1. Consider the second order differential equation

$$y'' + y = 0.$$

   (a) Find a power series solution for this differential equation.

   (b) Use your knowledge of what solutions to this differential equation look like to rearrange your power series into some Taylor series formulas for familiar functions.

2. So far, we've only done power series solutions center at 0. That is, our guess for a power-series solution looks like $y = \sum_{i \geq 0} a_i x^i$. This often times works, but sometimes it won't give all solutions because at $x = 0$ some information is lost.

   Consider the differential equation

$$xy'' + y' + xy = 0.$$

   (a) Classify this differential equation in terms of linear/autonomous/homogeneous/nth order etc.

   (b) What dimension do you expect the space of all solutions to be? That is, how many free parameters do you expect to have?

   (c) Find a power series solution to this differential equation centered at 0. How many parameters do you have in your solution?

   (d) The power series solution centered at x_0 is obtained by guessing a solutions of the form $y = \sum_{i \geq 0} a_i (x - x_0)^i$. Find the power series solution to this differential equation centered at $x_0 = 1$. How many parameters do you have in your solution? Can you explain what happened in the previous part?

3. We haven't spent a lot of time studying *degenerate* differential equations/systems. That is, differential equations/systems that have 0 as an eigenvalue. Let's fix that.

   (a) Find all solutions to the differential equation

$$y'' + y' = 0. \tag{1}$$

   Then, rewrite equation (1) as a system in matrix form and find all flows.

   (b) Consider the system of differential equations given in matrix form by

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{2}$$

   Find the characteristic polynomial for equation (2) and all eigenvalues.

   (c) Can the differential equation (2) be solved for all initial conditions $\begin{bmatrix} a'(0) \\ b'(0) \end{bmatrix} = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$? Explain.

   (d) Write down all flows for equation (2).

4. Use the Laplace transform to solve the following initial value problems.

   (a) $y'' - 10y' + 9y = 5t$ and $y(0) = -1$ and $y'(0) = 2$.

   (b) $2y'' + 3y' - 2y = te^{-2t}$ and $y(0) = 0$ and $y'(0) = -2$.

   (c) $y'' + 3ty' - 6y = 2$ and $y(0) = 0$ and $y'(0) = 0$. In solving this one, you might end up with a differential equation involving the Laplace transform. You might find the fact that $f(t) = 2/t^3$ is a solution to the differential equation $f' + (3/t - t/3)f = -\frac{2}{3s^2}$ useful.

5. Compute the following convolutions.

(a) $t^2 * 1$
(b) $t * e^{-t}$
(c) $t^2 * (t^2 - 1)$
(d) $\cos t * \cos t$

6. Without computing, estimate where the asymptotes are in the analytic continuation of the following Laplace transforms. (In other words, if we did compute the Laplace transform and interpreted it as a formula ignoring the restrictions on the domain, where would the asymptotes be).

   (a) $\mathscr{L}(e^t + e^{8t} - 14e^{-14t})$
   (b) $\mathscr{L}(1 + (x-2)^2)$
   (c) $\mathscr{L}(\sin^2(t)e^{-\pi t})$
   (d) $\mathscr{L}(\sin t)$
   (e) Compare your estimate for the asymptotes of $\mathscr{L}(\sin t)$ with the actual asymptotes of the formula. Are they where you thought? (Don't forget about complex numbers.)
   Let $F(s) = \mathscr{L}(\sin t)$. Explain why it makes sense that
   $$\lim_{s \to 0} F(s) \neq \infty.$$
   Explain why
   $$\lim_{s \to 0} F(i + s) = \pm\infty.$$
   You'll find Euler's formula helpful!

7. We can do some really cool things with the Laplace transform that don't seem related to ODEs. The function
   $$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$
   is called the *sinc* function and shows up a lot in science and engineering. We're going to find
   $$I = \int_0^\infty f(x) \mathrm{d}x.$$
   Since the sinc function resembles an alternating harmonic series, it's reasonable to assume that I actually exists, and in fact using the alternating series test, we can prove that $I < \infty$ exists. But integrating it directly is hard!

   (a) Write out the formula for $F(s) = \mathscr{L}(f(t))$, and compute $\frac{\mathrm{d}f}{\mathrm{d}s}$. Do you see see a Laplace transform that appears on your lookup table anywhere?
   (b) You now have a formula for $\mathrm{d}F/\mathrm{d}s$, so if you find its antiderivative you'll obtain $F + k$ for some constant k. Do so, and find the antiderivative that corresponds to $k = 0$. (Think about what this means. It's not as easy as not "adding k" when integrating.) Hint: consider any values of $F(s)$ or any limits of $F(s)$ that you might actually know exactly.
   (c) For what value of s does $F(s)$ coincide with I? Evaluate F at that value to obtain I.

8. There are several things that you need to know are associated with differential equations and solutions to ODEs. Though I won't make you use these things, it is important that you recognize them as associated with differential equations so that you can look them up when you need them.

   Look up in your textbook (or Wikipedia or a source of your choice) the following items and read about them until they are cemented in your mind as associated with differential equations.

   (a) Separation of Variables
   (b) Variation of Parameters
   (c) Bessel Functions