1. Suppose the matrix equation $A\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$ has the general solution

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

   (a) How many rows and how many columns does $A$ have?

   $A$ has 3 columns because the general solution we are given is a subset of $\mathbb{R}^3$, and $A$ has 3 rows because $A\vec{x} \in \mathbb{R}^3$.

   (b) Find $\text{null}(A)$.

   The general solution to a matrix equation $A\vec{x} = \vec{b}$ takes the form of $\vec{p} + N$ where $N$ is the null space of $A$. Thus,

$$\text{null}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

   (c) Find $\text{rank}(A)$.

   Since the null space of $A$ has dimension 2 and $A$ is a $3 \times 3$ matrix, by the rank-nullity theorem, the rank of $A$ must be 1.

   (d) Find $\text{col}(A)$.

   The column space of $A$ is the same as the range of $A$. Since the rank of $A$ is 1, the range of $A$ must be a one dimensional subspace, and in particular,

$$\begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix} \in \text{range}(A).$$

   Thus

$$\text{col}(A) = \text{span}\left\{ \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix} \right\}.$$

   (e) Find $\text{row}(A)$.

   The row space of $A$ is orthogonal to the null space of $A$ and by the rank-nullity theorem, the row space of $A$ must be dimension 1. By inspection we find that the vector $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is orthogonal to all vectors in $\text{null}(A)$, and so $\vec{v} \in \text{row}(A)$. We conclude

$$\text{row}(A) = \text{span}\{\vec{v}\}.$$

2. Let

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{b}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{b}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

   and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and $T(\vec{b}_1) = 2\vec{b}_1$, $T(\vec{b}_2) = 3\vec{b}_2$, and $T(\vec{b}_3) = -\vec{b}_3$.

   (a) Compute $[\vec{c}]_{\mathcal{B}}$.

Let $X = [\vec{b}_1|\vec{b}_2|\vec{b}_3]$. Computing we see

$$X^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Interpreted as a change of basis, $X$ converts from the $\mathcal{B}$ basis to the $\mathcal{S}$ basis and $X^{-1}$ converts from the $\mathcal{S}$ basis to the $\mathcal{B}$ basis.
Thus, we can express $\vec{c}$ in the $\mathcal{B}$ basis by computing

$$X^{-1}[\vec{c}]_{\mathcal{S}} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

and so

$$[\vec{c}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{B}}.$$

(b) Compute $[T\vec{c}]_{\mathcal{B}}$ and $[T\vec{c}]_{\mathcal{S}}$.

We have already computed that $\vec{c} = 2\vec{b}_1 + \vec{b}_3$ and so $T\vec{c} = T(2\vec{b}_1 + \vec{b}_3) = 2T\vec{b}_1 + T\vec{b}_3 = 4\vec{b}_1 - \vec{b}_3$. Thus, we see

$$[T\vec{c}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}_{\mathcal{B}},$$

and

$$[T\vec{c}]_{\mathcal{S}} = 4[\vec{b}_1]_{\mathcal{S}} - [\vec{b}_3]_{\mathcal{S}} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}.$$

(c) Find a matrix for $T$ in the $\mathcal{B}$ basis (i.e., the matrix $[T]_{\mathcal{B}}$) and a matrix for $T$ in the $\mathcal{S}$ basis (i.e., $[T]_{\mathcal{S}}$).

We know $T\vec{b}_1 = 2\vec{b}_1$, $T\vec{b}_2 = 3\vec{b}_2$, and $T\vec{b}_3 = -\vec{b}_3$, and so in the $\mathcal{B}$ basis, $T$ acts like a diagonal matrix. From this, we produce the matrix

$$[T]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}_{\mathcal{B}}.$$

Since we have already written $[T]_{\mathcal{B}}$, if we want to find $[T]_{\mathcal{S}}$, we may use the matrix $X^{-1}$ to first convert vectors from the $\mathcal{S}$ basis to the $\mathcal{B}$ basis, then use $[T]_{\mathcal{B}}$ to apply the transformation $T$, then use $X$ to convert back from the $\mathcal{B}$ basis to the $\mathcal{S}$ basis. Following this procedure, we get

$$[T]_{\mathcal{S}} = X[T]_{\mathcal{B}}X^{-1} = \frac{1}{3}\begin{bmatrix} 4 & -5 & 7 \\ -1 & 8 & -1 \\ 3 & 3 & 0 \end{bmatrix}.$$

3. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & x \end{bmatrix}$.

(a) Compute $\det(A)$.

Using the cofactor expansion on the bottom row of $A$, we see

$$\det(A) = \det\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - 0\det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 0\det\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = 1.$$

(b) Compute $\det(B)$. For what values of $x$ is $B$ not invertible?

Using the cofactor expansion on the bottom row of $B$, we see

$$\det(B) = \det\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - 0\det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + x\det\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = 1 - x.$$

We conclude that when $x = 1$, $B$ is not invertible.

4. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 9 \end{bmatrix}$.

(a) Find an equation for the function $p(x) = \det(A - xI)$ (this is called the *characteristic polynomial* of $A$).

$$p(x)\det(A - xI) = \det\left(\begin{bmatrix} 1 & 2 \\ 5 & 9 \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}\right) = \det\begin{bmatrix} 1-x & 2 \\ 5 & 9-x \end{bmatrix}$$
$$= (1-x)(9-x) - 10 = x^2 - 10x - 1$$

(b) For what values of $x$ is $A - xI$ non-invertible?

By using the quadratic formula, $p(x) = 0$ when $x = 5 \pm \sqrt{26}$.

(c) Compute $p(A)$, the polynomial $p$ with the matrix $A$ plugged into it. When you plug a matrix into a polynomial, replace any constant terms $k$ with the matrix $kI$. Can you guess why $p$ is called an *annihilating* polynomial for $A$?

$$p(A) = A^2 - 10A - I = \begin{bmatrix} 1 & 2 \\ 9 & 5 \end{bmatrix}^2 - 10\begin{bmatrix} 1 & 2 \\ 5 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 11 & 20 \\ 50 & 91 \end{bmatrix} - \begin{bmatrix} 10 & 20 \\ 50 & 90 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$p$ is called an annihilating polynomial for $A$ because $p(A)$ is the zero matrix!