1. Suppose the matrix equation $A\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$ has the general solution
$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$
   (a) How many rows and how many columns does A have?
   (b) Find null(A).
   (c) Find rank(A).
   (d) Find col(A).
   (e) Find row(A).

2. Let
$$\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{b}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{b}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$
and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and $T(\vec{b}_1) = 2\vec{b}_1$, $T(\vec{b}_2) = 3\vec{b}_2$, and $T(\vec{b}_3) = -\vec{b}_3$.
   (a) Compute $[\vec{c}]_{\mathcal{B}}$.
   (b) Compute $[T\vec{c}]_{\mathcal{B}}$ and $[T\vec{c}]_{\mathcal{S}}$.
   (c) Find a matrix for T in the $\mathcal{B}$ basis (i.e., the matrix $[T]_{\mathcal{B}}$) and a matrix for T in the $\mathcal{S}$ basis (i.e., $[T]_{\mathcal{S}}$).

3. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & x \end{bmatrix}$.
   (a) Compute $\det(A)$.
   (b) Compute $\det(B)$. For what values of x is B not invertible?

4. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 9 \end{bmatrix}$.
   (a) Find an equation for the function $p(x) = \det(A - xI)$ (this is called the *characteristic polynomial* of A).
   (b) For what values of x is $A - xI$ non-invertible?
   (c) Compute $p(A)$, the polynomial p with the matrix A plugged into it. When you plug a matrix into a polynomial, replace any constant terms k with the matrix kI. Can you guess why p is called an *annihilating* polynomial for A?

5. PLAYING WITH COORDINATE SYSTEMS Let
$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \mathcal{B} = \{\vec{u}, \vec{v}\} \qquad \mathcal{P} = \operatorname{span} \mathcal{B}.$$
   (a) For any point $\vec{x} \in \mathcal{P}$, $\vec{x} = \alpha\vec{u} + \beta\vec{v}$ for some α, β. So, we might say $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. Let $\|\cdot\|_{\mathcal{B}} : \mathbb{R}^2 \to \mathbb{R}$ be the *induced norm* on $\mathbb{R}^2$ defined by,
$$\|[\vec{x}]_{\mathcal{B}}\|_{\mathcal{B}} = \|\vec{x}\|.$$
   (Remember, $[\vec{x}]_{\mathcal{B}}$ is a list of two numbers—it isn't the same thing as the vector $\vec{x}$). Write down a formula for $\left\| \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\|_{\mathcal{B}}$.

(b) Just like an induced norm, we can also have an induced dot-product. Let the inclusion map $\iota : \mathbb{R}^2 \to \mathcal{P}$ be defined as

$$\iota \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha\vec{u} + \beta\vec{v}.$$

Define the $\mathcal{B}$-dot-product, $\odot$, as

$$\vec{a} \odot \vec{b} = (\iota\vec{a}) \cdot (\iota\vec{b})$$

for any $\vec{a}, \vec{b} \in \mathbb{R}^2$. Verify that $\|\vec{a}\|_{\mathcal{B}} = \sqrt{\vec{a} \odot \vec{a}}$ for $\vec{a} \in \mathbb{R}^2$ and draw the set of all unit vectors in $\mathbb{R}^2$ under the norm $\|\cdot\|_{\mathcal{B}}$.

(c) For $\vec{a}, \vec{b} \in \mathbb{R}^2$, we will *define* the angle between $\vec{a}$ and $\vec{b}$ to be the number θ so that $\vec{a} \odot \vec{b} = \|\vec{a}\|_{\mathcal{B}} \|\vec{b}\|_{\mathcal{B}} \cos\theta$. Let $\vec{c}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{c}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For each $\vec{c}_i$, draw the set of all vectors orthogonal to $\vec{c}_i$ with respect to $\odot$. Is the notion of angle coming from $\odot$ the same as from the standard dot product? Is it always different?

(d) An *inner product* on a vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that is symmetric, bilinear, and positive definite. That is, for any $\vec{u}, \vec{v}, \vec{w} \in V$ and $\alpha \in \mathbb{R}$,

i. $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ (symmetric)

ii. $\langle \alpha\vec{u}, \vec{v} \rangle = \alpha\langle \vec{u}, \vec{v} \rangle = \langle \vec{u}, \alpha\vec{v} \rangle$
and $\langle \vec{u} + \vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$
and $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$ (bilinear)

iii. $\langle \vec{u}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = 0$ (positive definite).

Show that both the standard dot product and $\odot$ are inner products on $\mathbb{R}^2$.

(e) Notice that for $\vec{x}, \vec{y} \in \mathbb{R}^2$, $\vec{x} \cdot \vec{y} = \vec{x}^T\vec{y} = \vec{x}^T I \vec{y}$. Find a matrix A such that $\vec{x} \odot \vec{y} = \vec{x}^T A \vec{y}$. Matrices like A show up in the study of relativity. When you are moving at relativistic speeds, the angle you perceive between two objects is different than someone at rest would perceive. Using a matrix like A or an inner product like $\odot$ allows you compensate for how relativistic effects change your perceptions.