Recurrence Relations

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Autumn 2020

Introduction

Recurrence Relations are Mathematical Equations: A recurrence relation is an equation which is defined in terms of itself.

Natural Computable Functions as Recurrences: Many natural functions are expressed using recurrence relations.

- (*linear*) $f(n) = f(n-1) + 1, f(1) = 1 \Rightarrow f(n) = n$
- (*polynomial*) $f(n) = f(n-1) + n, f(1) = 1 \Rightarrow f(n) = \frac{1}{2}(n^2 + n)$
- (*exponential*) $f(n) = 2.f(n-1), f(0) = 1 \Rightarrow f(n) = 2^n$
- (*factorial*) $f(n) = n.f(n-1), f(0) = 1 \Rightarrow f(n) = n!$

Recurrence is Mathematical Induction:

Recurrence: $T(n) = 2T(n-1) + 1$ with base condition, $T(0) = 0$.
Base-condition check: $T(0) = 2^0 - 1$
Induction Hypothesis: $T(n-1) = 2^{n-1} - 1$

Proof: $T(n) = 2T(n-1) + 1 = 2(2^{n-1} - 1) + 1 = 2^n - 1$

Types of Recurrence Relations:

- First Order, Second Order, ..., Higher Order
- Linear vs. Non-Linear
- Homogeneous vs. Non-Homogeneous
- Constant vs. Variable Coefficients

Applications: Algorithm Analysis, Counting, Problem Solving, Reasoning etc.

Recurrent Problems

Regions using Straight Lines in a Plane

Recurrent Problem: Maximum number of regions defined using n lines in a plane.

Recursive Solution: (Proposed by Jacob Steiner in 1826)

1. Observation-0: No line is parallel and co-linear with another.
2. Observation-1: $(n+1)^{th}$ line, when introduced into a plane with n lines, intersects with all n line exactly once.
3. Observation-2: When traversed from one endpoint to another of a newly introduced line, every time at crossing-point of intersection with another line, the new line has created one new region.
4. Observation-3: After last intersection, the line cuts the infinite ending region into two (that is, introducing the final new region).

Recurrence Relation: L_n = maximum number of regions created by n lines in a plane.

$$L_n = \begin{cases} L_{n-1} + n, & \text{if } n > 0 \\ 1, & \text{if } n = 0 \end{cases}$$

Number of Regions: $L_n = L_{n-1} + n = L_{n-2} + (n-1) + n = L_{n-3} + (n-2) + (n-1) + n$

$$= \cdots = L_0 + 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n = 1 + \sum_{i=1}^{n} i = \frac{n(n+1)}{2} + 1$$

Recurrent Problems

Regions using Bent Lines (V-shaped) in a Plane

Recurrent Problem: Maximum number of regions defined using n bent-lines (V-shaped) in a plane.

Recursive Solution: (Variant of Maximum Regions by Straight Lines Problem)

1. Observation-0: No V-shaped bent-line tip will coincide with the tip of another bent-line.
2. Observation-1: A bent-line is like two straight lines except that regions merge when the two lines do not extend past their intersection point.
3. Observation-2: The tip point must lie beyond the intersections with the other lines – that is all we lose; that is, we lose only two regions per line.

Recurrence Relation: $V_n =$ maximum number of regions created by n bent-lines.

$$V_n = \begin{cases} L_{2n} - 2n, & \text{if } n > 0 \\ 1, & \text{if } n = 0 \end{cases}$$

Number of Regions: $V_n = L_{2n} - 2n = \frac{2n(2n+1)}{2} + 1 - 2n = 2n^2 - n + 1$

Recurrent Problems

Tower of Hanoi: n Disk Transfer with 3 Pegs

Recurrent Problem: Number of steps required in transferring all n disks (having different sizes) from Peg-A to Peg-B using auxiliary Peg-C, such that –

- Always smaller sized disk is placed above larger sized disk.
- At start, all n disks are stacked together in Peg-A in their descending order of size (bottom-up).

Recursive Solution: (Proposed by François Édouard Anatole Lucas in 1883)

1. If $n = 1$, Move the disk from Peg-A to Peg-B.
2. If $n > 1$, Move top $(n-1)$ disks from Peg-A to Peg-C using Peg-B as auxiliary. Move Largest disk directly from Peg-A to Peg-B. Move $(n-1)$ disks from Peg-C to Peg-B using Peg-A as auxiliary.

Recurrence Relation: T_n = number of movements for transferring n disks.

$$T_n = \begin{cases} T_{n-1} + 1 + T_{n-1}, & \text{if } n > 1 \\ 1, & \text{if } n = 1 \end{cases} \Rightarrow T_n = 2T_{n-1} + 1 \ (n > 1),\ T_1 = 1$$

Number of Moves: $T_n = 2T_{n-1} + 1 = 2^2 T_{n-2} + 2 + 1 = 2^3 T_{n-3} + 2^2 + 2 + 1 = \cdots$

$$= 2^{n-1} T_1 + 2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2^1 + 2^0 = \sum_{i=0}^{n-1} 2^i = 2^n - 1$$

Recurrent Problems

Tower of Hanoi: n Disk Transfer with 4 Pegs

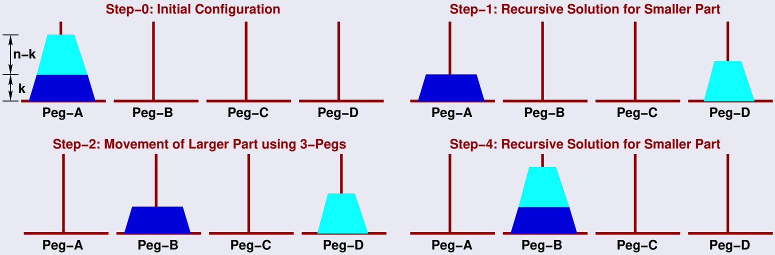
Recurrent Problem: Number of steps required in transferring n different-sized disks from Peg-A to Peg-B using auxiliary Peg-C and Peg-D, such that –

- Always smaller sized disk is placed above larger sized disk.
- At start, all n disks are stacked together in Peg-A in their descending order of size (bottom-up).

Recursive Solution: (Proposed by J.S. Frame and B.M. Stewart in 1941)

1. If $n \leq 3$, Solve the problem directly using 3 pegs.
2. Fix a value of k in the range $1 \leq k \leq n$.
3. Keep the k largest disks on Peg-A, and transfer the smallest $(n-k)$ disks from Peg-A to Peg-D.
4. Transfer the largest k disks from Peg-A to Peg-B without disturbing the smallest $(n-k)$ disks already sitting on Peg-D.
   (Since larger disk can never be above smaller disk, Peg-D is unusable in this part, that is, we solve 3-peg Tower-of-Hanoi problem on k disks.)
5. Transfer the smallest $(n-k)$ disks from Peg-D to Peg-B without disturbing the largest k disks on Peg-B.
   (In this step, all the four pegs can be used.)

Tower of Hanoi: n Disk Transfer with 4 Pegs



Recurrence Relation: H_n = number of movements for transferring n disks with 4-pegs.

T_n = number of movements for transferring n disks with 3-pegs.

$$\therefore\ H_n = \begin{cases} H_{n-k} + T_k + H_{n-k} = 2H_{n-k} + 2^k - 1, & \text{if } n > 3 \\ T_n = 2^n - 1, & \text{if } 0 \leq n \leq 3 \end{cases}$$

Number of Moves: Depends on best choice of k. For simplicity, let us assume $n = uk$.

$$\begin{aligned} U_n &\approx 2U_{n-k} + 2^k \approx 2^2 U_{n-2k} + (2+1).2^k \approx 2^3 U_{n-3k} + (2^2+2+1).2^k \\ &\approx \cdots \approx 2^{u-1} U_k + (2^{u-2} + 2^{u-3} + \cdots + 2^2 + 2^1 + 2^0).2^k \\ &\approx \left(\sum_{i=0}^{u-1} 2^i\right).2^k = 2^{u+k} = 2^{\frac{n}{k}+k} \end{aligned}$$

(by Paul Stockmeyer in 1994)

Since, $\left(\frac{n}{k} + k\right)$ can be minimized for $k = \sqrt{n}$, therefore $U_n \approx 2^{2\sqrt{n}}$.

