Warm up

Calculate
$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$$
Hint: Use the substitution $u = \sqrt{x}$.

Computation practice: integration by substitution

Use substitutions to compute:

1. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

2. $\displaystyle\int e^x \cos\left(e^x\right) dx$

3. $\displaystyle\int \cot x \, dx$

4. $\displaystyle\int x^2\sqrt{x+1}\,dx$

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3. $\displaystyle\int \cot x \, dx$
4. $\displaystyle\int x^2\sqrt{x+1}\, dx$
5. $\displaystyle\int \frac{e^{2x}}{\sqrt{e^x+1}}\, dx$
6. $\displaystyle\int \frac{(\ln\ln x)^2}{x\ln x}\, dx$
7. $\displaystyle\int xe^{-x^2}\, dx$
8. $\displaystyle\int e^{-x^2}\, dx$

Definite integral via substitution

This final answer is right, but the write-up is WRONG. Why?

Calculate $I = \int_0^2 \sqrt{x^3+1}\, x^2 dx$

Wrong answer

Substitution: $u = x^3 + 1,\ du = 3x^2 dx.$

$$
\begin{aligned}
I &= \frac{1}{3}\int_0^2 \sqrt{x^3+1}\,(3x^2 dx) & &= \frac{1}{3}\int_0^2 u^{1/2}\, du \\
&= \frac{1}{3}\,\frac{2}{3}\ \left. u^{3/2}\right|_0^2 & &= \frac{1}{9}\left. \left(x^3+1\right)^{2/3}\right|_0^2 \\
&= \frac{2}{9}\left(2^3+1\right)^{3/2} - \frac{2}{9}\left(0+1\right)^{3/2} & &= \frac{52}{9}
\end{aligned}
$$

Integral of products of sin and cos

We want to compute

$$I = \int \sin^3 x \cos^2 x \, dx$$

1. Attempt the substitution $u = \sin x$
2. Attempt the substitution $u = \cos x$
3. One worked better than the other. Which one? Why? Finish the problem.

Integral of products of sin and cos

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1. Attempt the substitution $u = \sin x$
2. Attempt the substitution $u = \cos x$
3. One worked better than the other. Which one? Why? Finish the problem.
4. Assume we want to compute

$$\int \sin^n x \cos^m x \, dx$$

When will the substitution $u = \sin x$ be helpful?
When will the substitution $u = \cos x$ be helpful?

Odd functions

Theorem

Let f be a continuous function. Let $a > 0$. IF f is odd, THEN

$$\int_{-a}^{a} f(x)dx = 0$$

Odd functions

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Let f be a continuous function. Let $a > 0$. IF f is odd, THEN

$$\int_{-a}^{a} f(x)dx = 0$$

1. Write down the definition of "odd function".
2. Draw a picture to interpret the theorem geometrically.
3. Prove the theorem!
   Hint: Write the integral as sum of two pieces. Use a substitution to show that one of the two pieces equals minus the other.

Computation practice: Integration by parts

Use integration by parts (possibly in combination with other methods) to compute:

1. $\displaystyle\int xe^{-2x}\,dx$
2. $\displaystyle\int x^2 \sin x\,dx$
3. $\displaystyle\int \ln x\,dx$
4. $\displaystyle\int \sin\sqrt{x}\,dx$

Computation practice: Integration by parts

Use integration by parts (possibly in combination with other methods) to compute:

1. $\int xe^{-2x}dx$
2. $\int x^2 \sin x \, dx$
3. $\int \ln x \, dx$
4. $\int \sin \sqrt{x} \, dx$
5. $\int x \arctan x \, dx$
6. $\int x^2 \arcsin x \, dx$
7. $\int e^{\cos x} \sin^3 x \, dx$
8. $\int e^{ax} \sin(bx) dx$

Persistence

Compute

- $\displaystyle\int_1^e (\ln x)^3 \, dx$

Persistence

Compute

- $\int_1^e (\ln x)^3 \, dx$
- $\int_1^e (\ln x)^{10} \, dx$

Persistence

Compute

- $\displaystyle\int_1^e (\ln x)^3\, dx$
- $\displaystyle\int_1^e (\ln x)^{10}\, dx$

