1. $\hat{u} = \frac{1}{\sqrt{2^2+1^2}}\langle 2,-1\rangle = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right\rangle$

$\hat{v} = \frac{1}{\sqrt{1^2+3^2}}\langle 1,3\rangle = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right\rangle$

$\hat{w} = \frac{1}{\sqrt{1^2+1^2+1^2}}\langle -1,-1,-1\rangle = \left\langle \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right\rangle$

$\vec{r} = \left\langle \frac{1}{\sqrt{10} - \frac{1}{\sqrt{3}}, \frac{3}{\sqrt{10}}} - \frac{1}{\sqrt{3}, \frac{-1}{\sqrt{3}}} \right\rangle = \left\langle \frac{\sqrt{3}-\sqrt{10}}{\sqrt{30}}, \frac{3\sqrt{3}-\sqrt{10}}{\sqrt{30}}, \frac{-\sqrt{10}}{\sqrt{30}} \right\rangle.$

Hence $\left|\vec{r}^2\right| = \frac{3+10-2\sqrt{30}+27+10-6\sqrt{30}+10}{30} = \frac{60-8\sqrt{30}}{30} = \frac{30-4\sqrt{30}}{15}$. Thus,

$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \sqrt{\frac{30-4\sqrt{30}}{15}} \left\langle \frac{\sqrt{3}-\sqrt{10}}{\sqrt{30}}, \frac{3\sqrt{3}-\sqrt{10}}{\sqrt{30}}, \frac{-\sqrt{10}}{\sqrt{30}} \right\rangle$

2. (a) $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^n a_i b_i.$

Assume $\vec{a} = \langle a_1, a_2, ..., a_n\rangle, \vec{b} = \langle b_1, b_2, ..., b_n\rangle$ and $\vec{c} = \langle c_1, c_2, ..., c_n\rangle$. Then,

(b) $\vec{a}.(\alpha\vec{b}) = \langle a_1, a_2, ..., a_n\rangle \cdot \langle \alpha b_1, \alpha b_2, ..., \alpha b_n\rangle$
$= \sum_{i=1}^n a_i \alpha b_i = \alpha \sum_{i=1}^n a_i bi = \alpha\left(\langle a_1, a_2, ..., a_n\rangle \cdot \langle b_1, b_2, ..., b_n\rangle\right) = \alpha\left(\vec{a}\cdot\vec{b}\right)$

(c) $\vec{a}.\left(\vec{b}+\vec{c}\right) = \langle a_1, a_2, ..., a_n\rangle \cdot \langle b_1+c_1, b_2+c_2, ..., b_n+c_n\rangle$
$= \sum_{i=1}^n a_i\left(b_i+c_i\right) = \sum_{i=1}^n a_i b_i + \sum_{i=1}^n a_i c_i$
$= \langle a_1, a_2, ..., a_n\rangle \cdot \langle b_1, b_2, ..., b_n\rangle + \langle a_1, a_2, ..., a_n\rangle \cdot \langle c_1, c_2, ..., c_n\rangle = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

3. (a) Suppose $\vec{p} = \langle x_p, y_p, z_p\rangle$ where $x_p, y_p, z_p \in \mathbb{R}$. Since $\vec{p}$ is a point in $\mathcal{P}$, x_p, y_p and z_p satisfy equation:

$$3x_p - 2y_p + z_p = 4$$

Hence $\vec{p}\cdot\langle 3,-2,1\rangle = \langle x_p, y_p, z_p\rangle \cdot \langle 3,-2,1\rangle = 3x_p - 2y_p + z_p = 4$.

(b) Plane $\mathcal{Q}$ is a translation of plane $\mathcal{P}$ in y-direction 1 unit. Hence it has equation $3x - 2(y-1) + z = 4$, or equivalently

$$3x - 2y + z = 2$$

Similar to part(a), $\vec{q}\cdot\langle 3,-2,1\rangle = 2$.

4. (a) Write $a\begin{bmatrix}3\\1\end{bmatrix} + b\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}107\\64\end{bmatrix}$ $(a, b \in \mathbb{R})$.

We do row reduction to augmented matrix to find solution for above equation:

$$\begin{bmatrix} 3 & 1 & 107 \\ 1 & 2 & 64 \end{bmatrix} \xrightarrow{(1)-3(2)} \begin{bmatrix} 0 & -5 & -85 \\ 1 & 2 & 64 \end{bmatrix} \xrightarrow{\frac{-1}{5}(1)} \begin{bmatrix} 0 & 1 & 17 \\ 1 & 2 & 64 \end{bmatrix} \xrightarrow{(2)-2(1)} \begin{bmatrix} 0 & 1 & 17 \\ 1 & 0 & 30 \end{bmatrix}$$

.

Hence $a = 30, b = 17$.

(b) We want to solve for $a, b \in \mathbb{R}$ in the equation $a\begin{bmatrix}3\\1\end{bmatrix} + b\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-99\\-100\end{bmatrix}$.

We do row reduction to augmented matrix to find solution for above equation:

$$\begin{bmatrix} 3 & 1 & -99 \\ 1 & 2 & -100 \end{bmatrix} \xrightarrow{(1)-3(2)} \begin{bmatrix} 0 & -5 & 201 \\ 1 & 2 & -100 \end{bmatrix} \xrightarrow{\frac{-1}{5}(1)} \begin{bmatrix} 0 & 1 & \frac{-201}{5} \\ 1 & 2 & -100 \end{bmatrix} \xrightarrow{(2)-2(1)} \begin{bmatrix} 0 & 1 & \frac{-201}{5} \\ 1 & 0 & \frac{-98}{5} \end{bmatrix}$$

Thus $a = \frac{-98}{5}$ and $b = \frac{-201}{5}$.

(c) Let $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, and $\vec{c} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$. A linear combination of $\vec{u}, \vec{v}, \vec{w}$ has the form $a\vec{u} + b\vec{v} + c\vec{w} = \begin{bmatrix} a+2b \\ 0 \\ a-c \end{bmatrix}$. The resulting vector has the form $\begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$ where $x, z \in \mathbb{R}$.

In fact any vector of that form is a linear combination of $\vec{u}, \vec{v}$ and $\vec{w}$. It is because for any $x, z \in \mathbb{R}$, the system of equation

$$\begin{cases} a + 2b = x \\ a - c = z \end{cases}$$

always has solution $\begin{cases} a = 0 \\ b = \frac{x}{2} \\ c = -z \end{cases}$.