

1. Show that the standard basis in \mathbb{R}^3 is linearly independent.

First recall that a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is *linearly independent* if the only way to satisfy the equation

$$\vec{0} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n$$

is when $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

The *standard basis* for \mathbb{R}^3 consists of the three vectors

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Considering the arbitrary linear combination $\vec{w} = \alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 + \alpha_3 \vec{e}_3$, we see that

$$\vec{w} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}.$$

Thus, if $\vec{w} = \vec{0}$, we must have that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and so $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is linearly independent.

2. Find the angle between the vectors $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

We can find the angle between \vec{v} and \vec{w} using the dot product formula.

The *dot product* of two vectors $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is defined as

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n.$$

The dot product of two vectors can also be computed by the formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta,$$

where θ is the angle between \vec{a} and \vec{b} .

In order to use these formulae with \vec{v} and \vec{w} , we must first compute some quantities.

$$\|\vec{v}\| = \sqrt{1^2 + 2^2} = \sqrt{5} \quad \|\vec{w}\| = \sqrt{3^2 + 4^2} = 5.$$

Now, our dot product formulae gives

$$\vec{v} \cdot \vec{w} = 1(3) + 2(4) = 11 = 5\sqrt{5} \cos \theta,$$

and so $\theta = \arccos\left(\frac{11}{5\sqrt{5}}\right) \approx 0.1799$ radians.

3. For vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, is it possible for $\text{span}\{\vec{u}, \vec{v}, \vec{w}\} = \text{span}\{\vec{u}, \vec{v}\}$? Explain when it is and when it isn't.

The *span* of a set of vectors is the set of all linear combinations of those vectors.

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ be vectors and suppose \vec{w} is a *linear combination* of \vec{u} and \vec{v} . This means there are scalars α, β so that

$$\vec{w} = \alpha \vec{u} + \beta \vec{v}.$$

Since \vec{w} is a linear combination of \vec{u} and \vec{v} and $\text{span}\{\vec{u}, \vec{v}\}$ is the set of *all* linear combinations of \vec{u} and \vec{v} , we conclude that $\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$, and so adding \vec{w} to $\{\vec{u}, \vec{v}\}$ cannot enlarge its span. Therefore,

$$\text{span}\{\vec{u}, \vec{v}\} = \text{span}\{\vec{u}, \vec{v}, \vec{w}\}.$$

On the other hand, if \vec{u} , \vec{v} , and \vec{w} were linearly independent, none could be written as a linear combinations of the others. Thus we would have $\vec{w} \notin \text{span}\{\vec{u}, \vec{v}\}$ and so

$$\text{span}\{\vec{u}, \vec{v}\} \subsetneq \text{span}\{\vec{u}, \vec{v}, \vec{w}\}.$$