1. Consider the second order linear homogeneous differential equation

$$y'' + y' - 6y = 0. (1)$$

(a) Write down the corresponding system of first order differential equations in matrix form. That is, you system should be written as

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = M \begin{bmatrix} a \\ b \end{bmatrix} \tag{2}$$

for some matrix M.

- (b) Find all eigenvalues of M.
- (c) An eigenvector for (1) is a solution ϕ such that $\phi' = k\phi$ for some ϕ . Use your knowledge of how to solve (1) to find two eigenvectors for (1).
- (d) Use your eigenvectors from part (c) to find two eigenvectors for M. That is, find vectors \vec{v} and \vec{w} so that $M\vec{v} = k_1\vec{v}$ and $M\vec{w} = k_2\vec{w}$ for scalars k_1 and k_2 .
- (e) Write down a flow for (2) that passes through the point $(a, b) = (a_0, b_0)$ at time t = 0.
- (f) Show that if $\vec{\varphi}_1 : \mathbb{R} \to \mathbb{R}^2$ and $\vec{\varphi}_2 : \mathbb{R} \to \mathbb{R}^2$ are flows for (2), then so is any linear combination of $\vec{\varphi}_1$ and $\vec{\varphi}_2$. (You may use facts about the linearity of matrix multiplication that you proved last homework).
- 2. Consider the second order linear homogeneous differential equation

$$y'' + \alpha y' + \beta y = 0 \tag{3}$$

where α and β are constants.

- (a) Write the corresponding system of first order linear differential equations in matrix form.
- (b) Classify the α, β that give rise to sources, sinks, spiral sources, spiral sinks, saddles, and swirls. Are there any $\alpha, \beta \in \mathbb{R}$ that don't fit into this classification?
- (c) Classify the α , β that give rise to forward-stable, backward-stable, bi-stable, and unstable solutions to (3).
- 3. The motion of a simple pendulum is governed by the equation

$$\theta'' + \frac{g}{\ell}\sin\theta = 0$$

where g is the acceleration due to gravity, ℓ is the length of the rigid pendulum arm, and θ is the angular displacement from vertical. Since we're doing math in this course, we may assume $q = \ell = 1$.

- (a) Write a system of first-order differential equations corresponding to the equation for a pendulum and draw a phase portrait for this system.
- (b) Analyze the critical points for the pendulum system (by linearizing the system near the critical points). Does their stability or instability make physical sense? Explain.
- (c) There are three fundamental behaviors for a simple pendulum: stationary, swinging back and forth, and spinning all the way around. Draw, and clearly label, a flow corresponding to each behavior on your phase portrait.
- (d) Given an initial condition $(t, \theta, \theta') = (0, \theta_0, 0)$, if θ_0 is small, a pendulum's motion is modeled well by simple harmonic oscillation (i.e., a $\theta = k \cos(\omega t)$ for some k, ω). We're going to see exactly how well.
 - Let $(a',b') = \vec{F}(a,b)$ be the system of ODEs corresponding to a simple pendulum, and let $(a',b') = \vec{L}(a,b)$ be the linearization of \vec{F} around the critical point (0,0). Let $\vec{\varphi}_{\theta_0}$ be the flow along \vec{F} with initial conditions $(t,\theta,\theta') = (0,\theta_0,0)$ and let $\vec{\gamma}_{\theta_0}$ be the flow long

 \vec{L} with initial conditions $(t, \theta, \theta') = (0, \theta_0, 0)$. Use numerical techniques to estimate the period of $\vec{\varphi}_{\theta_0}$ and $\vec{\gamma}_{\theta_0}$ for at least three choices of θ_0 . Give a range of θ_0 where the relative error between the period of $\vec{\varphi}_{\theta_0}$ and $\vec{\gamma}_{\theta_0}$ is less than 5%. (Recall, relative error between quantities a and b is |(a-b)/a|.)

- 4. Consider the vector field $\vec{F}(x,y) = (x+y^2, -y)$.
 - (a) For each of the following functions, show whether or not it is a flow for \vec{F} .

