Set Theory

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Set: Well-defined collection of distinct objects
(Ex: $\mathcal{S} = \{4, 9, 16 \ldots, 81, 100\} = \{x^2 \mid x \text{ is integer and } 1 < x \leq 10\}$)

- Membership: Element belonging to (or a member of) a set
  (Ex: $25, 64 \in \mathcal{S}$ and $50, 72 \notin \mathcal{S}$)
- Cardinality: Number of elements in a set (Ex: $|\mathcal{S}| = 9$)
- Finite Set: Set having finite cardinality (Ex: The set, $\mathcal{S}$)
- Infinite Set: Set having infinite (∞) cardinality
  (Ex: $\mathcal{T} = \{1, 2, 4, 8, 16, \ldots\} = \{2^y \mid y \text{ is integer and } y \geq 0\}$)

Subset: A set ($\mathcal{A}$) is a subset of another set ($\mathcal{B}$) iff each element of $\mathcal{A}$ is also a member of $\mathcal{B}$. Formally, $\mathcal{A} \subseteq \mathcal{B}$ iff $\forall x\ [x \in \mathcal{A} \Rightarrow x \in \mathcal{B}]$.

Hence, $\mathcal{A} \not\subseteq \mathcal{B}$ iff $\neg\forall x\ [x \in \mathcal{A} \Rightarrow x \in \mathcal{B}] \equiv \exists x\ [x \in \mathcal{A} \wedge x \notin \mathcal{B}]$.

(Ex: Let $\mathcal{R} = \{z \mid z \text{ is composite integer and } 2 \leq z \leq 100\}$, so $\mathcal{S} \subseteq \mathcal{R}$)

Equal Sets: $\mathcal{A} = \mathcal{B}$ iff $[(\mathcal{A} \subseteq \mathcal{B}) \wedge (\mathcal{B} \subseteq \mathcal{A})] \equiv \forall x\ [x \in \mathcal{A} \Leftrightarrow x \in \mathcal{B}]$

Proper Subset: $\mathcal{A} \subset \mathcal{B}$ iff $[\forall x\ (x \in \mathcal{A} \Rightarrow x \in \mathcal{B}) \wedge \exists y\ (y \in \mathcal{B} \wedge y \notin \mathcal{A})]$

Null Set: Set containing NO element, denoted using ϕ or $\{\}$
(Ex: $\mathcal{Q} = \{z \mid x + y = z \text{ and all } x, y, z \text{ are odd}\} = \phi$)

Note: $|\phi| = 0$, but $\phi \neq \{0\}$ and $\phi \neq \{\phi\}$ (since, $|\{0\}| = |\{\phi\}| = 1$)

Power Set and Set Properties

Power Set: Set of all possible subsets of a set ($\mathcal{A}$), denoted as $\mathcal{P}(\mathcal{A})$ or $2^{\mathcal{A}}$

(Ex: Let $\mathcal{A} = \{1, 2, 3\}$,
Thus, $\mathcal{P}(\mathcal{A}) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$)

Cardinality: $|\mathcal{P}(\mathcal{A})| = 2^{|\mathcal{A}|}$ (Why?)

Proof: Let $|\mathcal{A}| = n$. There are $\binom{n}{k}$ subsets of size k possible (for any k, $0 \leq k \leq n$). So, the total number of subsets $= \sum_{i=0}^{n} \binom{n}{k} = 2^n$.

Properties: For sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$, we have the following:

- $\mathcal{A} \subset \mathcal{B} \Rightarrow \mathcal{A} \subseteq \mathcal{B}$, but $\mathcal{A} \subseteq \mathcal{B} \not\Rightarrow \mathcal{A} \subset \mathcal{B}$.
- $(\mathcal{A} \subset \mathcal{B})$ if and only if $[(\mathcal{A} \subseteq \mathcal{B}) \wedge (\mathcal{A} \neq \mathcal{B})]$.
- $(\mathcal{A} \neq \mathcal{B})$ if and only if $(\mathcal{A} \not\subseteq \mathcal{B}) \vee (\mathcal{B} \not\subseteq \mathcal{A})$.
- $\phi \subseteq \mathcal{A}$. If $\mathcal{A} \neq \phi$, then $\phi \subset \mathcal{A}$. $\mathcal{A} \subseteq \mathcal{A}$ and $\mathcal{A} \in \mathcal{P}(\mathcal{A})$.
- If $(\mathcal{A} \subseteq \mathcal{B})$, then $|\mathcal{A}| \leq |\mathcal{B}|$.
  If $(\mathcal{A} \subset \mathcal{B})$, then $|\mathcal{A}| < |\mathcal{B}|$.
  If $(\mathcal{A} = \mathcal{B})$, then $|\mathcal{A}| = |\mathcal{B}|$.
- If $(\mathcal{A} \subseteq \mathcal{B})$ and $(\mathcal{B} \subseteq \mathcal{C})$, then $(\mathcal{A} \subseteq \mathcal{C})$.
  If $(\mathcal{A} \subset \mathcal{B})$ and $(\mathcal{B} \subset \mathcal{C})$, then $(\mathcal{A} \subset \mathcal{C})$.
  If $(\mathcal{A} \subseteq \mathcal{B})$ and $(\mathcal{B} \subset \mathcal{C})$, then $(\mathcal{A} \subset \mathcal{C})$.
  If $(\mathcal{A} \subset \mathcal{B})$ and $(\mathcal{B} \subseteq \mathcal{C})$, then $(\mathcal{A} \subset \mathcal{C})$.

