1. Let $\vec{u} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ -4 \\ -4 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 14 \\ 0 \\ d \end{bmatrix}$.

   (a) For what value(s) of d is $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$ a plane?

   $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$ is a plane precisely when $\{\vec{u}, \vec{v}, \vec{w}\}$ consists of exactly two linearly independent vectors. Since $\{\vec{u}, \vec{v}\}$ is a linearly independent set, we must choose d so that $\vec{w} \in \operatorname{span}\{\vec{u}, \vec{v}\}$. That is, we must choose a value of d so that there exist scalars $\alpha, \beta \in \mathbb{R}$ so

   $$\begin{bmatrix} 14 \\ 0 \\ d \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 4 \\ -4 \\ -4 \end{bmatrix}.$$

   Considering the top two coordinates, we see

   $$\begin{aligned} 14 &= 3\alpha + 4\beta \\ 0 &= 4\alpha - 4\beta, \end{aligned}$$

   and so $\alpha = \beta = 2$. Thus

   $$\begin{bmatrix} 14 \\ 0 \\ d \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ -4 \\ -4 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ -6 \end{bmatrix}$$

   and so $d = -6$ is the only value of d that makes $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$ a plane.

   (b) Is there a value of d so $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$ a line? Explain.

   Since $\{\vec{u}, \vec{v}\}$ is linearly independent, $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$ must be at least two dimensional. That is, it must be a plane or all of $\mathbb{R}^3$ and so could never be a line.

2. Let $G \subseteq \mathbb{R}^2$ be the graph of the line given by the equation $y = 3x + 2$. Find vectors $\vec{d}$ and $\vec{p}$ so that

   $$G = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \in \mathbb{R}\}.$$

   After graphing G, we see that the line with direction vector $\vec{d} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is parallel to G. Since G has a y-intercept of 2, we can obtain G by translating the line through the origin with direction $\vec{d}$ by the vector $\vec{p} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Verifying, we see that if

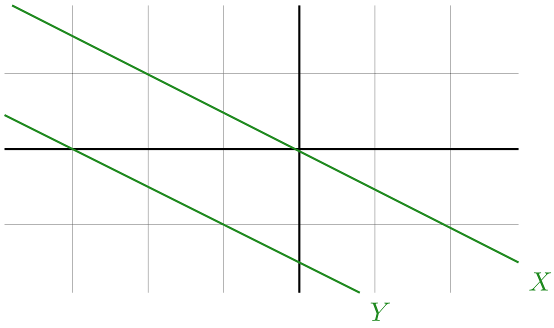
   $$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

   for some $t \in \mathbb{R}$, then x, y satisfy the equation $y = 3x + 2$.

3. Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

   $$\begin{aligned} X &= \{\vec{x} \in \mathbb{R}^2 : \vec{x} \cdot \vec{v} = 0\} \\ Y &= \{\vec{x} \in \mathbb{R}^2 : \vec{x} \cdot \vec{v} = -3\}. \end{aligned}$$

   (a) Draw X and Y. What do you notice? Are either of them subspaces?



X is a subspace since X is a line through the origin. We can verify, if $\vec{a}, \vec{b} \in X$, then $\vec{a} \cdot \vec{v} = \vec{b} \cdot \vec{v} = 0$. Thus, $(\vec{a} + \vec{b}) \cdot \vec{v} = \vec{a} \cdot \vec{v} + \vec{b} \cdot \vec{v} = 0 + 0 = 0$ and so property (i) of subspaces is satisfied. Further, $(k\vec{a}) \cdot \vec{v} = k(\vec{a} \cdot \vec{v}) = k0 = 0$ and so property (ii) of subspaces is satisfied, and so X is a subspace.

Y is not a "flat space through the origin" and so it is not a subspace. Verifying, we see $\begin{bmatrix} -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \in Y$, but $\begin{bmatrix} -3 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix} = \vec{c}$ has $\vec{c} \cdot \vec{v} = -5 \neq -3$ and so Y is not closed under addition and therefore fails to satisfy property (i) of subspaces.

(b) Find a vector $\vec{w}$ so that

$$Y = \{\vec{x} + \vec{w} : \vec{x} \in X\}.$$

We know that $\vec{x} \in X$ means that $\vec{x} \cdot \vec{v} = 0$. We are looking for a $\vec{w}$ so that $\vec{x} + \vec{w} \in Y$ whenever $\vec{x} \in X$. Thus, we must have $(\vec{x}+\vec{w}) \cdot \vec{v} = \vec{x} \cdot \vec{v} + \vec{w} \cdot \vec{v} = 0 + \vec{w} \cdot \vec{v} = \vec{w} \cdot \vec{v} = -3$. Picking $\vec{w} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$ achieves this.