Solving First-Order Recurrence Relations

First-Order Linear Homogeneous Recurrence with Constant Coefficients

General Form: $t_{n+1} = c.t_n$, where $n \geq 0$ and c is a constant

Boundary Condition: $t_0 = B$, where B is a constant

Solution: $t_n = c.t_{n-1} = c^2.t_{n-2} = \cdots = c^i.t_{n-i} = \cdots = c^n.t_0 = B.c^n$, for $n \geq 0$

Example

1. $a_n = 3.a_{n-1}$ where $n \geq 1$ and $a_2 = 18$. Clearly, $a_2 = 3^2.a_0 = 18 \Rightarrow a_0 = 2$. So, $a_n = 2.3^n$ for $n \geq 0$ is the unique solution.
2. *Number of Different Summands of n*: $s_{n+1} = 2.s_n$ where $n \geq 1$ with boundary condition $s_1 = 1$. To directly apply the formula proposed, let $t_n = s_{n+1}$, which formulates the reccurence as, $t_n = 2.t_{n-1}$ where $n \geq 0$ with $t_0 = 1$. So, $t_n = 1.2^n$. Now, $s_n = t_{n-1} = 2^{n-1}$ for $n \geq 1$.

Different Summands of 3		Different Summands of 4			
(1) 3	(2) $1+2$	(1′) 4	(2′) $1+3$	(3′) $2+2$	(4′) $1+1+2$
(3) $2+1$	(4) $1+1+1$	(1″) $3+1$	(2″) $1+2+1$	(3″) $2+1+1$	(4″) $1+1+1+1$

Solving First-Order Recurrence Relations

First-Order Linear Homogeneous Recurrence with **Variable Coefficients**

General Form: $t_{n+1} = f(n).t_n$, where $n \geq 0$

Boundary Condition: $t_0 = B$, where B is a constant

Solution: $t_n = f(n-1).t_{n-1} = f(n-2).f(n-1).t_{n-2} = \cdots = B.\left[\prod_{k=1}^{n} f(n-k)\right]$

Example: (Factorials) $f_n = n.f_{n-1},\ n \geq 1$ and $f_0 = 1$. Solution: $f_n = n!\ (n \geq 0)$.

First-Order **Non-Linear** Homogeneous Recurrence with Constant Coefficients

General Form: $t_{n+1}^k = c.t_n^k$, where $t_n > 0$ for $n \geq 0$ and c, k are constants

Boundary Condition: $t_0 = B$, where B is a constant

Solution: Let $r_n = t_n^k$. So, the recurrence becomes, $r_{n+1} = c.r_n$ for $n \geq 0$ and $r_0 = B^k$. Hence, $t_n^k = r_n = B^k.c^n$ implying $t_n = B.(\sqrt[k]{c})^n$ for $n \geq 0$.

Example (a small Variation): $\log_2 a_{n+1} = 2.\log_2 a_n$ for $n \geq 0$ and $a_0 = 2$.

Putting $b_n = \log_2 a_n$ gives $b_{n+1} = 2.b_n$ and $b_0 = 1$.

So, $b_n = 2^n$ and hence $a_n = 2^{2^n}$ for $n \geq 0$.

Solving First-Order Recurrence Relations

First-Order Linear Non-Homogeneous Recurrence with Constant Coefficients

General Form: $t_{n+1} + d.t_n = f(n)$ or alternatively, $t_{n+1} = c.t_n + f(n)$, where $f(n) \neq 0$ (which means non-homogeneous) for $n \geq 0$ and $c = -d$ is a constant

Boundary Condition: $t_0 = B$, where B is a constant

Solution: $t_n = c.t_{n-1} + f(n-1) = c^2.t_{n-2} + c^1.f(n-2) + f(n-1) = \cdots$

$$= c^i.t_{n-i} + \sum_{k=0}^{i-1} c^k.f(n-i+k) = \cdots = B.c^n + \sum_{k=0}^{n-1} c^k.f(k), \text{ for } n \geq 0$$

Example:

- (Comparisons in Sorting) – Bubble, Selection and Insertion
  $a_n = a_{n-1} + (n-1)$ where $n \geq 2$ and $a_1 = 0$.
  Hence, the solution, $a_n = 0 + \sum_{k=1}^{n-1} k = \frac{n^2-n}{2}$. $\Rightarrow O(n^2)$
- (n^{th} term in Sequence) $0, 2, 6, 12, 20, 30, 42, \ldots$
  $a_n = a_{n-1} + 2n$ where $n \geq 1$ and $a_0 = 0$. (How?)
  Since $a_1 - a_0 = 2$, $a_2 - a_1 = 4$, $a_3 - a_2 = 6$, $a_4 - a_3 = 8$, $a_5 - a_4 = 10$, $a_6 - a_5 = 12$,
  therefore $a_n - a_0 = 2 + 4 + \cdots + 2n = n^2 + n$, implies $a_n = n^2 + n$.

First-Order Linear Non-Homogeneous Recurrence with Variable Coefficients

General Form: $t_{n+1} = f(n).t_n + g(n)$, where $g(n) \neq 0$ for $n \geq 0$ and $t_0 = B$ (constant)

Generic Solution: $t_n = B.\left[\prod_{k=0}^{n-1} f(k)\right] + \sum_{k=1}^{n-1} \left[\prod_{j=1}^{k-1} f(n-j)\right].g(n-k)$, for $n \geq 0$

Solving Second-Order Recurrence Relations

Second-Order Linear Homogeneous Recurrence with Constant Coefficients

General Form: $C_0.t_n + C_1.t_{n-1} + C_2.t_{n-2} = 0$ $(n \geq 2)$ and $t_0 = D_0, t_1 = D_1$;
$C_0(\neq 0), C_1, C_2(\neq 0)$ and D_0, D_1 all are constants.

Characteristic Equation: Seeking a solution, $t_n = c.x^n$ $(c, x \neq 0)$, after substitution,
$C_0.c.x^n + C_1.c.x^{n-1} + C_2.c.x^{n-2} = 0 \quad \Rightarrow C_0.x^2 + C_1.x + C_2 = 0$

Equation Roots: **2 Distinct Real Roots** as, $R_1 = \frac{-C_1+\sqrt{C_1^2-4C_0C_2}}{2C_0}$, $R_2 = \frac{-C_1-\sqrt{C_1^2-4C_0C_2}}{2C_0}$

Exact Solution: As $t_n = A_1.R_1^n$ and $t_n = A_2.R_2^n$ are linearly independent solutions, so

$$t_n = A_1.R_1^n + A_2.R_2^n = A_1.\left(\frac{-C_1+\sqrt{C_1^2-4C_0C_2}}{2C_0}\right)^n + A_2.\left(\frac{-C_1-\sqrt{C_1^2-4C_0C_2}}{2C_0}\right)^n$$

(Here, A_1 and A_2 are arbitrary constants)

Constant Determination: $A_1 + A_2 = t_0 = D_0$ and $A_1 - A_2 = \frac{2C_0D_1+C_1D_0}{\sqrt{C_1^2-4C_0C_2}}$

because, $D_1 = t_1 = (A_1 + A_2).\left(-\frac{C_1}{2C_0}\right) + (A_1 - A_2).\left(\frac{\sqrt{C_1^2-4C_0C_2}}{2C_0}\right)$

$$\therefore A_1 = \frac{1}{2}\left(D_0 + \frac{2C_0D_1+C_1D_0}{\sqrt{C_1^2-4C_0C_2}}\right) \text{ and } A_2 = \frac{1}{2}\left(D_0 - \frac{2C_0D_1+C_1D_0}{\sqrt{C_1^2-4C_0C_2}}\right).$$

Unique Solution:

$$\therefore\ t_n = \frac{1}{2}\left[\left(D_0 + \frac{2C_0D_1+C_1D_0}{\sqrt{C_1^2-4C_0C_2}}\right).\left(\frac{-C_1+\sqrt{C_1^2-4C_0C_2}}{2C_0}\right)^n + \left(D_0 - \frac{2C_0D_1+C_1D_0}{\sqrt{C_1^2-4C_0C_2}}\right).\left(\frac{-C_1-\sqrt{C_1^2-4C_0C_2}}{2C_0}\right)^n\right]$$

Solving Second-Order Recurrence Relations

Example (Fibonacci Number)

Recurrence Relation: $F_{n+2} = F_{n+1} + F_n$, where $n \geq 0$ and $F_0 = 0, F_1 = 1$

Substituting with $F_n = c.x^n$ $(c, x \neq 0)$, we get $cx^{n+2} = cx^{n+1} + cx^n$.

Characteristic Equation $x^2 - x - 1 = 0$ has two distinct roots, $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Hence, $F_n = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n$, with the constants derived as, $c_1 = \frac{1}{\sqrt{5}}$, $c_2 = -\frac{1}{\sqrt{5}}$.

