Hunting Inspiration:[1]

Snow collects on the brim of your fur coat and musket as you stalk your prey through the white woods. A flash of orange, a rustling of branches, and then it's gone. You mutter a curse under your breath. Foxes are scarce this year, and you'll have to explain to the Dutch East India Trading Company why you've come up short. Worse yet, your rival, a trapper who only hunts rabbits, is having a terrific year. You shouldn't have teased him so much when rabbits were down and foxes were up just a few seasons ago.

If only you could somehow predict which game would be plentiful, you could always bid on the easier contract! But how?

Back at camp, amid the crackling of your lonely fire, the answer comes to you. Just two months ago you attended a talk by Dr. Lotka on autocatalytic chemical reactions. It was quite a spectacle when, after Dr. Lotka had finished talking, a Dr. Volterra stood up and proclaimed that he had applied the same model to predator-prey ecology. At the time you were rushed and didn't think much about the proclamation, but now the basic assumptions were making more and more sense:

(a) In the absence of foxes, the rabbit population grows at a rate proportional to the number of rabbits.

(b) In the absence of rabbits, the fox population declines at a rate proportional to the number of foxes.

(c) The population of rabbits declines at a rate proportional to the product of the rabbit and fox populations.

(d) The population of foxes grows at a rate proportional to the product of the rabbit and fox populations.

Pop! A hot coal explodes, snapping you out of your pondering state and into one of action. Grabbing a piece of paper from your limited supplies, you begin to grapple with the consequences of the Lotka–Volterra model.

Let R and F stand for the rabbit and fox populations, respectively, and let α, β, γ, and δ be the constants of proportionality for parts (a)–(d).

Work through the following before you begin your report.

1. Write down the Lotka-Volterra system of differential equations. For each of (a)–(d), explain whether or not the assumption is reasonable.

2. When is the fox population increasing or decreasing? Given R and F, could you predict which one is on the rise on which one is on the decline?

3. Is there a steady state for the fox population? Could the fox population remain steady while the rabbit population is changing?

4. Sketch an RF-phase portrait for the Lotka-Volterra system of differential equations with the following constants:

$$\begin{aligned}\alpha &= 0.2 \text{ rabbits per month per rabbit}\\ \beta &= 0.1 \text{ foxes per month per fox}\\ \gamma &= 0.002 \text{ rabbits per month per rabbit-fox}\\ \delta &= 0.001 \text{ foxes per month per rabbit-fox}\end{aligned}$$

   Hint: you will need to consider rabbit and fox populations of well over 100 to see interesting behavior in your phase portrait.

[1] "Hunting Inspiration" is a collaboration between Max Brugger and Jason Siefken

5. Does your phase portrait have any singular points? What do they mean?

6. Use technology to graph $R(t)$ and $F(t)$ for some initial conditions. Do the initial conditions affect the period of the population increase or decrease? Does this seem reasonable when looking at your phase portrait?

For this project, you may work alone or with a partner and turn in one $\LaTeX$ writeup between the two of you. Your writeup should include the following:

- An explanation of the Lotka-Volterra model along with a discussion of whether or not each assumption is reasonable.
- A description of what behavior you expect from which initial conditions. You may use the parameters specified in question 4. Include a phase portrait in your description as well as how to interpret the phase portrait, and make sure to point out any critical points.
- Suppose you wanted to legislate limits on the hunting of rabbits and foxes to ensure the population of either never dipped below a certain level. Based on the Lotka-Volterra model, propose legislation. Be specific and comment on whether a flat-out hunting ban would achieve the desired effect.

Be careful with your simulations. Euler's method loses accuracy quickly on Lotka-Volterra–based systems.