1. We know a system of linear equations can have $0, 1$, or infinitely many solutions.

   (a) Explain why a system of linear equations cannot have exactly 2 solutions.

   Let $\mathcal{E}$ be the system of linear equations given in vector form as

   $$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n = \vec{b},$$

   and suppose $\mathcal{E}$ has two solutions $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{y} = (y_1, y_2, \ldots, y_n)$. Then, $\vec{z} = \frac{1}{2}(\vec{x} + \vec{y})$ is another solution since

   $$\tfrac{1}{2}(x_1 + y_1)\vec{v}_1 + \tfrac{1}{2}(x_2 + y_2)\vec{v}_2 + \cdots + \tfrac{1}{2}(x_n + y_n)\vec{v}_n$$

   $$= \tfrac{1}{2}\Big(x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n\Big) + \tfrac{1}{2}\Big(y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_n\vec{v}_n\Big) = \tfrac{1}{2}\vec{b} + \tfrac{1}{2}\vec{b} = \vec{b}.$$

   Thus if a system as two solutions it also has at least three solutions!

   (b) For a homogeneous system of linear equations, what are the possibilities for the number of solutions? Explain, and make sure to define *homogeneous system*.

   Recall that a system of linear equations $\mathcal{S}$ is *homogeneous* if every equation equals zero. Let $\mathcal{S}$ be the homogeneous system of linear equations given in vector form by

   $$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n = \vec{0}.$$

   We see that this system always has at least one solution, namely the solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly dependent, $\mathcal{S}$ would have infinitely many solutions. Thus, the number of possible solutions for a homogeneous system is 1 or ∞, but never 0.

2. Suppose $\mathcal{M}$ is a homogeneous system of 4 linear equations and 3 variables. Let M be the non-augmented matrix of coefficients.

   (a) If $\operatorname{rank}(M) = 3$, how many solutions does the system $\mathcal{M}$ have? (You do not need to define *reduced row echelon form*, but include all other relevant definitions.)

   Recall that a *homogeneous* system of linear equations is one where every equation equals zero. The *rank* of a matrix is the number of leading ones in the reduced row echelon form of that matrix.
   A homogeneous system of linear equations has either 1 or infinitely many solutions, and this directly corresponds to the number of free variables in the system: if there are no free variables, the system has 1 solution; if there are any free variables, the system has infinitely many solutions. Since we choose free variables by picking columns of the reduced row echelon form of the coefficient matrix that do not have leading 1's and the matrix M has 3 columns and 3 leading 1's (because the rank is 3), we know there are no free variables for the system $\mathcal{M}$. Thus, $\mathcal{M}$ has a unique solution.

   (b) If $\operatorname{rank}(M) = 2$, how many solutions does the system $\mathcal{M}$ have?

   As in part (a), the number of solutions is determined by the number of free variables. Since M has 3 columns and 2 leading 1's in its reduced row echelon form (since the rank of M is 2), we have one free variable, and so $\mathcal{M}$ has infinitely many solutions.

   (c) Could $\operatorname{rank}(M) = 4$? Explain.

   In the reduced row echelon form of a matrix, every column can only have one leading 1. Since there are 3 columns in M, M has at most 3 leading ones and so

   $$\operatorname{rank}(M) \leq 3.$$

3. Consider the equation $\mathcal{E}$ given by

$$x_1 - x_2 + x_3 - x_4 = 0$$

and let $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ is a solution to } \mathcal{E} \right\}$. Find vectors $\vec{v}_1, \ldots, \vec{v}_n$ such that $V = \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$. Explain your process.

Since $\mathcal{E}$ is a homogeneous equation, we may row-reduce a non-augmented matrix of coefficients to produce the set of solutions. This gives us the matrix

$$\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$$

which has three free variable columns, namely x_2, x_3, x_4. Let $x_2 = t$, $x_3 = s$, and $x_4 = r$. We then see the solutions to this system are given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t - s + r \\ t \\ s \\ r \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

for $t, r, s \in \mathbb{R}$. However, the right hand side of the equation is exactly linear combinations of the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Thus,

$$V = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}.$$