1. Suppose the matrix equation $A\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$ has the general solution

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

   (a) How many rows and how many columns does A have?
   (b) Find $\text{null}(A)$.
   (c) Find $\text{rank}(A)$.
   (d) Find $\text{col}(A)$.
   (e) Find $\text{row}(A)$.

2. NOT ALL BASES ARE CREATED EQUALLY. Let $\mathcal{X} = \{\vec{x}_1, \vec{x}_2\}$ be a basis for $\mathbb{R}^2$ consisting of unit vectors. By definition, for any $\vec{v} \in \mathbb{R}^2$, we can find a unique $\alpha_1, \alpha_2 \in \mathbb{R}^2$ so that

$$\vec{v} = \alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2.$$

   We call the quantity $S_{\mathcal{X}}(\vec{v}) = \sqrt{\alpha_1^2 + \alpha_2^2}/\|\vec{v}\|$ the *size of the representation of of $\vec{v}$ in the basis $\mathcal{X}$*. For many applications, it is desirable for $S_{\mathcal{X}}$ to be as small as possible. We are going to explore what makes $S_{\mathcal{X}}$ small or large.

   (a) Let

$$\vec{a}' = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \qquad \vec{b}' = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

   and $\vec{a} = \vec{a}'/\|\vec{a}'\|$ and $\vec{b} = \vec{b}'/\|\vec{b}'\|$ and consider the basis $\mathcal{B} = \{\vec{a}, \vec{b}\}$.
   Use Matlab/Octave to compute the **average** of $S_{\mathcal{B}}(\vec{v})$ for at least 1000 vectors with random entries in the interval $[-1, 1]$. (Hint: you can use `(rand(2,1) - .5)*2` to create a random vector with entries in $[-1, 1]$).

   (b) Let $\mathcal{S} = \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ be the standard basis for $\mathbb{R}^2$. Use Matlab/Octave to compute the average of $S_{\mathcal{S}}(\vec{v})$ for at least 1000 vectors with random entries in the interval $[-1, 1]$.

   (c) Consider the basis $\mathcal{Q}_\theta = \{\hat{\mathbf{x}}, \vec{v}_\theta\}$ where $\vec{v}_\theta = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\theta \neq n\pi$.
   What happens to the average value of $S_{\mathcal{Q}_\theta}$ as $\theta \to 0$? Why?

   (d) We're going to numerically explore the space of unit-vector bases for $\mathbb{R}^2$. Let $\vec{b}_t = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ and let $\mathcal{B}_{\theta_1,\theta_2} = \left\{\vec{b}_{\theta_1}, \vec{b}_{\theta_2}\right\}$ where θ_1, θ_2 are randomly chosen from the interval $[0, 2\pi]$.
      i. What percentage of random bases for $\mathbb{R}^2$ satisfy $S_{\mathcal{B}} < 2$?
      ii. What percentage satisfy $S_{\mathcal{B}} > 10$?
      iii. What is the smallest that $S_{\mathcal{B}}$ can be?
      iv. What do bases achieving minimal values for $S_{\mathcal{B}}$ look like?

   (e) Now let's explore bases for $\mathbb{R}^3$. For a basis of unit vectors, $\mathcal{B}$, in $\mathbb{R}^3$, we define $S_{\mathcal{B}}$ in the same way. However, choosing a random basis of unit vectors is a little harder in $\mathbb{R}^3$. (We cannot just use spherical coordinates and randomly pick ϕ and θ since that would cause most of the vectors to collect at the north and south poles—remember what the volume form looks like for polar coordinates?) However, we can use a trick.
   The Matlab/Octave command `randn(3,1)` will create a random vector with three *normally distributed* components. If we turn this vector into a unit vector (something like `a = randn(3,1); a = a/norm(a)`) it will be randomly (and uniformly) distributed on the unit sphere in $\mathbb{R}^3$.

Find out what percentage of random bases for $\mathbb{R}^3$ satisfy $S_{\mathcal{B}} < 2$. Provide a histogram of $S_{\mathcal{B}}$ for random bases for $\mathbb{R}^3$. (The Matlab/Octave command `hist(v)` with give you a histogram if `v` is a list of numbers.)