

Probability and statistics

Lecture #3

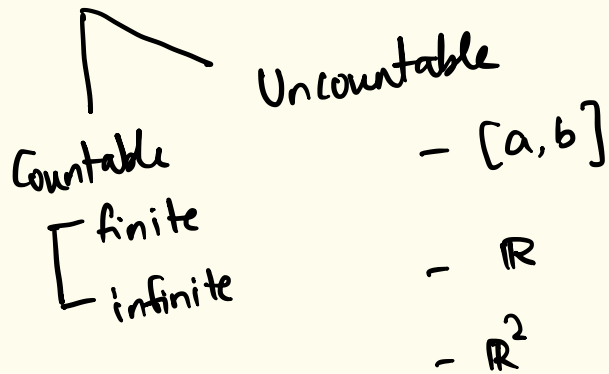
(August 23)



Model random experiments.

R : Random expt.

Ω : sample space



Probability assignment

- Uniform probability principle

Axiomatic definition
of probability

\mathcal{F} : collection of all
events to which
probability is
assigned.

In case finite countable
sample space

$$\mathcal{F} = 2^{\Omega}$$

In case of $[0, 1] = \Omega$

\mathcal{F} = Borel- σ -algebra on
[0,1]

Conditional probability :

R : Random expt.

Ω : sample space (Set of all possible outcomes).

R : Rolling a "fair" die

$\Omega = \{1, 2, 3, 4, 5, 6\}$; $P\{a\} = \frac{1}{6} \quad \forall a \in \Omega$

Now if I have additional information about the random expt.

{ An even number appears on the top face } = B
of the die.

$$B = \{2, 4, 6\}$$

$$P\{1\} = \frac{1}{6}$$

Under the assumption that B has occurred

$$P\{1\} = 0$$

$A_1 = \{1\}$ = 1 appears on the top face of the die.

$$P(A_1) = \frac{1}{6}$$

$$A_2 = \{2\}$$

$$P(A_2) = \frac{1}{6}$$

$$P(A_1|B) = 0$$

$$P(A_2|B) = \frac{1}{3}$$

$$P(A|B) = \frac{\#(A \cap B)}{\#(B)} = \frac{\#(A \cap B) / \#(\Omega)}{\#(B) / \#(\Omega)} = \frac{P(A \cap B)}{P(B)}$$

Definition (conditional probability)

Let $A, B \in \mathcal{F}$ s.t. $P(B) > 0$

conditional probability of A given B

is defined as:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\begin{aligned} P(A) &= P(A|\Omega) \\ &= \frac{P(A \cap \Omega)}{P(\Omega)} \end{aligned}$$

Product rule:

$$P(A \cap B) = P(A|B) P(B) \quad \text{--- (1)}$$

If $P(A) > 0$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(A \cap B) = P(B|A) P(A) \quad \text{--- (2)}$$

Given $P(A) > 0$ and $P(B) > 0$ and from (1) & (2)

$$P(A|B) P(B) = P(B|A) P(A)$$

$$\Rightarrow \boxed{\begin{aligned} P(A|B) &= \frac{P(B|A) P(A)}{P(B)} \\ P(B|A) &= \frac{P(A|B) P(B)}{P(A)} \end{aligned}}$$

Baye's
rule

Consider $\{B_1, \dots, B_n\}$ to be a disjoint cover

of Ω .

$$B_i \cap B_j = \emptyset \quad \text{for } i \neq j \quad \leftarrow \text{disjoint}$$

$$\bigcup_{i=1}^n B_i = \Omega \quad \leftarrow \text{cover}$$

Addition requirement:

$$P(B_i) \neq 0 \\ \forall i$$

For any $A \in \mathcal{F}$

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \dots \cup (A \cap B_n) \leftarrow \text{disjoint union}$$

$$\begin{aligned} P(A) &= P(A \cap B_1) + \dots + P(A \cap B_n) \\ &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \dots \\ &\quad \dots + P(A|B_n)P(B_n) \end{aligned}$$

$$P(A) = \sum_{i=1}^n P(A|B_i)P(B_i)$$

← Total law of probability.

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^n P(A|B_i)P(B_i)}$$

A = You score grade 'A' in Prob/stats

B_i = Attendance record

$B_1 = [0, 10]$; $B_2 = [10, 20]$

\vdots

Independence of events.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A|B) P(B)$$

Definition: A & $B \in \mathcal{F}$ are indep.

$$\text{if } P(A \cap B) = P(A) P(B)$$