Solution: *(Binet Form)* $F_n = \frac{1}{\sqrt{5}}\left[\alpha^n - \beta^n\right]$ $\quad$ ($\alpha = 1 - \beta$ is called the *Golden Ratio*)

Example (Count of Subsets with NO Consecutive Elements Chosen)

Let, the number of such subsets of $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$ is $= a_n$

If $n = 0 \Rightarrow \mathcal{S} = \phi$, $a_0 = 1$. $\qquad$ If $n = 1 \Rightarrow \mathcal{S} = \{x_1\}$, $a_1 = 2$.

Let $n \geq 2$ and $\mathcal{A} \subseteq \mathcal{S} = \{x_1, x_2, \ldots, x_{n-1}, x_n\}$, a_n can be contributed from:

- When $x_n \in \mathcal{A} \Rightarrow x_{n-1} \notin \mathcal{A}$, $\therefore \mathcal{A}$ may be counted in a_{n-2} ways.
- When $x_n \notin \mathcal{A}$, $\therefore \mathcal{A}$ may be counted in a_{n-1} ways.

Recurrence Relation: $a_n = a_{n-1} + a_{n-2}$ $(n \geq 2)$ and $a_0 = 1, a_1 = 2$.

Solution: $a_n = \frac{1}{\sqrt{5}}\left[(\frac{1+\sqrt{5}}{2})^{n+2} - (\frac{1-\sqrt{5}}{2})^{n+2}\right]$, $n \geq 0$ $\qquad$ (Note that, $a_n = F_{n+2}$)

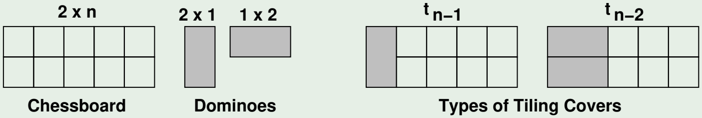
Solving Second-Order Recurrence Relations

Example (Count of Binary Strings having NO consecutive 0s)

Let, b_n = number of such binary strings of length n;
$b_n^{(0)}$ = count of such strings ending with 0 and $b_n^{(1)}$ = count of such strings ending with 1
Recurrence Relation: $b_n = 2.b_{n-1}^{(1)} + b_{n-1}^{(0)} = b_{n-1}^{(1)} + b_{n-1} = b_{n-2} + b_{n-1}$ $(n \geq 3)$ and
$b_1 = 2, b_2 = 3$, implying $b_0 = b_2 - b_1 = 1$.
Solution: $b_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right]$, $n \geq 0$ (Note that, $b_n = F_{n+2}$)

Example ($2 \times n$ Chessboard Tiling using Dominoes)

Let, t_n = number of ways to tile $2 \times n$ $(n \in \mathbb{Z}^+)$ chessboard.
Recurrence Relation: $t_n = t_{n-1} + t_{n-2}$ $(n \geq 2)$ and $t_1 = 1, t_2 = 2$



Solution: $t_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$, $n \geq 0$ (Note that, $t_n = F_{n+1}$)

Solving Second-Order Recurrence Relations

Example (Counting Legal Arithmetic Expressions without Parenthesis)

10 digit symbols: $0, 1, 2, \ldots, 9$ and 4 binary operation symbols: $+, -, *, /$
$e_n =$ number of legal arithmetic expressions with n symbols.
Note that, last symbol is always a digit. So, Two ways to construct recurrence for e_n:
$10e_{n-1}$ (last two symbols as digits) and $39e_{n-2}$ (last two symbol as operator and digit)
Recurrence Relation: $e_n = 10e_{n-1} + 39e_{n-2}$ $(n \geq 0)$ and $e_1 = 10, e_2 = 100 \Rightarrow e_0 = 0$
Characteristics Roots: $R_1 = 5 + 3\sqrt{6}$ *and* $R_2 = 5 - 3\sqrt{6}$
Solution: $e_n = \frac{5}{3\sqrt{6}}\left[(5 + 3\sqrt{6})^n - (5 - 3\sqrt{6})^n\right]$, $n \geq 0$

Example (Count of Transmission Words with Constraints)

$w_n =$ number of n-length words using a, b, c (three) letters that can be transmitted where no word having two consecutive a's
Two ways to construct recurrence for w_n:

- First letter is b or c: Number of words $= w_{n-1}$ (each)
- First letter is a, Second letter is b or c: Number of words $= w_{n-2}$ (each)

Recurrence Relation: $w_n = 2w_{n-1} + 2w_{n-2}$ $(n \geq 2)$ and $w_0 = 1, w_1 = 3$
Characteristics Roots: $R_1 = 1 + \sqrt{3}$ *and* $R_2 = 1 - \sqrt{3}$
Solution: $w_n = \left(\frac{2+\sqrt{3}}{2\sqrt{3}}\right)(1 + \sqrt{3})^n + \left(\frac{-2+\sqrt{3}}{2\sqrt{3}}\right)(1 - \sqrt{3})^n$, $n \geq 0$

Solving Second-Order Recurrence Relations

Example (Number of Palindromic Summands)

$p_n =$ number of palindromic summands of n.
Two ways to construct recurrence for p_n:

- Appending $+1$ at both sides of all the $(n-2)^{th}$ palindromic summands.
- Incrementing both ends of all the $(n-2)^{th}$ palindromic summands by $+1$.

For 3:	**For** 5:	**For** 4:	**For** 6:	
(1) 3 (2) $1+1+1$	$(1')$ 5 $(2')$ $2+1+2$ $(1'')$ $1+3+1$ $(2'')$ $1+1+1+1+1$	(1) 4 (2) $1+2+1$ (3) $2+2$ (4) $1+1+1+1$	$(1')$ 6 $(2')$ $2+2+2$ $(3')$ $3+3$ $(4')$ $2+1+1+2$	$(1'')$ $1+4+1$ $(2'')$ $1+1+2+1+1$ $(3'')$ $1+2+2+1$ $(4'')$ $1+1+1+1+1+1$

Recurrence Relation: $p_n = 2p_{n-2}$ $(n \geq 3)$ and $p_1 = 1, p_2 = 2$
Characteristics Roots: $R_1 = \sqrt{2}$ *and* $R_2 = -\sqrt{2}$
Solution: $p_n = \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)(\sqrt{2})^n + \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)(-\sqrt{2})^n$, $n \geq 1$

Observation: $p_n = 2^{\frac{n}{2}}$ (when n is even) and $p_n = 2^{\lfloor \frac{n}{2} \rfloor}$ (when n is odd) (How?)
Reason: For $n = 2k$ $(k \in \mathbb{Z}^+)$, $p_n = \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)(\sqrt{2})^{2k} + \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)(-\sqrt{2})^{2k} = 2^k = 2^{\frac{n}{2}}$
For $n = 2k-1$ $(k \in \mathbb{Z}^+)$, $p_n = \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)(\sqrt{2})^{2k-1} + \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)(-\sqrt{2})^{2k-1} = 2^{k-1} = 2^{\lfloor \frac{n}{2} \rfloor}$

Solving Second-Order Recurrence Relations

Example (Number of Divisions in Euclidean GCD Computation)

Computation of $GCD(a, b)$ is done as follows: (Let $r_0 = a$ and $r_1 = b$)

$r_0 = q_1 r_1 + r_2$ $(0 < r_2 < r_1, q_1 \geq 1)$, $r_1 = q_2 r_2 + r_3$ $(0 < r_3 < r_2, q_2 \geq 1)$, $r_2 = q_3 r_3 + r_4$ $(0 < r_4 < r_3, q_3 \geq 1)$

....