Frequently-Used Set Examples and Notations

Popular Set Examples:

$$
\begin{aligned}
\mathbb{N} &= \text{Set of Non-negative natural numbers} = \{0, 1, 2, \ldots\} \\
\mathbb{Z} &= \text{Set of Integers} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \\
\mathbb{Z}^+ &= \text{Set of Positive Integers} = \{x \in \mathbb{Z} \mid x > 0\} \\
\mathbb{Q} &= \text{Set of Rational Numbers} = \{\tfrac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\} \\
\mathbb{Q}^+ &= \text{Set of Positive Rational Numbers} = \{r \in \mathbb{Q} \mid r > 0\} \\
\mathbb{Q}^* &= \text{Set of Non-zero Rational Numbers} = \{r \in \mathbb{Q} \mid r \neq 0\} \\
\mathbb{R} &= \text{Set of Real Numbers} \\
\mathbb{R}^+ &= \text{Set of Positive Real Numbers} \\
\mathbb{R}^* &= \text{Set of Non-zero Real Numbers} \\
\mathbb{C} &= \text{Set of Complex Numbers} = \{a + ib \mid a, b \in \mathbb{R}, i^2 = -1\} \\
\mathbb{C}^* &= \text{Set of Non-zero Complex Numbers} = \{c \in \mathbb{C} \mid c \neq 0\}
\end{aligned}
$$

Frequently-Used Notations:

- For each $n \in \mathbb{Z}^+$, $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$
- For real numbers, a, b with $a < b$, we define intervals as follows:
  (Closed) $[a, b] = \{x \mid a \leq x \leq b\}$ (Open) $(a, b) = \{x \mid a < x < b\}$
  (Half-Open) $(a, b] = \{x \mid a < x \leq b\}$ and $[a, b) = \{x \mid a \leq x < b\}$

Counting using Set Theory

Prove that, $\binom{n}{r+1} = \frac{n-r}{r+1}\binom{n}{r}$

Counting: Total number of $(r+1)$-element subsets, formed from all r-element subsets by adding an element from $(n-r)$ remaining elements, is, $m = (n-r)\binom{n}{r}$.
Ex: Let $n = 4$ and $\mathcal{S} = \{1, 2, 3, 4\}$. All 2-element subsets are, $\mathcal{A}_1 = \{1, 2\}$, $\mathcal{A}_2 = \{1, 3\}$, $\mathcal{A}_3 = \{1, 4\}$, $\mathcal{A}_4 = \{2, 3\}$, $\mathcal{A}_5 = \{2, 4\}$, $\mathcal{A}_6 = \{3, 4\}$. From each $\mathcal{A}_i$s, a 3-element subset can be formed in *two* ways. So, total possibilities $= 2 \times \binom{4}{2} = 12$.
Repetition: Each $(r+1)$ element subset can be formed from $(r+1)$ different r-element subsets. So, the total choice reduces to, $\binom{n}{r+1} = \frac{m}{r+1} = \frac{n-r}{r+1}\binom{n}{r}$.
Ex: 3-element subset $\{1, 2, 3\}$ can formed from $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4$ by adding an element to each. So, reduced number of possibilities $= \frac{12}{3} = 4 = \binom{4}{3}$

Prove that, $\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n-1}{r} + \binom{n}{r} = \binom{n+1}{r+1}$

Let the $(n+1)$-element set be $= \{1, 2, \ldots, n, n+1\}$ From $(n+1)$-element set, choosing $(r+1)$-element subsets with smallest element i can be done in $\binom{n+1-i}{r}$ ways. So, all such possible choice leads to, $\sum_{i=1}^{(n+1)-(r+1)} \binom{n+1-i}{r} = \binom{n+1}{r+1}$, implying the proof.

