1. Find all solutions to the following linear differential equations.
   (a) $y'' - 4y' + 4y = 0$
   (b) $y'' + 4y = 0$
   (c) $y'' - y = t^2$
   (d) $y''' + 4y'' - 5y' = 0$

2. Consider the differential equation
$$y'' - 4y' + 4y = 0. \tag{1}$$
   (a) Find all values k so that this equation solution of the form e^{kt}.
   (b) Can you express all solutions to equation (1) in the form $Ae^{k_1 t} + Be^{k_2 t}$ for some fixed constants k_1 and k_2? Why or why not?
   (c) Find a k such that te^{kt} is a solution to equation (1).
   (d) Can you express all solutions to equation (1) in the form $Ae^{k_1 t} + Bte^{k_2 t}$ for some k_1 and k_2? Explain.
   (e) Can you express all solutions to equation (1) in the form $A(1+t)e^{k_1 t} + B(1-t)e^{k_2 t}$ for some k_1 and k_2? Explain.

3. The characteristic polynomial of a linear homogeneous differential equation $F(y) = 0$ is the polynomial $p(k)$ so that $F(e^{kt}) = p(k)e^{kt}$. (Recall here that F a linear function. For example, maybe $F(y) = y'' - 2y$.)
   (a) For each linear differential equations from part 1, write the characteristic polynomial (if it isn't homogeneous, consider the corresponding homogeneous equation).
   (b) Consider a second-order linear homogeneous ODE with characteristic polynomial $p(k)$ having roots α and β. Write the equation for this ODE.
   (c) Consider a second-order linear homogeneous ODE with characteristic polynomial $p(k)$. Suppose that $p(k)$ is two roots α, β with $\alpha \neq \beta$. Show that $Ae^{\alpha t} + Be^{\beta t}$ is always a solution. Further, show that all solutions to this differential equation can be written in this way.
   (d) Consider a second-order linear homogeneous ODE with characteristic polynomial $p(k)$ having one repeated root α. Show that $Ae^{\alpha t} + Bte^{\alpha t}$ is always a solution. Further, show that all solutions to this differential equation can be written in this way.
   (e) Consider a second-order linear homogeneous ODE with characteristic polynomial $p(k)$ having two distinct roots α, β. Show that there is no γ such that $te^{\gamma t}$ is a solution.
   (f) Consider a third-order linear homogeneous ODE with characteristic polynomial $p(k) = (k-2)^3$. Write the equation for this ODE. Then, find all solutions.

4. Consider $y'' + y = 0$.
   (a) Show that for any initial value problem, you can find $A, B \in \mathbb{C}$ so that
$$Ae^{i\theta} + Be^{-i\theta}$$
   is a solution. Could you solve any IVP if you restrict $A, B \in \mathbb{R}$?
   (b) Use part (a) along with Euler's formula and the fact that sin and cos are solutions to this differential equation to come up with a formula for sin and cos as a linear combination of complex exponentials.
   (c) Show that $A\cos\theta + B\sin\theta$ also solves every initial value problem.
   (d) Can you get any additional solutions by considering $A\cos\theta + B\sin\theta + Ae^{i\theta} + Be^{-i\theta}$? Explain.

(e) Use your formula from part (b) to find arccos and arcsin in terms of complex numbers and complex logarithms. (Hint: $e^{2t} = (e^t)^2$ and $e^t e^{-t} = 1$).

5. The matrices are coming! Read about *matrix multiplication* in the source of your choice (the Evans text talks about it) and how to compute 2×2 and 3×3 determinants.

(a) Compute $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, $\det\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\right)$, and $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}^2$.

(b) Let $\vec{a} = (1, 2, 3)$ and $\vec{b} = (1, 2, 2)$. Find matrices A and B so that $\vec{a} \cdot \vec{b}$ is equivalent to the matrix product AB. What are the dimensions of A and B?

(c) Let $M = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, and consider the function $F_M : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F_M(a, b) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Show that F_M is linear.