$r_{n-2} = q_{n-1} r_{n-1} + r_n$ $(0 < r_n < r_{n-1}, q_{n-1} \geq 1)$, $r_{n-1} = q_n r_n$ $(q_n \geq 2$ as $r_n < r_{n-1})$

Estimation of remainders are done as follows: ($F_n = n^{th}$ Fibonacci Number)

$$
\begin{array}{rcl}
(r_n > 0) & \Rightarrow & r_n \geq 1 = F_2 \\
(q_n \geq 2) \wedge (r_n \geq F_2) & \Rightarrow & r_{n-1} = q_n r_n \geq 2.1 = 2 = F_3 \\
(q_{n-1} \geq 1) \wedge (r_{n-1} \geq F_3) \wedge (r_n \geq F_2) & \Rightarrow & r_{n-2} = q_{n-1} r_{n-1} + r_n \geq 1.r_{n-1} + r_n = F_3 + F_2 = F_4 \\
\cdots \quad \cdots & & \cdots \quad \cdots \\
(q_3 \geq 1) \wedge (r_3 \geq F_{n-1}) \wedge (r_4 \geq F_{n-2}) & \Rightarrow & r_2 = q_3 r_3 + r_4 \geq 1.r_3 + r_4 = F_{n-1} + F_{n-2} = F_n \\
(q_2 \geq 1) \wedge (r_2 \geq F_n) \wedge (r_3 \geq F_{n-1}) & \Rightarrow & b = r_1 = q_2 r_2 + r_3 \geq 1.r_2 + r_3 = F_n + F_{n-1} = F_{n+1}
\end{array}
$$

Important Property of Fibonacci Numbers: $F_n > \alpha^{n-2}$ (for $n \geq 3$), where $\alpha = \frac{1+\sqrt{5}}{2}$

Let, $GCD(a, b)$ uses n Divisions ($a \geq b \geq 2$). So, $b \geq F_{n+1} > \alpha^{n-1} = (\frac{1+\sqrt{5}}{2})^{n-1}$.

$\therefore b > \alpha^{n-1} \Rightarrow \log_{10} b > (n-1)\log_{10}\alpha > \frac{n-1}{5}$ (as $\log_{10}\alpha = \log_{10}\left(\frac{1+\sqrt{5}}{2}\right) \approx 0.209 > \frac{1}{5}$).

If b is k-digit decimal number, $10^{k-1} \leq b < 10^k \Rightarrow k > \log_{10} b > \frac{n-1}{5} \Rightarrow n < 5k + 1$.

Lamé's Theorem: Number of divisions performed in Euclidean GCD computation $GCD(a, b)$ $(a \geq b \geq 2,\ a, b \in \mathbb{Z}+)$ is at most 5 times the number of decimal digits in b.

Corollary: Number of divisions, $n < 1 + 5\log_{10} b < 9\log_{10} b \quad \Rightarrow n = O(\log_{10} b)$
(as, $b \geq 2 \Rightarrow 4\log_{10} b \geq \log_{10} 2^4 > 1$)

Solving Second-Order Recurrence Relations

Second-Order Linear Homogeneous Recurrence with Constant Coefficients

General Form: $C_0.t_n + C_1.t_{n-1} + C_2.t_{n-2} = 0$ $(n \geq 2)$ and $t_0 = D_0, t_1 = D_1$;
$C_0(\neq 0), C_1, C_2(\neq 0)$ and D_0, D_1 all are constants.

Characteristic Equation: Seeking a solution, $t_n = c.x^n$ $(c, x \neq 0)$, after substitution,
$C_0.c.x^n + C_1.c.x^{n-1} + C_2.c.x^{n-2} = 0 \quad \Rightarrow C_0.x^2 + C_1.x + C_2 = 0$

Equation Roots: **Complex Conjugate Pair as Roots**, $R_1 = x + iy, R_2 = x - iy$
OR, $R_1 = r.(\cos\theta + i\sin\theta), R_2 = r.(\cos\theta - i\sin\theta)$
where, $r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}(\frac{y}{x})$ $\qquad (i = \sqrt{-1})$

Exact Solution: $t_n = A_1.R_1^n + A_2.R_2^n = A_1.(x + iy)^n + A_2.(x - iy)^n$
$= (\sqrt{x^2 + y^2})^n[A_1.(\cos(n\theta) + i\sin(n\theta)) + A_2.(\cos(n\theta) - i\sin(n\theta))]$
$= (\sqrt{x^2 + y^2})^n[B_1.\cos(n\theta) + B_2.\sin(n\theta)]$, where
$B_1 = (A_2 + A_2), B_2 = i(A_1 - A_2)$ (Here, A_1, A_2, B_1, B_2 are arbitrary constants)

Constant Determination: $t_0 = D_0 = B_1$ and $B_2 = \frac{D_1 - D_0.x}{y}$
because, $t_1 = D_1 = (\sqrt{x^2 + y^2}).(B_1.\cos\theta + B_2\sin\theta) = (B_1.x + B_2.y)$.

Unique Solution: $t_n = (\sqrt{x^2 + y^2})^n\left[D_0.\cos(n\theta) + \left(\frac{D_1 - D_0.x}{y}\right).\sin(n\theta)\right]$

Solving Second-Order Recurrence Relations

Example (Finding Value of $n \times n$ Determinant)

For $b \in \mathbb{R}^+$, $D_n = \begin{vmatrix} b & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ b & b & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & b & b & b & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & b & b & \cdots & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & \cdots & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \cdots & b & b & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b & b & b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & b & b \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b & b \end{vmatrix}$, for $n \geq 1$.

$D_1 = |b| = b$, $D_2 = \begin{vmatrix} b & b \\ b & b \end{vmatrix} = 0$, $D_3 = \begin{vmatrix} b & b & 0 \\ b & b & b \\ 0 & b & b \end{vmatrix} = -b^3$ and

Recurrence Relation: $D_n = b.D_{n-1} - b.b.D_{n-2}$ $(n \geq 3)$

Complex Conjugate Pair Roots: $R_1 = b\left[\frac{1}{2} + i.\frac{\sqrt{3}}{2}\right], R_2 = b\left[\frac{1}{2} - i.\frac{\sqrt{3}}{2}\right]$

Solution: $D_n = b^n.\left[A_1.(\frac{1}{2} + i.\frac{\sqrt{3}}{2})^n + A_2.(\frac{1}{2} - i.\frac{\sqrt{3}}{2})^n\right] = b^n\left[B_1 \cos(\frac{n\pi}{3}) + B_2 \sin(\frac{n\pi}{3})\right]$

Constants: $b = D_1 = b.\left[B_1.(\frac{1}{2}) + B_2.(\frac{\sqrt{3}}{2})\right]$; $\quad 0 = D_2 = b^2.\left[B_1.(-\frac{1}{2}) + B_2.(\frac{\sqrt{3}}{2})\right]$

Therefore, $\Rightarrow B_1 = 1, B_2 = \frac{1}{\sqrt{3}}$, implying $D_n = b^n\left[\cos(\frac{n\pi}{3}) + (\frac{1}{\sqrt{3}})\sin(\frac{n\pi}{3})\right]$, $n \geq 1$

Solving Second-Order Recurrence Relations

Second-Order Linear Homogeneous Recurrence with Constant Coefficients

General Form: $C_0.t_n + C_1.t_{n-1} + C_2.t_{n-2} = 0$ $(n \geq 2)$ and $t_0 = D_0, t_1 = D_1$;
$C_0(\neq 0), C_1(\neq 0), C_2(\neq 0)$ and D_0, D_1 all are constants.

Characteristic Equation: Seeking a solution, $t_n = c.x^n$ $(c, x \neq 0)$, after substitution,
$C_0.c.x^n + C_1.c.x^{n-1} + C_2.c.x^{n-2} = 0 \quad \Rightarrow C_0.x^2 + C_1.x + C_2 = 0$

Equation Roots: **2 Equal Roots**, $R = R_1 = R_2 = -\frac{C_1}{2C_0}$ (here, $C_1^2 = 4C_0C_2$)

Exact Solution: Forming two linearly independent solutions using,
$t_n = A_1.R^n = A_1.(-\frac{C_1}{2C_0})^n$ and $t_n = A_2.g(n).R^n = A_2.g(n).(-\frac{C_1}{2C_0})^n$
$\Rightarrow C_0.g(n).(-\frac{C_1}{2C_0})^n + C_1.g(n-1).(-\frac{C_1}{2C_0})^{n-1} + C_2.g(n-2).(-\frac{C_1}{2C_0})^{n-2} = 0$
$\Rightarrow g(n) - 2.g(n-1) + g(n-2) = 0$ (as, $C_1^2 = 4C_0C_2$ and $C_0, C_1, C_2 \neq 0$)
is satisfied by, $g(n) = an + b$ (constants $a(\neq 0), b$, with simplest $g(n) = n$)

$$\therefore \; t_n = (A_1 + A_2.n).(-\frac{C_1}{2C_0})^n$$

Constant Determination: $t_0 = D_0 = A_1$ and
$t_1 = D_1 = (A_1 + A_2).(-\frac{C_1}{2C_0}) \Rightarrow A_2 = -\frac{2C_0D_1 + C_1D_0}{C_1}$

Unique Solution: $t_n = \left[D_0 - (\frac{2C_0D_1 + C_1D_0}{C_1}).n\right].(-\frac{C_1}{2C_0})^n$

Generic Solution: $t_n = (A_1 + A_2.n + A_2.n^2 + \cdots + A_{k-1}.n^{k-1}).R^n$, for all k equal roots

