1. Show that the standard basis in $\mathbb{R}^3$ is linearly independent.

   First recall that a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is *linearly independent* if the only way to satisfy the equation

   $$\vec{0} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$$

   is when $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.
   The *standard basis* for $\mathbb{R}^3$ consists of the three vectors

   $$\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

   Considering the arbitrary linear combination $\vec{w} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3$, we see that

   $$\vec{w} = \alpha_1\begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha_2\begin{bmatrix}0\\1\\0\end{bmatrix} + \alpha_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}.$$

   Thus, if $\vec{w} = \vec{0}$, we must have that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and so $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is linearly independent.

2. Find the angle between the vectors $\vec{v} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}3\\4\end{bmatrix}$.

   We can find the angle between $\vec{v}$ and $\vec{w}$ using the dot product formula.

   The *dot product* of two vectors $\vec{a} = \begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$ is defined as

   $$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

   The dot product of two vectors can also be computed by the formula

   $$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta,$$

   where $\theta$ is the angle between $\vec{a}$ and $\vec{b}$.
   In order to use these formulae with $\vec{v}$ and $\vec{w}$, we must first compute some quantities.

   $$\|\vec{v}\| = \sqrt{1^2 + 2^2} = \sqrt{5} \qquad \|\vec{w}\| = \sqrt{3^2 + 4^2} = 5.$$

   Equating the two dot product formulae gives

   $$\vec{v}\cdot\vec{w} = 1(3) + 2(4) = 11 = 5\sqrt{5}\cos\theta,$$

   and so $\theta = \arccos\left(\frac{11}{5\sqrt{5}}\right) \approx 0.1799$ radians.

3. For vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, is it possible for $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\} = \operatorname{span}\{\vec{u}, \vec{v}\}$? Explain when it is and when it isn't.

   The *span* of a set of vectors is the set of all linear combinations of those vectors.
   Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ be vectors and suppose $\vec{w}$ is a *linear combination* of $\vec{u}$ and $\vec{v}$. This means there are scalars $\alpha, \beta$ so that

   $$\vec{w} = \alpha\vec{u} + \beta\vec{v}.$$

Since $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$ and $\operatorname{span}\{\vec{u}, \vec{v}\}$ is the set of *all* linear combinations of $\vec{u}$ and $\vec{v}$, we conclude that $\vec{w} \in \operatorname{span}\{\vec{u}, \vec{v}\}$, and so adding $\vec{w}$ to $\{\vec{u}, \vec{v}\}$ cannot enlarge its span. Therefore,

$$\operatorname{span}\{\vec{u}, \vec{v}\} = \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}.$$

On the other hand, if $\vec{u}$, $\vec{v}$, and $\vec{w}$ were linearly independent, none could be written as a linear combinations of the others. Thus we would have $\vec{w} \notin \operatorname{span}\{\vec{u}, \vec{v}\}$. But $\vec{w} \in \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$, so

$$\operatorname{span}\{\vec{u}, \vec{v}\} \subsetneq \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}.$$