1. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$. Explain whether the set $A = \{\vec{u}, \vec{v}, \vec{w}\}$ is a basis for $\mathbb{R}^3$. Make sure to include all relevant definitions.

   Recall a *basis* for a subspace $V \subset \mathbb{R}^3$ is a linearly independent set of vectors B such that $\operatorname{span} B = V$. Let

   $$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \qquad \text{and} \qquad \vec{w} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

   We will consider the question of whether $A = \{\vec{u}, \vec{v}, \vec{w}\}$ is a basis for $\mathbb{R}^3$.
   First, we will check to see if A is a linearly independent set. By definition, A is *linearly independent* if the only way to write $\vec{0}$ as a linear combinations of vectors in A is the trivial linear combination (the linear combination where all the coefficients are 0). That is, we would like to know if

   $$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

   has a solution other than $c_1 = c_2 = c_3 = 0$. Rewriting this, we see

   $$\begin{aligned} 0 &= c_1 + 4c_2 + 7c_3 \\ 0 &= 2c_1 + 5c_2 + 8c_3 \\ 0 &= 3c_1 + 6c_2 + 9c_3. \end{aligned}$$

   Solving this system, we deduce that any c_i's satisfying

   $$c_2 = -2c_3 = -2c_1$$

   give a solution. Choosing $c_1 = 1$, we see that $(c_1, c_2, c_3) = (1, -2, 1)$ is a solution. That is,

   $$\vec{0} = \vec{u} - 2\vec{v} + \vec{w}.$$

   Since $\vec{0}$ could be written as a non-trivial linear combination of vectors in A, A is a linearly dependent set and therefore is not a basis for $\mathbb{R}^3$ (or any subspace).

2. Fix $\vec{u}, \vec{v} \in \mathbb{R}^n$. Show that $\operatorname{span}(\operatorname{span}\{\vec{u}, \vec{v}\}) = \operatorname{span}\{\vec{u}, \vec{v}\}$. Make sure to include all relevant definitions.

   The *span* of a set of vectors X is the set of all linear combinations of vectors in X. We will show that for any two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $\operatorname{span}(\operatorname{span}\{\vec{u}, \vec{v}\}) = \operatorname{span}\{\vec{u}, \vec{v}\}$.
   Fix $\vec{u}, \vec{v} \in \mathbb{R}^n$. Let $S_1 = \operatorname{span}\{\vec{u}, \vec{v}\}$. By definition,

   $$S_1 = \{\vec{x} \in \mathbb{R}^n : \vec{x} = \alpha\vec{u} + \beta\vec{v} \text{ for some } \alpha, \beta \in \mathbb{R}\}.$$

   Let $\vec{w} \in \operatorname{span} S_1$. Again, by definition,

   $$\vec{w} = \sum_{i=1}^{n} \gamma_i \vec{s}_i$$

   for some $s_i \in S_1$ and $\gamma_i \in \mathbb{R}$. Expanding based on the definition of S_1, we see

   $$\vec{w} = \sum_{i=1}^{n} \gamma_i(\alpha_i\vec{u} + \beta_i\vec{v}) = \left(\sum_{i=1}^{n} \gamma_i\alpha_i\right)\vec{u} + \left(\sum_{i=1}^{n} \gamma_i\beta_i\right)\vec{v}$$

   for some $\alpha_i, \beta_i \in \mathbb{R}$. However, the right hand sides expresses $\vec{w}$ as a linear combination of $\vec{u}$ and $\vec{v}$. Thus, $\vec{w} \in S_1$. This shows $\operatorname{span} S_1 \subseteq S_1$. Since $S_1 \subseteq \operatorname{span} S_1$ is clear, we conclude $S_1 = \operatorname{span} S_1$.

3. The worksheets define $\text{proj}_{\vec{v}}\vec{u}$ as the vector in the direction $\vec{v}$ such that $\vec{u}-\text{proj}_{\vec{v}}\vec{u}$ is orthogonal to $\vec{v}$. Call this definition (a). Your textbook defined $\text{proj}_{\vec{v}}\vec{u}$ as the vector $\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$. Call this definition (b). Show that definitions (a) and (b) are equivalent by showing that that the vector arising from definition (b) must be the same as the vector arising from definition (a). In your answer, elaborate on definition (a) by including the definition of *vector in the direction of* $\vec{v}$ and *orthogonal*.

Recall that $\vec{u}$ is *in the same direction* as $\vec{v} \neq \vec{0}$ if $\vec{u} = t\vec{v}$ for some scalar t. Further, $\vec{u}$ is *orthogonal* to $\vec{v}$ if $\vec{u}\cdot\vec{v} = 0$.
Define
$$\text{proj}_{1,\vec{v}}\vec{u}$$
to be the vector in the direction $\vec{v} \neq \vec{0}$ such that $\vec{u} - \text{proj}_{1,\vec{v}}\vec{u}$ is orthogonal to $\vec{v}$.
Define
$$\text{proj}_{2,\vec{v}}\vec{u} = \tfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v},$$
provided $\vec{v} \neq \vec{0}$.
We aim to show that $\text{proj}_{2,\vec{v}}\vec{u} = \text{proj}_{1,\vec{v}}\vec{u}$. Computing,
$$\vec{v}\cdot(\vec{u} - \text{proj}_{2,\vec{v}}\vec{u}) = \vec{v}\cdot\left(\vec{u} - \tfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\right) = \vec{v}\cdot\vec{u} - \tfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}(\vec{v}\cdot\vec{v}) = 0,$$
and so $\vec{u} - \text{proj}_{2,\vec{v}}\vec{u}$ is orthogonal to $\vec{v}$. Since $\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}$ is a scalar, $\text{proj}_{2,\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$ is a vector in the direction of $\vec{v}$. Therefore, $\text{proj}_{2,\vec{v}}\vec{u}$ satisfies the properties of $\text{proj}_{1,\vec{v}}\vec{u}$. Since $\text{proj}_{1,\vec{v}}\vec{u}$ is unique provided $\vec{v} \neq 0$, we conclude
$$\text{proj}_{1,\vec{v}}\vec{u} = \text{proj}_{2,\vec{v}}\vec{u}.$$