

1. Find all solutions to the following linear differential equations.

- (a)  $y'' + 4y = 0$
- (b)  $y'' - y = t^2$
- (c)  $y''' + 4y'' - 5y' = 0$

2. Consider the differential equation

$$y'' - 4y' + 4y = 0. \quad (1)$$

- (a) Find all values  $k$  so that this equation has a solution of the form  $e^{kt}$ .
  - (b) Can you express all solutions to equation (1) in the form  $Ae^{k_1 t} + Be^{k_2 t}$  for some fixed constants  $k_1$  and  $k_2$ ? Why or why not?
  - (c) Find a  $k$  such that  $te^{kt}$  is a solution to equation (1).
  - (d) Can you express all solutions to equation (1) in the form  $Ae^{k_1 t} + Bte^{k_2 t}$  for some  $k_1$  and  $k_2$ ? Explain.
  - (e) Can you express all solutions to equation (1) in the form  $A(1+t)e^{k_1 t} + B(1-t)e^{k_2 t}$  for some  $k_1$  and  $k_2$ ? Explain.
3. The *characteristic polynomial* of a linear homogeneous differential equation  $F(y) = 0$  is the polynomial  $p(k)$  so that  $F(e^{kt}) = p(k)e^{kt}$ . (Recall here that  $F$  a linear function. For example, maybe  $F(y) = y'' - 2y$ .)
- (a) For each linear differential equations from part 1, write the characteristic polynomial (if it isn't homogeneous, consider the corresponding homogeneous equation).
  - (b) Consider a second-order linear homogeneous ODE with characteristic polynomial  $p(k)$  having roots  $\alpha$  and  $\beta$ . Write the equation for this ODE.
  - (c) Consider a second-order linear homogeneous ODE with characteristic polynomial  $p(k)$ . Suppose that  $p(k)$  is two roots  $\alpha, \beta$  with  $\alpha \neq \beta$ . Show that  $Ae^{\alpha t} + Be^{\beta t}$  is always a solution. Further, show that all solutions to this differential equation can be written in this way.
  - (d) Consider a second-order linear homogeneous ODE with characteristic polynomial  $p(k)$  having one repeated root  $\alpha$ . Show that  $Ae^{\alpha t} + Bte^{\alpha t}$  is always a solution. Further, show that all solutions to this differential equation can be written in this way.
  - (e) Consider a second-order linear homogeneous ODE with characteristic polynomial  $p(k)$  having two distinct roots  $\alpha, \beta$ . Show that there is no  $\gamma$  such that  $te^{\gamma t}$  is a solution.
  - (f) Consider a third-order linear homogeneous ODE with characteristic polynomial  $p(k) = (k-2)^3$ . Write the equation for this ODE. Then, find all solutions.

4. Consider  $y'' + y = 0$ .

- (a) Show that for any initial value problem, you can find  $A, B \in \mathbb{C}$  so that

$$Ae^{i\theta} + Be^{-i\theta}$$

is a solution. Could you solve any IVP if you restrict  $A, B \in \mathbb{R}$ ?

- (b) Use part (a) along with Euler's formula and the fact that  $\sin$  and  $\cos$  are solutions to this differential equation to come up with a formula for  $\sin$  and  $\cos$  as a linear combination of complex exponentials.
- (c) Show that  $A \cos \theta + B \sin \theta$  also solves every initial value problem.
- (d) Can you get any additional solutions by considering  $A \cos \theta + B \sin \theta + Ae^{i\theta} + Be^{-i\theta}$ ? Explain.

- (e) Use your formula from part (b) to find arccos and arcsin in terms of complex numbers and complex logarithms. (Hint:  $e^{2t} = (e^t)^2$  and  $e^t e^{-t} = 1$ ).
5. The matrices are coming! Read about *matrix multiplication* in the source of your choice (the Evans text talks about it) and how to compute  $2 \times 2$  and  $3 \times 3$  determinants.

(a) Compute  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ ,  $\det \left( \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \right)$ , and  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}^2$ .

- (b) Let  $\vec{a} = (1, 2, 3)$  and  $\vec{b} = (1, 2, 2)$ . Find matrices  $A$  and  $B$  so that  $\vec{a} \cdot \vec{b}$  is equivalent to the matrix product  $AB$ . What are the dimensions of  $A$  and  $B$ ?

- (c) Let  $M = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ , and consider the function  $F_M : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by

$$F_M(a, b) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Show that  $F_M$  is linear.