Solving Second-Order Recurrence Relations

Example (Finding Value of $n \times n$ Determinant)

$$D_n = \begin{vmatrix} 2 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & \cdots & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 2 \end{vmatrix}, \text{ for } n \geq 1.$$

$D_1 = |2| = 2$, $D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$, $D_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4$ and

Recurrence Relation: $D_n = 2D_{n-1} - D_{n-2}$ $(n \geq 3)$

Equal Real Roots: $R = 1$

Solution: $D_n = (A_1 + A_2.n).1^n = (A_1 + A_2.n)$

Constants: $2 = D_1 = A_1 + A_2$; $3 = D_2 = A_1 + 2A_2$ $\Rightarrow A_1 = A_2 = 1$

Therefore, $D_n = 1 + n$, $n \geq 1$

General Form: $C_0.t_n + C_1.t_{n-1} + C_2.t_{n-2} + \cdots + C_k.t_{n-k} = f(n) = 0$, for $n \geq k$
where the order $k \in \mathbb{Z}^+$, $C_0(\neq 0), C_1, C_2, \ldots, C_k(\neq 0)$ are real constants,
and t_n $(n \geq 0)$ be a discrete function. ($f(n) \neq 0$ for non-homogeneous)

Boundary Condition: $t_j = D_j$, for each $0 \leq j \leq k-1$ and every D_j is a constant

Characteristic Equation: Seeking a solution as, $t_n = c.x^n$ $(c, x \neq 0)$
After substitution, $C_0.c.x^n + C_1.c.x^{n-1} + \cdots + C_k.c.x^{n-k} = 0$
Since $c, x \neq 0$, so $C_0.x^k + C_1.x^{k-1} + \cdots + C_{k-1}.x + C_k = 0$

Characteristic Roots: k roots as, $R_1, R_2, \ldots, R_k$, such that
$C_0.R_i^k + C_1.R_i^{k-1} + \cdots + C_{k-1}.R_i + C_k = 0$, where $1 \leq i \leq k$

Classification of Roots: $(u + 2v + w = k$ and $1 \leq \alpha_i, \beta_i, \beta'_i, \gamma_i \leq k)$

1. *Real Distinct Roots*: u such roots, $R_{\alpha_1}, R_{\alpha_2}, \ldots, R_{\alpha_u}$
2. *Complex Conjugate Pair Roots*: v such root pairs,
   $\langle R_{\beta_1}, R_{\beta'_1}\rangle, \langle R_{\beta_2}, R_{\beta'_2}\rangle, \ldots, \langle R_{\beta_v}, R_{\beta'_v}\rangle$ having the form,
   $\langle R_{\beta_l}, R_{\beta'_l}\rangle = x_l \pm iy_l = r_l(\cos\theta_l \pm i\sin\theta_l)$, where $r_l = \sqrt{x_l^2 + y_l^2}, \theta_l = \tan^{-1}(\frac{y_l}{x_l})$
3. *Real Equal Roots*: w such roots, $R_\gamma = R_{\gamma_1} = R_{\gamma_2} = \cdots = R_{\gamma_w}$

Generic Solution:
$$t_n = \sum_{l=1}^{u} A_{\alpha_l}.R_{\alpha_l}^n + \sum_{l=1}^{v}\left(A_{\beta_l}.R_{\beta_l}^n + A_{\beta'_l}.R_{\beta'_l}^n\right) + R_\gamma^n.\sum_{l=1}^{w} A_{\gamma_l}.n^{l-1}$$
$$= \sum_{l=1}^{u} A_{\alpha_l}.R_{\alpha_l}^n + \sum_{l=1}^{v} r_l^n.\left(B_{\beta_l}.\cos n\theta_l + B_{\beta'_l}.\sin n\theta_l\right) + R_\gamma^n.\sum_{l=1}^{w} A_{\gamma_l}.n^{l-1}$$

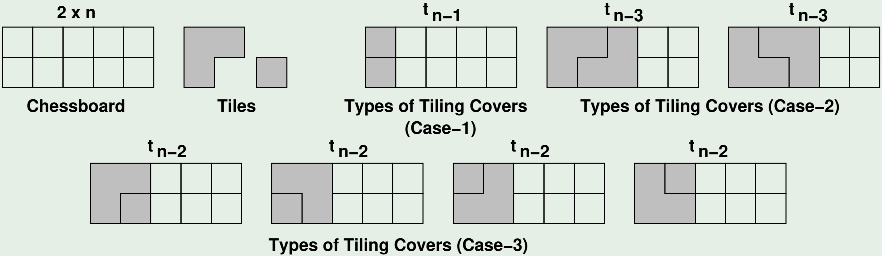
$(A_{\alpha_l}, A_{\beta_l}, A_{\beta'_l}, A_{\gamma_l}, B_{\beta_l}, B_{\beta'_l}$ are constants and $B_{\beta_l} = A_{\beta_l} + A_{\beta'_l}, B_{\beta_l} = i(A_{\beta_l} - A_{\beta'_l}), i = \sqrt{-1}$

Solving Third-Order Recurrence Relations

Example (Tiling Problem)

Let, $t_n =$ number of ways to tile $2 \times n$ ($n \in \mathbb{Z}^+$) chessboard

Tile Types: one L-shaped and one 1×1



Recurrence Relation: $t_n = t_{n-1} + 4t_{n-2} + 2t_{n-3}$ $(n \geq 4)$ and $t_1 = 1, t_2 = 5, t_3 = 11$

Characteristics Roots: $R_1 = -1, R_2 = 1 + \sqrt{3}, R_3 = 1 - \sqrt{3}$

Solution: $t_n = 1.(-1)^n + (\frac{1}{\sqrt{3}}).(1 + \sqrt{3})^n + (-\frac{1}{\sqrt{3}}).(1 - \sqrt{3})^n$

$$= (-1)^n + (\frac{1}{\sqrt{3}}).\left[(1 + \sqrt{3})^n - (1 - \sqrt{3})^n\right], \quad n \geq 1$$

Solving Non-Homogeneous Recurrence Relations

First-Order Linear Non-Homogeneous Recurrence with Constant Coefficients

General Form: $t_n + C.t_{n-1} = K.B^n$ $(n \geq 1)$ and $t_0 = D$
(Here, $B(\neq 0), C(\neq 0), D, K$ are all arbitrary constants)

Homogeneous Solution Part: $t_n^{(h)} = A.(-C)^n$ (A is an arbitrary constant)

Particular Solution Part: $t_n^{(p)} = \begin{cases} A_1.B^n, & \text{if } B^n \neq (-C)^n \\ A_2.n.B^n, & \text{if } B^n = (-C)^n \end{cases}$ (A_1, A_2 are constants)

Exact Solution: $t_n = t_n^{(h)} + t_n^{(p)} = \begin{cases} A.(-C)^n + A_1.B^n, & \text{if } B^n \neq (-C)^n \\ (A + A_2.n).B^n, & \text{if } B^n = (-C)^n \end{cases}$

Constant Determination: $A_1.B^n + C.A_1.B^{n-1} = K.B^n \Rightarrow A_1 = \frac{K.B}{B+C}$

$A_2.n.B^n + C.A_2.(n-1).B^{n-1} = K.B^n \Rightarrow A_2 = K$

Finally, $t_0 = D = \begin{cases} A + A_1 & \Rightarrow A = \frac{DB+DC-KB}{B+C} \\ A & \Rightarrow A = D \end{cases}$

Unique Solution: $t_n = \begin{cases} \left(\frac{DB+DC-KB}{B+C}\right).(-C)^n + \left(\frac{KB}{B+C}\right)B^n \\ (D + K.n).B^n = (D + K.n).(-C)^n \end{cases}$, $n \geq 1$

Solving Non-Homogeneous Recurrence Relations

Example (Towers of Hanoi Problem)

Strategy for T_n: Moving n disks with 3 pegs requires – (i) twice the movement of $(n-1)$ disks, and (ii) once the movement of the largest disk.

Recurrence Relation: $T_n = 2T_{n-1} + 1$ $(n \geq 1)$ and $T_0 = 0$

Homogeneous Solution: $T_n^{(h)} = A.2^n$

Particular Solution: $T_n^{(p)} = A_1.1^n = A_1$, hence $A_1 = 2A_1 + 1 \Rightarrow A_1 = -1$

Final Solution: $T_n = A.2^n - 1$, with $T_0 = 0 = A.2^0 - 1 \Rightarrow A = 1$,

implying $T_n = 2^n - 1$, $n \geq 0$.