i.
$$\vec{\varphi}(t) = (-e^{-2t}, \sqrt{3}e^{-t})$$

ii.
$$\vec{\varphi}(t) = (-e^{-4t}, \sqrt{3}e^{-2t})$$

iii.
$$\vec{\varphi}(t) = (\frac{4}{3}e^t - \frac{1}{3}e^{-2t}, e^{-t})$$

iv. $\vec{\varphi}(t) = (e^t - e^{-2t}, e^{-t})$

iv.
$$\vec{\varphi}(t) = (e^t - e^{-2t}, e^{-t})$$

- (b) A constant flow is a flow $\vec{\varphi}$ such that $\vec{\varphi}(t_1) = \vec{\varphi}(t_2)$ for all $t_1, t_2 \in \mathbb{R}$. Find all constant flows for \vec{F} . (Hint, think about $\vec{\varphi}'$ in this situation.)
- (c) Classify all critical points of the system of differential equations given by (a'(t), b'(t)) = $\vec{F}(a(t),b(t)).$
- (d) Find all flows of \vec{F} that pass through the point (1,0). Remember, if $\vec{\varphi}$ is a flow with this property, it is not a requirement that $\vec{\varphi}(0) = (1,0)$, only that $\vec{\varphi}(t_0) = (1,0)$ for some t_0 .
- (e) We call the flows $\vec{\varphi}_1$ and $\vec{\varphi}_2$ time shifts of each other if $\vec{\varphi}_1(t) = \vec{\varphi}_2(t+t_0)$ for some t_0 . Show that the flows from part (d) are time shifts of each other.
- 5. I hope we haven't forgotten about vector calculus! Consider the vector field \vec{F} : $\mathbb{R}^2 \to \mathbb{R}^2$ and imagine it describes the velocity of some fluid spread out on a plane (maybe we'll call this whole setup a laminar flow).

Let $\vec{\varphi}_{\vec{x}}$ be the flow along \vec{F} with initial conditions $\vec{\varphi}_{\vec{x}}(0) = \vec{x}$. That is, $\vec{\varphi}_{\vec{x}}(t)$ describes the position of a particle that starts at \vec{x} at time zero and flows for t seconds.

Viewing $\vec{\varphi}_{\vec{x}}$ as a function of \vec{x} , we can view $\vec{\varphi}_{\vec{x}}(t)$ as inducing a different coordinate change for each t. Specifically, for a fixed t_0 , we could think of the coordinate change

$$\vec{x} \mapsto \vec{\varphi}_{\vec{x}}(t_0).$$

Let $c_{t_0}: \mathbb{R}^2 \to \mathbb{R}^2$ be this coordinate change.

Our goal will be to compute the volume form for this coordinate change and learn some fluid mechanics along the way!

- (a) Suppose the volume form for the change of coordinates induced by c_t was $V_t(x,y)=3^t$. If you released a drop of die into the laminar flow governed by \vec{F} and at time 0 it had area 4, what would the area be after 1 second?
- (b) We can compute the volume form by estimating the change in volume of a tiny square under the function c_{t_0} and taking the limit as the square gets infinitely tiny (just like we did before). Let $R_{\varepsilon}^{(a,b)}$ be the square of side lengths ε and lower left corner at (a,b). Compute

$$V_{t_0}(a,b) = \lim_{\varepsilon \to 0} \frac{\text{area of } c_{t_0}(R_{\varepsilon}^{(a,b)})}{\text{area of } R_{\varepsilon}^{(a,b)}}.$$

You may express your answer in terms of partial derivatives. You may also find it helpful to assume that if ε is tiny, then $c_{t_0}(R_{\varepsilon}^{(a,b)})$ is a parallelogram. Further, recall that determinants can be used to compute the area of a parallelogram and that the determinant is a continuous function, and so you may move limits in and out of a determinant.

(c) The change of coordinates function c_t has some fantastic properties. Explain why $c_{t_1} \circ$ $c_{t_2} = c_{t_1+t_2}$. Using this fact, explain why $V_{t_1+t_2}(x,y) = V_{t_1}(x,y)V_{t_2}(c_{t_1}(x,y))$.

(d) Write down the limit definition of the derivative of $\frac{\mathrm{d}V_t}{\mathrm{d}t}(x,y)$ at $t=t_0$ and at t=0. Then, use your knowledge from the previous part to write down an expression for

$$\left. \frac{\mathrm{d}V_t}{\mathrm{d}t}(x,y) \right|_{t=t_0}$$

in terms of $\frac{\mathrm{d}V_t}{\mathrm{d}t}\Big|_{t=0}$.

(e) Use \vec{F} to write down a linear approximation to c_t when t is very close to zero. Using this approximation, compute $\frac{\mathrm{d}V_t}{\mathrm{d}t}$ at t=0 and show that

$$\left. \frac{\mathrm{d}V_t}{\mathrm{d}t} \right|_{t=0} = \nabla \cdot \vec{F}.$$

- (f) A flow is called *incompressible* if $\nabla \cdot \vec{F} = 0$. Explain why. In particular, your explanation should in some way combine knowledge from parts (d) and (e) to draw strong conclusions about the volume form.
- (g) Let $F(x,y)=(x+y^2,-y)$ from the previous problem. If you released a drop of die into the laminar flow governed by \vec{F} and at time 0 it had area 4, what would the area be after 1 second?