*We are nearly experts at typing by now. Though you are not required to type your homework, it's strongly encouraged. You can even download the* `tex` *file for this homework and type your answers below each problem. Using the* `\begin{quote}` *and* `\end{quote}` *environment will indent anything you type inbetween. Perfect for typing answers!*

1. Let $A$ and $B$ be $n \times n$ invertible matrices and let $X = AB$. Does $X^{-1} = A^{-1}B^{-1}$ or does $X^{-1} = B^{-1}A^{-1}$ or neither? Explain.

   Observe that $B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$. Further observe that $ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$. By definition, $B^{-1}A^{-1}$ must be the inverse of $AB$. Therefore, $X^{-1} = B^{-1}A^{-1}$.

2. For each of the following sets, determine whether or not it is a subspace. Explain your answer.

   (a) $A = \left\{ \vec{x} : \vec{x} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 0 \right\} \subseteq \mathbb{R}^2$

   Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ and let $\alpha \in \mathbb{R}$ be arbitrary. Then,

   - If $\vec{u} \in A$, then

   $$\vec{u} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 0 \implies (\alpha \vec{u}) \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \alpha \left( \vec{u} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) = \alpha \cdot 0 = 0 \implies \alpha \vec{u} \in A.$$

   - If $\vec{u}, \vec{v} \in A$, then

   $$\vec{u} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \vec{v} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 0.$$

   Hence

   $$(\vec{u} + \vec{v}) \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \vec{u} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \vec{v} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 0 + 0 = 0 \implies \vec{u} + \vec{v} \in A.$$

   By definition, $A$ is a subspace of $\mathbb{R}^2$.

   (b) $B \subseteq \mathbb{R}^3$ is the $x$-axis.

   By similar arguments to part (a), we have that $B$ is a subspace of $\mathbb{R}^3$.

   (c) $C \subseteq \mathbb{R}^3$ is the plane given in vector form as $\vec{x} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$.

   We begin with row reduction

   $$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & -24 \end{bmatrix}.$$

The above matrix has 3 pivot columns, hence its columns are linearly independent. Therefore,

$$\begin{bmatrix}0\\1\\5\end{bmatrix} \notin \operatorname{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-2\\3\\-1\end{bmatrix}\right\}.$$

Thus the equation

$$t\begin{bmatrix}1\\1\\1\end{bmatrix} + s\begin{bmatrix}-2\\3\\1\end{bmatrix} + \begin{bmatrix}0\\1\\5\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

has no solution. Then $\vec{0} \notin C$ so $C$ is not a subspace of $\mathbb{R}^3$.

(d) $D \subseteq \mathbb{R}^3$ is the plane with normal vector $\vec{n} = \begin{bmatrix}1\\1\\-1\end{bmatrix}$ passing through the point $\begin{bmatrix}2\\2\\4\end{bmatrix}$.

$D$ is a plane with the equation $(x-2)+(y-2)-(z-4)=0$, so $D$ is equivalently represented by a plane with the equation $x+y-z=0$. Therefore, $D$ is a plane passing through the origin, and is thus a subspace of $\mathbb{R}^3$.

(e) $E = \{(x,y) : y = 3x+4\} \subseteq \mathbb{R}^2$.

$E$ is not a subspace of $\mathbb{R}^2$ because $\vec{0} \notin E$.

(f) $F = \operatorname{span}\{\vec{u}_1, \vec{u}_2\} \subseteq \mathbb{R}^2$ where $\vec{u}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix}2\\0\end{bmatrix}$.

A span of any collection of vectors in $\mathbb{R}^2$ is always a subspace of $\mathbb{R}^2$. Hence $F$ is a subspace of $\mathbb{R}^2$.

(g) $G \subseteq \mathbb{R}^4$ is the set of all solutions to the matrix equation $\begin{bmatrix}1 & 1 & 0 & -2\\0 & 3 & 0 & 0\\0 & 1 & 0 & 0\end{bmatrix}\vec{x} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

$G$ is the null space of the matrix

$$\begin{bmatrix}1 & 1 & 0 & -2\\0 & 3 & 0 & 0\\0 & 1 & 0 & 0\end{bmatrix}.$$

All null spaces are subspaces, and therefore $G$ is a subspace of $\mathbb{R}^4$.

