1. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be some unknown function and let $\mathcal{S} \subseteq \mathbb{R}^3$ be the surface described by $z = g(x, y)$. You know the following information: the point $\vec{A} = (2, -3, 1) \in \mathcal{S}$ and the tangent plane to $\mathcal{S}$ at the point $\vec{A}$ has normal vector $\vec{n} = (7, 1, 11)$.

   (a) Give an equation for the tangent plane to $\mathcal{S}$ at $\vec{A}$. Is your equation the only equation describing the tangent plane?

   (b) Write down a linear approximation to g at the point $(2, -3)$. Is your linear approximation the only linear approximation to g at the point $(2, -3)$?

   (c) Compute $\nabla g(2, -3)$.

2. Let $f(x, y) = (x - 3)^3 - 7y^2$, let $\vec{a} = (a_x, a_y) = (1, 2)$, and let $\mathcal{S} \subseteq \mathbb{R}^3$ be the surface given by the equation $z = f(x, y)$.

   (a) Write $\mathcal{S}$ in set-builder notation.

   (b) Draw level curves for $\mathcal{S}$.

   (c) Let $\vec{A} = (a_x, a_y, f(a_x, a_y))$. Compute four tangent vectors to $\mathcal{S}$ at the point $\vec{A}$ such that none of your four tangent vectors is parallel to another. Call your vectors $\vec{d}_1$, $\vec{d}_2$, $\vec{d}_3$, $\vec{d}_4$.

   (d) Geometrically, what is the object

   $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\vec{d}_1 + s\vec{d}_2 + r\vec{d}_3 + q\vec{d}_4 + \vec{A}$$

   where t, s, r, q range over $\mathbb{R}$?

   (e) Let $T_{\vec{A}}$ be the tangent plane to the surface $\mathcal{S}$ at the point $\vec{A}$. Write down $T_{\vec{A}}$ in normal form using a normal vector of $\vec{n}_1 = \vec{d}_1 \times \vec{d}_2$ and using a normal vector of $\vec{n}_2 = \vec{d}_3 \times \vec{d}_4$. Are your equations the same? Do they describe the same object?

   (f) Solve both equations you computed above for z. Now are the equations the same? Can you rewrite these equations using ∇f?

3. (a) Let $\mathcal{P} \subseteq \mathbb{R}^3$ be a plane. Prove that there is at most one triplet of numbers (a, b, c) so that $z = ax + by + c$ describes the plane $\mathcal{P}$. (*Hint: when trying to show something is unique, it is often useful to assume there are two such things and then show that actually those two things were the same all along!*)

   (b) Consider the statement: *For any plane $\mathcal{P} \subseteq \mathbb{R}^3$, you can always find numbers a, b, c so that $\mathcal{P}$ is uniquely described by the equation $z = ax + by + c$.* Is this statement true or false? Explain.

4. MULTIVARIABLE CHAIN RULE. The idea of this problem is to discover and prove the multi-variable chain rule. For this problem, you may assume that the linear approximation to a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\vec{a}$ is given by $L_{\vec{a}}(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$. However, please don't use any formula for the multi-variable chain rule that is in your notes. If we use that formula, the derivation of the chain rule would be circular!

   (a) Let $L(x, y) = ax + by + c$ for some unknown constants $a, b, c \in \mathbb{R}$. Notice the graph of $z = L(x, y)$ is a plane. Let $\vec{p}(t) = (p_x(t), p_y(t))$ be a parameterization of a path in $\mathbb{R}^2$. Compute $(L \circ \vec{p})'(t_0)$. Express your answer as dot product involving ∇L and $\vec{p}\,'$.

   (b) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be an unknown but differentiable function and let $\vec{a} \in \mathbb{R}^2$. Write an equation for the linear approximation, $L_{\vec{a}}$, of g at the point $\vec{a}$.

   (c) The great thing about linear approximations to differentiable functions is that not only do they capture all information about the value of a function at a point, they capture all information about the first derivative of a function at a point. That is, if $\vec{p} : \mathbb{R} \to \mathbb{R}^2$ is a parameterization of a path so that $\vec{p}(t_0) = \vec{a}$ and $L_{\vec{a}}$ is a linear approximation of g at the point $\vec{a}$, then

$$(g \circ \vec{p})'(t_0) = (L_{\vec{a}} \circ \vec{p})'(t_0).$$

Use this equality and your results from part (a) and (b) to write down a formula for $(g \circ \vec{p})'(t_0)$ involving ∇g and $\vec{p}\,'$.

(d) Complete the formula

$$(g \circ \vec{p})'(t) = ?$$

where ? consists only of the symbols ∇g, $\vec{p}$, $\vec{p}\,'$, t and whatever math operations you want (multiplication, dot product, composition, etc.). In particular, $\vec{a}$ should *not* show up in your formula. Congratulations, you've discovered the multivariable chain rule!

(e) The professor across the hall from me claims the multivariable chain rule is

$$(g \circ \vec{p})'(t) = \left[D_{\vec{p}(t)} g\right](\vec{p}\,'(t)).$$

Is she right? Explain.