1. Let $\vec{a} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{b} = \begin{bmatrix}4\\5\\6\end{bmatrix}$, $\vec{c} = \begin{bmatrix}d\\1\\1\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}$ be vectors where $d, w_1, w_2, w_3 \in \mathbb{R}$ are unknown constants.

(a) For what values of $d$ is $\{\vec{a}, \vec{b}, \vec{c}\}$ linearly independent? For which values of $d$ is $\{\vec{a}, \vec{b}, \vec{c}\}$ linearly dependent?

We begin with row reduction.

$$\left[\begin{array}{ccc|c} \vec{a} & \vec{b} & \vec{c} & \vec{w} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 4 & d & w_1 \\ 2 & 5 & 1 & w_2 \\ 3 & 6 & 1 & w_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & d & w_1 \\ 0 & -3 & 1-2d & w_2 - 2w_1 \\ 0 & -6 & 1-3d & w_3 - 3w_1 \end{array}\right].$$

Continuing gives

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & d & w_1 \\ 0 & -3 & 1-2d & w_2 - 2w_1 \\ 0 & 0 & (1-3d) - 2(1-2d) & w_3 - 3w_1 - 2(w_2 - 2w_1) \end{array}\right].$$

This is thus equal to

$$\left[\begin{array}{ccc|c} 1 & 4 & d & w_1 \\ 0 & -3 & 1-2d & w_2 - 2w_1 \\ 0 & 0 & d-1 & w_1 - 2w_2 + w_3 \end{array}\right].$$

Therefore, based on the above row-equivalent matrix to the augmented matrix, we have that $\vec{a}, \vec{b}, \vec{c}$ are independent if and only if the first three columns are all pivot columns. That means $d - 1 \neq 0$ or $d \neq 1$. Vice versa, $\vec{a}, \vec{b}, \vec{c}$ are dependent if and only if $d = 1$.

(b) Write down the system of equations coming from the rows of the vector equation

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{w}.$$

$$\begin{cases} x + 4y + d_z & = w_1 \\ 2x + 5y + z & = w_2 \\ 3x + 6y + z & = w_3. \end{cases}$$

(c) Give three numeric examples of different vectors $\vec{w}$ such that the above system is consistent no matter what $d$ is. Explain.

For the system to be consistent no matter what $d$ is, $w_1 - 2w_2 + w_3 = 0$. Therefore, pick

$$\vec{w} = \begin{bmatrix}0\\0\\0\end{bmatrix} \text{ or } \begin{bmatrix}1\\1\\1\end{bmatrix} \text{ or } \begin{bmatrix}1\\2\\3\end{bmatrix}.$$

(d) Give a numeric example of a vector $\vec{w}$ such that the above system is only consistent for some $d$. Explain.

When $d = 1$, the system is consistent only if $w_1 - 2w_2 + w_3 = 0$. Hence if $w_1 - 2w_2 + w_3 \neq 0$, then the system is consistent only when $d \neq 1$. We can pick

$$\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

for this situation to happen.

2. (a) Use an augmented matrix to solve

$$\begin{aligned} x + y &= 7 \\ 2x - 3y &= 13. \end{aligned}$$

Are there any values you could replace the right hand side of the equations with such that there would be no solution? Explain both *geometrically* (using vectors, span, etc.) and *algebraically* (using systems, consistency, etc.) using technical linear algebra terms.

We begin with row reduction.

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 2 & -3 & -13 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & -5 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 34/5 \\ 0 & 1 & 1/5 \end{array}\right].$$

Thus, the solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 34/5 \\ 1/5 \end{bmatrix}.$$

There are no values we could replace the right hand side of the equations with such that there would be no solution because:

(a) Geometrically, the column vectors of the coefficient matrix

$$\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$$

are linearly independent. Hence any vector in $\mathbb{R}^2$ can be written as their linear combination.

(b) Algebraically,

$$\text{rref} \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

has all non-zero rows. Hence the system

$$\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

is consistent for all $a, b \in \mathbb{R}$.

