1. Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} d \\ 1 \\ 1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ be vectors where $d, w_1, w_2, w_3 \in \mathbb{R}$ are unknown constants.

   (a) For what values of d is $\{\vec{a}, \vec{b}, \vec{c}\}$ linearly independent? For which values of d is $\{\vec{a}, \vec{b}, \vec{c}\}$ linearly dependent?

   (b) Write down the system of equations coming from the rows of the vector equation
   $$x\vec{a} + y\vec{b} + z\vec{c} = \vec{w}.$$

   (c) Give three numeric examples of different vectors $\vec{w}$ such that the above system is consistent no matter what d is. Explain.

   (d) Give a numeric example of a vector $\vec{w}$ such that the above system is only consistent for some d. Explain.

2. (a) Use an augmented matrix to solve
   $$\begin{aligned} x + y &= 7 \\ 2x - 3y &= 13. \end{aligned}$$
   Are there any values you could replace the right hand side of the equations with such that there would be no solution? Explain both *geometrically* (using vectors, span, etc.) and *algebraically* (using systems, consistency, etc.) using technical linear algebra terms.

   (b) Consider the system given by the augmented matrix
   $$C = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right].$$
   and call the variables in this system x_1, x_2, x_3, x_4, x_5. Write all solutions to this system in vector form. How many free variables are there?

   (c) There are 10 ways to pick two things from the set $\{x_1, x_2, x_3, x_4, x_5\}$. For each of the ten ways, determine whether that pair is a valid choice of free variables for C.

   (d) Write down all solutions to the homogeneous system corresponding to C (i.e., when the right-hand side is replaced with all zeros). How does this set of solutions compare to the set of solutions of C?

3. Let $M = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

   (a) Find solutions $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ to the three matrix equations
   $$M\vec{v}_1 = \vec{e}_1 \qquad M\vec{v}_2 = \vec{e}_2 \qquad M\vec{v}_3 = \vec{e}_3.$$

   (b) Compute $M(a\vec{v}_1)$ and $M(a\vec{v}_1 + b\vec{v}_2)$ (where $\vec{v}_i$ are from above) where $a, b \in \mathbb{R}$ are unknown scalars. Was what happened a surprise?

   (c) Express the solution to the matrix equation $M\vec{x} = \vec{w}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. (Recall, $\vec{w}$ is defined at the beginning of the problem.)

   (d) Let $V = [\vec{v}_1 | \vec{v}_2 | \vec{v}_3]$ be the matrix whose columns are $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. Compute the matrix product MV. Explain why you got the result you did.

(e) Can you use V to help you solve the system

$$M\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}?$$

If so, explain how and do so.

(f) Compute the matrix product VM. Can you explain why you got what you did? (*Hint: you might have to think about linear transformations for this one.*)

4. Consider the transformations $A : \mathbb{R}^2 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^2$ given by the formulas

$$A(x, y) = (x - y, x + y) \qquad \text{and} \qquad B(x, y) = (x^2, y^2).$$

(a) Compute $A(\vec{e}_1)$, $A(\vec{e}_2)$, $A(\vec{e}_1 + \vec{e}_2)$, $B(\vec{e}_1)$, $B(\vec{e}_2)$, and $B(\vec{e}_1 + \vec{e}_2)$.

(b) Find a matrix M_A so that A is given by matrix multiplication, or explain why it is impossible.

(c) Find a matrix M_B so that B is given by matrix multiplication, or explain why it is impossible.

(d) A function $X : \mathbb{R}^n \to \mathbb{R}^m$ is called *linear* if it satisfies:

i. $X(\alpha\vec{v}) = \alpha X(\vec{v})$ for all $\vec{v} \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$ **and**

ii. $X(\vec{v} + \vec{w}) = X(\vec{v}) + X(\vec{w})$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$.

For each of A and B, determine whether or not it is a linear function. Prove your answers.

5. (a) Let $\vec{v} \in \mathbb{R}^n$ and define the function $d_{\vec{v}} : \mathbb{R}^n \to \mathbb{R}$ by $d_{\vec{v}}(\vec{w}) = \vec{v} \cdot \vec{w}$. Prove that $d_{\vec{v}}$ is linear. (*Hint: you should be having flashbacks to homework 1.*)

(b) For a 2×2 matrix M, let $f_M : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f_M(\vec{v}) = M\vec{v}$, where $\vec{v}$ is a column vector. Prove that f_M is a linear transformation.

(c) Make a conjecture about functions that can be computed using matrix multiplication and their linearity.