Before next class:

- **Watch videos 11.3, 11.4**

Warm up

Write a formula for the general term of these sequences

(A) $\{a_n\}_{n=0}^{\infty} = \{\, 1,\ 4,\ 9,\ 16,\ 25,\ \ldots \,\}$

(B) $\{b_n\}_{n=1}^{\infty} = \{\, 1,\ -2,\ 4,\ -8,\ 16,\ -32,\ \ldots \,\}$

(C) $\{c_n\}_{n=1}^{\infty} = \left\{ \frac{2}{1!}, \frac{3}{2!}, \frac{4}{3!}, \frac{5}{4!}, \ldots \right\}$

(D) $\{d_n\}_{n=1}^{\infty} = \{\, 1,\ 4,\ 7,\ 10,\ 13,\ \ldots \,\}$

Sequences vs functions – convergence

For any function f with domain $[0, \infty)$,
we define a sequence as $a_n = f(n)$.
Let $L \in \mathbb{R}$. Which of these implications is true?

(A) IF $\lim_{x\to\infty} f(x) = L$, THEN $\lim_{n\to\infty} a_n = L$.

(B) IF $\lim_{n\to\infty} a_n = L$, THEN $\lim_{x\to\infty} f(x) = L$.

(C) IF $\lim_{n\to\infty} a_n = L$, THEN $\lim_{n\to\infty} a_{n+1} = L$.

Definition of limit of a sequence

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$.
Which statements are equivalent to "$\{a_n\}_{n=0}^{\infty} \longrightarrow L$"?

(A) $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$

(B) $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\quad n > n_0 \implies |L - a_n| < \varepsilon.$

(C) $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{R},\ \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$

(D) $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{R},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$

(E) $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| \leq \varepsilon.$

(F) $\forall \varepsilon \in (0,1),\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$

(G) $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \dfrac{1}{\varepsilon}.$

(H) $\forall k \in \mathbb{Z}^+,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < k.$

(I) $\forall k \in \mathbb{Z}^+,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \dfrac{1}{k}.$

Before next class:

- **Watch videos 11.5, 11.6**

Sequences vs functions – monotonicity and boundness

For any function f with domain $[0, \infty)$,
we define a sequence as $a_n = f(n)$.
Which of these implications is true?

(A) IF f is increasing, THEN $\{a_n\}_{n=0}^{\infty}$ is increasing.

(B) IF $\{a_n\}_{n=0}^{\infty}$ is increasing, THEN f is increasing.

(C) IF f is bounded, THEN $\{a_n\}_{n=0}^{\infty}$ is bounded.

(D) IF $\{a_n\}_{n=0}^{\infty}$ is bounded, THEN f is bounded.

Examples

Construct 8 examples of sequences.
If any of them is impossible, cite a theorem to justify it.

		convergent	divergent
monotonic	bounded		
	unbounded		
not monotonic	bounded		
	unbounded		

A sequence defined by recurrence

Consider the sequence $\{R_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} & R_0 = 1 \\ \forall n \in \mathbb{N}, & R_{n+1} = \dfrac{R_n + 2}{R_n + 3} \end{cases}$$

Compute R_1, R_2, R_3.

True or False - convergence, monotonicity, and boundedness

(A) If a sequence is convergent, then it is bounded above.

(B) If a sequence is bounded, then it is convergent

(C) If a sequence is convergent, then it is eventually monotonic.

(D) If a sequence is positive and converges to 0, then it is eventually monotonic.

(E) If a sequence diverges to ∞, then it is eventually monotonic.

(F) If a sequence diverges, then it is unbounded.

(G) If a sequence diverges and is unbounded above, then it diverges to ∞.

(H) If a sequence is eventually monotonic, then it is either convergent, divergent to ∞, or divergent to $-\infty$.

Convergence and divergence

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence.
Write the formal definition of the following concepts:

(A) $\{a_n\}_{n=0}^{\infty}$ is convergent.

(B) $\{a_n\}_{n=0}^{\infty}$ is divergent.

(C) $\{a_n\}_{n=0}^{\infty}$ is divergent to ∞.

Before next class:

- **Watch videos 11.7, 11.8**

True or False - Rapid fire

(A) (convergent) $\implies$ (bounded)

(B) (convergent) $\implies$ (monotonic)

(C) (convergent) $\implies$ (eventually monotonic)

(D) (bounded) $\implies$ (convergent)

(E) (monotonic) $\implies$ (convergent)

(F) (bounded + monotonic) $\implies$ (convergent)

(G) (divergent to ∞) $\implies$ (eventually monotonic)

(H) (divergent to ∞) $\implies$ (unbounded above)

(I) (unbounded above) $\implies$ (divergent to ∞)

Proof of Theorem 3

Write a proof for the following Theorem

Theorem 3

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence.

