Recall the definitions

1. **Type-1 improper integrals.** Let f be a bounded, continuous function on $[c, \infty)$. How do we define the improper integral

$$\int_c^\infty f(x)dx\,?$$

2. **Type-2 improper integrals.** Let f be a continuous function on $(a, b]$, possibly with $x = a$ as a vertical asymptote. How do we define the improper integral

$$\int_a^b f(x)dx\,?$$

Computation

Calculate, using the definition of improper integral

$$\int_1^\infty \frac{1}{x^2+x}dx$$

Hint: $\dfrac{1}{x^2+x} = \dfrac{(x+1)-(x)}{x(x+1)}$

The most important improper integrals

Use the definition of improper integral to determine for which values of $p \in \mathbb{R}$ each of the following improper integrals converges.

1. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$

2. $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$

3. $\displaystyle\int_0^\infty \frac{1}{x^p}\,dx$

Quick review

For which values of $p \in \mathbb{R}$ is each of the following improper integrals convergent?

1. $\displaystyle\int_1^{\infty} \frac{1}{x^p}\, dx$

2. $\displaystyle\int_0^{1} \frac{1}{x^p}\, dx$

3. $\displaystyle\int_0^{\infty} \frac{1}{x^p}\, dx$

Examples

1. Let f be continuous on $[a, \infty)$. Let $A = \int_a^\infty f(x)dx$

   $$\text{Then } A \text{ may be } \begin{cases} \text{convergent (a number)} \\ \text{divergent} \begin{cases} \text{to } \infty \\ \text{to } -\infty \\ \text{“oscillating”} \end{cases} \end{cases}$$

   Give one example of each of the four results.

Examples

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   Give one example of each of the four results.

2. Now do the same thing for “type 2” improper integrals.

Positive functions

- Let f be continuous on $[a, \infty)$. Let $A = \displaystyle\int_a^\infty f(x)dx$

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- Assume $\forall x \geq a, f(x) \geq 0$.

  $\boxed{\text{Which of the four options are still possible?}}$

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  $\boxed{\text{Which of the four options are still possible?}}$

- Assume $\exists M \geq a,$ s.t. $\forall x \geq M, f(x) \geq 0$.

  $\boxed{\text{Which of the four options are still possible?}}$

A "simple" integral

What is $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx$?

A "simple" integral

What is $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx$?

1. $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx \;=\; (\ln|x|)\Big|_{-1}^{1} \;=\; \ln|1| - \ln|-1| = 0$

2. $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx = 0$ because $f(x) = \dfrac{1}{x}$ is an odd function.

3. $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx$ is divergent.

What is wrong with this computation?

$$
\begin{aligned}
\int_{-1}^{1} \frac{1}{x}\,dx &= \lim_{\varepsilon\to 0^+} \left[\int_{-1}^{-\varepsilon} \frac{1}{x}\,dx + \int_{\varepsilon}^{1} \frac{1}{x}\,dx \right] \\
&= \lim_{\varepsilon\to 0^+} \left[\ln|x|\Big|_{-1}^{-\varepsilon} + \ln|x|\Big|_{\varepsilon}^{1} \right] \\
&= \lim_{\varepsilon\to 0^+} \left[\ln|-\varepsilon| - \ln|\varepsilon| \right] \\
&= \lim_{\varepsilon\to 0^+} [\,0\,] = 0
\end{aligned}
$$

Probability

A nonnegative function f defined on $(-\infty, \infty)$ is called a **probability density function** if

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

The *mean* of a probability density function is defined as

$$\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx.$$

Let $f(x) = \begin{cases} Ce^{-kx} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

1. For $k > 0$, find a constant C such that the function f is a probability density function.
2. Calculate the mean μ.

Collection of antiderivatives

Let f be a positive, continuous function with domain $\mathbb{R}$.
We know two ways to describe a collection of antiderivatives:

1. $G(x) + C$ for $C \in \mathbb{R}$, where G is any one antiderivative.
2. The collection of functions F_a for $a \in \mathbb{R}$, where

$$F_a(x) = \int_a^x f(t)dt$$

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These two collections are not always the same. Why not? Are they the same for some functions f? When are they the same?