(b) Consider the system given by the augmented matrix

$$C = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right].$$

and call the variables in this system $x_1, x_2, x_3, x_4, x_5$. Write all solutions to this system in vector form. How many free variables are there?

Observe that this matrix corresponds to the following systems of equations

$$\begin{cases} x_1 + x_3 + 2x_4 & = -1 \\ x_2 + x_3 & = 3 \\ x_5 & = 4 \end{cases} \iff \begin{cases} x_1 & = -x_3 - 2x_4 - 1 \\ x_2 & = -x_3 + 3 \\ x_5 & = 4 \end{cases}.$$

Therefore, the solutions given in vector form are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \\ 4 \end{bmatrix}$$

with $x_3, x_4 \in \mathbb{R}$. Thus, there are two free variables in this system of equations.

(c) There are 10 ways to pick two things from the set $\{x_1, x_2, x_3, x_4, x_5\}$. For each of the ten ways, determine whether that pair is a valid choice of free variables for $C$.

Valid choices are: $(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_4),$ and $(x_3, x_4)$.

(d) Write down all solutions to the homogeneous system corresponding to $C$ (i.e., when the right-hand side is replaced with all zeros). How does this set of solutions compare to the set of solutions of $C$?

Solutions to the homogeneous system corresponding to $C$ satisfy

$$\begin{cases} x_1 + x_3 + 2x_4 & = 0 \\ x_2 + x_3 & = 0 \\ x_5 = 0 \end{cases} \iff \begin{cases} x_1 & = -x_3 - 2x_4 \\ x_2 & = -x_3 \\ x_5 & = 0 \end{cases}.$$

Therefore, we have that the set of solutions to the homogeneous system given in vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

The solution set of the homogeneous system is the solution set of the original system translated by the vector

$$\begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \\ 4 \end{bmatrix}.$$

3. Let $M = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

(a) Find solutions $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ to the three matrix equations

$$M\vec{v}_1 = \vec{e}_1 \qquad M\vec{v}_2 = \vec{e}_2 \qquad M\vec{v}_3 = \vec{e}_3.$$

We begin with row reduction, first with $M\vec{v}_1 = \vec{e}_1$.

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -6 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 0 & -9 & -3 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Thus, we find that

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

We compute $v_2$ and $v_3$ similarly, and find that

$$\vec{v}_2 = \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix} \text{ and } \vec{v}_3 = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}.$$

(b) Compute $M(a\vec{v}_1)$ and $M(a\vec{v}_1 + b\vec{v}_2)$ (where $\vec{v}_i$ are from above) where $a, b \in \mathbb{R}$ are unknown scalars. Was what happened a surprise?

We compute.

$$M(a\vec{v}_1) = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -a \\ 2a \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a\vec{e}_1.$$

Further,

$$M(a\vec{v}_1 + b\vec{v}_2) = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -a + 3b \\ 2a - 6b \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = a\vec{e}_1 + b\vec{v}_2.$$

(c) Express the solution to the matrix equation $M\vec{x} = \vec{w}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. (Recall, $\vec{w}$ is defined at the beginning of the problem.)

We row reduce
$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & -7 \\ 0 & 0 & 1 & 2 \end{array}\right].$$
Therefore, we find that a solution is
$$\vec{x} = \begin{bmatrix} 3 \\ -7 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix},$$
so then $\vec{x} = \vec{v}_1 + 2\vec{v}_2 - \vec{v}_3$.

(d) Let $V = [\vec{v}_1|\vec{v}_2|\vec{v}_3]$ be the matrix whose columns are $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. Compute the matrix product $MV$. Explain why you got the result you did.

We compute $MV$.
$$MV = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 3 & 2 \\ 2 & -6 & -3 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
We get that $MV = I$ because $MV = M\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix} = \begin{bmatrix}M\vec{v}_1 & M\vec{v}_2 & M\vec{v}_3\end{bmatrix} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\end{bmatrix}$.

(e) Can you use $V$ to help you solve the system
$$M\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}?$$
If so, explain how and do so.

