Assignment - 2

Linear Algebra for AI & ML

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A.1) The random surfer model is represented by the matrix M as follows:
A user will typically follow connections from a page: for example, from page i a user will follow any outbound links and go on to one of i's neighbours. A lesser, but significant portion of the time, the user will "teleport" to another page on the web after abandoning the present one. The likelihood that the user will leave the current page and "teleport" to another is represented by the damping factor p. Each page has a $\frac{1}{n}$ probability of being picked since he or she can teleport to any website. This supports the matrix B's structure.

The page rank matrix M is given by

$$M = (1-p)A + pB$$

where $p \in (0,1)$

A = transition matrix

$$B = \frac{1}{n}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

By definition of column stochastic matrix, we know that matrix A representing transitions, is a column stochastic matrix.

$$A_{ij} \geq 0$$

Now for any column A_i,

$$\sum_{i=1}^{n} (A_i)_j = 1 \quad \text{——} \quad ①$$

For any column j of M,

$$M_j = (1-p)\, A_j + p B_j$$

$$= (1-p)\, A_j + p \frac{1}{n} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Adding up all we get,

$$(M_i)_j = (1-p)(A_i)_j + \frac{p}{n} \quad \text{——} \quad ②$$

$$\sum_{i=1}^{n} (M_i)_j = \sum_{i=1}^{n} \left((1-p)(A_i)_j + \frac{p}{n} \right)$$

$$= (1-p) \times 1 + \sum_{i=1}^{n} \frac{p}{n} \qquad \text{(from ①)}$$

$$= 1 - \cancel{p} + \cancel{p}$$

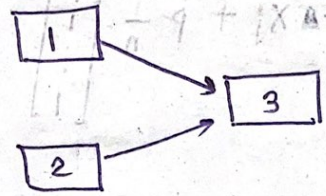
$$= 1$$

Since it is clear from eq^n ② that if $p < 1$, then $(M_i)_j > 0$ (as generally $p \neq 0$).

Therefore, M is a column stochastic matrix.

A.2) Problems encountered in Page Rank

① Nodes with no outgoing edges:



for the above graph,

$$M = (1-p)A + pB$$

$$= (1-p)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \frac{p}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{p}{3} & \frac{p}{3} & \frac{p}{3} \\ \frac{p}{3} & \frac{p}{3} & \frac{p}{3} \\ 1-\frac{2p}{3} & 1-\frac{2p}{3} & \frac{p}{3} \end{bmatrix}$$

Now, $v_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$v_1 = Mv_0 = \begin{bmatrix} \frac{p}{3} & \frac{p}{3} & \frac{p}{3} \\ \frac{p}{3} & \frac{p}{3} & \frac{p}{3} \\ 1-\frac{2p}{3} & 1-\frac{2p}{3} & \frac{p}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

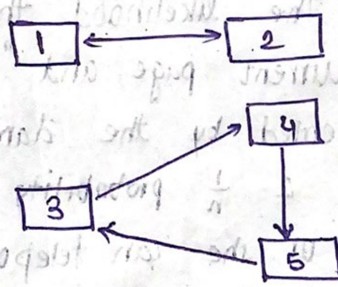
$$= \begin{bmatrix} p \\ p \\ 2-p \end{bmatrix}, \text{ where } 0<p<1$$

$$v_2 = Mv_1 = \begin{bmatrix} \frac{p}{3} & \frac{p}{3} & \frac{p}{3} \\ \frac{p}{3} & \frac{p}{3} & \frac{p}{3} \\ 1-\frac{2p}{3} & \frac{p}{3} & \frac{p}{3} \end{bmatrix} \begin{bmatrix} p \\ p \\ 2p \end{bmatrix}$$

$$= \begin{bmatrix} \frac{p(p+2)}{3} \\ \frac{p(p+2)}{3} \\ \frac{p(8-5p)}{3} \end{bmatrix}$$

" We can uniquely assign rank till it converges

② Disconnected components



for the above graph,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M = (1-p)\,A + pB$$

$$= (1-p) \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} + \frac{p}{5} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{p}{5} & 1-\frac{4p}{5} & \frac{p}{5} & \frac{p}{5} & \frac{p}{5} \\ 1-\frac{4p}{5} & \frac{p}{5} & \frac{p}{5} & \frac{p}{5} & \frac{p}{5} \\ \frac{p}{5} & \frac{p}{5} & \frac{p}{5} & \frac{p}{5} & 1-\frac{4p}{5} \\ \frac{p}{5} & \frac{p}{5} & 1-\frac{4p}{5} & \frac{p}{5} & \frac{p}{5} \\ \frac{p}{5} & \frac{p}{5} & \frac{p}{5} & 1-\frac{4p}{5} & \frac{p}{5} \end{bmatrix}$$

Without loss of generality, we take $p = 0.15$

$$M = \begin{bmatrix} 0.03 & 0.88 & 0.03 & 0.03 & 0.03 \\ 0.88 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 & 0.03 & 0.88 \\ 0.03 & 0.03 & 0.88 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 & 0.88 & 0.03 \end{bmatrix}$$

Now $v_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$v_1 = Mv_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

This will go on till some k

$$v_k = M^k v_{k-1} \quad \text{(By power method)}$$

The above converges ~~for som~~ to $[1 \; 1 \; .. \; 1]^T$

The above linear transformation on matrix M. Intutively, the matrix M connects the network and eliminata the dangling nodes. A node with no incoming edge has no reason to relocate to another node.

By "Power Method" convergence theorem, since M is a positive, column stochastic $n \times n$ matrix. The probabilistic eigenvector corresponding to eigenvalue 1 be v^*. Let z be a column vector with all entries equal to $\frac{1}{n}$. Then the sequence $z, Mz, \ldots \ldots, M^k z$ converges to vector z^*. So, the above discussed problems are overcame via use of M.