These problems will not be turned in nor graded. However, a non-empty subset of these problems will appear on the final skills check/takehome. The numbers may change when they appear on the exam, but it behooves you to have a thorough understanding of every problem on this homework.

1. Suppose the matrix equation $A\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$ has the general solution

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

   (a) How many rows and how many columns does A have?

      Based on the size of the input and output vectors, A must be a 3×3 matrix

   (b) Find null(A).

      The complete solution takes the form of $\vec{a} + \text{null}(A)$ where $\vec{a}$ is a particular solution. Therefore $\text{null}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

   (c) Find rank(A).

      Since the nullity of A is 2 and A is a 3×3 matrix, the rank of A must be 1.

   (d) Find col(A).

      Since the rank of A is 1, the dimension of the column space is 1. Since $\begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$ is contained in the column space, the column space must be $\text{span}\left\{ \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix} \right\}$.

   (e) Find row(A).

      Again, by the rank-nullity theorem, the row space of A is one dimensional. The row space is orthogonal to the null space. By inspection we see that $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is orthogonal to the null space. Thus, the row space is $\text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

2. Let

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{b}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{b}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

   and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and $T(\vec{b}_1) = 2\vec{b}_1$, $T(\vec{b}_2) = 3\vec{b}_2$, and $T(\vec{b}_3) = -\vec{b}_3$.

   (a) Compute $[\vec{c}]_{\mathcal{B}}$.

      Since $\vec{c} = 2\vec{b}_1 + \vec{b}_3$, $[\vec{c}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

   (b) Compute $[T\vec{c}]_{\mathcal{B}}$ and $[T\vec{c}]_{\mathcal{S}}$.

We have $T\vec{c} = T(2\vec{b_1}) + T(\vec{b_3}) = 4\vec{b_1} - \vec{b_3}$, so $[T\vec{c}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$. Converting this to the standard basis using given values of $\vec{b_1}$ and $\vec{b_3}$, we find $[T\vec{c}]_{\mathcal{S}} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$.

(c) Find a matrix for T in the $\mathcal{B}$ basis and a matrix for T in the $\mathcal{S}$ basis. (In class, we might have said, "A matrix for T in the $\mathcal{B}$ coordinate system." This is another way of saying, "A matrix for T in the $\mathcal{B}$ basis.")

In the $\mathcal{B}$ basis, T takes the basis vectors to $2\vec{b_1}$, $3\vec{b_2}$, and $-\vec{b_3}$, so its matrix is $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

In the standard basis, computing the matrix for T is much harder. Note that $\vec{e_1} = \frac{1}{3}\vec{b_1} + \frac{1}{3}\vec{b_2} - \frac{1}{3}\vec{b_3}$, so $T\vec{e_1} = \frac{2}{3}\vec{b_1} + \vec{b_2} + \frac{1}{3}\vec{b_3} = \begin{bmatrix} 4/3 \\ -1/3 \\ 5/3 \end{bmatrix}$.

Also, $\vec{e_2} = \frac{1}{3}\vec{b_1} - \frac{2}{3}\vec{b_2} - \frac{1}{3}\vec{b_3}$, so $T\vec{e_2} = \frac{2}{3}\vec{b_1} - 2\vec{b_2} + \frac{1}{3}\vec{b_3} = \begin{bmatrix} -5/3 \\ 8/3 \\ 1 \end{bmatrix}$.

Finally, $\vec{e_3} = \frac{1}{3}\vec{b_1} + \frac{1}{3}\vec{b_2} + \frac{2}{3}\vec{b_3}$, so $T\vec{e_3} = \frac{2}{3}\vec{b_1} + \vec{b_2} - \frac{2}{3}\vec{b_3} = \begin{bmatrix} 7/3 \\ -1/3 \\ 0 \end{bmatrix}$.

This means that the matrix for T in the standard basis is $\begin{bmatrix} 4/3 & -5/3 & 7/3 \\ -1/3 & 8/3 & -1/3 \\ 5/3 & 1 & 0 \end{bmatrix}$.

(There are other methods for computing the matrix for T in the standard basis. Try doing it using the PAP^{-1} method from class, for instance.)

3. Read section 3.1 and 3.2 in your textbook about computing the determinant of a matrix. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & x \end{bmatrix}$.

(a) Compute $\det(A)$.

We'll expand along the bottom row, since it has two 0's in it. The determinant is $1 \cdot (2 \cdot 1 - 1 \cdot 1) = 1$.

(b) Compute $\det(B)$. For what values of x is B not invertible?

We'll expand along the bottom row again. The determinant is $1 \cdot (2 \cdot 1 - 1 \cdot 1) + x \cdot (1 \cdot 1 - 1 \cdot 2) = 1 - x$.

B is not invertible if the determinant is zero; that is, if $x = 1$. For all other values of x, B is invertible.

4. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 9 \end{bmatrix}$.

(a) Find an equation for the function $p(x) = \det(A - xI)$ (this is called the *characteristic polynomial* of A).

$\text{char}(A) = p(x) = x^2 - 10x - 1$.

(b) For what values of x is $A - xI$ non-invertible?

Exactly when $x = 5 \pm \sqrt{26}$

(c) Compute $p(A)$, the polynomial p with the matrix A plugged into it. When you plug a matrix into a polynomial, replace any constant terms k with the matrix kI. Can you guess why p is called an *annihilating* polynomial for A?

If we compute $A^2 - 10A - I$ we get the zero matrix. $p(x)$ is called annihilating because when you "plug in A" you get the zero matrix.

5. For each of the following, either give an example or a reason why it is impossible.

(a) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that is invertible.

