1. For each of the following statements, produce a counterexample to show that the statement is **false**.

   (a) If A and B are square matrices, $AB = BA$.

$$\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix} \neq \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$$

   (b) If $AB = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$, then A and B are 2×2 matrices.

$$\begin{bmatrix}1\\ 1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

   (c) If $AB = I$ then $BA = I$.

$$\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{bmatrix} = [1] = I_{1\times 1},$$

   but

$$\begin{bmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \neq I_{2\times 2}.$$

   (d) If $A^2 = 0$, then $A = 0$.

$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}^2 = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = 0_{2\times 2},$$

   but

$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \neq 0_{2\times 2}.$$

2. Let $R = \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}$.

   (a) Find all solutions to the matrix equation $R\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}2\\ 5\\ 8\end{bmatrix}$.

   The matrix equation $R\vec{x} = \begin{bmatrix}2\\ 5\\ 8\end{bmatrix}$ corresponds to the system of linear equations given by the augmented matrix

$$A = \left[\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 4 & 5 & 6 & 5\\ 7 & 8 & 9 & 8\end{array}\right].$$

   Row reducing, we find

$$\operatorname{rref}(A) = \left[\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 1 & 2 & 1\\ 0 & 0 & 0 & 0\end{array}\right].$$

   Identifying x_3 as a free variable and adding the equation $x_3 = t$, we produce a general solution of

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = t\begin{bmatrix}1\\ -2\\ 1\end{bmatrix} + \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$$

   for $t \in \mathbb{R}$.

(b) Prove that the set $X = \{\vec{x} \in \mathbb{R}^3 : R\vec{x} = \vec{0}\}$ is a subspace.

Suppose $\vec{u}, \vec{v} \in X$. That means $R\vec{u} = R\vec{v} = \vec{0}$. Considering $\vec{u} + \vec{v}$ we see

$$R(\vec{u} + \vec{v}) = R\vec{u} + R\vec{v} = \vec{0} + \vec{0} = \vec{0},$$

and so $\vec{u} + \vec{v} \in X$. Further, if $k \in \mathbb{R}$, then

$$R(k\vec{u}) = kR\vec{u} = k\vec{0} = \vec{0},$$

and so $k\vec{u} \in X$, showing that X is a subspace.

3. Suppose E is a 4×3 matrix with columns $\vec{c}_1, \vec{c}_2, \vec{c}_3$ and rows $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4$. Let $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

(a) Express $E\vec{v}$ as a linear combination of $\vec{c}_1, \vec{c}_2, \vec{c}_3$.

$$E\vec{v} = [\vec{c}_1|\vec{c}_2|\vec{c}_3] \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 2\vec{c}_1 - \vec{c}_2 + \vec{c}_3.$$

(b) Supposing $\vec{r}_1 \cdot \vec{v} = 1$, $\vec{r}_2 \cdot \vec{v} = 6$, $(\vec{r}_3 + \vec{r}_4) \cdot \vec{v} = 2$, and $(\vec{r}_3 - \vec{r}_4) \cdot \vec{v} = -2$, compute $E\vec{v}$.

Considering matrix multiplication as dot products of rows of the first matrix with columns of the second, we see

$$E\vec{v} = \begin{bmatrix} \vec{r}_1 \cdot \vec{v} \\ \vec{r}_2 \cdot \vec{v} \\ \vec{r}_3 \cdot \vec{v} \\ \vec{r}_4 \cdot \vec{v} \end{bmatrix}.$$

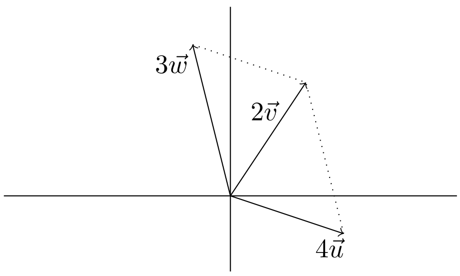
The only values we haven't been given are $\vec{r}_3 \cdot \vec{v}$ and $\vec{r}_4 \cdot \vec{v}$, but by using the equations $(\vec{r}_3 + \vec{r}_4) \cdot \vec{v} = 2$ and $(\vec{r}_3 - \vec{r}_4) \cdot \vec{v} = -2$, we deduce that

$$\vec{r}_3 \cdot \vec{v} = 0 \qquad \text{and} \qquad \vec{r}_4 \cdot \vec{v} = 2.$$

Thus,

$$E\vec{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 2 \end{bmatrix}.$$

4. Suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^2$ that are related by the following diagram.



Let $A = [\vec{u}|\vec{v}|\vec{w}]$ be the matrix with columns $\vec{u}$, $\vec{v}$, and $\vec{w}$.

(a) What is the rank of A?

Notice that $\{\vec{u},\vec{v}\}$ are linearly independent, but $4\vec{u}-2\vec{v}+3\vec{w}=\vec{0}$ and so $\{\vec{u},\vec{v},\vec{w}\}$ is linearly dependent. Thus, there are two linearly independent columns in A and so $\operatorname{rank}(A)=2$.

(b) Find all solutions to the equation $A\vec{x}=\vec{0}$.

Since $4\vec{u}-2\vec{v}+3\vec{w}=\vec{0}$, we know that

$$A\begin{bmatrix}4\\-2\\3\end{bmatrix}=[\vec{u}|\vec{v}|\vec{w}]\begin{bmatrix}4\\-2\\3\end{bmatrix}=4\vec{u}-2\vec{v}+3\vec{w}=\vec{0}.$$

In fact, any multiple of $\begin{bmatrix}4\\-2\\3\end{bmatrix}$ is also a solution to $A\vec{x}=\vec{0}$. Since $\operatorname{nullity}(A)+\operatorname{rank}(A)=\#$of columns of A, we know $\operatorname{nullity}(A)=1$. Thus, all solutions to $A\vec{x}=\vec{0}$ are given by

$$x=t\begin{bmatrix}4\\-2\\3\end{bmatrix}$$

for some $t\in\mathbb{R}$.

(c) Find a basis for the subspace $V=\{\vec{x}\in\mathbb{R}^3 : A\vec{x}=\vec{0}\}$.

As noted earlier, V consists precisely of multiples of $\begin{bmatrix}4\\-2\\3\end{bmatrix}$ and so a basis is $\left\{\begin{bmatrix}4\\-2\\3\end{bmatrix}\right\}$.