- IF $\{a_n\}_{n=0}^{\infty}$ is increasing AND unbounded above,
- THEN $\{a_n\}_{n=0}^{\infty}$ is divergent to ∞

Proof of Theorem 3

Write a proof for the following Theorem

Theorem 3

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence.

- IF $\{a_n\}_{n=0}^{\infty}$ is increasing AND unbounded above,
- THEN $\{a_n\}_{n=0}^{\infty}$ is divergent to ∞

(A) Write the definitions of "increasing", "unbounded above", and "divergent to ∞"

(B) Using the definition of what you want to prove, write down the structure of the formal proof.

(C) Do some rough work if necessary.

(D) Write a formal proof.

Proof feedback

(A) Does your proof have the correct structure?
(B) Are all your variables fixed (not quantified)? In the right order? Do you know what depends on what?
(C) Is the proof self-contained? Or do I need to read the rough work to understand it?
(D) Does each statement follow logically from previous statements?
(E) Did you explain what you were doing? Would your reader be able to follow your thought process without reading your mind?

Critique this proof - #1

- $\forall M \in \mathbb{R},\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies x_n > M$

- M is not an upper bound: $\exists n_0 \in \mathbb{N}$ s.t. $x_{n_0} > M$

- $n \geq n_0 \implies x_n \geq x_{n_0} > M$

Critique this proof - #2

- WTS $a_n \to \infty$. This means:
  $\forall M \in \mathbb{R},\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies x_n > M$

- bounded above: $\exists M \in \mathbb{R},\ \forall n \in \mathbb{N},\ x_n \leq M$

- negation: $\forall M \in \mathbb{R},\ \exists n \in \mathbb{N},\ x_n > M$

- $\forall n \in \mathbb{N}$, take $n = n_0$.

Before next class:

- **Watch videos 12.1, 12.4, 12.5**

Calculations

(A) $$\lim_{n\to\infty} \frac{n! + 2e^n}{3n! + 4e^n}$$

(B) $$\lim_{n\to\infty} \frac{2^n + (2n)^2}{2^{n+1} + n^2}$$

(C) $$\lim_{n\to\infty} \frac{5n^5 + 5^n + 5n!}{n^n}$$

True or False – The Big Theorem

Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be positive sequences.

(A) IF $a_n << b_n$, THEN $\forall m \in \mathbb{N},\ a_m < b_m$.

(B) IF $a_n << b_n$, THEN $\exists m \in \mathbb{N}$ s.t. $a_m < b_m$.

(C) IF $a_n << b_n$, THEN $\exists n_0 \in \mathbb{N}$ s.t.
$\forall m \in \mathbb{N},\ m \geq n_0 \implies a_m < b_m$.

True or False – The Big Theorem

Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be positive sequences.

(A) IF $a_n << b_n$, THEN $\forall m \in \mathbb{N},\ a_m < b_m$.

(B) IF $a_n << b_n$, THEN $\exists m \in \mathbb{N}$ s.t. $a_m < b_m$.

(C) IF $a_n << b_n$, THEN $\exists n_0 \in \mathbb{N}$ s.t. $\forall m \in \mathbb{N},\ m \geq n_0 \implies a_m < b_m$.

(D) IF $\forall m \in \mathbb{N},\ a_m < b_m$, THEN $a_n << b_n$.

(E) IF $\exists m \in \mathbb{N}$ s.t. $a_m < b_m$, THEN $a_n << b_n$.

(F) IF $\exists n_0 \in \mathbb{N}$ s.t. $\forall m \in \mathbb{N},\ m \geq n_0 \implies a_m < b_m$, THEN $a_n << b_n$.

Refining the Big Theorem - 1

(A) Construct a sequence $\{u_n\}_n$ such that

$$\begin{cases} \forall a < 0, & n^a \ll u_n \\ \forall a \geq 0, & u_n \ll n^a \end{cases}$$

(B) Construct a sequence $\{v_n\}_n$ such that

$$\begin{cases} \forall a \leq 0, & n^a \ll v_n \\ \forall a > 0, & v_n \ll n^a \end{cases}$$

Refining the Big Theorem - 2

(A) Construct a sequence $\{u_n\}_n$ such that

$$\begin{cases} \forall a < 2, & n^a \ll u_n \\ \forall a \geq 2, & u_n \ll n^a \end{cases}$$

(B) Construct a sequence $\{v_n\}_n$ such that

$$\begin{cases} \forall a \leq 2, & n^a \ll v_n \\ \forall a > 2, & v_n \ll n^a \end{cases}$$