Math 240
Homework 2 Solution (*October 13, 2016*)

1. (a) $\begin{bmatrix}1 & 4 & 1 & -2\\ 3 & 11 & -4 & 0\\ 2 & 2 & 1 & 1\end{bmatrix} \xrightarrow[R2-3R1]{R3-2R1} \begin{bmatrix}1 & 4 & 1 & -2\\ 0 & -1 & -7 & 6\\ 0 & -6 & -1 & 5\end{bmatrix} \xrightarrow{-1R2} \begin{bmatrix}1 & 4 & 1-2 & \\ 0 & 1 & 7 & -6\\ 0 & -6 & -1 & 5\end{bmatrix} \xrightarrow[R1-4R2]{R3+6R2}$
$\begin{bmatrix}1 & 0 & -27 & 22\\ 0 & 1 & 7 & -6\\ 0 & 0 & 41 & -31\end{bmatrix} \xrightarrow{1/41R3} \begin{bmatrix}1 & 0 & -27 & 22\\ 0 & 1 & 7 & -6\\ 0 & 0 & 1 & -\frac{31}{41}\end{bmatrix} \xrightarrow[R1+27R3]{R2-7R3} \begin{bmatrix}1 & 0 & 0 & \frac{65}{41}\\ 0 & 1 & 0 & -\frac{29}{41}\\ 0 & 0 & 1 & -\frac{31}{41}\end{bmatrix}$

   (b) $\begin{bmatrix}1 & 6 & 4 & 2\\ 1 & 3 & 1 & 4\\ 1 & 8 & 6 & -1\end{bmatrix} \xrightarrow[R2-R1]{R3-R1} \begin{bmatrix}1 & 6 & 4 & 2\\ 0 & -3 & -3 & 2\\ 0 & 2 & 2 & -3\end{bmatrix} \xrightarrow{-1/3R2} \begin{bmatrix}1 & 6 & 4 & 2\\ 0 & 1 & 1 & -\frac{2}{3}\\ 0 & 2 & 2 & -3\end{bmatrix} \xrightarrow[R1-6R2]{R3-2R2}$
$\begin{bmatrix}1 & 0 & -2 & 6\\ 0 & 1 & 1 & -\frac{2}{3}\\ 0 & 0 & 0 & -\frac{5}{3}\end{bmatrix} \xrightarrow{-3/5R3} \begin{bmatrix}1 & 0 & -2 & 6\\ 0 & 1 & 1 & -\frac{2}{3}\\ 0 & 0 & 0 & 1\end{bmatrix} \xrightarrow[R1-6R3]{R2+2/3R3} \begin{bmatrix}1 & 0 & -2 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$

2. (a) $\begin{bmatrix}2 & 1\\ 1 & -1\end{bmatrix} \cdot \begin{bmatrix}1 & 3\\ 2 & 2\end{bmatrix} = \begin{bmatrix}4 & 8\\ -1 & 1\end{bmatrix}$

   (b) $\begin{bmatrix}1 & 1 & 1 & 1\\ 2 & 1 & 1 & -1\end{bmatrix} \cdot \begin{bmatrix}2 & 3 & 4\\ 3 & 4 & 5\\ 4 & 5 & 6\\ -1 & -1 & -1\end{bmatrix} = \begin{bmatrix}8 & 11 & 14\\ 12 & 16 & 20\end{bmatrix}$

3. (a) We need to find $a, b \in \mathbb{R}$ such that:

$$a\begin{bmatrix}2\\ -1\\ -1\end{bmatrix} + b\begin{bmatrix}1\\ 3\\ 1\end{bmatrix} = \begin{bmatrix}1\\ 10\\ 4\end{bmatrix}$$

   Augmented matrix for above equation and its row-reduced-echelon form:
$\begin{bmatrix}2 & 1 & 1\\ -1 & 3 & 10\\ -1 & 1 & 4\end{bmatrix} \xrightarrow{R1-R3} \begin{bmatrix}-1 & 1 & 4\\ -1 & 3 & 10\\ 2 & 1 & 1\end{bmatrix} \xrightarrow{-1R1} \begin{bmatrix}1 & -1 & -4\\ -1 & 3 & 10\\ 2 & 1 & 1\end{bmatrix} \xrightarrow[R2+R1]{R3-2R1} \begin{bmatrix}1 & -1 & -4\\ 0 & 2 & 6\\ 0 & 3 & 9\end{bmatrix} \xrightarrow[1/2R2]{1/3R3}$
$\begin{bmatrix}1 & -1 & -4\\ 0 & 1 & 3\\ 0 & 1 & 3\end{bmatrix} \xrightarrow[R1+R2]{R3-R2} \begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 3\\ 0 & 0 & 0\end{bmatrix}$
   Hence $a = -1$ and $b = 3$.