(h) $H = \{(x,y) : xy = 0\} \subseteq \mathbb{R}^2$.

Observe that

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \in H \text{ but } \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \notin H.$$

Thus, $H$ is not a subspace of $R^2$.

3. For every set in problem 2 that is a vector space, find a basis.

(a) Let $[x, y] \in A$. Then

$$\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 0 \iff 2x + 3y = 0 \iff y = -\frac{2}{3}x.$$

Therefore,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -2/3x \end{bmatrix} = x \begin{bmatrix} 1 \\ -2/3 \end{bmatrix} \implies A = \text{span}\left\{\begin{bmatrix} 1 \\ -2/3 \end{bmatrix}\right\} \implies \left\{\begin{bmatrix} 1 \\ -2/3 \end{bmatrix}\right\}$$

is a basis for $A$.

(b) Observe that

$$B = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\} \implies \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

is a basis for $B$.

(d) $D$ is described by the plane with equation $x + y - z = 0$. Therefore, $\dim(D) = 2$, so we need to find two linearly independent vectors on $D$ to form a basis for $D$. Pick

$$\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$$

as a basis for $D$.

(f) Observe that

$$F = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \implies \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

is a basis for $F$.

(g) We begin by row reduction on an augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, we have a system of equations

$$\begin{cases} x_1 - 2x_4 & = 0 \\ x_2 = 0 \end{cases} \iff \begin{cases} x_1 & = 2x_4 \\ x_2 & = 0 \end{cases}.$$

Thus, the we have that $G$ is represented in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

with $x_3, x_4 \in \mathbb{R}$. Therefore, a basis for $G$ is the set

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

4. For every set in problem 2 that is a vector space, find its dimension.

   We have that

$$\begin{aligned} \dim A &= 1 \\ \dim A &= 1 \\ \dim D &= 2 \\ \dim F &= 1 \\ \dim G &= 2 \end{aligned}$$

5. Let $\mathcal{P}$ be the plane in $\mathbb{R}^3$ given in vector form by $\vec{x} = t\vec{d_1} + s\vec{d_2} + \vec{p}$ for unknown vectors $\vec{d_1}, \vec{d_2}, \vec{p}$.

   (a) Show that if $\vec{p} = \vec{0}$, then $\mathcal{P}$ is a subspace.

   If $\vec{p} = \vec{0}$, then $\mathcal{P} = \{\vec{x} = t\vec{d_1} + s\vec{d_2} | t, s \in \mathbb{R}\} = \text{span}\{\vec{d_1}, \vec{d_2}\}$. As we know the span of any collection of vectors in $\mathbb{R}^3$ is a subspace, $\mathcal{P}$ is a subspace of $\mathbb{R}^3$.

   (b) What if $\vec{p} \neq \vec{0}$? Could $\vec{p}$ still be a subspace? Give conditions on $\vec{p}$ that determine whether or not $\mathcal{P}$ is a subspace. That is, give a condition so that if it is true, $\mathcal{P}$ is a subspace and if it is false, $\mathcal{P}$ is not a subspace.

   - If $\mathcal{P}$ is a subspace of $R^3$, then $\vec{0} \in \mathcal{P}$. Therefore, there exist some $t, s \in \mathbb{R}$ so that $t\vec{d_1} + s\vec{d_2} + \vec{p} = \vec{0}$. Thus, $\vec{p} = -t\vec{d_1} - s\vec{d_2}$, so $\vec{p} \in \text{span}\{\vec{d_1}, \vec{d_2}\}$.
   - On the other hand, if $p \in \text{span}\{\vec{d_1}, \vec{d_2}\}$, then there exists some $t_0, s_0 \in \mathbb{R}$ so that $\vec{p} = t_0\vec{d_1} + s_0\vec{d_2}$. Therefore, $\vec{x} = t\vec{d_1} + s\vec{d_2} + \vec{p} = (t+t_0)\vec{d_1} + (s+s_0)\vec{d_2}$. Therefore, $\mathcal{P} = \{\vec{x} = (t+t_0)\vec{d_1} + (s+s_0)\vec{d_2} | t, s \in \mathbb{R}\} = \text{span}\{\vec{d_1}, \vec{d_2}\}$. Therefore, $\mathcal{P}$ is a subspace of $\mathbb{R}^3$.

