1. Consider the second order linear homogeneous differential equation

$$y'' + y' - 6y = 0. \tag{1}$$

   (a) Write down the corresponding system of first order differential equations in matrix form. That is, you system should be written as

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = M \begin{bmatrix} a \\ b \end{bmatrix} \tag{2}$$

   for some matrix M.

   (b) Find all eigenvalues of M.

   (c) An eigenvector for (1) is a solution ϕ such that $\phi' = k\phi$ for some k. Use your knowledge of how to solve (1) to find two eigenvectors for (1).

   (d) Use your eigenvectors from part (c) to find two eigenvectors for M. That is, find vectors $\vec{v}$ and $\vec{w}$ so that $M\vec{v} = k_1\vec{v}$ and $M\vec{w} = k_2\vec{w}$ for scalars k_1 and k_2.

   (e) Write down a flow for (2) that passes through the point $(a, b) = (a_0, b_0)$ at time $t = 0$.

   (f) Show that if $\vec{\varphi}_1 : \mathbb{R} \to \mathbb{R}^2$ and $\vec{\varphi}_2 : \mathbb{R} \to \mathbb{R}^2$ are flows for (2), then so is any linear combination of $\vec{\varphi}_1$ and $\vec{\varphi}_2$. (You may use facts about the linearity of matrix multiplication that you proved last homework).

2. Consider the second order linear homogeneous differential equation

$$y'' + \alpha y' + \beta y = 0 \tag{3}$$

   where α and β are constants.

   (a) Write the corresponding system of first order linear differential equations in matrix form.

   (b) Classify the α, β that give rise to sources, sinks, spiral sources, spiral sinks, saddles, and swirls. Are there any $\alpha, \beta \in \mathbb{R}$ that don't fit into this classification?

   (c) Classify the α, β that give rise to forward-stable, backward-stable, bi-stable, and unstable solutions to (3).

3. The motion of a simple pendulum is governed by the equation

$$\theta'' + \frac{g}{\ell} \sin\theta = 0$$

   where g is the acceleration due to gravity, ℓ is the length of the rigid pendulum arm, and θ is the angular displacement from vertical. Since we're doing math in this course, we may assume $g = \ell = 1$.

   (a) Write a system of first-order differential equations corresponding to the equation for a pendulum and draw a phase portrait for this system.

   (b) Analyze the critical points for the pendulum system (by linearizing the system near the critical points). Does their stability or instability make physical sense? Explain.

   (c) There are three fundamental behaviors for a simple pendulum: stationary, swinging back and forth, and spinning all the way around. Draw, and clearly label, a flow corresponding to each behavior on your phase portrait.

   (d) Given an initial condition $(t, \theta, \theta') = (0, \theta_0, 0)$, if θ_0 is small, a pendulum's motion is modeled well by simple harmonic oscillation (i.e., a $\theta = k\cos(\omega t)$ for some k, ω). We're going to see exactly how well.

   Let $(a', b') = \vec{F}(a, b)$ be the system of ODEs corresponding to a simple pendulum, and let $(a', b') = \vec{L}(a, b)$ be the linearization of $\vec{F}$ around the critical point $(0, 0)$. Let $\vec{\varphi}_{\theta_0}$ be the flow along $\vec{F}$ with initial conditions $(t, \theta, \theta') = (0, \theta_0, 0)$ and let $\vec{\gamma}_{\theta_0}$ be the flow long

$\vec{L}$ with initial conditions $(t, \theta, \theta') = (0, \theta_0, 0)$. Use numerical techniques to estimate the period of $\vec{\varphi}_{\theta_0}$ and $\vec{\gamma}_{\theta_0}$ for at least three choices of θ_0. Give a range of θ_0 where the relative error between the period of $\vec{\varphi}_{\theta_0}$ and $\vec{\gamma}_{\theta_0}$ is less than 5%. (Recall, relative error between quantities a and b is $|(a-b)/a|$.)

4. Consider the vector field $\vec{F}(x, y) = (x + y^2, -y)$.

   (a) For each of the following functions, show whether or not it is a flow for $\vec{F}$.

      i. $\vec{\varphi}(t) = (-e^{-2t}, \sqrt{3}e^{-t})$

      ii. $\vec{\varphi}(t) = (-e^{-4t}, \sqrt{3}e^{-2t})$

      iii. $\vec{\varphi}(t) = (\frac{4}{3}e^{t} - \frac{1}{3}e^{-2t}, e^{-t})$

      iv. $\vec{\varphi}(t) = (e^{t} - e^{-2t}, e^{-t})$

   (b) A *constant flow* is a flow $\vec{\varphi}$ such that $\vec{\varphi}(t_1) = \vec{\varphi}(t_2)$ for all $t_1, t_2 \in \mathbb{R}$. Find all constant flows for $\vec{F}$. (Hint, think about $\vec{\varphi}'$ in this situation.)