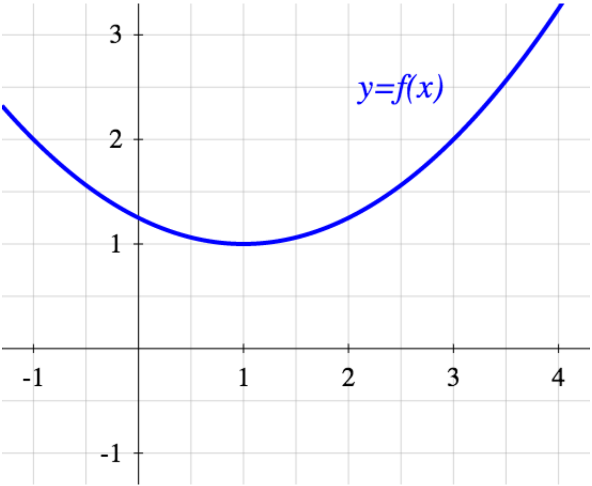
There is a more efficient approach. Call

$$I_n = \int_1^e (\ln x)^n\, dx$$

Use integration by parts on I_n. You will get an equation with I_n and I_{n-1}. Now solve the previous questions.

▸ OEIS

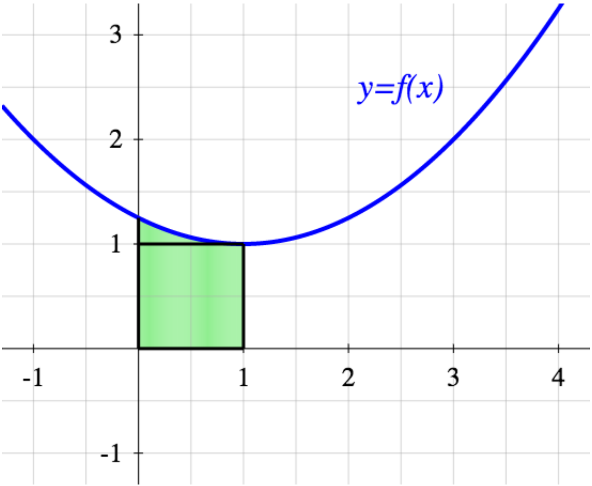
Integrals from a graph



Estimate:

1. $\int_0^1 f(x)dx$
2. $\int_0^1 f'(x)dx$
3. $\int_0^3 x\, f'(x)dx$
4. $\int_0^1 f(3x)dx$

Integrals from a graph



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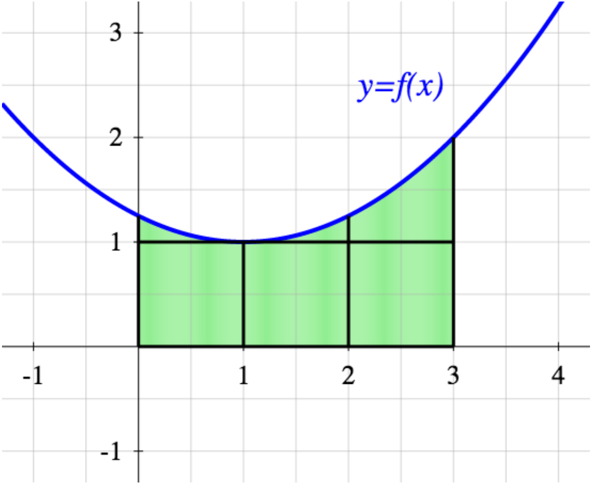

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The error function

The following function is tabulated.

$$E(x) = \int_0^x e^{-t^2} dt.$$

Write the following quantities in terms of E:

1. $\displaystyle\int_1^2 e^{-t^2} dt$
2. $\displaystyle\int_0^x t^2 e^{-t^2} dt$
3. $\displaystyle\int_0^x e^{-2t^2} dt$

The error function

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2. $\displaystyle\int_0^x t^2 e^{-t^2} dt$
3. $\displaystyle\int_0^x e^{-2t^2} dt$
4. $\displaystyle\int_0^1 e^{-t^2+6t} dt$
5. $\displaystyle\int_{x_1}^{x_2} e^{-\frac{(t-\mu)^2}{\sigma^2}} dt$
6. $\displaystyle\int_1^2 \frac{e^{-t}}{\sqrt{t}} dt$

Exp-trig antiderivative

We want to compute

$$I = \int e^{ax} \sin(bx)\, dx$$

- Try once integration by parts choosing $u = e^{ax}$. Stop.
- Go back to I. Now try integration by parts once choosing $u = \sin(bx)$ instead. Stop.
- Look at what you did. Think.

