1. Remember tangent planes and linear approximations? Given a function $f(x, y)$, we approximated it at the point $\vec{p}$ as $f(\vec{p}+\vec{x}) \approx L(\vec{x}) + f(\vec{p})$, where L was a *linear function*. To be more precise, $L(x_1, \ldots, x_n) = \sum \alpha_i x_i$ for some choice of α_i, and that choice of α_i happens to be the directional derivative of f in the $\hat{x}, \hat{y}$, etc. directions.

   (a) Let $f(x) = x^3$. Find a linear approximation to f at $x = 2$. That is, find a linear function L so that $f(2 + x) \approx L(x) + f(2)$.

   (b) Let $f(x, y) = -yx^2$. Find a linear approximation to f at $(x, y) = (2, 3)$. That is, find a linear function L so that $f((2, 3) + (x, y)) \approx L(x, y) + f(2, 3)$.

   (c) Let $f(x, y, z) = -yx^2 + z^3$. Find a linear approximation to f at $(x, y, z) = (2, 3, 1)$.

   (d) Now let's do something we've never done before. Consider the vector field

   $$\vec{f}(x, y, z) = \begin{bmatrix} xy \\ -z^2 \\ zx \end{bmatrix} = \begin{bmatrix} p(x, y, z) \\ q(x, y, z) \\ r(x, y, z) \end{bmatrix}.$$

   We'd like to find a linear approximation of $\vec{f}$ at the point $(x, y, z) = (2, 3, 1)$. The easiest way to do this is to find linear approximations for p, q, and r, and stick them in as the components of our linear approximation of f. I'd like your answer to look like $\vec{f}((2, 3, 1) + (x, y, z)) = \vec{L}(x, y, z) + \vec{f}(2, 3, 1)$ where each component of $\vec{L}$ is a linear function.

   (e) If a vector field is given by a linear function $\vec{L}$, must the curl of $\vec{L}$ be zero? Prove or give a counter example.

2. In the definition of curl, we found the circulation around a rectangle and shrunk that rectangle to zero. What happens if we tried another shape, like a circle? Let's test this in the simple case of finding $\text{curl}_{\hat{z}}(\vec{f})$ where $\vec{f}$ is linear.

   For this problem, we will assume

   $$\vec{f}(x, y, z) = \begin{bmatrix} A_x(x, y, z) \\ B_y(x, y, z) \\ D_z(x, y, z) \end{bmatrix} = \begin{bmatrix} \alpha_x x + \alpha_y y + \alpha_z z \\ \beta_x x + \beta_y y + \beta_z z \\ \delta_x x + \delta_y y + \delta_z z \end{bmatrix}$$

   (a) Find $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ of $\vec{f}$.

   (b) Parameterize C, a counter-clockwise oriented circle with radius r, centered at the origin and lying in the xy-plane.

   (c) Set up and evaluate an integral for the circulation of $\vec{f}$ around C.

   (d) Evaluate $\displaystyle\lim_{r\to 0} \frac{\text{circulation around } C}{\text{area of } C}$. Is this the same as the definition of curl using a rectangle? Explain.

3. Let $\vec{f}(x, y, z) = (-y, zx, x + z)$.

   (a) Find the curl of $\vec{f}$.

   (b) Imagine $\vec{f}$ represents the force of the wind at every point in space. If you place a tiny ball at $\vec{a} = (1, 2, 3)$, will it start spinning? What will be its axis of rotation?

   (c) Let H be the upper half of a sphere of radius 2 centered at the origin (i.e., the part of the sphere with positive z component). Compute the circulation around ∂H and the flux of $\nabla \times \vec{f}$ through H. How do they compare?

4. Plot each of the following vector fields (your picture can be 2d). From your picture, estimate divergence and curl, then compute the divergence and curl.

   (a) $\vec{F}(x, y, z) = (-y, x, 0)$

(b) $\vec{G}(x, y, z) = (x, 2y, 0)$

(c) $\vec{H}(x, y, z) = (x, -y, 0)$

(d) $\vec{J}(x, y, z) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0)$

(e) A vector field is called *closed* if it has zero curl. The Poincaré lemma states that: *If a vector field $\vec{A} : \mathbb{R}^3 \to \mathbb{R}^3$ is closed on the interior of a sphere $\mathcal{S}$, then $\vec{A}$ is conservative on the interior of $\mathcal{S}$.* For each vector field above, explain what the Poincaré lemma says about the existence of a potential function. If a global potential function exists (that is, there is some f so that $\nabla f = \vec{A}$ everywhere), write it down.

If only a local potential function exists at the point $\vec{x}$, find a sphere $\mathcal{S}_{\vec{x}}$ and a potential $f_{\vec{x}}$ so that $\nabla f_{\vec{x}} = \vec{A}$ when restricted to the interior of $\mathcal{S}_{\vec{x}}$. For simplicity, you may assume $\mathcal{S}_{\vec{x}}$ is centered at $\vec{x}$ and just give its radius.

Hint: pay special attention to where each vector field is defined and where your potential function(s) are defined, and look at the previous homework set for if you need extra inspiration.