Transforming to/from an ~~arbir~~ arbitrary "observer" frame (XYZ) and the frame where the ray lies on the $z'=0$ plane.

A worked out example:

In the observer's (XYZ) frame assume a ray emanates from

$$\bar{r}_0 = (X_0 = 5,\ Y_0 = 3,\ Z_0 = 6).$$

Furthermore, assume that the ray emanated from $\bar{r}_0$ makes an angle of $\alpha_0 = 60^\circ$ with the +x-axis, $\beta_0 = 60^\circ$ with the +y-axis.

Then the angle γ_0 that the ray makes to the +z-axis can be calculated from the relationship

$$\cos^2\alpha_0 + \cos^2\beta_0 + \cos^2\gamma_0 = 1 \quad \text{-----} (1)$$

which in this case becomes

$$\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \cos^2\gamma_0 = 1$$

$$\Rightarrow \cos^2\gamma_0 = \tfrac{1}{2} \quad \Rightarrow \gamma_0 = 45^\circ \text{ or } 180^\circ - 45^\circ = 135^\circ$$

<u>NOTE</u> : Since there is an ambiguity in determining the third angle when only two are given, it may be best for the code to ast the user to input all three angles, and then the code should verify ~~at~~ than equation (1) is indeed valid before proceeding.

In this example let me <u>assume</u> that $\gamma_0 = 135^\circ$.

The $+X'$-axis will be in the direction of $\bar{r}_0$.

Therefore $\hat{i}'$, the unit vector in the $+X'$-direction is

$$\hat{i}' = \frac{\bar{r}_0}{|r_0|} = \frac{5\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{5^2+3^2+6^2}}$$

$$= \frac{1}{\sqrt{70}}\left(5\hat{i} + 3\hat{j} + 6\hat{k}\right)$$

$$= \frac{5}{\sqrt{70}}\hat{i} + \frac{3}{\sqrt{70}}\hat{j} + \frac{6}{\sqrt{70}}\hat{k}$$

The direction of the $+Z'$-axis will be

$$\bar{Z}' = \bar{r}_0 \times \left[\cos\alpha_0\,\hat{i} + \cos\beta_0\,\hat{j} + \cos\gamma_0\,\hat{k}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 3 & 6 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \end{vmatrix}$$

$$= -5.12132\,\hat{i} + 6.53553\,\hat{j} + \hat{k}$$

So the unit vector in the direction of $+Z'$-axis, i.e. $\hat{k}'$ is

$$\hat{k}' = \frac{\bar{Z}'}{|Z|} = \frac{1}{\sqrt{70}}\bar{Z}'$$

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[Since the vector $\cos\alpha_0 \hat{i} + \cos\beta_0 \hat{j} + \cos\gamma_0 \hat{k}$ is already normalized]

Another way to ~~come~~ calculate $\hat{k}$ would be

$$\hat{k}' = \frac{1}{|\bar{r}_0|} \bar{r}_0 \times \left(\cos\alpha_0 \hat{i} + \cos\beta_0 \hat{j} + \cos\gamma_0 \hat{k}\right)$$

$$= \frac{1}{|\bar{r}_0|} \bar{Z}'$$

$$= \frac{1}{\sqrt{70}} \bar{Z}'$$

$$= \frac{1}{\sqrt{70}} \left(-5.12132 \hat{i} + 6.53553 \hat{j} + \hat{k}\right)$$

Once $\hat{i}'$ and $\hat{k}'$ are known, then $\hat{j}'$ is simply

$$\hat{j}' = \hat{k}' \times \hat{i}'$$

$$= \frac{1}{|\bar{r}_0|} \Big[\bar{r}_0 \times \underbrace{\left(\cos\alpha_0 \hat{i} + \cos\beta_0 \hat{j} + \cos\gamma_0 \hat{k}\right)}_{\bar{p}}\Big] \times \frac{\bar{r}_0}{|\bar{r}_0|}$$

$$= \frac{1}{|\bar{r}_0|^2} \left(\bar{r}_0 \times \bar{p}\right) \times \bar{r}_0$$

$$\boxed{(\bar{a} \times \bar{b}) \times \bar{c} = (\bar{c} \cdot \bar{a})\bar{b} - (\bar{c} \cdot \bar{b})\bar{a}}$$

$$= \frac{1}{|\bar{r}_0|^2} \left[(\bar{r}_0 \cdot \bar{r}_0)\bar{p} - (\bar{r}_0 \cdot \bar{p})\bar{r}_0\right]$$

$$\Rightarrow \hat{j}' = \bar{p} - \frac{(\bar{r}_o \cdot \bar{p})\,\bar{r}_o}{|\bar{r}_o|^2}$$

Since $\bar{p} = (\cos\alpha_o, \cos\beta_o, \cos\gamma_o) = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}\right)$ and $\bar{r}_o = (r_{ox}, r_{oy}, r_{oz}) = (5, 3, 6)$

$$\hat{j}' = (\cos\alpha_o, \cos\beta_o, \cos\gamma_o) - \frac{(r_{ox}\cos\alpha_o + r_{oy}\cos\beta_o + r_{oz}\cos\gamma_o)}{|\bar{r}_o|^2}(r_{ox}, r_{oy}, r_{oz})$$

Say $$f = \frac{r_{ox}\cos\alpha_o + r_{oy}\cos\beta_o + r_{oz}\cos\gamma_o}{|\bar{r}_o|^2}$$

Then

$$\hat{j}' = (\cos\alpha_o - f r_{ox},\ \cos\beta_o - f r_{oy},\ \cos\gamma_o - f r_{oz})$$

Plugging the values, we get

$$\hat{j}' = (0.51733,\ 0.5104,\ -0.6863)$$

Thus we can compute the three unit vectors in the $(X'Y'Z')$ system.

In this $X'Y'Z'$ coordinate system the ray is launched from $(|\bar{r}_o|, 0, 0)$ coordinates, lies on the X'-Y' plane, and makes an angle δ_o counter-clockwise from $+X'$-axis such that $\hat{i}' \cdot \bar{p} = \cos\delta_o$

In this particular case, the $X'Y'Z'$ coordinates of the ~~[illegible]~~ launching point is $(\sqrt{70}, 0, 0)$, and

$$\cos \delta_0 = \hat{i}' \cdot \bar{p}$$

$$= \frac{\bar{r}_0}{r_0} \cdot \bar{p}$$

$$= \frac{1}{\sqrt{70}} \left(5 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} - 6 \cdot \frac{1}{\sqrt{2}} \right)$$

$$\Rightarrow \delta_0 = 91.662 \text{ degrees}$$

At this point we can use the 2D codes we've already developed to calculate the trajectory of the ray and find the X'- and Y'-coordinates of the ray.

Q. Knowing $(X', Y', 0)$ of some point on the ray, what is its corresponding (X, Y, Z) coordinate.

Ans. The position vector of this point is

$$\bar{P} = X' \hat{i}' + Y' \hat{j}' + 0 \hat{k}'$$

$$= X' \left(\frac{5}{\sqrt{70}} \hat{i} + \frac{3}{\sqrt{70}} \hat{j} + \frac{6}{\sqrt{70}} \hat{k} \right)$$

$$+ Y' \left(0.51733 \hat{i} + 0.5104 \hat{j} + (-0.6863) \hat{k} \right)$$

$$= \left\{ X' \cdot \frac{5}{\sqrt{70}} + Y'(0.51733) \right\} \hat{i} + \left\{ X' \cdot \frac{3}{\sqrt{70}} + Y'(0.5104) \right\} \hat{j} + \left\{ X' \cdot \frac{6}{\sqrt{70}} + Y'(-0.6863) \right\} \hat{k}$$

$\rightarrow$ These are the required (X, Y, Z) coordinates