Practice: Integrals with trigonometric functions

Compute the following antiderivatives. (Once you get them to a form from where you see a path to finish them, even if long, you may stop.)

1. $\int \sin^{10} x \cos x \; dx$
2. $\int \sin^{10} x \cos^7 x \, dx$
3. $\int e^{\cos x} \cos x \sin^3 x \, dx$
4. $\int \cos^2 x \, dx$
5. $\int \cos^4 x \, dx$
6. $\int \csc x \, dx$

Useful trig identities

$$\sin^2 x + \cos^2 x = 1 \qquad \sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\tan^2 x + 1 = \sec^2 x \qquad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

Integral of products of secant and tangent

To integrate

$$\int \sec^n x \tan^m x \, dx$$

- If $\boxed{}$, then use the substitution $u = \tan x$.
- If $\boxed{}$, then use the substitution $u = \sec x$.

Hint: You will need

- $\dfrac{d}{dx}[\tan x] = \dots$
- $\dfrac{d}{dx}[\sec x] = \dots$
- The trig identity involving sec and tan

A reduction formula

Let $I_n = \int_0^{2\pi} \sin^n x \, dx$.

1. Compute I_0 and I_1.
2. Write an equation for I_n in terms of I_{n-2}. This is called a reduction formula.

   Hint: Starting with I_n, use integration by parts once. Then use $\sin^2 x + \cos^2 x = 1$ to rewrite the new integral in terms of I_n and I_{n-2}.
3. Write a a formula for I_n for all natural numbers n.

A different kind of substitution

Calculate

$$\int_0^1 \sqrt{1-x^2}\, dx$$

using the substitution

$$\begin{cases} x = \sin\theta \\ dx = ?? \end{cases}$$

Rational integrals

1. Calculate $\int \frac{1}{x+a}\,dx$
2. Reduce to common denominator $\frac{2}{x} - \frac{3}{x+3}$
3. Calculate $\int \frac{-x+6}{x^2+3x}\,dx$
4. Calculate $\int \frac{1}{x^2+3x}\,dx$
5. Calculate $\int \frac{1}{x^3-x}\,dx$

Repeated factors

1. Calculate $\displaystyle\int \frac{1}{(x+1)^n}\,dx \quad$ for $n > 1$
2. Calculate $\displaystyle\int \frac{(x+1)-1}{(x+1)^2}\,dx$
3. Calculate $\displaystyle\int \frac{2x+6}{(x+1)^2}\,dx$
4. Calculate $\displaystyle\int \frac{x^2}{(x+1)^3}\,dx$

Irreducible quadratics

1. Calculate $\displaystyle\int \frac{1}{x^2+1}\,dx$ and $\displaystyle\int \frac{x}{x^2+1}\,dx$.

   Hint: These two are very short.

2. Calculate $\displaystyle\int \frac{2x+3}{x^2+1}\,dx$

3. Calculate $\displaystyle\int \frac{x^2}{x^2+1}\,dx$

4. Calculate $\displaystyle\int \frac{x}{x^2+x+1}\,dx$

   Hint: Complete the square in the denominator and use a substitution to transform into one of the previous ones.

Repeated quadratics

1. Calculate

$$\frac{d}{dx}\left[\arctan x\right], \qquad \frac{d}{dx}\left[\frac{x}{1+x^2}\right].$$

2. Use the previous answer to calculate

$$\int \frac{1}{\left(1+x^2\right)^2}\,dx$$

The integral of secant

Compute

$$\int \sec x \, dx$$

using the substitution $u = \sin x$.

Messier rational functions

1. How could we compute an integral of the form

$$\int \frac{\text{polynomial}}{(x+1)^3(x+2)}\,dx\ ?$$

2. How could we compute an integral of the form

$$\int \frac{\text{polynomial}}{x^4(x+1)^3(x+2)(x^2+1)(x^2+4)}\,dx\ ?$$