Counting using Set Theory

Prove that, $\sum_{i=0}^{n} i\binom{n}{i} = n.2^{n-1}$

From an n-element set, Size of a subset with i elements + Size of its complement subset $= i + (n - i) = n$ and there are $\binom{n}{i}$ number of these each.
Therefore, $2\sum_{i=0}^{n} i\binom{n}{i} = n\sum_{i=0}^{n}\binom{n}{i} = n.2^n$, implying the proof.

Prove that, Number of Summands of n is 2^{n-1}.

Consider, $n = 4$.

Summand	Subset Correspondence
$1+1+1+1 = 1+1+1+1$	ϕ
$2+1+1 = (1+1)+1+1$	$\{1\}$
$1+2+1 = 1+(1+1)+1$	$\{2\}$
$1+1+2 = 1+1+(1+1)$	$\{3\}$
$3+1 = (1+1+1)+1$	$\{1,2\}$
$2+2 = (1+1)+(1+1)$	$\{1,3\}$
$1+3 = 1+(1+1+1)$	$\{2,3\}$
$4 = (1+1+1+1)$	$\{1,2,3\}$

$\therefore$ Number of summands of n = Number of subsets of an $(n-1)$-element set $= 2^{n-1}$.

Set Operations

For two sets, $\mathcal{A}, \mathcal{B} \in \mathcal{U}$ (universal set), the following operations are defined:

(Ex: Let, $\mathcal{A} = \{1, 2, 3\}$ and $\mathcal{B} = \{2, 3, 4\}$)

Union: $\mathcal{A} \cup \mathcal{B} = \{x \mid x \in \mathcal{A} \vee x \in \mathcal{B}\}$ (Ex: $\mathcal{A} \cup \mathcal{B} = \{1, 2, 3, 4\}$)

Intersection: $\mathcal{A} \cap \mathcal{B} = \{x \mid x \in \mathcal{A} \wedge x \in \mathcal{B}\}$ (Ex: $\mathcal{A} \cap \mathcal{B} = \{2, 3\}$)

Complement: $\overline{\mathcal{A}} = \{x \mid x \in \mathcal{U} \wedge x \notin \mathcal{A}\}$

Relative Complement: $\mathcal{A} - \mathcal{B} = \{x \mid x \in \mathcal{A} \wedge x \notin \mathcal{B}\} = \mathcal{A} \cap \overline{\mathcal{B}}$ (Ex: $\mathcal{A} - \mathcal{B} = \{1\}$)

Symmetric Difference:

$$\begin{aligned}
\mathcal{A} \,\Delta\, \mathcal{B} &= \{x \mid (x \in \mathcal{A} \vee x \in \mathcal{B}) \wedge x \notin \mathcal{A} \cap \mathcal{B}\} \\
&= \{x \mid x \in \mathcal{A} \cup \mathcal{B} \wedge x \notin \mathcal{A} \cap \mathcal{B}\} = (\mathcal{A} \cup \mathcal{B}) - (\mathcal{A} \cap \mathcal{B}) \\
&= \{x \mid x \in \mathcal{A} \cap \overline{\mathcal{B}} \wedge x \in \overline{\mathcal{A}} \cap \mathcal{B}\} = (\mathcal{A} \cap \overline{\mathcal{B}}) \cup (\overline{\mathcal{A}} \cap \mathcal{B}) \\
&= (\mathcal{A} - \mathcal{B}) \cup (\mathcal{B} - \mathcal{A}) = \mathcal{B} \,\Delta\, \mathcal{A}
\end{aligned}$$

(Ex: $\mathcal{A} \,\Delta\, \mathcal{B} = \{1, 4\}$)

Mutual Disjoint: Sets, $\mathcal{A}$ and $\mathcal{B}$, are mutually disjoint (or disjoint), when $\mathcal{A} \cap \mathcal{B} = \phi$.
In such a case, $\mathcal{A} \,\Delta\, \mathcal{B} = \mathcal{A} \cup \mathcal{B}$, $\mathcal{A} \cap \overline{\mathcal{B}} = \mathcal{A}$ and $\overline{\mathcal{A}} \cap \mathcal{B} = \mathcal{B}$.

