1. A first order differential equation is called *separable* if it can be written in the form $f(y)\frac{\mathrm{d}y}{\mathrm{d}x} = h(x)$. Separable equations are easy to implicitly solve. By integrating both sides, we have $\int f(y)\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int h(x)\mathrm{d}x$. Applying the chain rule to the left hand side gives us

$$\int f(y)\mathrm{d}y = \int h(x)\mathrm{d}x.$$

For each differential equation, state whether or not it is separable. If it is separable, find the set of implicit solutions. Further, find explicit solutions (i.e., solutions where y is a function of x) to the initial value problem $y(x_0) = y_0$ for every applicable (x_0, y_0). (For example, if the implicit solution were $x^2 + y^2 = K$, then the solution would be $y = \sqrt{(x_0^2 + y_0^2) - x^2}$ if $y_0 > 0$ and $y = -\sqrt{(x_0^2 + y_0^2) - x^2}$ if $y_0 < 0$. If you encounter a similar situation, you must include both such solutions and describe which initial conditions give rise to which solution.)

   (a) $y' = \frac{1-2y}{x}$

   (b) $y' = \frac{-xy}{x+1}$

   (c) $y' = 3\sqrt[3]{y^2}$

   (d) $xy' + y = y^2$

2. Consider the differential equation $y' = Ky - y^3$ where K is constant.

   (a) Draw a slope field for the differential equation for your choice of K. Make sure it's detailed enough for you to see what's going on.

   (b) Partition the set of initial conditions (x_0, y_0) into sets that give rise to "similar looking" solutions and describe the types of solutions you get from each set. Warning: your partitions may depend on K!

   (c) Let $\vec{x}_0 = (x_0, y_0)$ and $\vec{x}_0^* = (x_0^*, y_0^*)$ with $\|\vec{x}_0 - \vec{x}_0^*\| < \varepsilon$. Let $y(x)$ and $y^*(x)$ be solutions to the respective initial value problems. The initial condition $\vec{x}_0$ is called *bi-stable* if $|y(x) - y^*(x)| < C\varepsilon$ for some fixed C. Otherwise it is called *unstable*. It is called *forward stable* if $|y(x) - y^*(x)| < C\varepsilon$ when $x \geq x_0$ and *backwards stable* if $|y(x) - y^*(x)| < C\varepsilon$ when $x \leq x_0$ (so stable implies both forward and backwards stable).

   Identify each region of of initial conditions as forward/backward/bi stable or unstable.

   (d) For $K = 2$, use Euler's method with 5 steps to estimate $y(5)$ where y is a solution to the initial value problem $(0, 1)$. Plot your estimate. What is going on?

   (e) Repeat your calculation for $K = 2$ using Euler's method with 50 steps. (You can use a computer for the 50 steps.)

   (f) Use Euler's method and this differential equation (with an appropriate value for K) to approximate $\sqrt{7}$.