Example (Comparisons to find Min-Max from 2^n Element Set)

Strategy for M_n: Divide 2^n-element set into two. Find Min-Max from both sets + two comparisons (Max-vs-Max and Min-vs-Min) from chosen Min-Max elements of each set.

Recurrence Relation: $M_n = 2M_{n-1} + 2$ $(n \geq 2)$ and $M_1 = 1$

Homogeneous Solution: $M_n^{(h)} = A.2^n$

Particular Solution: $M_n^{(p)} = A_1.1^n = A_1$, hence $A_1 = 2A_1 + 2 \Rightarrow A_1 = -2$

Final Solution: $M_n = A.2^n - 2$, with $M_1 = 1 = A.2^1 - 2 \Rightarrow A = \frac{3}{2}$,

implying $M_n = (\frac{3}{2}).2^n - 2$, $n \geq 1$.

Solving Non-Homogeneous Recurrence Relations

Example (Strings with Digits containing Even Number of 1s)

$S_n =$ number of n-length strings constructed using $\Sigma = \{0, 1, 2, \ldots, 9\}$ having even 1s.
Two ways to contribute to S_n:

- n^{th} symbol is not 1: S_{n-1} ways for each 9 such cases.
- n^{th} symbol is 1: Odd number of 1s in $(n-1)$-length part $= (10^{n-1} - S_{n-1})$

Recurrence Relation: $S_n = 9S_{n-1} + (10^{n-1} - S_{n-1}) = 8S_{n-1} + 10^{n-1}$ $(n \geq 2)$ and
$S_1 = 9$ (all digits except 1)

Homogeneous Solution: $S_n^{(h)} = A.8^n$

Particular Solution: $S_n^{(p)} = A_1.10^{n-1}$, hence $10A_1 = 8A_1 + 10 \Rightarrow A_1 = 5$

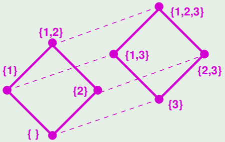
Final Solution: $S_n = A.8^n + 5.10^{n-1}$, with $S_1 = 9 = 8A + 5 \Rightarrow A = \frac{1}{2}$,
implying $S_n = (\frac{1}{2}).8^n + 5.10^{n-1}$, $n \geq 1$.

Example (Edges in Hasse Diagram)

$\mathcal{P}(\mathcal{S}) =$ Power Set of n-element set S forming Poset $(\mathcal{P}(\mathcal{S}), \subseteq)$.
$E_n =$ number of edges in Hasse Diagram in poset $(\mathcal{P}(\mathcal{S}), \subseteq)$

Recurrence Relation: $E_{n+1} = 2E_n + 2^n$ $(n \geq 1)$ and $E_1 = 1$

Solution: $E_n = E_n^{(h)} + E_n^{(p)} = A.2^n + A_1.n.2^n$ with $A = 0, A_1 = \frac{1}{2}$
implies $E_n = n.2^{n-1}$, $n \geq 1$

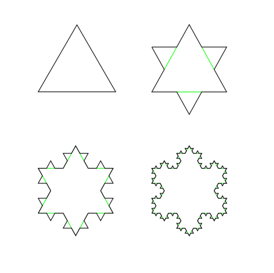


Solving Non-Homogeneous Recurrence Relations

Example (Area under a Snowflake – Concept of Fractals)

$a_n =$ area of 3-sided regular polygon after n transforms

(Koch's Snowflake, 1904)



Formulating the Recurrence Relation:

$a_0 = \frac{\sqrt{3}}{4}$ (3-sided),

$a_1 = \frac{\sqrt{3}}{4} + 3.(\frac{\sqrt{3}}{4}).[\frac{1}{3}]^2 = \frac{\sqrt{3}}{3}$ $(4 \times 3 = 12\text{-sided})$,

$a_2 = \frac{\sqrt{3}}{3} + 4^1.3.(\frac{\sqrt{3}}{4}).[\frac{1}{3^2}]^2 = \frac{10\sqrt{3}}{27}$ $(4^2 \times 3 = 48\text{-sided})$

$a_3 = \frac{10\sqrt{3}}{27} + 4^2.3.(\frac{\sqrt{3}}{4}).[\frac{1}{3^3}]^2$ $(4^3 \times 3 = 192\text{-sided})$

Recurrence Relation:

$a_{n+1} = a_n + 4^n.3.(\frac{\sqrt{3}}{4}).[\frac{1}{3^{n+1}}]^2 = a_n + (\frac{1}{4\sqrt{3}}).(\frac{4}{9})^n \quad (n \geq 0)$

Solution: $a_n = a_n^{(h)} + a_n^{(p)} = A.1^n + B.(\frac{4}{9})^n = A + B.(\frac{4}{9})^n$

So, $B = (-\frac{9}{5})(\frac{1}{4\sqrt{3}})$ and $a_n = A + (-\frac{9}{5})(\frac{1}{4\sqrt{3}})(\frac{4}{9})^n = A - (\frac{1}{5\sqrt{3}})(\frac{4}{9})^{n-1}$

Now, $a_0 = \frac{\sqrt{3}}{4} = A - (\frac{1}{5\sqrt{3}}).(\frac{4}{9})^{-1} \Rightarrow A = \frac{6}{5\sqrt{3}}$

Finally, $a_n = \frac{6}{5\sqrt{3}} - (\frac{1}{5\sqrt{3}})(\frac{4}{9})^{n-1} = (\frac{1}{5\sqrt{3}})[6 - (\frac{4}{9})^{n-1}], \quad n \geq 0$

Generalized Recurrence Relations for Area under Regular Polygon Fractals

For 4-sided (unit-length) Regular Polygon: $a_{n+1} = a_n + 5^n.4.1.[\frac{1}{3^{n+1}}]^2 = a_n + (\frac{4}{9}).(\frac{5}{9})^n$

For k-sided (m-length) Regular Polygon: $a_{n+1} = a_n + (k+1)^n.k.\left[\frac{m^2.k}{4.\tan(\frac{180^o}{k})}\right].[\frac{1}{3^{n+1}}]^2$

Solving Non-Homogeneous Recurrence Relations

Second-Order Linear Non-Homogeneous Recurrence with Constant Coefficients

General Form: $t_n + C_1.t_{n-1} + C_2.t_{n-2} = K.B^n$ $(n \geq 1)$ and $t_0 = D_0, t_1 = D_1$
(Here, $B(\neq 0), C_1, C_2(\neq 0), D_0, D_1, K$ are all arbitrary constants)

Homogeneous Solution Part: (A_1, A_2 are constants)

$$t_n^{(h)} = \begin{cases} A_1.R_1^n + A_2.R_2^n, & \text{for distinct roots} \\ (A_1 + A_2.n).R^n, & \text{for equal roots} \end{cases}$$

Particular Solution Part: (A', A'', A''' are constants)

$$t_n^{(p)} = \begin{cases} A'.B^n, & \text{for distinct roots when } R_1 \neq B \neq R_2 \\ A''.n.B^n, & \text{for distinct roots when } R = R_1 \text{ or } R = R_2 \\ A'.B^n, & \text{for equal roots when } B \neq R \\ A'''.n^2.B^n, & \text{for equal roots when } B = R \end{cases}$$

Exact Solution: $t_n = t_n^{(h)} + t_n^{(p)} =$

$$\begin{cases} (A_1.R_1^n + A_2.R_2^n) + A'.B^n, & \text{for distinct roots when } R_1 \neq B \neq R_2 \\ (A_1.R_1^n + A_2.R_2^n) + A''.n.B^n, & \text{for distinct roots when } R = R_1 \text{ or } R = R_2 \\ (A_1 + A_2.n).R^n + A'.B^n, & \text{for equal roots when } B \neq R \\ (A_1 + A_2.n).R^n + A'''.n^2.B^n, & \text{for equal roots when } B = R \end{cases}$$

Constant Determination: *Left For You as an Exercise!*
Unique Solution: *Left For You as an Exercise!*

Homework: *What happens for Complex Conjugate Pair Roots ?*

Solving Non-Homogeneous Recurrence Relations

Example (Solve: $t_{n+2} - 4t_{n+1} + 3t_n = -200$ $(n \geq 0)$, $t_0 = 3000, t_1 = 3300$)

