1. For each of the following statements, produce a counterexample to show that the statement is **false**.

   (a) If A and B are square matrices, $AB = BA$.

   (b) If $AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then A and B are 2×2 matrices.

   (c) If $AB = I$ then $BA = I$.

   (d) If $A^2 = 0$, then $A = 0$.

2. Let $R = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

   (a) Find all solutions to the matrix equation $R \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$.

   (b) Prove that the set $X = \{\vec{x} \in \mathbb{R}^3 : R\vec{x} = \vec{0}\}$ is a subspace.

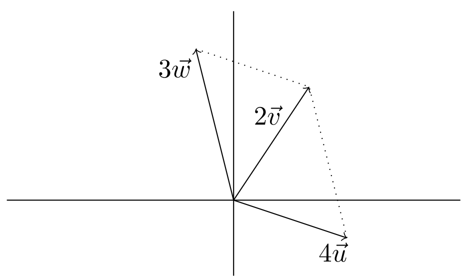
   (c) Prove that the set $Y = \{\vec{y} \in \mathbb{R}^3 : \vec{y} = R\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^3\}$ is a subspace.

3. Suppose E is a 4×3 matrix with columns $\vec{c}_1, \vec{c}_2, \vec{c}_3$ and rows $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4$. Let $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

   (a) Express $E\vec{v}$ as a linear combination of $\vec{c}_1, \vec{c}_2, \vec{c}_3$.

   (b) Supposing $\vec{r}_1 \cdot \vec{v} = 1$, $\vec{r}_2 \cdot \vec{v} = 6$, $(\vec{r}_3 + \vec{r}_4) \cdot \vec{v} = 2$, and $(\vec{r}_3 - \vec{r}_4) \cdot \vec{v} = -2$, compute $E\vec{v}$.

4. Suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^2$ that are related by the following diagram.



   Let $A = [\vec{u}|\vec{v}|\vec{w}]$ be the matrix with columns $\vec{u}$, $\vec{v}$, and $\vec{w}$.

   (a) What is the rank of A?

   (b) Find all solutions to the equation $A\vec{x} = \vec{0}$.

   (c) Find a basis for the subspace $V = \{\vec{x} \in \mathbb{R}^3 : A\vec{x} = \vec{0}\}$.

5. LINEAR TRANSFORMATIONS WERE HERE THE WHOLE TIME! For each transformation, determine whether or not it is a linear transformation. If it is a linear transformation, give its null space.

   (a) Let $V = \text{span}\{1, x, x^2\}$ where $1, x, x^2$ are (as usual) polynomials. Let $D : V \to V$ be differentiation. (Use the limit definition of the derivative. However, you can do limits like you did in Calc I; you don't have to do ε–δ proofs).

   (b) Let $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$ and let $D^s : \mathcal{F} \to \mathcal{F}$ be left-translation by s. That is, $D^s f(t) = f(t + s)$.

   (c) Let $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$ and let $D^s : \mathcal{F} \to \mathcal{F}$ be vertical-translation by s. That is, $D^s f(t) = f(t) + s$.

(d) Let $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$ and let $D^s : \mathcal{F} \to \mathcal{F}$ be horizontal-stretching by s. That is, $D^s f(t) = f(t/s)$.

(e) Let $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$ and let $D^s : \mathcal{F} \to \mathcal{F}$ be vertical-stretching by s. That is, $D^s f(t) = sf(t)$.

(f) Let $\mathcal{I} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is increasing}\}$ and let $D : \mathcal{I} \to \mathcal{I}$ be so that $D(f) = f^{-1}$ takes a function to its inverse.

(g) Let $\mathcal{L} = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) = mx + b \text{ for some } m, b \in \mathbb{R}\}$ be the set of all functions whose graphs are non-vertical lines and let $D : \mathcal{L} \to \mathbb{R}$ be evaluation at the point $x = 1$. That is, $D(f) = f(1)$.

6. ONCE UPON A TIME WE WERE DOING VECTOR CALCULUS!

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \cos(x) + \sin(y)$. Let $D^f : \mathbb{R}^2 \to \mathbb{R}$ be such that $D^f(\vec{v})$ is the directional derivative of f in at the point $(0, 0)$ in the direction $\vec{v}$. Show, using the definition of directional derivative, that D^f is linear.

(b) Let $\mathcal{F} = \{\text{functions from } \mathbb{R}^2 \text{ to } \mathbb{R}\}$ and let $\vec{v} \in \mathbb{R}^2$. Define $D^{\vec{v}} : \mathcal{F} \to \mathbb{R}$ to be the function such that $D^{\vec{v}}(h)$ is the directional derivative of h at the point $(0, 0)$ in the direction $\vec{v}$. Show that $D^{\vec{v}}$ is a linear transformation.

(c) Let $D_{\vec{x}} : \mathcal{F} \times \mathbb{R}^2 \to \mathbb{R}$ be so that $D_{\vec{x}}(h, \vec{v})$ is the directional derivative of h at the point $\vec{x}$ in the direction $\vec{v}$. Take a moment to appreciate that $D_{\vec{x}}$ is *bilinear*. That is, it is linear in both its first and second arguments. Let $\vec{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field. Notice that $D_{\vec{x}}(h, \vec{f}(\vec{x}))$ is still a linear function of h. The function $d_{\vec{x},\vec{f}}(h) = D_{\vec{x}}(h, \vec{f}(\vec{x}))$ is called a *derivation*. Much of what we did in multi-variable calculus can be rephrased in terms of derivations. Bilinearity of $D_{\vec{x}}$ ensures that for a fixed $\vec{x}$, the set of all derivations is itself a vector space. This vector space is called the *tangent space at* $\vec{x}$, and it is equivalent to the notion of tangent planes we developed first term.

Think about the functions defined in the previous paragraph until their definitions make sense. Then, contemplate whether a differential geometry course (where these ideas are studied in depth) would interest you. You do not need to write anything for this part.

(d) In Math 281-1, homework 3, problem 5, we considered the function $f(x, y) = \frac{x^2 y}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Let $D^f : \mathbb{R}^2 \to \mathbb{R}$ be so that $D^f(\vec{v})$ is the directional derivative of f at the point $(0, 0)$ in the direction $\vec{v}$. Is D^f linear? Make a conjecture about the relationship between linearity of the directional derivative operator and differentiability of a function.