Verifying, we see indeed if $\vec{y} \cdot \vec{v} = -3$, then $\left(\vec{y} - \begin{bmatrix} -3 \\ 0 \end{bmatrix}\right) \cdot \vec{v} = 0$ and so every vector in Y can be represented as $\vec{x} + \begin{bmatrix} -3 \\ 0 \end{bmatrix}$ for some $\vec{x} \in X$.

4. Let $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Prove that $\{\vec{x}, \vec{y}\}$ is a basis for $\mathbb{R}^2$.

For $\{\vec{x}, \vec{y}\}$ to be a basis for $\mathbb{R}^2$, $\{\vec{x}, \vec{y}\}$ must be linearly independent and we must have $\operatorname{span}\{\vec{x}, \vec{y}\} = \mathbb{R}^2$. First we will show linear independence. Suppose that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for some α, β. This means $0 = \alpha + \beta$ and $0 = \alpha - \beta$. Solving this system of equations, we find $\alpha = \beta = 0$, and so the only way to obtain $\vec{0}$ as a linear combination of $\vec{x}, \vec{y}$ is as the trivial linear combination. Thus $\{\vec{x}, \vec{y}\}$ is linearly independent.

Since $\{\vec{x}, \vec{y}\}$ is linearly independent, $\operatorname{span}\{\vec{x}, \vec{y}\} \subseteq \mathbb{R}^2$ must be a two dimensional space and so must be all of $\mathbb{R}^2$. We may further verify that $\operatorname{span}\{\vec{x}, \vec{y}\} = \mathbb{R}^2$ by picking an arbitrary vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ and noticing that

$$\begin{bmatrix} a \\ b \end{bmatrix} = \tfrac{a+b}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \tfrac{a-b}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in \operatorname{span}\{\vec{x}, \vec{y}\}.$$

5. Let

$$U = \operatorname{span}\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}4\\4\\4\end{bmatrix} \right\},$$
$$V = \left\{ \vec{x} \in \mathbb{R}^3 : \vec{x} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = 0 \right\},$$
$$W = U \cup V.$$

For each subset U, V, W of $\mathbb{R}^3$, show whether it is a subspace or not. If it is a subspace, classify it as a point, line, plane, or all of $\mathbb{R}^3$. Further, if it is a subspace, give a basis for it.

U:

Since the span of any set is always a subspace, U must be a subspace. Since $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}3\\2\\1\end{bmatrix}$ are linearly independent, but

$$\begin{bmatrix}4\\4\\4\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}3\\2\\1\end{bmatrix},$$

we see that $U = \operatorname{span}\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix} \right\}$ and that a basis for U is $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix} \right\}$.

V:

We will verify the subspace properties for V. Suppose $\vec{u}, \vec{v} \in V$. This means $\vec{u} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = \vec{v} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = 0$. Thus

$$(\vec{u}+\vec{v}) \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = \vec{u} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} + \vec{v} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = 0 + 0 = 0.$$

Further,

$$(k\vec{u}) \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = k\left(\vec{u} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}\right) = k0 = 0,$$

and so V is a subspace. Since $\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = 3 \neq 0$, $\begin{bmatrix}1\\1\\1\end{bmatrix} \notin V \subseteq \mathbb{R}^3$. Thus, V cannot be all of $\mathbb{R}^3$ and therefore must be a point, line, or plane. Observe that $\vec{v}_1 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}0\\1\\-1\end{bmatrix}$ are both in V and are linearly independent. Since $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$ is a plane, we conclude that V must be a plane and $\{\vec{v}_1, \vec{v}_2\}$ is a basis.

W:
Intuitively, W is the union of two non-parallel planes, and so it doesn't "look" like a flat space. Thus, we will search for a violation of one of the rules of subspaces. Let $\vec{u} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and notice $\vec{u}, \vec{v} \in W$. Consider

$$\vec{w} = \vec{u} + \vec{v} = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}.$$

We know $\vec{w} \in W$ if $\vec{w} \in U$ or $\vec{w} \in V$. If $\vec{w} \in U$, then there are scalars $\alpha, \beta \in \mathbb{R}$ so that

$$\begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

The first two coordinates give us the system of equations

$$\begin{aligned} 4 &= \alpha + 3\beta \\ 5 &= 2\alpha + 2\beta \end{aligned}$$

and so $\alpha = \frac{7}{4}$ and $\beta = \frac{3}{4}$. However,

$$\frac{7}{4} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{3}{4} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \neq \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix},$$

and so $\vec{w} \notin U$. To check if $\vec{w} \in V$ we compute $\begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 12 \neq 0$, and so $\vec{w} \notin V$. Thus $\vec{w} \notin U \cup V = W$.