Midterm 1

MATH 211 (A01), Spring 2015 (Siefken)

Date: ____________

Name: ________________________________

ID Number: ________________________________

This is a 50 minute test. It has 6 pages including this cover page.

Q1		/10
Q2		/10
Q3		/10
Q4		/10
Q5		/10
Total		/50

1 (10pts) Complete each of the following sentences with a mathematically precise definition.

(a) (2pts) A non-empty subset $V \subseteq \mathbb{R}^n$ is a **subspace** if

for all $\vec{v}, \vec{w} \in V$ we have $\vec{v} + \vec{w} \in V$ and $k\vec{v} \in V$ for all scalars k.

(b) (2pts) The set of vectors $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is **linearly independent** if

the only solution to

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 + \alpha_4\vec{v}_4 = \vec{0}$$

is $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$.

(c) (2pts) The set of vectors $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a **basis for the subspace** V if

B is linearly independent and $\operatorname{span} B = V$.

(d) (2pts) The vector $\vec{a}$ is a **unit vector** if

$\|\vec{a}\| = 1$.

(e) (2pts) The vector $\vec{a}$ is **orthogonal** to the vector $\vec{b}$ if

$\vec{a} \cdot \vec{b} = 0$.

2 (10pts) Let

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{c} = \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix}.$$

(a) (3pts) Compute $\vec{a} \cdot (\vec{b} - \vec{c})$.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = 10.$$

(b) (2pts) Compute $-\vec{a} + 2\vec{b}$.

$$-\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix}.$$

(c) (5pts) Is the set of vectors $X = \{\vec{a}, \vec{b}, \vec{c}\}$ linearly independent or linearly dependent? Explain your answer.

X is linearly dependent because $-\vec{a} + 2\vec{b} - \vec{c} = \vec{0}$ is a non-trivial linear combination of vectors in X that equals $\vec{0}$.

3 (10pts)

(a) (7pts) Use an augmented matrix to solve the following system of equations

$$\begin{array}{ccccccc} & & y & + & z & = & 1 \\ x & + & 2y & + & z & = & 1 \\ 2x & + & 4y & + & 3z & = & 4 \end{array}.$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 4 & 3 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & 4 & 3 & 4 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

and so $(x, y, z) = (1, -1, 2)$.

(b) (3pts) Let $\vec{a} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$. Express $\vec{v}$ as a linear combination of $\vec{a}$, $\vec{b}$, and $\vec{c}$.
Hint, these vectors closely relate to the system of equations in part (a).

$\vec{v} = \vec{a} - \vec{b} + 2\vec{c}$.

4 (10pts) Below you are given the matrix A and its reduced row echelon form.

$$A = \begin{bmatrix} 1 & 2 & -1 & -1 & 4 \\ 2 & 4 & 1 & 0 & 4 \\ 1 & 2 & 1 & -1 & 4 \end{bmatrix} \qquad \mathrm{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

(a) (2pts) What is the rank of A?

$\mathrm{rank}(A) = 3.$

(b) (2pts) Write down the *homogeneous system* of linear equations corresponding to A.

$$\begin{array}{rrrrrl} x_1 & +2x_2 & -x_3 & -x_4 & +4x_5 & = 0 \\ 2x_1 & +4x_2 & +x_3 & & +4x_5 & = 0 \\ x_1 & +2x_2 & +x_3 & -x_4 & +4x_5 & = 0 \end{array} .$$

(c) (6pts) Write down the *general solution to* the homogeneous system of linear equations corresponding to A.

Since we've been given the row reduced form of A, we immediately indentify x_2 and x_5 as the only free variables.

Let $x_2 = t$ and $x_5 = s$. We then see

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t - 2s \\ t \\ 0 \\ 2s \\ s \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} .$$

5 (10pts) For each of the following subsets of $\mathbb{R}^2$, either prove that the subset is a subspace or provide an example that shows it is not a subspace.

(a) (5pts) $U = \{\vec{x} \in \mathbb{R}^2 : \vec{x} \cdot \vec{e}_1 + \vec{x} \cdot \vec{e}_2 = 1\}$.

U is not a subspace because it is not closed under vector addition. Notice $\vec{e}_1, \vec{e}_2 \in U$, but $\vec{w} = \vec{e}_1 + \vec{e}_2$ satisfies $\vec{w} \cdot \vec{e}_1 + \vec{w} \cdot \vec{e}_2 = 2$ and so $\vec{w} \notin U$.

(b) (5pts) $V = \{\vec{x} \in \mathbb{R}^2 : \vec{x} \cdot \vec{e}_1 + \vec{x} \cdot \vec{e}_2 = 0\}$.

V is a subspace. Let $\vec{u}, \vec{v} \in V$. Then $\vec{u} \cdot \vec{e}_1 + \vec{u} \cdot \vec{e}_2 = 0$ and $\vec{v} \cdot \vec{e}_1 + \vec{v} \cdot \vec{e}_2 = 0$. We then have

$$(\vec{u} + \vec{v}) \cdot \vec{e}_1 + (\vec{u} + \vec{v}) \cdot \vec{e}_2 = \vec{u} \cdot \vec{e}_1 + \vec{u} \cdot \vec{e}_2 + \vec{v} \cdot \vec{e}_1 + \vec{v} \cdot \vec{e}_2 = 0 + 0 = 0,$$

and so V is closed under vector addition. Further, if $\vec{v} \in V$, we have

$$(k\vec{v}) \cdot \vec{e}_1 + (k\vec{v}) \cdot \vec{e}_2 = k(\vec{v} \cdot \vec{e}_1 + \vec{v} \cdot \vec{e}_2) = k0 = 0$$

for any scalar k, and so V is closed under scalar multiplication.