The following statements are equivalent: *(Proof Left as an Exercise!)*

(a) $\mathcal{A} \subseteq \mathcal{B}$, (b) $\mathcal{A} \cup \mathcal{B} = \mathcal{B}$, (c) $\mathcal{A} \cap \mathcal{B} = \mathcal{A}$, (d) $\overline{\mathcal{B}} \subseteq \overline{\mathcal{A}}$

Laws of Set Theory

For three sets, $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{U}$, the rules given as follows:

Name of the Law	Mathematical Expressions
Double Complement:	$\overline{\overline{\mathcal{A}}} = \mathcal{A}$
DeMorgan's Laws:	$\overline{\mathcal{A} \cup \mathcal{B}} = \overline{\mathcal{A}} \cap \overline{\mathcal{B}}, \quad \overline{\mathcal{A} \cap \mathcal{B}} = \overline{\mathcal{A}} \cup \overline{\mathcal{B}}$
Commutative Laws:	$\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}, \quad \mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}$
Associative Laws:	$\mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C}, \quad \mathcal{A} \cap (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C}$
Distributive Laws:	$\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C}), \mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$
Idempotent Laws:	$\mathcal{A} \cup \mathcal{A} = \mathcal{A}, \quad \mathcal{A} \cap \mathcal{A} = \mathcal{A}$
Identity Laws:	$\mathcal{A} \cup \phi = \mathcal{A}, \quad \mathcal{A} \cap \mathcal{U} = \mathcal{A}$
Inverse Laws:	$\mathcal{A} \cup \overline{\mathcal{A}} = \mathcal{U}, \quad \mathcal{A} \cap \overline{\mathcal{A}} = \phi$
Domination Laws:	$\mathcal{A} \cup \mathcal{U} = \mathcal{U}, \quad \mathcal{A} \cap \phi = \phi$
Absorption Laws:	$\mathcal{A} \cup (\mathcal{A} \cap \mathcal{B}) = \mathcal{A}, \quad \mathcal{A} \cap (\mathcal{A} \cup \mathcal{B}) = \mathcal{A}$

An Example Proof Sketch: $\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$

$$\begin{aligned} x \in \mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) &\Leftrightarrow (x \in \mathcal{A}) \vee (x \in \mathcal{B} \cap \mathcal{C}) \Leftrightarrow (x \in \mathcal{A}) \vee ((x \in \mathcal{B}) \wedge (x \in \mathcal{C})) \\ &\Leftrightarrow ((x \in \mathcal{A}) \vee (x \in \mathcal{B})) \wedge ((x \in \mathcal{A}) \vee (x \in \mathcal{C})) \\ &\Leftrightarrow (x \in \mathcal{A} \cup \mathcal{B}) \wedge (x \in \mathcal{A} \cup \mathcal{C}) \Leftrightarrow x \in (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C}) \end{aligned}$$

Some Derived Laws and Observations

$\mathcal{A}_1 = \mathcal{A}_1 \cup (\mathcal{A}_1 \cap \mathcal{A}_2) \cup (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3) \cup (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4) \cup \cdots \quad (\forall i,\ \mathcal{A}_i \in \mathcal{U})$

Proof: $\mathcal{A}_1 \cup (\mathcal{A}_1 \cap \mathcal{A}_2) = \mathcal{A}_1, \quad (\mathcal{A}_1 \cap \mathcal{A}_2) \cup (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3) = (\mathcal{A}_1 \cap \mathcal{A}_2),$
$(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3) \cup (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4) = (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3)$, *and so on ...*

Similarly, $\mathcal{A}_1 = \mathcal{A}_1 \cap (\mathcal{A}_1 \cup \mathcal{A}_2) \cap (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3) \cap (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4) \cap \cdots$

$\overline{\mathcal{A} \,\Delta\, \mathcal{B}} = \mathcal{A} \,\Delta\, \overline{\mathcal{B}} = \overline{\mathcal{A}} \,\Delta\, \mathcal{B}$

Proof: As, $\mathcal{A} \,\Delta\, \mathcal{B} = (\mathcal{A} \cup \mathcal{B}) - (\mathcal{A} \cap \mathcal{B})$ *and* $\mathcal{A} \,\Delta\, \mathcal{B} = (\mathcal{A} \cap \overline{\mathcal{B}}) \cup (\overline{\mathcal{A}} \cap \mathcal{B})$, *so*

$\overline{\mathcal{A} \,\Delta\, \mathcal{B}} = \overline{(\mathcal{A} \cap \overline{\mathcal{B}}) \cup (\overline{\mathcal{A}} \cap \mathcal{B})} = (\overline{\mathcal{A}} \cup \mathcal{B}) \cap \overline{(\overline{\mathcal{A}} \cap \mathcal{B})} = (\overline{\mathcal{A}} \cup \mathcal{B}) - (\overline{\mathcal{A}} \cap \mathcal{B}) = \overline{\mathcal{A}} \,\Delta\, \mathcal{B}$ *and*

