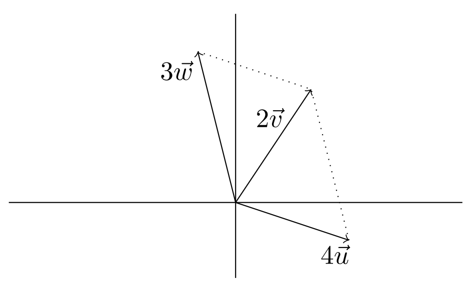
1. For each of the following statements, produce a counterexample to show that the statement is **false** in general.
   (a) If A and B are square matrices, $AB = BA$.
   (b) If $AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then A and B are 2×2 matrices.
   (c) If $AB = I$, then $BA = I$.
   (d) If $A^2 = 0$, then $A = 0$.

2. Let $R = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.
   (a) Find all solutions to the matrix equation $R \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$.
   (b) Prove that the set $X = \{\vec{x} \in \mathbb{R}^3 : R\vec{x} = \vec{0}\}$ is a subspace.
   (c) Prove that the set $Y = \{\vec{y} \in \mathbb{R}^3 : \vec{y} = R\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^3\}$ is a subspace.

3. This problem explores two interpretations of matrix multiplication: *linear combinations of columns* and *dot products with rows*. If you need ideas, please review these two interpretations.

   Suppose E is a 4×3 matrix with columns $\vec{c}_1, \vec{c}_2, \vec{c}_3$ and rows $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4$. Let $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.
   (a) Express $E\vec{v}$ as a linear combination of $\vec{c}_1, \vec{c}_2, \vec{c}_3$.
   (b) Supposing $\vec{r}_1 \cdot \vec{v} = 1$, $\vec{r}_2 \cdot \vec{v} = 6$, $(\vec{r}_3 + \vec{r}_4) \cdot \vec{v} = 2$, and $(\vec{r}_3 - \vec{r}_4) \cdot \vec{v} = -2$, compute $E\vec{v}$.

4. Suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^2$ that are related by the following diagram of a parallelogram.



   Let $A = [\vec{u}|\vec{v}|\vec{w}]$ be the matrix with columns $\vec{u}$, $\vec{v}$, and $\vec{w}$.
   (a) What is the rank of A?
   (b) Find all solutions to the equation $A\vec{x} = \vec{0}$.
   (c) Find a basis for the subspace $V = \{\vec{x} \in \mathbb{R}^3 : A\vec{x} = \vec{0}\}$.

5. (a) For each of the following linear transformations, find a matrix corresponding to the transformation.
      i. $\mathcal{A}$: Rotate 90° counterclockwise around the origin (in $\mathbb{R}^2$).
      ii. $\mathcal{B}$: Send every vector in $\mathbb{R}^2$ to the zero vector.
      iii. $\mathcal{C}$: Project (in $\mathbb{R}^3$) onto the yz-plane.
      iv. $\mathcal{D}$: Project (in $\mathbb{R}^3$) onto the x-axis.

v. $\mathcal{E}$: Reflect (in $\mathbb{R}^3$) across the xy-plane.
vi. $\mathcal{F}$: Stretch by a factor of 2 in the y-direction (in $\mathbb{R}^2$).
vii. $\mathcal{G}$: Stretch by a factor of 2 in the y-direction (in $\mathbb{R}^3$).
viii. $\mathcal{H}$: Rotate 90° counterclockwise (as viewed looking "down" from the positive y-axis towards the origin) around the y-axis (in $\mathbb{R}^3$).

(b) For each transformation in part (a), explain geometrically whether or not the transformation is invertible. (Recall, a function f is invertible if there exists another function g so that $f \circ g$ *and* $g \circ f$ are both the identity function.)

(c) Let

$$X = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad W = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For each matrix $M \in \{X, Y, Z, W\}$, define a transformation T_M where $T_M(\vec{v}) = M\vec{v}$. For each M, explain how to obtain T_M as a combination of linear transformations from part (a). (Hint: you may need to combine two, three, or more transformations from part (a).)

6. Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subset \mathbb{R}^3$ be a linearly independent set. Suppose a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $T(\vec{v}_i) = \vec{v}_i$ for $i \in \{1, 2, 3\}$. Prove that T is the identity transformation.

7. LINEAR TRANSFORMATIONS WERE HERE THE WHOLE TIME! For each transformation, determine whether or not it is a linear transformation. If it is a linear transformation, give its null space, i.e., all solutions to $\mathcal{F}\vec{x} = \vec{0}$.

(a) Let $V = \operatorname{span}\{1, x, x^2\}$ where $1, x, x^2$ are (as usual) polynomials. Let $D : V \to V$ be differentiation. (Use the limit definition of the derivative. However, you can do limits like you did in Calc I; you don't have to do ε–δ proofs).

(b) Let $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$ and let $D^s : \mathcal{F} \to \mathcal{F}$ be left-translation by s. That is, $D^s f(t) = f(t + s)$.

(c) Let $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$ and let $D^s : \mathcal{F} \to \mathcal{F}$ be vertical-translation by s. That is, $D^s f(t) = f(t) + s$.

(d) Let $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$ and let $D^s : \mathcal{F} \to \mathcal{F}$ be horizontal-stretching by s. That is, $D^s f(t) = f(t/s)$.

(e) Let $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$ and let $D^s : \mathcal{F} \to \mathcal{F}$ be vertical-stretching by s. That is, $D^s f(t) = sf(t)$.

(f) Let $\mathcal{I} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is increasing}\}$ and let $D : \mathcal{I} \to \mathcal{I}$ be so that $D(f) = f^{-1}$ takes a function to its inverse.

(g) Let $\mathcal{L} = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) = mx + b \text{ for some } m, b \in \mathbb{R}\}$ be the set of all functions whose graphs are non-vertical lines and let $D : \mathcal{L} \to \mathbb{R}$ be evaluation at the point $x = 1$. That is, $D(f) = f(1)$.