1. (a) Find f so that $\vec{F}(x,y) = \nabla f(x,y) = (x^2, y^2)$ and use this knowledge to compute the amount of work done moving along the parabola $y = 2x^2$ from $(-1,2)$ to $(2,8)$.

   (b) Find g so that $\vec{G}(x,y) = \nabla g(x,y) = (\frac{y^2}{1+x^2}, 2y \arctan x)$ and use this knowledge to compute the amount of work done moving along the parabola $y = 2x^2$ from $(-1,2)$ to $(2,8)$.

2. Let $\mathcal{S}$ be the surface of the unit sphere in $\mathbb{R}^3$. $\mathcal{S}$ is parameterized by

$$\vec{p}(\theta,\phi) = \Big(f(\theta,\phi), g(\theta,\phi), h(\theta,\phi)\Big)$$

and is oriented outwards. Additionally you are told that for a vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$, the flux of $\vec{F}$ through $\mathcal{S}$ is given by

$$\iint \vec{F} \circ \vec{p}(\theta,\phi) \cdot \left(\frac{\partial \vec{p}}{\partial \theta}(\theta,\phi) \times \frac{\partial \vec{p}}{\partial \phi}(\theta,\phi) \right) \mathrm{d}\theta \mathrm{d}\phi.$$

   (a) For a fixed (θ_0, ϕ_0), does $\frac{\partial \vec{p}}{\partial \theta}(\theta_0,\phi_0) \times \frac{\partial \vec{p}}{\partial \phi}(\theta_0,\phi_0)$ point inwards or outwards? How about $\frac{\partial \vec{p}}{\partial \phi}(\theta_0,\phi_0) \times \frac{\partial \vec{p}}{\partial \theta}(\theta_0,\phi_0)$?

   (b) Using the component functions f, g, and h, come up with a new parameterization of $\mathcal{S}$ called $\vec{q}(\theta,\phi)$ such that

$$\frac{\partial \vec{q}}{\partial \phi}(\theta_0,\phi_0) \times \frac{\partial \vec{q}}{\partial \theta}(\theta_0,\phi_0)$$

   points outwards. (Be creative; draw pictures; think about the orientation of tangent vectors).

3. *A new way to do volume forms.* Let's consider polar coordinates again. We know the volume form for polar coordinates is $r\mathrm{d}r\mathrm{d}\theta$, which we computed using the geometry of circles. However, the parameterization

$$\vec{p}(r,\theta) = (r\cos\theta, r\sin\theta, 0),$$

parameterizes a surface (the xy-plane) in three dimensions in a very similar way to polar coordinates.

Let $R = \{(x,y,z) : x^2 + y^2 \leq 1 \text{ and } z = 0\}$.

   (a) Use the parameterization $\vec{p}$ to set up an integral to find the surface area of R. (I know there are easier ways in this case, but set up the integral like a surface integral).

   (b) Do you see volume form from polar coordinates in your answer to part (a)?

   (c) Consider the skewed coordinate system from homework 5 given by

$$x = a - b \qquad \text{and} \qquad y = 2a + b.$$

   Imagine the xy-plane parameterized by $\vec{s}(a,b) = (a-b, 2a+b, 0)$. What would it look like if you set up a surface integral to compute the area of regions of the xy-plane? How does $\left\| \frac{\partial \vec{s}}{\partial a} \times \frac{\partial \vec{s}}{\partial b} \right\|$ relate to the volume form for skewed coordinates?

   (d) Consider the stretched polar coordinates from a million assignments ago given by

$$x = \rho\cos\theta \qquad \text{and} \qquad y = 2\rho\sin\theta.$$

   Imagine again that you parameterize the xy-plane with stretched polar coordinates and decided to do surface integrals. Where and how does the volume form for stretched polar coordinates appear in your answer?

   (e) Consider weirdo polar coordinates $\mathcal{WP}$ give by by

$$x = \rho^2\cos\theta^2 \qquad \text{and} \qquad y = \rho^2\sin\theta^2.$$

   Compute the volume form for weirdo polar coordinates both from the definition of the volume form and by pretending its a surface integral. Which way is easier to compute? Which way makes the most sense in your head?

4. *Conservative can be complicated...* Let $\vec{f}(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$.

   (a) Plot the vector field $\vec{f}$. Is it conservative? Why or why not?

   (b) Let $F(x,y) = \arctan(y/x)$. Compute ∇F. What's going on here?

   (c) Let $R_{(x,y)}$ be the square oriented counter clockwise with side lengths 1 and lower-left corner at the point (x,y). Find the work done by $\vec{f}$ on a particle traversing the paths $R_{(1,1)}$, $R_{(-1,1)}$, and $R_{(-4,-3)}$. Will your answer always be the same? (Doing these integrals by hand isn't important. After breaking them up into appropriate pieces, you may use a compute to evaluate, if you like).

   (d) Graph the function

   $$Q(x,y) = \text{work done by } \vec{f} \text{ traversing } R_{(x,y)}.$$

   How many values does this function take? Is it defined everywhere?

   (e) A subset $X \subset \mathbb{R}^2$ is called *simply connected* if any two circles in X can be deformed into each other without having to leave X. If $\vec{g} = \nabla G$ for some $G : \mathbb{R}^2 \to \mathbb{R}$ and the domain of $\vec{g}$ is simply connected, then the work done by $\vec{g}$ traversing any closed loop is zero. Explain how this property relates to the current situation with $\vec{f}$.