1. **Reading Assignment:** Read and practice the algorithm for finding the inverse of a matrix, found at the end of section 2.2 of the textbook.

2. For each of the following transformations, find a matrix corresponding to the transformation.

   (a) $\mathcal{A}$: Rotate 90° counterclockwise around the origin (in $\mathbb{R}^2$).

   Let M_A be the corresponding matrix for A. Then

   $$M_A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } M_A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \implies M_A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

   We find the rest of the matrices in this problem similarly.

   (b) $\mathcal{B}$: Send every vector in $\mathbb{R}^2$ to the zero vector.

   $$M_B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

   (c) $\mathcal{C}$: Project (in $\mathbb{R}^3$) onto the yz-plane.

   $$M_C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (d) $\mathcal{D}$: Project (in $\mathbb{R}^3$) onto the x-axis.

   $$M_D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   (e) $\mathcal{E}$: Reflect (in $\mathbb{R}^3$) across the xy-plane.

   $$M_E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

   (f) $\mathcal{F}$: Stretch by a factor of 2 in the y-direction (in $\mathbb{R}^2$).

   $$M_F = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

   (g) $\mathcal{G}$: Stretch by a factor of 2 in the y-direction (in $\mathbb{R}^3$).

   $$M_G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (h) $\mathcal{H}$: Rotate 90° counterclockwise (as viewed looking "down" from the positive y-axis towards the origin) around the y-axis (in $\mathbb{R}^3$).

$$M_H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

3. (a) For each transformation in problem 2, explain geometrically whether or not the transformation is invertible.

   - Since we can rotate 90° clockwise to undo the effect of $\mathcal{A}$, $\mathcal{A}$ must be invertible.
   - Since multiple vectors in $\mathbb{R}^2$ are transformed to $\vec{0}$, it is impossible to invert the transformation $\mathcal{B}$.
   - Similarly, $\mathcal{C}$ and $\mathcal{D}$ are not invertible, while $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ are invertible.

   (b) Which of the matrices you generated in problem 2 are invertible? (You may wish to use the algorithm you read about.)

   By using the algorithm for finding the inverse of a matrix, we see that matrices M_A, M_E, M_F, M_G, M_H are invertible, while M_B, M_C, M_D are not.

   (c) Suppose f is a transformation that is represented by the matrix M. Is it possible for f to be invertible and M not invertible? Is it possible for M to be invertible and f not be invertible? Explain your reasoning.

   Assume that f is a transformation on $\mathbb{R}^n$ represented by the matrix M. Hence, for any vector $\vec{v} \in \mathbb{R}^n$, $f(\vec{v}) = M\vec{v}$.

   - If M is invertible, then there must exist some matrix M^{-1} so that $MM^{-1} = M^{-1}M = I$. Let g be a transformation represented by the matrix M^{-1}. Then, for all $\vec{v} \in \mathbb{R}^n$, we have that $g(f(\vec{v})) = M^{-1}M\vec{v} = I\vec{v} = \vec{v}$ and also that $f(g(\vec{v})) = MM^{-1}\vec{v} = I\vec{v} = \vec{v}$. By definition, therefore, g is the inverse of f, so f must be invertible.
   - If f is invertible, then the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one. The same is true for the mapping $M : \mathbb{R}^n \to \mathbb{R}^n$. Therefore, $\vec{x} = \vec{0}$ is the unique solution for the equation $M\vec{x} = \vec{0}$. By the Inverse Matrix Theorem, M is invertible.

4. Each of the following matrices gives a transformation of $\mathbb{R}^2$ or $\mathbb{R}^3$. Describe the effect of that transformation in terms of the transformations from problem 2. (Hint: you may need to combine two, three, or more transformations from problem 2 for some examples.)

$$X = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad W = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   - $$X = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$$

Observe that

$$\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ 2x \end{bmatrix}.$$

X rotates $90°$ counterclockwise and then stretches by a factor of 2 in the y-direction (in $\mathbb{R}^2$). Therefore, $X = \mathcal{F} \circ \mathcal{A}$. We check

$$M_F M_A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} = M_X.$$

- Similarly, we can deduce
  - $Y = \mathcal{C} \circ \mathcal{E}$ since
$$M_C M_E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$
  - $Z = \mathcal{G} \circ \mathcal{H}$ since
$$M_G M_H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix} = M_Z.$$
  - $W = \mathcal{D} \circ \mathcal{H}$ since
$$M_D M_H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = M_W.$$

5. Use the definition of a linear transformation to show that if T is a linear transformation, then $T(\vec{0}) = \vec{0}$. Do not appeal to the matrix representation of T.

   Proof. Assume that T is a linear transformation. Then by definition, $T(\vec{0}+\vec{0}) = T(\vec{0}) + T(\vec{0})$. However, $\vec{0}+\vec{0} = \vec{0}$. Therefore, $T(\vec{0}+\vec{0}) = T(\vec{0})$. Thus, we have that $T(\vec{0}) = T(\vec{0}) + T(\vec{0})$. Therefore, $T(\vec{0}) = \vec{0}$. ☐

6. Consider the matrices $A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

   (a) Describe the geometric effect of A. Feel free to use the language of the in-class worksheets.

   Let
$$\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2.$$
   Then,
$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}.$$
   By looking at the action of A on the vector $[x, y]$, we see A is a "sheer transformation", which keeps the first component of the vector the same and adds a k-multiple of the first component to the second component.

(b) Compute the product BA.

$$BA = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ k & 1 \end{bmatrix}.$$

(c) Find a matrix C such that $AC = BA$.

Let C be the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $a, b, c, d \in \mathbb{R}$. Then, we want

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ k & 1 \end{bmatrix}.$$

This gives a system of equations.

$$\begin{cases} a & = 2 \\ b & = 0 \\ ka + c & = k \\ kb + d & = 1 \end{cases} \iff \begin{cases} a & = 2 \\ b & = 0 \\ c & = -k \\ d & = 1 \end{cases} \implies C = \begin{bmatrix} 2 & 0 \\ -k & 1 \end{bmatrix}.$$

(d) Describe geometrically why $C \neq B$.

Let $[x, y] \in \mathbb{R}^2$. Then,

$$C \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -k & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ -kx + y \end{bmatrix},$$

but

$$B \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}.$$

Both transformations B and C double the first component of the vector $\vec{x} = [x, y]$, but while B doesn't change the second component, C subtracts k multiples of the first component from the second component. Hence, unless $k = 0$, B and C must be different.

7. Suppose A is an unknown matrix, but $A^{-1} = \begin{bmatrix} 3 & 2 & -1 \\ -2 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

(a) Find a solution to the matrix equation $A\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$.

$$A\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix} \iff A^{-1}A\vec{x} = A^{-1}\begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix} \iff I\vec{x} = A^{-1}\begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}.$$

Thus,

$$\vec{x} = \begin{bmatrix} 3 & 2 & -1 \\ -2 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 11 \end{bmatrix} \implies \vec{x} = \begin{bmatrix} 2 \\ 2 \\ 11 \end{bmatrix}.$$

(b) Are there other solutions besides the one you found in (a)? Explain how you know.

There are no other solutions. The matrix A has an inverse, and is thus invertible. Therefore, by the Invertible Matrix Theorem, the equation

$$A\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$$

has a unique solution.