1. For each situation, set up a differential equation. Explain what each function in your equation represents and explicitly specify what symbols represent constants.

   (a) Newton's law of cooling states that the change of temperature of an object is proportional to the difference between its temperature and the ambient temperature. Model temperature with respect to time.

   (b) Your pickle plant has a brine making machine. Water with a salinity of 10% flows into a 100 liter tank at a rate of 0.2 liters/second. The incoming liquid mixes fully in the tank and then the mixed liquid flows out at 0.2 liters/second. Model the salinity with respect to time.

   (c) An object is falling. The second derivative with respect to time is proportional to (gravity $-F_R$) where F_R is the force of air resistance. Air resistance is proportional to the product of surface area and velocity. Model velocity with respect to time.

   (d) The logistic model of population growth states that the growth in population is proportional to the product of the population and the remaining resources. Further, resources decrease linearly with population. Model population with respect to time.

   (e) Fibonacci's model for bunny populations stated that the growth of the population is equal to the number of adult bunnies. Assuming it takes the bunnies 1 year to mature and that they never die, model the population of bunnies with respect to time.

2. A first order differential equation is called *separable* if it can be written in the form $f(y)\frac{\mathrm{d}y}{\mathrm{d}x} = h(x)$. Separable equations are easy to implicitly solve. By integrating both sides, we have $\int f(y)\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int h(x)\mathrm{d}x$. Applying the chain rule to the left hand side gives us

$$\int f(y)\mathrm{d}y = \int h(x)\mathrm{d}x.$$

   For each differential equation, state whether or not it is separable. If it is separable, find the set of implicit solutions. Further, find explicit solutions (i.e., solutions where y is a function of x) to the initial value problem $y(x_0) = y_0$ for every applicable (x_0, y_0). (For example, if the implicit solution were $x^2 + y^2 = K$, then the solution would be $y = \sqrt{(x_0^2 + y_0^2) - x^2}$ if $y_0 > 0$ and $y = -\sqrt{(x_0^2 + y_0^2) - x^2}$ if $y_0 < 0$. If you encounter a similar situation, you must include both such solutions and describe which initial conditions give rise to which solution.)

   (a) $y' = \frac{1-2y}{x}$

   (b) $y' = \frac{-xy}{x+1}$

   (c) $y' = 3\sqrt[3]{y^2}$

   (d) $xy' + y = y^2$

3. Consider the differential equation $y' = Ky - y^3$ where K is constant.

   (a) Draw a slope field for the differential equation for your choice of K. Make sure it's detailed enough for you to see what's going on.

   (b) Partition the set of initial conditions (x_0, y_0) into sets that give rise to "similar looking" solutions and describe the types of solutions you get from each set. Warning: your partitions may depend on K!

   (c) Let $\vec{x}_0 = (x_0, y_0)$ and $\vec{x}_0^* = (x_0^*, y_0^*)$ with $\|\vec{x}_0 - \vec{x}_0^*\| < \varepsilon$. Let $y(x)$ and $y^*(x)$ be solutions to the respective initial value problems. The initial condition $\vec{x}_0$ is called *bi-stable* if $|y(x) - y^*(x)| < C\varepsilon$ for some fixed C. Otherwise it is called *unstable*. It is called *forward stable* if $|y(x) - y^*(x)| < C\varepsilon$ when $x \geq x_0$ and *backwards stable* if $|y(x) - y^*(x)| < C\varepsilon$ when $x \leq x_0$ (so stable implies both forward and backwards stable).

   Identify each region of of initial conditions as forward/backward/bi stable or unstable.

(d) For $K = 2$, use Euler's method with 5 steps to estimate $y(5)$ where y is a solution to the initial value problem $(0, 1)$. Plot your estimate. What is going on?

(e) Repeat your calculation for $K = 2$ using Euler's method with 50 steps. (You can use a computer for the 50 steps.)

(f) Use Euler's method and this differential equation (with an appropriate value for K) to approximate $\sqrt{7}$.

4. Consider the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x$$

and the initial values $x(0) = 1$ and $y(0) = 0$.

(a) Show that $(x, y) = (\cos t, \sin t)$ is a solution to this initial value problem.

(b) Euler's method would suggest you approximate a solution to this differential equation iteratively with the formulas

$$x_n = x_{n-1} - y_{n-1}\Delta t \qquad \text{and} \qquad y_n = y_{n-1} + x_{n-1}\Delta t.$$

What happens when you do so? Do you get a closed curve?

(c) Use the modified Euler's method

$$x_n = x_{n-1} - y_{n-1}\Delta t \qquad \text{and} \qquad y_n = y_{n-1} + x_n\Delta t.$$

What happens now? Explain.