1. Read about spherical coordinates in the Evans text (or another source of your choosing).
   (a) Write $(\rho, \theta, \phi) = (1, \pi/6, \pi/8)$ in rectangular coordinates.
   (b) Write down the volume form for spherical coordinates (you don't need to derive it, just write it down).
   (c) Let $H = \{(x, y, z) : x^2 + y^2 + z^2 \le 9 \text{ and } z \ge 0\}$. Describe H geometrically.
   (d) Let $f(x, y, z) = 9 - x^2 - y^2$. Find $\displaystyle\int_H f \, \mathrm{d}V$.

2. A *torus* is the surface of a doughnut. Let $\mathcal{T}$ be a torus with major radius $R = 3$ and minor radius $r = 1$. To visualize how R and r relate to the torus, imagine slicing it in half. Looking from the side, you see two circles of radius r and the centers of those two circles are separated by the diameter $2R$.
   (a) Find a parameterization $\vec{t} : [0, 2\pi) \times [0, 2\pi) \to \mathcal{T} \subset \mathbb{R}^3$ of the torus with the properties that: (i) the speed with respect to the first parameter is constant, (ii) the speed with respect to the second parameter is constant, and (iii) $\vec{t}(\theta, 0)$ traces out a circle of radius $R - r$. (Note that although the speed with respect to each parameter is constant as that parameter varies and the other parameter is held constant, it certainly doesn't need to be the same constant for different values of your constant parameter!)
   (b) Is your parameterization an isometric parameterization? Why or why not?
   (c) Let $X_1 = \{(\theta, \phi) : 0 \le \phi \le \pi/2 \text{ and } 0 \le \theta < 2\pi\}$ and $X_2 = \{(\theta, \phi) : \pi/2 \le \phi \le \pi \text{ and } 0 \le \theta < 2\pi\}$. You meet a strange baker who looks at your parameterization $\vec{t}$ and says, "I will bake you a doughnut, fine student. And, I will frost this doughnut! But, you have a choice. . . I with either cover the region $\vec{t}(X_1)$ or $\vec{t}(X_2)$ with a layer of frosting 0.1in thick." Being hungry, you crave the most possible frosting. Which choice should you take, or does it even matter?
   (d) Let $Y_1 = \{(\theta, \phi) : \phi = \pi/200, 2\pi/200, \ldots, 100\pi/200 \text{ and } \theta = \pi/200, 2\pi/200, \ldots, 400\pi/200\}$ and $Y_2 = \{(\theta, \phi) : \phi = 101\pi/200, 102\pi/200, \ldots, 200\pi/200 \text{ and } \theta = \pi/200, 2\pi/200, \ldots, 400\pi/200\}$. Again the baker announces, "Fine student, I will bake you a doughnut. And, I will sprinkle this doughnut! But, you get a choice. . . I will either place one sprinkle at each point in $\vec{t}(Y_1)$ or one sprinkle at each point in $\vec{t}(Y_2)$." If you want the most sprinkles, which option should you take, or does it even matter?
   (e) The baker finally goes to make your doughnut but is sloppy and spills icing over the entire top of the doughnut. However, the icing is slippery and deposits on the doughnut with a depth given by $0.1\hat{n} \cdot \hat{z}$ where $\hat{n}$ is the unit normal vector to a point on the doughnut (you may assume the doughnut is laying on a table and $\hat{z}$ is the unit vector perpendicular to the table). Does this situation make any physical sense? Set up an integral to find the total amount of icing on the doughnut. You may use a computer to evaluate this integral.

3. Surface area is subtle. So subtle that for a long, long time the prevailing mathematical definition of surface area was wrong! Here's what is was:

   > A *triangulation* of a surface $\mathcal{S}$ is a polyhedron P whose faces are all triangles such that the vertices of P lie on $\mathcal{S}$.
   >
   > Given a triangle in $\mathbb{R}^3$, its *diameter* is the smallest diameter of a disk that contains the triangle. Given a triangulation P, the diameter of P is the maximum diameter of every triangular face in P.
   >
   > The *bogus surface area* of $\mathcal{S}$ can then be defined as follows: Let P_n be a sequence of triangulations of $\mathcal{S}$ whose diameter tends to zero. The surface area of $\mathcal{S}$ is the limit of the surface area of P_n as $n \to \infty$ (if the limit exists).

   In 1890, H. A. Schwartz showed this definition didn't even work for something as simple as a cylinder!

Let $\mathcal{C}$ be the cylinder of radius 1 and height 1 and no top and no bottom (i.e., $\mathcal{C}$ is just the curvy part).

(a) Let $A_{m,n}$ be a triangulation of $\mathcal{C}$ defined as follows: Cut $\mathcal{C}$ into m cylinders of height $1/m$. Place n points $\{p_i\}$ around the top of each cylinder slice and n points $\{p'_i\}$ around the bottom each cylinder slice where the ith point is at the angle $\frac{2\pi i}{n}$. Form triangles by connecting the points p_i, p'_i, p_{i+1} and by connecting the points p'_i, p_{i+1}, p'_{i+1}.

Convince yourself that every triangle in $A_{m,n}$ has the same area. Find the surface area of the triangulation $A_{m,n}$ as a function of m and n. What happens as $m, n \to \infty$? Does this agree with what you know the surface area of a cylinder should be?

(b) Let $B_{m,n}$ be a triangulation of $\mathcal{C}$ defined as follows: Cut $\mathcal{C}$ into m cylinders of height $1/m$. For the jth slice, place n points $\{p_i\}$ around the top of the cylinder and n points $\{p'_i\}$ around the bottom of the cylinder where the point p_i is placed at an angle $\frac{2\pi i}{n} + \frac{j\pi}{n}$ and the point p'_i is placed at an angle $\frac{2\pi i}{n} + \frac{(j+1)\pi}{n}$. Form triangles by connecting the points p_i, p'_i, p_{i+1} and by connecting the points p'_i, p_{i+1}, p'_{i+1}.

Notice that $B_{m,n}$ is like $A_{m,n}$ but twisted a bit.

Convince yourself that every triangle in $B_{m,n}$ has the same area. Find the surface area of the triangulation $B_{m,n}$ as a function of m and n (I don't want any approximations here, I want the *exact* surface area). What happens as $m, n \to \infty$? Remember, this is a 2d limit! You may use the approximations

$$\sin \tfrac{1}{k} \approx \tfrac{1}{k} \qquad \cos \tfrac{1}{k} \approx 1 - \tfrac{1}{2k^2}$$

when k is large.

(c) Is all hope lost? Can you think of a modification to bogus surface area that would make it work? Propose your own definition of surface area.