Hint: https://tinyurl.com/137antiderivatives

A simple BCT application

We want to determine whether $\displaystyle\int_1^\infty \frac{1}{x+e^x}\,dx$ is convergent or divergent.

We can try at least two comparisons:

1. Compare $\dfrac{1}{x}$ and $\dfrac{1}{x+e^x}$.
2. Compare $\dfrac{1}{e^x}$ and $\dfrac{1}{x+e^x}$.

Try both. What can you conclude from each one of them?

True or False - Comparisons

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, \infty)$.
Assume that $\boxed{\forall x \geq a, \quad 0 \leq f(x) \leq g(x)}$.
What can we conclude?

1. IF $\int_a^\infty f(x)dx$ is convergent, THEN $\int_a^\infty g(x)dx$ is convergent.
2. IF $\int_a^\infty f(x)dx = \infty$, THEN $\int_a^\infty g(x)dx = \infty$.
3. IF $\int_a^\infty g(x)dx$ is convergent, THEN $\int_a^\infty f(x)dx$ is convergent.
4. IF $\int_a^\infty g(x)dx = \infty$, THEN $\int_a^\infty f(x)dx = \infty$.

True or False - Comparisons II

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, \infty)$.
Assume that $\boxed{\forall x \geq a, \quad f(x) \leq g(x)}$.
What can we conclude?

1. IF $\int_a^\infty f(x)dx$ is convergent, THEN $\int_a^\infty g(x)dx$ is convergent.
2. IF $\int_a^\infty f(x)dx = \infty$, THEN $\int_a^\infty g(x)dx = \infty$.
3. IF $\int_a^\infty g(x)dx$ is convergent, THEN $\int_a^\infty f(x)dx$ is convergent.
4. IF $\int_a^\infty g(x)dx = \infty$, THEN $\int_a^\infty f(x)dx = \infty$.

True or False - Comparisons III

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, \infty)$.
Assume that $\boxed{\exists M \geq a \text{ s.t. } \forall x \geq M, \quad 0 \leq f(x) \leq g(x)}$.
What can we conclude?

1. IF $\int_a^\infty f(x)dx$ is convergent, THEN $\int_a^\infty g(x)dx$ is convergent.
2. IF $\int_a^\infty f(x)dx = \infty$, THEN $\int_a^\infty g(x)dx = \infty$.
3. IF $\int_a^\infty g(x)dx$ is convergent, THEN $\int_a^\infty f(x)dx$ is convergent.
4. IF $\int_a^\infty g(x)dx = \infty$, THEN $\int_a^\infty f(x)dx = \infty$.

What can you conclude?

Let $a \in \mathbb{R}$. Let f be a continuous, **positive** function on $[a, \infty)$. In each of the following cases, what can you conclude about $\int_a^\infty f(x)dx$? Is it convergent, divergent, or we do not know?

1. $\forall b \geq a, \ \exists M \in \mathbb{R}$ s.t. $\quad \int_a^b f(x)\,dx \leq M.$
2. $\exists M \in \mathbb{R}$ s.t. $\forall b \geq a, \quad \int_a^b f(x)\,dx \leq M.$
3. $\exists M > 0$ s.t. $\quad \forall x \geq a, \ f(x) \leq M.$
4. $\exists M > 0$ s.t. $\quad \forall x \geq a, \ f(x) \geq M.$

BCT calculations

Use BCT to determine whether each of the following is convergent or divergent

1. $\displaystyle\int_1^\infty \frac{1+\cos^2 x}{x^{2/3}}\,dx$
2. $\displaystyle\int_1^\infty \frac{1+\cos^2 x}{x^{4/3}}\,dx$
3. $\displaystyle\int_0^\infty \frac{\arctan x^2}{1+e^x}\,dx$

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4. $\displaystyle\int_0^\infty e^{-x^2}\,dx$
5. $\displaystyle\int_2^\infty \frac{(\ln x)^{10}}{x^2}\,dx$

Rapid questions: convergent or divergent?

1. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$
3. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
4. $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$
5. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$
6. $\displaystyle\int_0^1 \frac{1}{x}\,dx$
7. $\displaystyle\int_1^\infty \frac{3}{x^2}\,dx$
8. $\displaystyle\int_1^\infty \frac{1}{x^2+3}\,dx$
9. $\displaystyle\int_1^\infty \left(\frac{1}{x^2}+3\right)dx$

Slow questions: convergent or divergent?

1. $\displaystyle\int_1^\infty \frac{x^3+2x+7}{x^5+11x^4+1}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2+x+1}}\,dx$
3. $\displaystyle\int_0^1 \frac{3\cos x}{x+\sqrt{x}}\,dx$
4. $\displaystyle\int_0^1 \sqrt{\cot x}\,dx$
5. $\displaystyle\int_0^1 \frac{\sin x}{x^{3/2}}\,dx$
6. $\displaystyle\int_0^\infty e^{-x^2}\,dx$
7. $\displaystyle\int_2^\infty \frac{(\ln x)^{10}}{x^2}\,dx$

A harder calculation

For which values of $a > 0$ is the integral

$$\int_0^\infty \frac{\arctan x}{x^a}\, dx$$

convergent?

A variation on LCT

This is the theorem you have learned:

Theorem (Limit-Comparison Test)

Let $a \in \mathbb{R}$. Let f and g be positive, continuous functions on $[a, \infty)$.

- IF the limit $L = \lim_{x \to \infty} \dfrac{f(x)}{g(x)}$ exists and $L > 0$
- THEN $\displaystyle\int_a^\infty f(x)dx$ and $\displaystyle\int_a^\infty g(x)dx$ are both convergent or both divergent.

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What if we change the hypotheses to $L = 0$?

1. Write down the new theorem (different conclusion).
2. Prove it.

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What if we change the hypotheses to $L = 0$?

1. Write down the new theorem (different conclusion).
2. Prove it.

Hint: If $\lim_{x \to \infty} \dfrac{f(x)}{g(x)} = 0$, what is larger $f(x)$ or $g(x)$?

A variation on LCT - 2

This is the theorem you have learned:

Theorem (Limit-Comparison Test)

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- THEN $\displaystyle\int_a^\infty f(x)dx$ and $\displaystyle\int_a^\infty g(x)dx$ are both convergent or both divergent.

What if we change the hypotheses to $L = \infty$?

1. Write down the new theorem (different conclusion).
2. Prove it.

Absolute Convergence

Definition

The integral $\int_a^\infty f(x)\,dx$ is called **absolutely convergent** when $\int_a^\infty |f(x)|\,dx$ converges.

Prove that

- IF an improper integral is absolutely convergent
- THEN it is convergent

Hint: Consider the functions

$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) \leq 0 \end{cases} \qquad f_-(x) = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ |f(x)| & \text{if } f(x) \leq 0 \end{cases}$$

Write $f(x)$ and $|f(x)|$ in terms of $f_+(x)$ and $f_-(x)$. Use BCT.

Dirichlet integral

Let $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

1. Is $\displaystyle\int_0^1 f(x)\,dx$ an inproper integral?
2. Show that $\displaystyle\int_1^\infty \frac{\cos x}{x^2}\,dx$ is absolutely convergent.
   Hint: Use BCT.
3. The same argument is inconclusive for $\displaystyle\int_1^\infty f(x)\,dx$. Why?
4. Show that $\displaystyle\int_1^\infty f(x)dx$ is convergent
   Hint: Use the definition of improper integral, not comparison tests. Use integration by parts with $u = \frac{1}{x}$ and $dv = \sin x\,dx$.

Note: It is possible to prove that $\displaystyle\int_1^\infty \frac{\sin x}{x}\,dx$ is not absolutely convergent.