1. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

   (a) Show that the null space of T is a subspace of $\mathbb{R}^n$.

   Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. By definition, T is *linear* if

   $$T(\vec{u} + \vec{v}) = T\vec{u} + T\vec{v} \qquad \text{and} \qquad T(k\vec{u}) = kT\vec{u}$$

   for any $\vec{u}, \vec{v} \in \mathbb{R}^n$ and any scalar k. By definition, the *null space* of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set $\mathrm{null}(T) = \{\vec{x} \in \mathbb{R}^n : T\vec{x} = \vec{0}\}$. We will show $\mathrm{null}(T)$ satisfies the two conditions of a subspace, namely that it is closed under vector addition and scalar multiplication.
   Pick $\vec{u}, \vec{v} \in \mathrm{null}(T)$. That means, $T\vec{u} = T\vec{v} = \vec{0}$. Now

   $$T(\vec{u} + \vec{v}) = T\vec{u} + T\vec{v} = \vec{0} + \vec{0} = \vec{0},$$

   with the first equality following from the linearity of T. We conclude $\vec{u}+\vec{v} \in \mathrm{null}(T)$.
   Further, we see that

   $$T(k\vec{u}) = kT\vec{u} = k\vec{0} = \vec{0},$$

   again with the first equality following from linearity. From this we may conclude that $k\vec{u} \in \mathrm{null}(T)$ for any scalar k, and so $\mathrm{null}(T)$ is a subspace.

   (b) Show that the range of T is a subspace of $\mathbb{R}^m$.

   By definition, the *range* of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set $\mathrm{range}(T) = \{\vec{x} \in \mathbb{R}^m : \vec{x} = T\vec{y} \text{ for some } \vec{y} \in \mathbb{R}^n\}$.
   Proceeding in the standard way, pick $\vec{x}, \vec{y} \in \mathrm{range}(T)$. By definition, this means there are vectors $\vec{x}', \vec{y}' \in \mathbb{R}^n$ such that $\vec{x} = T(\vec{x}')$ and $\vec{y} = T(\vec{y}')$.
   By linearity of T, we have

   $$T(\vec{x}' + \vec{y}') = T(\vec{x}') + T(\vec{y}') = \vec{x} + \vec{y}.$$

   Therefore $\vec{x} + \vec{y} \in \mathrm{range}(T)$ since we have found a vector $\vec{w}' = \vec{x}' + \vec{y}'$ so that $\vec{x} + \vec{y} = T(\vec{w}')$.
   Similarly, linearity of T gives us that

   $$T(k\vec{x}') = kT(\vec{x}') = k\vec{x},$$

   and so $k\vec{x} \in \mathrm{range}(T)$ because the vectors $\vec{w}' = k\vec{x}'$ satisfies $k\vec{x} = T(\vec{w}')$.

2. (a) For a 4×3 matrix M, must the column space of M be identical to the column space of $\mathrm{rref}(M)$?

   For a 4×3 matrix, the column space of M and $\mathrm{rref}(M)$ may be different. By definition, the column space of a matrix is the span of the columns. Consider the matrix

   $$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathrm{rref}(M) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Here we have the column space of M is the span of $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ but the column space of $\mathrm{rref}(M)$ is the span of $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, which is a completely different subspace!

(b) For a 3×3 matrix N with $\text{rank}(N) = 3$, must the column space of N be identical to the column space of $\text{rref}(N)$? Can the assumption that $\text{rank}(N) = 3$ be dropped?

The *rank* of a matrix N is the number of ones in its reduced row echelon form. Equivalently, it is the number of linearly independent columns of N.
Let $\vec{c}_1, \vec{c}_2, \vec{c}_3$ be the columns of N. Since N has rank 3, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ is a linearly independent set. Since N is a 3×3 matrix, $\vec{c}_1, \vec{c}_2, \vec{c}_3 \in \mathbb{R}^3$. Therefore $\mathcal{C}$ is a basis for $\mathbb{R}^3$ and so the column space of N is all of $\mathbb{R}^3$.
Alternatively, since the rank of N is 3 and N is a 3×3 matrix,

$$\text{rref}(N) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and so the column space of $\text{rref}(N)$ is the span of the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and so is all of $\mathbb{R}^3$. Thus, the column spaces of N and $\text{rref}(N)$ must be the same.
The assumption that $\text{rank}(N) = 3$ cannot be dropped. If $\text{rank}(N) < 3$, a counter-example like that in part (a) could be constructed.

3. For a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$, we have the following information:

$$L \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad L \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} \qquad L \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Write down a matrix for L.

$L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation, which by definition means $L(\vec{u}+\vec{v}) = L\vec{u}+L\vec{v}$ and $L(k\vec{u}) = kL\vec{u}$. Since L is linear, we know that we can write down a matrix for L.
The easiest way to write down a matrix for L is to computer $L\vec{e}_1, L\vec{e}_2$, and $L\vec{e}_3$ which will give the first, second, and third columns of a matrix for L. We will use linearity to our advantage.
First, notice that

$$\vec{e}_1 = \tfrac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \tfrac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{e}_2 = \tfrac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Using linearity, we now compute

$$L\vec{e}_1 = L\left(\tfrac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \tfrac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) = \tfrac{1}{2} L \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \tfrac{1}{2} L \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \tfrac{1}{2} \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -1 \\ -1 \end{bmatrix}$$

$$L\vec{e}_2 = L\left(\tfrac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) = \tfrac{1}{2} L \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \tfrac{1}{2} L \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 2 \\ 2 \end{bmatrix}$$

$$L\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This means a matrix for L is

$$\begin{bmatrix} \frac{5}{2} & -\frac{1}{2} & 0 \\ -1 & 2 & 0 \\ -1 & 2 & 0 \end{bmatrix}.$$

(b) Describe the range of L as a point, line, plane, or hyperplane and give a basis for the range of L.

From part (a), we can see that the rank of L is 2. Therefore, the range of L will be a two dimensional subspace, making it a plane.

(c) Describe the null space of L as a point, line, plane, or hyperplane and give a basis for the null space of L.

The *rank-nullity* theorem states that $\mathrm{rank}(L) + \mathrm{nullity}(L) = \#$ of columns in a matrix representation of L, or equivalently, the dimension of the target space. In this case, the target space is $\mathbb{R}^3$. Thus $\mathrm{rank}(L) + \mathrm{nullity}(L) = 3$. Since $\mathrm{rank}(L) = 2$, we know $\mathrm{nullity}(L) = 1$ and so the dimension of the null space of L is 1, making it a line.