   (b) $a\begin{bmatrix}2\\ -1\\ -1\end{bmatrix} + b\begin{bmatrix}1\\ 3\\ 1\end{bmatrix} = \begin{bmatrix}6 & -4 & 2\end{bmatrix}$

   Augmented matrix and its row reduced echelon:
$\begin{bmatrix}2 & 1 & 6\\ -1 & 3 & -4\\ -1 & 1 & 2\end{bmatrix} \xrightarrow{R1\leftrightarrow R3} \begin{bmatrix}-1 & 1 & 2\\ -1 & 3 & -4\\ 2 & 1 & 6\end{bmatrix} \xrightarrow{-1R1} \begin{bmatrix}1 & -1 & -2\\ -1 & 3 & -4\\ 2 & 1 & 6\end{bmatrix} \xrightarrow[R2+R1]{R3-2R1} \begin{bmatrix}1 & -1 & -2\\ 0 & 2 & -6\\ 0 & 3 & 10\end{bmatrix} \xrightarrow{1/3R2}$
$\begin{bmatrix}1 & -1 & -2\\ 0 & 1 & -3\\ 0 & 3 & 10\end{bmatrix} \xrightarrow[R1+R2]{R3-3R2} \begin{bmatrix}1 & 0 & -5\\ 0 & 1 & -3\\ 0 & 0 & 19\end{bmatrix}$

   The row reduced echelon matrix is inconsistent. Thus $\begin{bmatrix}6\\ -4\\ 2\end{bmatrix}$ cannot be written as a linear combination of $\vec{u}$ and $\vec{v}$.

   (c) It is impossible to write $\begin{bmatrix}0\\ 0\end{bmatrix}$ as a linear combination of $\vec{u}$ and $\vec{v}$ because $\begin{bmatrix}0\\ 0\end{bmatrix} \in \mathbb{R}^2$ while $\vec{u}, \vec{v} \in \mathbb{R}^3$.

4. (a) Neither $\vec{u}$ nor $\vec{v}$ is a multiple of the other. Hence $\vec{u}$ and $\vec{v}$ are linearly independent. Therefore, $\text{span}\{\vec{u}, \vec{v}\}$ is a plane.

   (b)
   - An example of vector in span of $\{\vec{u}, \vec{v}\}$ is vector $2\vec{u} = \begin{bmatrix} 4 \\ -2 \\ -2 \end{bmatrix}$.
   - Example of vector not in span of $\{\vec{u}, \vec{v}\}$ is vector $\begin{bmatrix} 6 \\ -4 \\ 2 \end{bmatrix}$ as proved in Question 3.
   - No, since span of $\{\vec{u}, \vec{v}\}$ is a subset of $\mathbb{R}^3$ while $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$.

5. Assume $\vec{u} = a\vec{v} + b\vec{w}$ where $a, b \in \mathbb{R}$.

   (a) They are linearly dependent because zero vector is a non trivial linear combination of them: $0 = -\vec{u} + a\vec{v} + b\vec{w}$.

   (b) No, we cannot guarantee that $\vec{v}$ is a linear combination of $\vec{u}$ and $\vec{w}$. To be more precise,
   - If $a \neq 0$, then $\vec{v} = \frac{1}{a}\vec{u} - \frac{b}{a}\vec{w}$. Thus $\vec{v}$ is a linear combination of $\vec{u}$ and $\vec{w}$.
   - However if $a = 0$, i.e $\vec{u} = b\vec{w}$, then nothing can be said about $\vec{v}$.

6. (a) $\text{span}\{\vec{u}, \vec{v}\} = \mathbb{R}^2$.

   (b) $\begin{bmatrix} 2 \\ 18 \end{bmatrix} \notin \text{span}\{\vec{w}, \vec{v}\}$.

   (c) The vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent.