6.

(a) (5pts) Give an example to show that if V and W are subspaces of $\mathbb{R}^n$, the set $V \cup W$ need not be a subspace.

Let $V \subseteq \mathbb{R}^2$ be $\text{span}\{\vec{e}_1\}$ and $W \subseteq \mathbb{R}^2$ be $\text{span}\{\vec{e}_2\}$. Since V and W are both spans, they are subspaces.

Let $X = V \cup W$. The vector $\vec{e}_1, \vec{e}_2 \in V \cup W$ since $\vec{e}_1 \in V$ and $\vec{e}_2 \in W$. However, $\vec{e}_1 + \vec{e}_2$ is neither in V nor W since it cannot be written as a linear combination of $\vec{e}_1$ nor as a linear combination of $\vec{e}_2$. Thus X is not closed under addition and so cannot be a subspace.

(b) (5pts) Let $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove that the range of $\mathcal{T}$ is a subspace.

By definition, range$(T) = \{\vec{y} : \vec{y} = T\vec{x}$ for some $\vec{x} \in \mathbb{R}^n\}$. Let $\vec{a}, \vec{b} \in$range(T). By definition there exists $\vec{a}', \vec{b}' \in \mathbb{R}^n$ so that $\vec{a} = T(\vec{a}')$ and $\vec{b} = T(\vec{b}')$. Now,

$$\vec{a} + \vec{b} = T(\vec{a}') + T(\vec{b}') = T(\vec{a}' + \vec{b}')$$

with the last equality following from the linearity of T. Thus, $\vec{a} + \vec{b} \in$rage(T).

Similarly,

$$k\vec{a} = kT(\vec{a}') = T(k\vec{a}')$$

with the last equality following from the linearity of T. Thus, $k\vec{a} \in$rage(T), and so range(T) is a subspace.

7. Let $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects vectors across the line $\text{span}\{\vec{\mathbf{e}}_1\}$, let $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation counter-clockwise by $90°$, and let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto the vector $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

(a) (6pts) Find matrices F, R, and P corresponding to $\mathcal{F}$, $\mathcal{R}$, and $\mathcal{P}$.

For F, note that $\mathcal{F}$ takes $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Thus $F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

For R, $\mathcal{R}$ takes $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, respectively. Thus $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

For P, we'll be a little more creative. $\mathcal{P}$ takes $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ to itself and $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (since $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is perpendicular to the vector we're projecting onto). Thus if $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we should have $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. From this, we get four equations:

$$\begin{aligned} 3a + 2b &= 3 \\ 3c + 2d &= 2 \\ 2a - 3b &= 0 \\ 2c - 3d &= 0 \end{aligned}$$

Adding three times the first equation to twice the third equation gives $13a = 9$, and thus $a = \frac{9}{13}$; plugging this into either of those equations then gives $b = \frac{6}{13}$. Similarly, the second and fourth equations give $13c = 6$, so $c = \frac{6}{13}$ and $d = \frac{4}{13}$. Thus $P = \begin{bmatrix} \frac{9}{13} & \frac{6}{13} \\ \frac{6}{13} & \frac{4}{13} \end{bmatrix}$.

(b) (6pts) Classify the transformations $\mathcal{F}$, $\mathcal{R}$, and $\mathcal{P}$ as invertible or not. Explain your reasoning.

The transformation $\mathcal{F}$ is invertible, since $\mathcal{F}$ is its own inverse; reflecting a second time across $\text{span}\{\vec{e}_1\}$ reverses the effect of $\mathcal{F}$.

The transformation $\mathcal{R}$ is also invertible; its inverse is a clockwise rotation by $90°$ around the origin.

The transformation $\mathcal{P}$ is not invertible. It is not one-to-one, since $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ both map to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. A transformation that is not one-to-one cannot be invertible.

(c) (4pts) Let M be the matrix corresponding to $\mathcal{F} \circ \mathcal{P}$. Explain two ways to compute M. Clearly label your methods "Method 1" and "Method 2." There is no need to actually compute M.

Method 1:

As we know, composition of transformations corresponds to matrix multiplication. Thus $M = FP$, using the matrices we computed before.

Method 2:

We could also compute M by finding the effect of $\mathcal{F} \circ \mathcal{P}$ on two chosen vectors (as long as the chosen vectors are linearly indenpendent). For example, $\mathcal{F} \circ \mathcal{P}$ takes $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. As in the method for computing P, let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; this will give four equations with a, b, c, d as the variables. We can solve this system and thus obtain M.

(d) (4pts) A linear transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ is said to have *finite order* if $\mathcal{T}^k = \mathrm{id}$ for some $k \geq 1$. (Here, $\mathcal{T}^k = \mathcal{T} \circ \mathcal{T} \circ \cdots \circ \mathcal{T}$ repeated k times, and id is the identity function.) Out of $\mathcal{F}$, $\mathcal{R}$, and $\mathcal{P}$, which transformations have finite order? Explain.