As we computed, $MV = I = \begin{bmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\end{bmatrix}$. Hence,
$$M(V\begin{bmatrix} a \\ b \\ c \end{bmatrix}) = (MV)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = I\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$
Therefore,
$$\vec{x} = V\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$
is a solution to the equation
$$M\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$
This is a unique solution since all columns of $\mathrm{rref}(M)$ are pivot columns.

(f) Compute the matrix product $VM$. Can you explain why you got what you did? (*Hint: you might have to think about linear transformations for this one.*)

$$VM = \begin{bmatrix} -1 & 3 & 2 \\ 2 & -6 & -3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, consider the equation from part (e):

$$M\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \implies VM\vec{x} = V \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Then, from part (e), we know that

$$\vec{x} = V \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

is a solution for the above equation. This implies that

$$VM\vec{x} = \vec{x} \quad \left( \vec{x} = V \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right).$$

Because $V$ is invertible, for any vector $\vec{x} \in \mathbb{R}^3$, $\vec{x}$ can be written as

$$\vec{x} = V \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

for some $a, b, c \in \mathbb{R}$. Therefore, $VM\vec{x} = \vec{x}$ for all $\vec{x} \in \mathbb{R}^3$, so then $VM$ is the identity transformation on $\mathbb{R}^3$. In other words, $VM = I$.

4. Consider the transformations $A : \mathbb{R}^2 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^2$ given by the formulas

$$A(x, y) = (x - y, x + y) \qquad \text{and} \qquad B(x, y) = (x^2, y^2).$$

(a) Compute $A(\vec{e}_1)$, $A(\vec{e}_2)$, $A(\vec{e}_1 + \vec{e}_2)$, $B(\vec{e}_1)$, $B(\vec{e}_2)$, and $B(\vec{e}_1 + \vec{e}_2)$.

$$\begin{aligned} A(\vec{e}_1) &= A(1, 0) = (1 - 0, 1 + 0) = (1, 1) \\ A(\vec{e}_2) &= A(0, 1) = (0 - 1, 0 + 1) = (-1, 1) \\ A(\vec{e}_1 + \vec{e}_2) &= A(1, 1) = (1 - 1, 1 + 1) = (0, 2) \\ B(\vec{e}_1) &= B(1, 0) = (1^2, 0^2) = (1, 0) \\ B(\vec{e}_2) &= B(0, 1) = (0^2, 1^2) = (0, 1) \\ B(\vec{e}_1 + \vec{e}_2) &= B(1, 1) = (1^2, 1^2) = (1, 1). \end{aligned}$$

(b) Find a matrix $M_A$ so that $A$ is given by matrix multiplication, or explain why it is impossible.

Let
$$M_A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
Then
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$$
for all $x, y \in \mathbb{R}$. Equivalently, for all $x, y \in \mathbb{R}$, we have that
$$\begin{cases} ax+by &= x-y \\ cx+dy &= x+y \end{cases} \iff \begin{cases} a &= 1 \\ b &= -1 \\ c &= 1 \\ d &= -1 \end{cases} \implies M_A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

(c) Find a matrix $M_B$ so that $B$ is given by matrix multiplication, or explain why it is impossible.

Let
$$M_B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
Then for all $x, y \in \mathbb{R}$,
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}.$$
Equivalently
$$\begin{cases} ax+by &= x^2 \\ cx+dy &= y^2 \end{cases}.$$
However, it is impossible to find $a, b$ so that $ax + by = x^2$ for all $x \in \mathbb{R}$ because the right hand side has the term $x^2$ while the left hand side does not have this term. Similarly, it is impossible to find $c, d$.

(d) A function $X : \mathbb{R}^n \to \mathbb{R}^m$ is called *linear* if it satisfies:

i. $X(\alpha\vec{v}) = \alpha X(\vec{v})$ for all $\vec{v} \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$ **and**

ii. $X(\vec{v} + \vec{w}) = X(\vec{v}) + X(\vec{w})$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$.

For each of $A$ and $B$, determine whether or not it is a linear function. Prove your answers.