Characteristic Roots (Homogeneous Consideration): $R_1 = 3, R_2 = 1$

Homogeneous Solution: $t_n^{(h)} = A_1.3^n + A_2.1^n = A_1.3^n + A_2$

Particular Solution: $t_n^{(p)} = A.n.1^n = A.n$

Hence, $(n+2)A - 4(n+1)A + 3nA = -200 \quad \Rightarrow A = 100$

Final Solution: $t_n = A_1.3^n + A_2 + 100n = 100.3^n + 2900 + 100n$, $n \geq 0$

(as $t_0 = 3000 = A_1 + A_2$, $t_1 = 3300 = 3.A_1 + A_2 + 100$ produces $A_1 = 100, A_2 = 2900$)

Example (Total Additions to Compute Fibonacci Number)

$a_n =$ total number of additions to compute n^{th} Fibonacci number

Recurrence Relation: $a_n = a_{n-1} + a_{n-2} + 1$ $(n \geq 2)$ and $a_0 = a_1 = 0$ (initial cases)

Homogeneous Solution: $a_n^{(h)} = A_1.(\frac{1+\sqrt{5}}{2})^n + A_2.(\frac{1-\sqrt{5}}{2})^n$

Particular Solution: $a_n^{(p)} = A.1^n = A$, hence $A = A + A + 1 \Rightarrow A = -1$

Final Solution: $a_n = A_1.(\frac{1+\sqrt{5}}{2})^n + A_2.(\frac{1-\sqrt{5}}{2})^n - 1$, with $A_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}, A_2 = -\frac{1-\sqrt{5}}{2\sqrt{5}}$,

$\Rightarrow a_n = (\frac{1+\sqrt{5}}{2\sqrt{5}}).(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2\sqrt{5}}).(\frac{1-\sqrt{5}}{2})^n - 1 = \frac{1}{\sqrt{5}}.(\frac{1+\sqrt{5}}{2})^{n+1} - \frac{1}{\sqrt{5}}.(\frac{1-\sqrt{5}}{2})^{n+1} - 1$, $n \geq 0$

General Form: $C_0.t_n + C_1.t_{n-1} + C_2.t_{n-2} + \cdots + C_k.t_{n-k} = f(n) \neq 0$, for $n \geq k$
where the order $k \in \mathbb{Z}^+$, $C_0(\neq 0), C_1, C_2, \ldots, C_k(\neq 0)$ are real constants.

Boundary Condition: $t_j = D_j$, for each $0 \leq j \leq k-1$ and every D_j is a constant

Homogeneous Solution: $t_n^{(h)}$ (computed assuming $f(n) = 0$ as earlier)

Particular Solution: Three cases to consider while constructing $t_n^{(p)}$:

1. Format of $f(n)$ is a constant multiple of following table (middle column) and is NOT associated with form of $t_n^{(h)}$:

Types	Format of $f(n)$	Format for $t_n^{(p)}$
Type-1	$n^m.R^n$ $(m \in \mathbb{N}, R \in \mathbb{R})$	$R^n.(\sum_{i=0}^{m} A_i.n^i)$
Type-2	$R^n.\sin(n\theta)$ or $R^n.\cos(n\theta)$	$R^n.(A_1.\sin(n\theta) + A_2.\cos(n\theta))$

2. Format of $f(n)$ is the sum of constant multiples of above table (middle column) and is NOT associated with form of $t_n^{(h)}$:
   Take $t_n^{(p)}$ as the sum of above table entries (right columns)
3. A summand $f'(n)$ from $f(n)$ is an associated solution in $t_n^{(h)}$:
   - *Format of $f'(n)$ is of Type-1 from above table*:
     $t_n^{(p)} \leftarrow n^s.t_n^{(p)}$, i.e. multiply with smallest s so that no summand of $n^s.f'(n)$ is associated with $t_n^{(h)}$.
   - *Format of $f'(n)$ is of Type-2 from above table*: Left as Exercise!

Solving Non-Homogeneous Recurrence Relations

Example (Distinct Handshakes with n Persons)

$H_n =$ number of total distinct pairwise handshakes among n persons.

Recurrence Relation: $H_{n+1} = H_n + n$ $(n \geq 2)$ and $H_1 = 0$ (no handshakes with oneself)

Homogeneous Solution: $H_n^{(h)} = A.1^n = A$

Particular Solution: $H_n^{(p)} = n^1.(A_1.n + A_0)$ (with A (const.) in $H_n^{(h)}$, $H_n^{(p)} \leftarrow n^1.H_n^{(p)}$)

Hence, $(n+1)^2.A_1 + (n+1).A_0 = n^2.A_1 + n.A_0 + n \quad \Rightarrow A_1 = \frac{1}{2}, A_0 = -\frac{1}{2}$

Final Solution: $H_n = A + \frac{1}{2}.n^2 - \frac{1}{2}.n$, with $H_1 = 0 = A$,

implying, $H_n = \frac{1}{2}.n^2 - \frac{1}{2}.n = \frac{n(n-1)}{2} = \binom{n}{2}$, $\quad n \geq 1$.

Example (Regions formed by Non-parallel Non-colinear Straight Lines)

$L_n =$ number of regions formed by n non-parallel and non-colinear straight lines.

Recurrence Relation: $L_{n+1} = L_n + (n+1)$ $(n \geq 1)$ and $L_0 = 1$ (whole 2-D plane)

Homogeneous Solution: $L_n^{(h)} = A.1^n = A$

Particular Solution: $L_n^{(p)} = n^1.(A_1.n + A_0)$ (with A (const.) in $L_n^{(h)}$, $L_n^{(p)} \leftarrow n^1.L_n^{(p)}$)

Hence, $(n+1)^2.A_1 + (n+1).A_0 = n^2.A_1 + n.A_0 + (n+1) \quad \Rightarrow A_1 = \frac{1}{2} = A_0$

Final Solution: $L_n = A + \frac{1}{2}.n^2 + \frac{1}{2}.n$, with $L_1 = 1 = A$,

implying, $H_n = 1 + \frac{1}{2}.n^2 + \frac{1}{2}.n = \frac{n(n+1)}{2} + 1$, $\quad n \geq 0$.

Solving Non-Homogeneous Recurrence Relations

Example (Deriving Formula for $S_n = \sum\limits_{i=0}^{n} i^2$)

Recurrence Relation: $S_{n+1} = S_n + (n+1)^2$ $(n \geq 0)$ and $S_0 = 0$

Homogeneous Solution: $S_n^{(h)} = A.1^n = A$

Particular Solution: $S_n^{(p)} = n.(A_0 + A_1.n + A_2.n^2) = (A_0.n + A_1.n^2 + A_2.n^3)$

Hence, $(n+1).A_0 + (n+1)^2.A_1 + (n+1)^3.A_2 = (n.A_0 + n^2.A_1 + n^3.A_2) + (n^2 + 2n + 1)$

implies, $3A_2 + A_1 = A_1 + 1 \quad \Rightarrow A_2 = \frac{1}{3}$ (comparing coefficients of n^2)

$3A_2 + 2A_1 + A_0 = A_0 + 2 \quad \Rightarrow A_1 = \frac{1}{2}$ (comparing coefficients of n)

$A_2 + A_1 + A_0 = 1 \quad \Rightarrow A_0 = \frac{1}{6}$ (comparing constant coefficients)

Final Solution: $S_n = A + \frac{1}{6}.n + \frac{1}{3}.n^2 + \frac{1}{2}.n^3$, with $S_0 = 0 = A$,

implying, $H_n = \frac{1}{6}.n + \frac{1}{2}.n^2 + \frac{1}{3}.n^3 = \frac{n(n+1)(2n+1)}{6}$, $\quad n \geq 0$.