   In conclusion, $\mathcal{P}$ is a subspace of $\mathbb{R}^3$ if and only if $\vec{p} \in \text{span}\{\vec{d_1}, \vec{d_2}\}$.

6. Let $\mathcal{V}$ be the subspace spanned by $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} -5 \\ 4 \\ 9 \end{bmatrix}$.

   (a) Find a basis for $\mathcal{V}$ and call your basis vectors $\vec{b}_1, \vec{b}_2$, etc.

Let $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{bmatrix}$. We row reduce $V$.

$$\begin{bmatrix} 1 & 1 & 4 & 2 & -5 \\ 2 & 1 & 0 & 3 & 4 \\ 1 & 0 & -4 & 1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 & 2 & -5 \\ 0 & -1 & -8 & -1 & 14 \\ 0 & -1 & -8 & -1 & 14 \end{bmatrix}.$$

Continuing gives

$$\sim \begin{bmatrix} 1 & 1 & 4 & 2 & -5 \\ 0 & 1 & 8 & 1 & -14 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 1 & 9 \\ 0 & 1 & 8 & 1 & -14 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$V$ only has two pivot columns, columns 1 and 2. Therefore, a basis for $\mathcal{V}$ is a basis for $\text{col}(V)$. Thus, a basis is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(b) Describe $\mathcal{V}$ geometrically.

$\mathcal{V}$ is spanned by two vectors in $\mathbb{R}^3$. Hence $\mathcal{V}$ is a plane going through the origin.

(c) Let $V = [\vec{v}_1|\vec{v}_2|\vec{v}_3|\vec{v}_4|\vec{v}_5]$ be the matrix whose columns are the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$, let $B = [\vec{b}_1|\vec{b}_2|\cdots]$ be the matrix whose columns are your basis vectors from part (a), and let $\vec{v} \in \mathcal{V}$.
Without computing, how many solutions does the equation $V\vec{x} = \vec{v}$ have? How about $B\vec{x} = \vec{v}$?

Assume that

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

- Then

$$V\vec{x} = \vec{v} \iff \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \vec{v}.$$

  Therefore, $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 + x_5\vec{v}_5 = \vec{v}$. Because $\vec{v} \in \mathcal{V} = \text{col}(V)$, $\vec{v}$ can be written as a linear combination of $\vec{v}_1, \cdots, \vec{v}_5$. Hence, the equation $V\vec{x} = \vec{v}$ has a solution.

  Since $V$ has 5 columns and only 3 rows, there must be free variables in the equation $V\vec{x} = \vec{v}$. This implies that $V\vec{x} = \vec{v}$ has infinitely many solutions.

- Note that $\text{col}(B) = \text{col}(V)$, so $\vec{v} \in \text{col}(B)$. By the same argument as above, the equation $B\vec{x} = \vec{v}$ has infinitely many solutions.

7. Suppose $A$ is an invertible matrix and $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n$ are its columns. Is $\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ a basis? Describe $\text{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$. Explain your reasoning.

   By the Invertible Matrix Theorem, since $A$ is invertible, its columns $\vec{c}_1, \cdots, \vec{c}_n$ are linearly independent. Hence $\{\vec{c}_1, \cdots, \vec{c}_n\}$ is a basis.

   Further, because $A$ is an invertible matrix, it is a square matrix. So, it must have $n$ columns and $n$ rows, so $\vec{c}_1, \cdots, \vec{c}_n \in \mathbb{R}^n$. Because $\{\vec{c}_1, \cdots, \vec{c}_n\}$ is a set of linearly independent vectors, $\text{span}\{\vec{c}_1, \cdots, \vec{c}_n\} = \mathbb{R}^n$.