   (c) Classify all critical points of the system of differential equations given by $(a'(t), b'(t)) = \vec{F}(a(t), b(t))$.

   (d) Find *all* flows of $\vec{F}$ that pass through the point $(1, 0)$. Remember, if $\vec{\varphi}$ is a flow with this property, it is not a requirement that $\vec{\varphi}(0) = (1, 0)$, only that $\vec{\varphi}(t_0) = (1, 0)$ for some t_0.

   (e) We call the flows $\vec{\varphi}_1$ and $\vec{\varphi}_2$ time shifts of each other if $\vec{\varphi}_1(t) = \vec{\varphi}_2(t + t_0)$ for some t_0. Show that the flows from part (d) are time shifts of each other.

5. I HOPE WE HAVEN'T FORGOTTEN ABOUT VECTOR CALCULUS! Consider the vector field $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ and imagine it describes the velocity of some fluid spread out on a plane (maybe we'll call this whole setup a laminar flow).

   Let $\vec{\varphi}_{\vec{x}}$ be the flow along $\vec{F}$ with initial conditions $\vec{\varphi}_{\vec{x}}(0) = \vec{x}$. That is, $\vec{\varphi}_{\vec{x}}(t)$ describes the position of a particle that starts at $\vec{x}$ at time zero and flows for t seconds.

   Viewing $\vec{\varphi}_{\vec{x}}$ as a function of $\vec{x}$, we can view $\vec{\varphi}_{\vec{x}}(t)$ as inducing a different coordinate change for each t. Specifically, for a fixed t_0, we could think of the coordinate change

$$\vec{x} \mapsto \vec{\varphi}_{\vec{x}}(t_0).$$

   Let $c_{t_0} : \mathbb{R}^2 \to \mathbb{R}^2$ be this coordinate change.

   Our goal will be to compute the volume form for this coordinate change and learn some fluid mechanics along the way!

   (a) Suppose the volume form for the change of coordinates induced by c_t was $V_t(x, y) = 3^t$. If you released a drop of die into the laminar flow governed by $\vec{F}$ and at time 0 it had area 4, what would the area be after 1 second?

   (b) We can compute the volume form by estimating the change in volume of a tiny square under the function c_{t_0} and taking the limit as the square gets infinitely tiny (just like we did before). Let $R_\varepsilon^{(a,b)}$ be the square of side lengths ε and lower left corner at (a, b). Compute

$$V_{t_0}(a, b) = \lim_{\varepsilon \to 0} \frac{\text{area of } c_{t_0}(R_\varepsilon^{(a,b)})}{\text{area of } R_\varepsilon^{(a,b)}}.$$

   You may express your answer in terms of partial derivatives. You may also find it helpful to assume that if ε is tiny, then $c_{t_0}(R_\varepsilon^{(a,b)})$ is a parallelogram. Further, recall that determinants can be used to compute the area of a parallelogram and that the determinant is a continuous function, and so you may move limits in and out of a determinant.

   (c) The change of coordinates function c_t has some fantastic properties. Explain why $c_{t_1} \circ c_{t_2} = c_{t_1+t_2}$. Using this fact, explain why $V_{t_1+t_2}(x, y) = V_{t_1}(x, y)V_{t_2}(c_{t_1}(x, y))$.

(d) Write down the limit definition of the derivative of $\dfrac{\mathrm{d}V_t}{\mathrm{d}t}(x,y)$ at $t = t_0$ and at $t = 0$. Then, use your knowledge from the previous part to write down an expression for

$$\left.\frac{\mathrm{d}V_t}{\mathrm{d}t}(x,y)\right|_{t=t_0}$$

in terms of $\left.\frac{\mathrm{d}V_t}{\mathrm{d}t}\right|_{t=0}$.

(e) Use $\vec{F}$ to write down a linear approximation to c_t when t is very close to zero. Using this approximation, compute $\dfrac{\mathrm{d}V_t}{\mathrm{d}t}$ at $t = 0$ and show that

$$\left.\frac{\mathrm{d}V_t}{\mathrm{d}t}\right|_{t=0} = \nabla \cdot \vec{F}.$$

(f) A flow is called *incompressible* if $\nabla \cdot \vec{F} = 0$. Explain why. In particular, your explanation should in some way combine knowledge from parts (d) and (e) to draw strong conclusions about the volume form.

(g) Let $F(x,y) = (x + y^2, -y)$ from the previous problem. If you released a drop of die into the laminar flow governed by $\vec{F}$ and at time 0 it had area 4, what would the area be after 1 second?