Midterm 2

MATH 211 (A01), Spring 2015 (Siefken)

Date: ____________

Name: ________________________________

ID Number: ________________________________

This is a 50 minute test. It has 6 pages including this cover page.

Q1		/10
Q2		/10
Q3		/10
Q4		/10
Q5		/10
Total		/50

1 (10pts) Complete each of the following sentences with a mathematically precise definition.

(a) (2pts) A non-empty subset $V \subseteq \mathbb{R}^n$ is a **subspace** if

(i) $\vec{u}, \vec{v} \in V$ implies $\vec{u} + \vec{v} \in V$ and

(ii) $\vec{u} \in V$ implies $k\vec{u} \in V$ for any scalar k.

(b) (2pts) A transfomration $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** if

(i) $T(\vec{u} + \vec{v}) = T\vec{u} + T\vec{v}$ and

(ii) $T(k\vec{u}) = kT(\vec{u})$ for any scalar k.

(c) (2pts) The **null space** of a matrix M is

$$\{\vec{x} : M\vec{x} = \vec{0}\}.$$

(d) (2pts) The **inverse** of a matrix A is
(Hint: you will get no points if all you write is A^{-1})

a matrix B such that $AB = BA = I$.

(e) (2pts) The **range** (or **image**) of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is

$$\{\vec{x} \in \mathbb{R}^m : \vec{x} = T(\vec{y}) \text{ for some } \vec{y} \in \mathbb{R}^n\}.$$

2 (10pts) Given

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \qquad C^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

(a) (2pts) Compute AB.

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

(b) (2pts) Compute A^{-1}.

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}.$$

(c) (2pts) Compute C^T.

$$C^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ -1 & 4 & -3 \end{bmatrix}.$$

(d) (4pts) Solve the equation $C\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$C\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is solved by

$$\vec{x} = C^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix}.$$

3 (10pts) For each of the following, indicate **true** or **false**. You *do not* need to explain how you arrived at your answer.

Let M be an $n \times n$ matrix. If M is invertible, then we *must* have

(i) $\mathrm{row}(M) = \{\vec{0}\}$

False

(ii) $\mathrm{col}(M) = \mathbb{R}^n$

True

(iii) $\mathrm{null}(M) = \{\vec{0}\}$

True

(iv) $\mathrm{rank}(M) = n$

True

(v) $M = M^T$

False

4 (10pts) Let $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects vectors across the line with direction vector $\vec{e}_2$. Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(a) (4pt) Compute $\mathcal{F} \begin{bmatrix} 7 \\ -3 \end{bmatrix}$ and $\mathcal{P} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\mathcal{F} \begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \end{bmatrix} \text{ and } \mathcal{P} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \tfrac{1}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(b) (6pts) Find a matrix for $\mathcal{F} \circ \mathcal{P}$.

Computing $\mathcal{F} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\mathcal{F} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we see that a matrix for $\mathcal{F}$ is

$$F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Computing $\mathcal{P} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \tfrac{1}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathcal{P} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \tfrac{2}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, we see that a matrix for $\mathcal{P}$ is

$$P = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Since the matrix representing a coposition of linear functions is the same as the product of the matrices representing that linear transformation, we see a matrix for $\mathcal{F} \circ \mathcal{P}$ is

$$FP = \frac{1}{5} \begin{bmatrix} -1 & -2 \\ 2 & 4 \end{bmatrix}.$$

5 (10pts) For each of the following transformations, either prove that the transformation is linear or provide an example that shows it is not linear.

(a) (5pts) $U : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $U \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x+y \end{bmatrix}$.

We will verify that U distributes with respect to vector addition and scalar multiplication. Pick $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$. Then

$$U\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = U \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_1 + y_1 + x_2 + y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_1 + y_2 \end{bmatrix} = U \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + U \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Similarly,

$$U\left(k \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = U \begin{bmatrix} kx_1 \\ kx_2 \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_1 + kx_2 \end{bmatrix} = k \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix} = kU \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

and so U is linear.

(b) (5pts) $V : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $V \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x-1 \end{bmatrix}$.

V is not linear because $V(\vec{0} + \vec{0}) = V(\vec{0}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, but $V(\vec{0}) + V(\vec{0}) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$, and so V does not distribute with respect to vector addition.