1. (a) Find all values of x, y that satisfy the following relationships:

$$\begin{aligned} x + y &= 7 \\ 2x - 3y &= 13. \end{aligned}$$

What can you say about span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$ and the vector $\begin{bmatrix} 7 \\ 13 \end{bmatrix}$?

(b) Find values of x, y, z that satisfy the following relationships (your answer may involve ugly fractions):

$$\begin{aligned} x + 2y + 8z &= 1 \\ 4x + 5y + 8z &= 2. \end{aligned}$$

What can you say about span $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \end{bmatrix} \right\}$ and the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$?

(c) Let $\vec{w} = \begin{bmatrix} 5 \\ -12 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Express $\vec{w}$ as a linear combination of $\vec{u}$ and $\vec{v}$.

2. (a) Let

$$S = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Is S a point, line, plane, or all of $\mathbb{R}^3$? Explain.

(b) Let $\vec{u} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ -4 \\ -4 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 14 \\ 0 \\ d \end{bmatrix}$.

i. For what value(s) of d is span$\{\vec{u}, \vec{v}, \vec{w}\}$ a plane?

ii. Is there a value of d so span$\{\vec{u}, \vec{v}, \vec{w}\}$ a line? Explain.

3. Let $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Prove that $\{\vec{x}, \vec{y}\}$ is a basis for $\mathbb{R}^2$.

4. Let

$$\begin{aligned} U &= \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \right\}, \\ V &= \left\{ \vec{x} \in \mathbb{R}^3 : \vec{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \right\}, \\ W &= U \cup V. \end{aligned}$$

For each subset U, V, W of $\mathbb{R}^3$, show whether it is a subspace or not. If it is a subspace, classify it as a point, line, plane, or all of $\mathbb{R}^3$. Further, if it is a subspace, give a basis for it.

5. Notice that no definitions we've used so far have referred to coordinates, just vectors and scalars (we haven't even demanded our scalars be $\mathbb{R}$). This is on purpose because any collection of objects that can be added and scalar multiplied can be treated as a "vector."

Let $\vec{u} = 1$, $\vec{v} = x$, and $\vec{w} = x^2$ be polynomials and let $Q = \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

(a) Is $4x + 3 \in Q$? What about $x^3 - 3x + 2$?

(b) Describe span$\{\vec{u}, \vec{v}, \vec{w}\}$ and span$\{\vec{v}, \vec{w}\}$.

(c) Consider the differential equation

$$y'' - 4y = 0.$$

Express all solutions as a span.

(d) Is the set $X = \{e^{it}, e^{-it}, \sin t, \cos t\}$ linearly independent if we allow complex scalars? (You may reference homework from last term to answer this question.) Explain.

(e) Let $S = \operatorname{span}\{\sin t, \cos t\}$ (we're back to real scalars now). Is $\sin(t + \pi/6) \in S$? Show that $S = \{\alpha \sin(t + \beta) : \alpha, \beta \in \mathbb{R}\}$.

(f) Define the map $P : \mathbb{R}^2 \to \mathbb{R}^2$ in the following way. Given $(\alpha, \beta) \in \mathbb{R}^2$ with $(\alpha, \beta) \neq \vec{0}$, find the unique $\alpha' \in [0, \infty)$ and $\beta' \in [0, 2\pi)$ such that $\alpha \sin t + \beta \cos t = \alpha' \sin(t + \beta')$. Define $P(\alpha, \beta) = (\alpha', \beta')$ and $P(0, 0) = (0, 0)$.

Find a formula for P. Is P invertible? If so, find a formula for its inverse? Do these formulas look familiar?