These problems will not be turned in nor graded. However, a non-empty subset of these problems will appear on the final skills check/takehome. The numbers may change when they appear on the exam, but it behooves you to have a thorough understanding of every problem on this homework.

1. Suppose the matrix equation $A\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$ has the general solution

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

   (a) How many rows and how many columns does A have?
   (b) Find null(A).
   (c) Find rank(A).
   (d) Find col(A).
   (e) Find row(A).

2. Let

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{b}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{b}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

   and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and $T(\vec{b}_1) = 2\vec{b}_1$, $T(\vec{b}_2) = 3\vec{b}_2$, and $T(\vec{b}_3) = -\vec{b}_3$.

   (a) Compute $[\vec{c}]_{\mathcal{B}}$.
   (b) Compute $[T\vec{c}]_{\mathcal{B}}$ and $[T\vec{c}]_{\mathcal{S}}$.
   (c) Find a matrix for T in the $\mathcal{B}$ basis and a matrix for T in the $\mathcal{S}$ basis. (In class, we might have said, "A matrix for T in the $\mathcal{B}$ coordinate system." This is another way of saying, "A matrix for T in the $\mathcal{B}$ basis.")

3. Read section 3.1 and 3.2 in your textbook about computing the determinant of a matrix. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & x \end{bmatrix}$.

   (a) Compute det(A).
   (b) Compute det(B). For what values of x is B not invertible?

4. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 9 \end{bmatrix}$.

   (a) Find an equation for the function $p(x) = \det(A - xI)$ (this is called the *characteristic polynomial* of A).
   (b) For what values of x is $A - xI$ non-invertible?
   (c) Compute $p(A)$, the polynomial p with the matrix A plugged into it. When you plug a matrix into a polynomial, replace any constant terms k with the matrix kI. Can you guess why p is called an *annihilating* polynomial for A?

5. For each of the following, either give an example or a reason why it is impossible.

   (a) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that is invertible.
   (b) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that is not invertible.

(c) A non-linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$.

(d) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose null space equals its range.

(e) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that when represented as a matrix in the standard basis has a column space equal to its row space.

(f) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ so that T^2 is the identity, but T is not invertible.

(g) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ so that T^3 is the identity, but T^2 is not the identity.

(h) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with exactly *one* eigenvector.

(i) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with exactly *one* eigen direction (i.e., all eigenvectors lie on a single line).

(j) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with exactly *two* eigen directions (i.e., all eigenvectors lie on one of two lines).

(k) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with *infinitely many* eigen directions.

(l) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with no real eigenvectors.

(m) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with eigenvalues 3 and -2.

(n) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 2 and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue 0.

6. *This problem will not be on the exam but is included for you to stress-test your mathematical thinking.*

Let $T : \mathbb{R}^n \to \mathbb{R}^n$. A subspace $X \subseteq \mathbb{R}^n$ is called *invariant with respect to* T if $T(X) = X$. That is, $\{\vec{v} : \vec{v} = T\vec{x} \text{ for some } \vec{x} \in X\} = X$. Note, this is *different* than saying $T\vec{x} = \vec{x}$ for all $\vec{x} \in X$.

(a) Describe all invariant subspaces of the linear transformation given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(b) Describe all invariant subspaces of the linear transformation given by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

(c) Describe all invariant subspaces of the sheer given by the matrix $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ where $a \neq 0$.

(d) Your friend from another university proposes the following addition to the invertible matrix theorem:

> An $n \times n$ matrix A is invertible if and only if $\mathbb{R}^n$ is an invariant subspace of the transformation given by A.

Is he right? If so, prove it is correct. If not, give a counterexample.

(e) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ has $\{\vec{0}\}$, $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = 2x \right\}$, and $\mathbb{R}^2$ as its *only* invariant subspaces. Give an example of a vector $\vec{v}$ that is an eigenvector for T and a vector $\vec{w}$ that is *not* an eigenvector for T. Explain how you know.