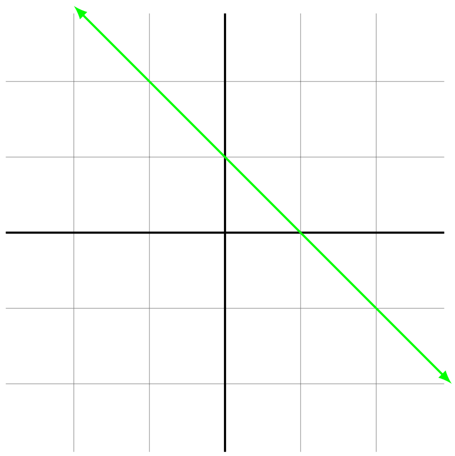
$\mathcal{F}$ and $\mathcal{R}$ have finite order, since $\mathcal{F}^2 = \mathcal{R}^4 = \mathrm{id}$. However, $\mathcal{P}$ does not have finite order. A transformation $\mathcal{T}$ that has finite order k is also invertible, since then $\mathcal{T}^{k-1}$ is the inverse of $\mathcal{T}$. But $\mathcal{P}$ is not invertible, so it cannot have finite order.

(e) (3pts) Give an example of a transformation that is invertible, but does not have finite order.

Let $\mathcal{T}$ be the transformation that multiplies the first component (or x-coordinate) of an input vector by 2. This is a linear transformation, since it corresponds to the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Its inverse is the transformation corresponding to the matrix $\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$, which divides the first component of an input vector by 2. However, $\mathcal{T}^k$ has the effect of multiplying the first component of input vectors by 2^k, and this will never be the identity function for $k > 0$.

8. Let $X = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x + y = 1 \right\}$ and $Y = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x - 2y = 0 \right\}$.

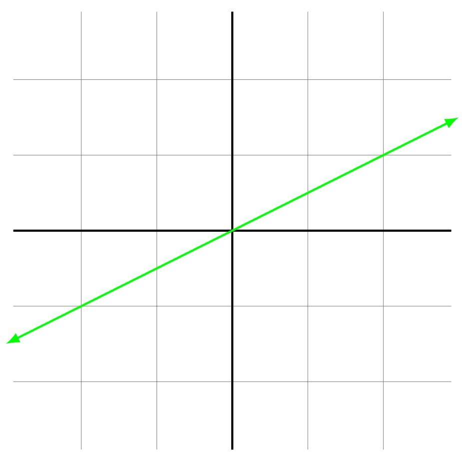
(a) (2pts) Draw X.



(b) (3pts) Is X a subspace of $\mathbb{R}^2$? Prove your answer directly from the definition of a subspace.

X is not a subspace. The vector $\vec{e}_1$ is in X, but $0\vec{e}_1 = \vec{0} \notin X$, so X fails to be closed under scalar multiplication.

(c) (2pts) Draw Y.



(d) (3pts) Is Y a subspace of $\mathbb{R}^2$? Prove your answer directly from the definition of a subspace.

Y is a subspace. Let $A = \begin{bmatrix} 1 & -2 \end{bmatrix}$. Now $\vec{x} \in Y$ if and only if $A\vec{x} = \vec{0}$.

Consider $\vec{x}, \vec{y} \in Y$. We now have

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0},$$

and so $(\vec{x} + \vec{y}) \in Y$. Similarly,

$$A(k\vec{x}) = kA\vec{x} = k\vec{0} = \vec{0},$$

and so $k\vec{x} \in Y$. Since $Y \subseteq \mathbb{R}^2$ satisfies both conditions for being a subspace, it is a subspace of $\mathbb{R}^2$.

9. Suppose $\{\vec{u}, \vec{v}, \vec{w}\}$ is a basis for $\mathbb{R}^3$.

(a) (5pts) Is $\{\vec{u}+\vec{v}, \vec{u}+\vec{w}, \vec{v}+\vec{w}\}$ a basis for $\mathbb{R}^3$? Explain.

Yet. Let $V = [\vec{u}|\vec{v}|\vec{w}]$ be the matrix whose columns are $\vec{u}$, $\vec{v}$, and $\vec{w}$, let $W = [\vec{u}+\vec{v}|\vec{u}+\vec{w}|\vec{v}+\vec{w}]$ be the matrix whose columns are $\vec{u}+\vec{v}$, $\vec{u}+\vec{w}$, and $\vec{v}+\vec{w}$, and let $X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. By the definition of matrix multiplication we see

$$W = XV.$$

Notice that X is invertible and since the columns of V are linearly independent and V is a square matrix V is invertible. Therefore, W, as the product of two invertible matrices is invertible.

By the invertible matrix theorem, the columns of W are linearly independent and the column space of W is $\mathbb{R}^3$. Thus the columns of W (that is, the vectors $\vec{u}+\vec{v}$, $\vec{u}+\vec{w}$, and $\vec{v}+\vec{w}$) form a basis for $\mathbb{R}^3$.

(b) (5pts) Suppose A is a 3-by-3 matrix, and $\{A\vec{u}, A\vec{v}, A\vec{w}\}$ is a linearly independent set. Show that A is invertible. (You may reference the Invertible Matrix Theorem.)

Let $V = [\vec{u}|\vec{v}|\vec{w}]$ be the matrix whose columns are $\vec{u}$, $\vec{v}$, and $\vec{w}$, and let $W = [A\vec{u}|A\vec{v}|A\vec{w}]$ be the matrix whose columns are $A\vec{u}$, $A\vec{v}$, and $A\vec{w}$.

From the definition of matrix multiplication, $W = AV$. By the invertible matrix theorem, V^{-1} exists and so $A = WV^{-1}$. Further, by the invertible matrix theorem, W^{-1} exists and so

$$B = (WV^{-1})^{-1} = VW^{-1}$$

is a well defined matrix. Computing, we see that

$$AB = A(VW^{-1}) = WV^{-1}(VW^{-1}) = I$$

and

$$BA = VW^{-1}A = VW^{-1}(WV^{-1}) = I,$$

and so $B = A^{-1}$. In particular, A is invertible.