Example (Deriving Other Summation Formulas: **Try Yourself!**)

(1) $\sum\limits_{i=0}^{n} i = L_n = L_{n-1} + n$ (2) $\sum\limits_{i=0}^{n} i^3 = C_n = C_{n-1} + n^3$

(3) $\sum\limits_{i=0}^{n} i^4 = Q_n = Q_{n-1} + n^4$ (4) $\sum\limits_{i=0}^{n} i^k = G_n = G_{n-1} + n^k$ $\quad (k \in \mathbb{Z}^+)$

(Here, $n \geq 1$ and $L_0 = C_0 = Q_0 = G_0 = 0$)

Solving Recurrences using Generating Functions

Example (Select r Objects from n Distinct Objects with Repetition)

$a(n, r) =$ number of ways to select r objects (repetition allowed) from n distinct objects

1. *A particular object is never selected:* r objects chosen from $(n-1)$ objects
2. *A particular object is at least once selected:* $(r-1)$ objects chosen from n objects

Recurrence Relation: $a(n, r) = a(n-1, r) + a(n, r-1), \quad (n \geq r$ and $n, r \in \mathbb{N})$
and $a(n, 0) = 1$ for $n \geq 0, \quad a(0, r) = 0$ for $r > 0$

Generating Function: Let, $f_n(x) = \sum_{r=0}^{\infty} a(n, r) x^r$ generates sequence $a(n, 0),\ a(n, 1),\ \ldots$

Derivation: $a(n, r) = a(n-1, r) + a(n, r-1) \quad (n, r \geq 1)$

$$\Rightarrow \quad \sum_{r=1}^{\infty} a(n, r) x^r = \sum_{r=1}^{\infty} a(n-1, r) x^r + \sum_{r=1}^{\infty} a(n, r-1) x^r$$

$$\Rightarrow \quad f_n(x) - a(n, 0) = f_{n-1}(x) - a(n-1, 0) + x.\sum_{r=1}^{\infty} a(n, r-1) x^{r-1}$$

$$\Rightarrow \quad f_n(x) - 1 = f_{n-1}(x) - 1 + x.f_n(x) \quad \Rightarrow \quad f_n(x) = \frac{f_{n-1}(x)}{1-x} = \frac{f_0(x)}{(1-x)^n}$$

So, $a(n, r)$ is the coefficient of x^r in $f_n(x) = \frac{f_0(x)}{(1-x)^n} = \frac{1}{(1-x)^n} = (1-x)^{-n}$

$$\Rightarrow \quad a(n, r) = (-1)^r.\binom{-n}{r} = \binom{n+r-1}{r}$$

Solving Recurrences using Generating Functions

Example (Select r Objects from n Distinct Objects w/o Repetition)

$a(n, r)$ = number of ways to select r objects (w/o repetition) from n distinct objects

1. *A particular object is never selected:* r objects chosen from $(n-1)$ objects
2. *A particular object is once selected:* $(r-1)$ objects chosen from $(n-1)$ objects

Recurrence Relation: $a(n, r) = a(n-1, r) + a(n-1, r-1), \quad (n \geq r \text{ and } n, r \in \mathbb{N})$
and $a(n, 0) = 1$ for $n \geq 0$, $\quad a(0, r) = 0$ for $r > 0$

Generating Function: Let, $f_n(x) = \sum_{r=0}^{\infty} a(n, r)x^r$ generates sequence $a(n, 0),\ a(n, 1),\ \ldots$

Derivation: $a(n, r) = a(n-1, r) + a(n-1, r-1) \quad (n, r \geq 1)$

$$\Rightarrow \quad \sum_{r=1}^{\infty} a(n, r)x^r = \sum_{r=1}^{\infty} a(n-1, r)x^r + \sum_{r=1}^{\infty} a(n-1, r-1)x^r$$

$$\Rightarrow \quad f_n(x) - a(n, 0) = f_{n-1}(x) - a(n-1, 0) + x.\sum_{r=1}^{\infty} a(n-1, r-1)x^{r-1}$$

$$\Rightarrow \quad f_n(x) - 1 = f_{n-1}(x) - 1 + x.f_{n-1}(x)$$

$$\Rightarrow \quad f_n(x) = (1+x).f_{n-1}(x) = (1+x)^n.f_0(x)$$

So, $a(n, r)$ is the coefficient of x^r in $f_n(x) = (1+x)^n.f_0(x) = (1+x)^n$

$$\Rightarrow \quad a(n, r) = \binom{n}{r}$$

Solving Recurrences using Generating Functions

Example (Solving a System of Recurrence Relations)

Upon interaction with a nucleus of fissionable material, the following activities happen:

1. A high-energy neutron releases *two* high-energy and *one* low-energy neutrons.
2. A low-energy neutron releases *one* high-energy and *one* low-energy neutron.

After $n \geq 0$ interactions, let $a_n =$ number of high-energy neutrons, and $b_n =$ number of low-energy neutrons. Assume, at beginning, $a_0 = 1, b_0 = 0$.

Recurrence Relation: $a_{n+1} = 2a_n + b_n, \quad b_{n+1} = a_n + b_n \quad (n \geq 0)$

Generating Function: $f(x) = \sum_{n=0}^{\infty} a_n.x^n$, $g(x) = \sum_{n=0}^{\infty} b_n.x^n$ generates $\{a_n\}$, $\{b_n\}$ $\quad (n \geq 0)$

Derivation:

$$\sum_{n=0}^{\infty} a_{n+1}.x^{n+1} = 2x \sum_{n=0}^{\infty} a_n.x^n + x \sum_{n=0}^{\infty} b_n.x^n \quad \Rightarrow \quad f(x) - a_0 = 2xf(x) + xg(x)$$

$$\sum_{n=0}^{\infty} b_{n+1}.x^{n+1} = x \sum_{n=0}^{\infty} a_n.x^n + x \sum_{n=0}^{\infty} b_n.x^n \quad \Rightarrow \quad g(x) - b_0 = xf(x) + xg(x)$$

Solving these system of recurrence equations and using generating functions,

$$f(x) = \frac{1-x}{x^2-3x+1} = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1}{\frac{3+\sqrt{5}}{2}-x}\right) + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1}{\frac{3-\sqrt{5}}{2}-x}\right) \quad \text{and}$$

$$g(x) = \frac{x}{x^2-3x+1} = \left(\frac{-5-3\sqrt{5}}{10}\right)\left(\frac{1}{\frac{3+\sqrt{5}}{2}-x}\right) + \left(\frac{-5+3\sqrt{5}}{10}\right)\left(\frac{1}{\frac{3-\sqrt{5}}{2}-x}\right)$$

$$a_n = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{3-\sqrt{5}}{2}\right)^{n+1} + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{3+\sqrt{5}}{2}\right)^{n+1} \quad \text{and}$$

$$b_n = \left(\frac{-5-3\sqrt{5}}{10}\right)\left(\frac{3-\sqrt{5}}{2}\right)^{n+1} + \left(\frac{-5+3\sqrt{5}}{10}\right)\left(\frac{3+\sqrt{5}}{2}\right)^{n+1}, \quad n \geq 0$$

Solving Special Recurrence Relations

Example (Solving Non-linear Recurrences using Generating Functions)

Some Recurrent Problems leading to non-linear recurrences:

- Number of ways to parenthesize an n length expressions
- Number of different ordered unlabelled rooted n-node binary trees
- Number of non-overlapping handshakes among n persons seated in round table
- Number of non-intersecting chords of circle with n points located in perimeter
- Number of paths in $a \times b$ grid from bottom-left $\rightarrow$ top-right corner not crossing diagonal
- Number of Triangulations of an n-sided regular polygon
- Number of *Stacky Sequences* [For $n \in \mathbb{Z}^+$, Push $1, 2, \ldots, n$ in order into stack, but Pop (from top) + Print anytime in between from unempty stack. All stack-realizable permutations of $1, 2, 3, \ldots, n$ are 'stacky sequences'.]

Catalan Numbers solving Non-linear Recurrences

Number of ways to parenthesize $(n+1)$-length string or construct $(n+1)$-node binary trees,

$$a_{n+1} = a_0 a_n + a_1 a_{n-1} + \cdots + a_{n-1} a_1 + a_n a_0 = \sum_{i=0}^{n} a_i a_{n-i}, \ (n \geq 0) \text{ and } a_0 = 1$$

Applying generating function, $f(x) = \sum_{n=0}^{\infty} a_n.x^n$ (to generate sequence $\{a_n\}$), we get –

$$\sum_{n=0}^{\infty} a_{n+1}.x^{n+1} = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i a_{n-i}\right).x^{n+1} \quad \Rightarrow [f(x) - a_0] = x[f(x)]^2 \quad \Rightarrow f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Now, $\sqrt{1-4x} = (1-4x)^{\frac{1}{2}} = \binom{\frac{1}{2}}{0} + \binom{\frac{1}{2}}{1}(-4x) + \binom{\frac{1}{2}}{2}(-4x)^2 + \cdots$, so the coefficient of x^{n+1} is:

$$\binom{\frac{1}{2}}{n+1}(-4)^{n+1} = \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-(n+1)+1)}{(n+1)!}(-4)^{n+1} = \left[\frac{-1}{2(n+1)-1}\right].\binom{2(n+1)}{n+1}$$

As $f(x) = \frac{1-\sqrt{1-4x}}{2x}$ (taking $-$ve sign to get $a_n \geq 0$), so $a_n = \frac{1}{2}\left[\frac{-1}{2(n+1)-1}\right].\binom{2(n+1)}{n+1} = \frac{1}{(n+1)}\binom{2n}{n}$

Thank You!