We are nearly experts at typing by now. Though you are not required to type your homework, it's strongly encouraged. You can even download the `tex` *file for this homework and type your answers below each problem. Using the* `\begin{quote}` *and* `\end{quote}` *environment will indent anything you type inbetween. Perfect for typing answers!*

1. Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 & 3 \\ 1 & 2 & 3 & -1 & -1 \\ 3 & 1 & 4 & 2 & 7 \end{bmatrix}$. Let $\mathcal{C}$, $\mathcal{R}$, and $\mathcal{N}$ be the column, row, and null spaces of A, respectively.

   (a) Find a basis for $\mathcal{C}$

   We begin with row reduction.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 3 \\ 1 & 2 & 3 & -1 & -1 \\ 3 & 1 & 4 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 3 \\ 0 & 2 & 2 & -2 & -4 \\ 0 & 1 & 1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   Since $\mathrm{rref}(A)$ has two pivot columns, columns 1 and 2, a basis for the column space of A is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

   (b) Find a basis for $\mathcal{R}$.

   Since $\mathrm{rref}(A)$ has two non-zero rows, rows 1 and 2, a basis for the row space of A is either

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ -2 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

   (c) Find a basis for $\mathcal{N}$.

   In order to find a basis for $\mathcal{N}$, we begin by solving the matrix equation

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

   Therefore, we have that

$$\begin{cases} x_1 + x_3 + x_4 + 3x_5 & = 0 \\ x_2 + x_3 - x_4 - 2x_5 & = 0 \end{cases} \iff \begin{cases} x_1 & = -x_3 - x_4 - 3x_5 \\ x_2 & = -x_3 + x_4 + 2x_5 \end{cases}.$$

Therefore, the null space given in vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, a basis for null(A) is

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(d) Find the dot product of each of your basis vectors for $\mathcal{R}$ with each of your basis vectors for $\mathcal{N}$. What can you say geometrically about $\mathcal{R}$ and $\mathcal{N}$?

Observe that $[1,0,1,1,3] \cdot [-1,-1,1,0,0] = -1+0+1+0+0 = 0$. Similarly, the dot products of each other basis vector of $\mathcal{R}$ dotted with each other basis vector for $\mathcal{N}$ are all 0. Hence, $\mathcal{R}$ and $\mathcal{N}$ are perpendicular.

2. Let L be the line $x = y = z$ in $\mathbb{R}^3$.

(a) Find a 3×3 matrix B_1 whose column space is the xy-plane and whose null space is L.

Let

$$B_1 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & l \end{bmatrix}.$$

Since the column space is the xy-plane, we have that $g = h = l = 0$. Next, we seek to find null(B_1). To do so, we solve the matrix equation

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{cases} ax + by + cz &= 0 \\ dx + ey + fz &= 0 \end{cases}.$$

Further note that null(B_1) is a line, and thus has dimension 1. Therefore, row(B_1) must have dimension 2, and thus is a plane. We also know that row(B_1) and null(B_1) are perpendicular, and that row(B_1) is

$$\operatorname{span}\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix} \right\}.$$

Thus, we need to find 2 linearly independent vectors that are perpendicular to the direction vector of the line $x = y = z$. This direction vector is $[1,1,1]$. Thus, we pick

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \implies B_1 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) Find a 3×3 matrix B_2 whose column space is the xz-plane and whose null space is L.

By a similar idea to part (a), we find that a possible B_2 is

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

(c) Find the row space of B_1 and the row space of B_2. How do they compare? Explain.

Both $\text{row}(B_1)$ and $\text{row}(B_2)$ are equal to

$$\text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

Thus, they are identical. This is further evidenced by the fact that since $\text{null}(B_1)$ and $\text{null}(B_2)$ have dimension 1, and thus $\text{row}(B_1)$ and $\text{row}(B_2)$ both have dimension 2. Further, both row spaces are perpendicular to the line $x = y = z$. Since the plane that goes through the origin and is perpendicular to the line $x = y = z$ is unique, we have that $\text{row}(B_1) = \text{row}(B_2)$.

3. A linear transformation T has the following effects:

- Along the line $y = x$, it "shrinks" everything by a factor of 2—points along this line move halfway to the origin.
- Along the line $y = -3x$, it reflects everything over the origin—if $\vec{v}$ is the position vector of a point on this line, then it moves to $-\vec{v}$.

(a) Find a matrix A such that $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^2$.

First, we pick a vector on the line $y = x$. Such a vector is $[1, 1]$. Therefore, we know that

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

Further, we pick a vector on the line $y = -3x$. Such a vector is $[1, -3]$. Then, we know that

$$A \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We set up a system of equations using the above vectors

$$\begin{cases} a+b &= 1/2 \\ c+d &= 1/2 \\ a-3b &= -1 \\ c-3d &= 3 \end{cases} \iff \begin{cases} a &= 1/8 \\ b &= 3/8 \\ c &= 9/8 \\ d &= 5/8 \end{cases} \implies A = \begin{bmatrix} 1/8 & 3/8 \\ 9/8 & -5/8 \end{bmatrix}.$$

(b) Find A^{-1}.

We row reduce an augmented matrix.

$$[A \mid I] = \left[\begin{array}{cc|cc} 1/8 & 3/8 & 1 & 0 \\ 9/8 & -5/8 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 3 & 8 & 0 \\ 0 & -32 & -72 & 8 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 3 & 8 & 0 \\ 9 & -5 & 0 & 8 \end{array}\right]$$

Further reduction gives

$$\sim \left[\begin{array}{cc|cc} 1 & 3 & 8 & 0 \\ 9 & -5 & 0 & 8 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 5/4 & 3/4 \\ 0 & 1 & 9/4 & -1/4 \end{array}\right] \implies A^{-1} = \begin{bmatrix} 5/4 & 3/4 \\ 9/4 & -1/4 \end{bmatrix}.$$

(c) What are the "stretch factors" and "stretch directions" of the transformation given by A^{-1}?

If $\lambda \neq 0$ is an eigenvalue for A with corresponding vector $\vec{x}$, then $A\vec{x} = \lambda\vec{x} \implies A^{-1}A\vec{x} = A^{-1}\lambda\vec{x}$. Therefore, $\vec{x} = \lambda A^{-1}\vec{x}$. Thus,

$$\frac{1}{\lambda}\vec{x} = A^{-1}\vec{x}.$$

Thus, $\vec{x}$ is also an eigenvector for A^{-1}, with corresponding eigenvalue $1/\lambda$. In this problem A^{-1} has stretch factor $1/2$ along the direction $[1, 1]$ and a stretch factor of -1 along the direction $[1, -3]$.

4. Recall the "italicizing N" matrix that you found in class: $A = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & \frac{4}{3} \end{bmatrix}$. Find all eigenvalues ("stretch factors") and eigenvectors ("stretch directions") of A.

Observe that A is a triangular matrix, and therefore its eigenvalues are the entries on its main diagonal: $\lambda = 1$ and $\lambda = 4/3$.

- $\lambda = 1$.

$$(A - \lambda I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1/3 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \frac{1}{3}y = 0 \iff y = 0.$$

Thus, the eigenvectors are vectors of the form

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

with $x \neq 0$.

- $\lambda = 4/3$

$$(A-\lambda I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1/3 & 1/3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff -\frac{1}{3}x + \frac{1}{3}y = 0 \iff y = x.$$

Thus, the eigenvectors are vectors of the form

$$\begin{bmatrix} x \\ x \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with $x \neq 0$.