{Z}$, and λ a fixed (constant) positive integer. Define two operations $\oplus$ and $\odot$ on A as

$$\begin{aligned} (a,b) \oplus (c,d) &= (a+c, b+d), \\ (a,b) \odot (c,d) &= (ad+bc, bd+\lambda ac). \end{aligned}$$

A is a commutative ring with identity under these two operations. You do not have to verify the ring axioms, but only mention what the additive and the multiplicative identities are in A (no need to prove their identity properties). Also, prove that A is an integral domain if and only if λ is ***not*** a perfect square. **(2 + 4)**

(b) Let $(G, \circ)$ be a group, and c a fixed element of G. Define a binary operation $*$ on G by $a * b = a \circ c \circ b$ for all $a, b \in G$. Prove that $(G, *)$ is a group, clearly showing that all the properties of a group are satisfied. **(4)**