$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}.$

(b) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that is not invertible.

$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

(c) A non-linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$.

$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

(d) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose null space equals its range.

The linear transformation given by multiplication by the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(e) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that when represented as a matrix in the standard basis has a column space equal to its row space.

The linear transformation given by multiplication by the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(f) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ so that T^2 is the identity, but T is not invertible.

This is impossible. If $T^2 = id$, then $T = T^{-1}$, and so T is invertible.

(g) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ so that T^3 is the identity, but T^2 is not the identity.

Rotation counter clockwise by $120°$.

(h) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with exactly *one* eigenvector.

This is impossible. If $T\vec{v} = \lambda\vec{v}$ and T is linear, then $T(2\vec{v}) = 2\lambda\vec{v}$ and so $2\vec{v}$ is another eigenvector.

(i) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with exactly *one* eigen direction (i.e., all eigenvectors lie on a single line).

The linear transformation given by multiplication by the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(j) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with exactly *two* eigen directions (i.e., all eigenvectors lie on one of two lines).

The linear transformation given by multiplication by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(k) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with *infinitely many* eigen directions.

The linear transformation given by multiplication by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(l) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with no real eigenvectors.

Rotation counter clockwise by $90°$.

(m) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with eigenvalues 3 and -2.

The linear transformation given by multiplication by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$.

(n) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 2 and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue 0.

The linear transformation given by multiplication by the matrix $\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}^{-1}$.

6. *This problem will not be on the exam but is included for you to stress-test your mathematical thinking.*

Let $T : \mathbb{R}^n \to \mathbb{R}^n$. A subspace $X \subseteq \mathbb{R}^n$ is called *invariant with respect to* T if $T(X) = X$. That is, $\{\vec{v} : \vec{v} = T\vec{x} \text{ for some } \vec{x} \in X\} = X$. Note, this is *different* than saying $T\vec{x} = \vec{x}$ for all $\vec{x} \in X$.

Note that $\{\vec{0}\}$ is an invariant subspace for any linear transformation. In subsequent parts, we'll list only the other invariant subspaces.

(a) Describe all invariant subspaces of the linear transformation given by the matrix $\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$.

The (nontrivial) invariant subspaces are $\mathbb{R}^2$, span$\{\vec{e_1}\}$, and span$\{\vec{e_2}\}$. (That is, $\mathbb{R}^2$ itself and the coordinate axes.)

(b) Describe all invariant subspaces of the linear transformation given by the matrix $\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$.

Every subspace of $\mathbb{R}^2$ is an invariant subspace of this transformation! This is because if T is the transformation given, then $T\vec{x} = 2\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$. $\mathbb{R}^2$ itself will be an invariant subspace, and any line through the origin will be "stretched" by this operation, making it also invariant.

(c) Describe all invariant subspaces of the shear given by the matrix $\begin{bmatrix}1 & a\\0 & 1\end{bmatrix}$ where $a \neq 0$.

If T is this transformation, then $T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x + ay\\y\end{bmatrix}$. This is never in the same 1-dimensional subspace as $\begin{bmatrix}x\\y\end{bmatrix}$ unless $y = 0$, so the only 1-dimensional invariant subspace is the x-axis, or span$\{\vec{e_1}\}$. $\mathbb{R}^2$ is also an invariant subspace, since this matrix is invertible, as described below.

(d) Your friend from another university proposes the following addition to the invertible matrix theorem:

An $n \times n$ matrix A is invertible if and only if $\mathbb{R}^n$ is an invariant subspace of the transformation given by A.

Is he right? If so, prove it is correct. If not, give a counterexample.

He is correct. Let $\mathcal{A}$ be the corresponding transformation. If $\mathbb{R}^n$ is an invariant subspace for $\mathcal{A}$, then the transformation must be onto (otherwise $\mathcal{A}(\mathbb{R}^n)$ would be smaller than $\mathbb{R}^n$). If $\mathcal{A}$ is onto, then by the invertible matrix theorem, A is invertible.
On the other hand, if A is invertible, then $\mathcal{A}$ is onto. This means that $\mathcal{A}(\mathbb{R}^n)$ contains all of $\mathbb{R}^n$. But $\mathcal{A}(\mathbb{R}^n)$ can't contain anything outside of $\mathbb{R}^n$, so it must equal $\mathbb{R}^n$, which means $\mathbb{R}^n$ is an invariant subspace for $\mathcal{A}$, as desired.

(e) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ has $\{\vec{0}\}$, $\left\{\begin{bmatrix}x\\y\end{bmatrix} : y = 2x\right\}$, and $\mathbb{R}^2$ as its *only* invariant subspaces. Give an example of a vector $\vec{v}$ that is an eigenvector for T and a vector $\vec{w}$ that is *not* an eigenvector for T. Explain how you know.

The vector $\vec{v} = \begin{bmatrix}1\\2\end{bmatrix}$ must be an eigenvector for T. $\vec{v}$ lies in a 1-dimensional invariant subspace of T, so $T\vec{v}$ must also be in this subspace. But everything in this subspace is a scalar multiple of $\vec{v}$ (since the subspace is 1-dimensional), so $\vec{v}$ is an eigenvector.

On the other hand, $\vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ cannot be an eigenvector for T. If it is, with a nonzero eigenvalue, then span$\{\vec{w}\}$ would be an invariant subspace of T (since T would either "stretch" or "shrink" this line). If it is with an eigenvalue of 0, then by the Invertible Matrix Theorem, T is not invertible, and by our extension to the theorem, $\mathbb{R}^2$ is not an invariant subspace. Neither of these is actually the case, so $\vec{w}$ can't be an eigenvalue for T.