*Proof.* We will prove that $A$ is a linear function. For all
$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2 \text{ and } \vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^2$$
and $\alpha \in \mathbb{R}$, we have that
$$A(\alpha\vec{v}) = A \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \end{bmatrix} = \begin{bmatrix} \alpha v_1 - \alpha v_2 \\ \alpha v_1 + \alpha v_2 \end{bmatrix} = \alpha \begin{bmatrix} v_1 - v_2 \\ v_1 + v_2 \end{bmatrix} = \alpha(A\vec{v}).$$

Further, we have that

$$A(\vec{v}+\vec{w}) = A\begin{bmatrix} v_1+w+1 \\ v_2+w_2 \end{bmatrix} = \begin{bmatrix} v_1-v_2+w_1-w_2 \\ v_1+v_2+w_1+w_2 \end{bmatrix} = \begin{bmatrix} v_1-v_2 \\ v_1+v_2 \end{bmatrix}+\begin{bmatrix} w_1-w_2 \\ w_1+w_2 \end{bmatrix} = A\vec{v}+A\vec{w}.$$

Thus, $A$ is a linear function, since it satisfies both requirements. □

*Proof.* We will prove that $B$ is not a linear function. Consider

$$B\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1^2 \\ 1^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Further consider

$$B\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2^2 \\ 2^2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

Thus, we have that

$$B\begin{bmatrix} 2 \\ 2 \end{bmatrix} \neq 2B\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so $B$ fails property i, and cannot be a linear function. □

5. (a) Let $\vec{v} \in \mathbb{R}^n$ and define the function $d_{\vec{v}} : \mathbb{R}^n \to \mathbb{R}$ by $d_{\vec{v}}(\vec{w}) = \vec{v}\cdot\vec{w}$. Prove that $d_{\vec{v}}$ is linear. (*Hint: you should be having flashbacks to homework 1.*)

   *Proof.* For any vectors $\vec{u}, \vec{w} \in \mathbb{R}^n$, and scalar $\alpha \in \mathbb{R}$, we have that

   (i) $d_{\vec{v}}(\alpha\vec{u}) = \vec{v}\cdot(\alpha\vec{w}) = \alpha(\vec{v}\cdot\vec{w}) = \alpha d_{\vec{v}}(\vec{u})$

   (ii) $d_{\vec{v}}(\vec{u}+\vec{w}) = \vec{v}\cdot(\vec{u}+\vec{w}) = \vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{w} = d_{\vec{v}}(\vec{u}) + d_{\vec{v}}(\vec{w})$.

   Therefore, $d_{\vec{v}}$ is linear. □

   (b) For a $2\times 2$ matrix $M$, let $f_M : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f_M(\vec{v}) = M\vec{v}$, where $\vec{v}$ is a column vector. Prove that $f_M$ is a linear transformation.

   *Proof.* Assume that

   $$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

   and let

   $$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

   be arbitrary vectors in $\mathbb{R}^2$. Further, let $\alpha \in \mathbb{R}$ be an arbitrary scalar. Then

   (i) $M(\alpha\vec{V}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha v_1 \\ \alpha v_2 \end{bmatrix} = \begin{bmatrix} a\alpha v_1 + b\alpha v_2 \\ c\alpha v_1 + d\alpha v_2 \end{bmatrix} = \alpha\begin{bmatrix} av_1+bv_2 \\ cv_1+dv_2 \end{bmatrix} = \alpha\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \alpha M\vec{v}$.

   (ii) $M(\vec{v}+\vec{w}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a(v_1+w_1)+b(v_2+w_2) \\ c(v_1+w_1)+d(v_2+w_2) \end{bmatrix} = \begin{bmatrix} av_1+bv_2 \\ cv_1+dv_2 \end{bmatrix}+\begin{bmatrix} aw_1+bw_2 \\ cw_1+dw_2 \end{bmatrix} = M\vec{v}+M\vec{w}$.

Therefore, $f_M$ is linear. $\square$

(c) Make a conjecture about functions that can be computed using matrix multiplication and their linearity.

Functions from $\mathbb{R}^n \to \mathbb{R}^m$ that can be computed using matrix multiplication are linear functions.