$\overline{\mathcal{A} \,\Delta\, \mathcal{B}} = \overline{(\overline{\mathcal{A}} \cap \mathcal{B}) \cup (\mathcal{A} \cap \overline{\mathcal{B}})} = (\mathcal{A} \cup \overline{\mathcal{B}}) \cap \overline{(\mathcal{A} \cap \overline{\mathcal{B}})} = (\mathcal{A} \cup \overline{\mathcal{B}}) - (\mathcal{A} \cap \overline{\mathcal{B}}) = \mathcal{A} \,\Delta\, \overline{\mathcal{B}}$

$$\mathcal{A} - (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} - \mathcal{B}) \cap (\mathcal{A} - \mathcal{C}) \qquad (\mathcal{A} \cup \mathcal{B}) - \mathcal{C} = (\mathcal{A} - \mathcal{C}) \cup (\mathcal{B} - \mathcal{C})$$
$$\mathcal{A} - (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} - \mathcal{B}) \cup (\mathcal{A} - \mathcal{C}) \qquad (\mathcal{A} \cap \mathcal{B}) - \mathcal{C} = (\mathcal{A} - \mathcal{C}) \cap (\mathcal{B} - \mathcal{C})$$
$$(\mathcal{A} \cap \mathcal{B}) - (\mathcal{A} \cap \mathcal{C}) = \mathcal{A} \cap (\mathcal{B} - \mathcal{C})$$

$$\overline{\mathcal{A}} \,\Delta\, \overline{\mathcal{B}} = \mathcal{A} \,\Delta\, \mathcal{B} = \mathcal{B} \,\Delta\, \mathcal{A} = \overline{\mathcal{B}} \,\Delta\, \overline{\mathcal{A}}$$
$$\mathcal{A} \,\Delta\, (\mathcal{B} \,\Delta\, \mathcal{C}) = (\mathcal{A} \,\Delta\, \mathcal{B}) \,\Delta\, \mathcal{C} \qquad \mathcal{A} \cap (\mathcal{B} \,\Delta\, \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \,\Delta\, (\mathcal{A} \cap \mathcal{C})$$

Index Set and Partitions

Index Set

Definition: Let $\mathcal{I} \neq \phi$ and $\forall i \in \mathcal{I}$, let $\mathcal{A}_i \subseteq \mathcal{U}$ (universal set). Then, $\mathcal{I}$ is called an *index set*, and each $i \in \mathcal{I}$ is an index.

Set Operations: (Union) $\bigcup_{i \in \mathcal{I}} \mathcal{A}_i = \{x \mid \exists i \in \mathcal{I},\ x \in \mathcal{A}_i\}$

(Intersection) $\bigcap_{i \in \mathcal{I}} \mathcal{A}_i = \{x \mid \forall i \in \mathcal{I},\ x \in \mathcal{A}_i\}$

Generalized DeMorgan's Law: $\overline{\bigcup_{i \in \mathcal{I}} \mathcal{A}_i} = \bigcap_{i \in \mathcal{I}} \overline{\mathcal{A}_i}$ and $\overline{\bigcap_{i \in \mathcal{I}} \mathcal{A}_i} = \bigcup_{i \in \mathcal{I}} \overline{\mathcal{A}_i}$

Partition of a Set

Definition: Let $\mathcal{S}$ be a non-empty set. A family of non-empty subsets, $\{\mathcal{S}_i \mid i \in \mathcal{I}\}$ ($\mathcal{I}$ being the index set) is said to form a partition of $\mathcal{S}$ if the following two condition holds:

- $\bigcup_{i \in \mathcal{I}} \mathcal{S}_i = \mathcal{S}$ (Complete Set Cover), and
- $\mathcal{S}_i \cap \mathcal{S}_j = \phi, \forall i, j \in \mathcal{I}$ and $i \neq j$ (Pairwise Disjoint).

Example: Let $\mathcal{Z}_0 = \{3m \mid m \text{ is an integer}\} = \{0, \pm 3, \pm 6, \ldots\}$,
$\mathcal{Z}_1 = \{3m + 1 \mid m \text{ is an integer}\} = \{\ldots, -8, -5, -2, +1, +4, +7, \ldots\}$
$\mathcal{Z}_2 = \{3m + 2 \mid m \text{ is an integer}\} = \{\ldots, -7, -4, -1, +2, +5, +8, \ldots\}$
Now, $\mathcal{Z}_0 \cup \mathcal{Z}_1 \cup \mathcal{Z}_2 = \mathbb{Z}$ and $\mathcal{Z}_0 \cap \mathcal{Z}_1 = \mathcal{Z}_1 \cap \mathcal{Z}_2 = \mathcal{Z}_2 \cap \mathcal{Z}_0 = \phi$

Thank You!