Warm-up: sums

Compute

1. $\sum_{i=2}^{4}(2i+1)$

2. $\sum_{i=2}^{4} 2i+1$

3. $\sum_{j=2}^{4}(2i+1)$

Write these sums with Σ notation

1. $1^5 + 2^5 + 3^5 + 4^5 + \ldots + 100^5$

2. $\dfrac{2}{4^2} + \dfrac{2}{5^2} + \dfrac{2}{6^2} + \dfrac{2}{7^2} + \ldots + \dfrac{2}{N^2}$

3. $\cos 0 - \cos 1 + \cos 2 - \cos 3 + \ldots \pm \cos(N+1)$

4. $\dfrac{1}{0!} + \dfrac{1}{2!} + \dfrac{1}{4!} + \dfrac{1}{6!} + \ldots + \dfrac{1}{(2N)!}$

5. $\dfrac{1}{1!} - \dfrac{1}{3!} + \dfrac{1}{5!} - \dfrac{1}{7!} + \ldots + \dfrac{1}{81!}$

6. $\dfrac{2x^3}{4!} + \dfrac{3x^4}{5!} + \dfrac{4x^5}{6!} + \ldots + \dfrac{999x^{1000}}{1001!}$

Re-writing sums

1. $$\sum_{i=1}^{100} \tan i - \sum_{i=1}^{50} \tan i = \sum_{\boxed{}}^{\boxed{}} \boxed{}$$

2. $$\sum_{i=1}^{N} (2i-1)^5 = \sum_{i=0}^{N-1} \boxed{}$$

3. $$\left[\sum_{k=1}^{N} x^k\right] + \left[\sum_{k=0}^{N} k\, x^{k+1}\right] = \left[\sum_{k=\boxed{}}^{\boxed{}} \boxed{}\, x^k\right] + \boxed{}$$

Hint: Write out the sums on the left hand side first, simplify if possible, then write them back into sigma notation.

Telescopic sum

- Calculate the exact value of

$$\sum_{i=1}^{137}\left[\frac{1}{i}-\frac{1}{i+1}\right]$$

Hint: Write down the first few terms.

Telescopic sum

- Calculate the exact value of

$$\sum_{i=1}^{137} \left[\frac{1}{i} - \frac{1}{i+1}\right]$$

Hint: Write down the first few terms.

- Calculate the exact value of

$$\sum_{i=1}^{10{,}000} \frac{1}{i(i+1)}$$

Double sums

Compute:

1. $$\sum_{i=1}^{N}\sum_{k=1}^{N} 1$$
2. $$\sum_{i=1}^{N}\sum_{k=1}^{i} 1$$
3. $$\sum_{i=1}^{N}\sum_{k=1}^{i} i$$
4. $$\sum_{i=1}^{N}\sum_{k=1}^{i} k$$
5. $$\sum_{i=1}^{N}\sum_{k=1}^{i} (ik)$$

Useful formulas:

$$\sum_{j=1}^{N} j = \frac{N(N+1)}{2}, \quad \sum_{j=1}^{N} j^2 = \frac{N(N+1)(2N+1)}{6}, \quad \sum_{j=1}^{N} j^3 = \frac{N^2(N+1)^2}{4}$$

Harmonic sums

We define the N-th Harmonic term as the sum

$$H_N = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N} = \sum_{i=1}^{N} \frac{1}{i}.$$

Write the following sums in terms of harmonic terms.

1. $\displaystyle\sum_{i=k}^{N} \frac{1}{i}$
2. $\displaystyle\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2N}$
3. $\displaystyle\sum_{i=1}^{N} \frac{1}{2i-1}$
4. $\displaystyle\sum_{i=1}^{2N} \frac{(-1)^{i+1}}{i}$

Fubini-Tonelli

- $A_{i,k}$ is a function of 2 variables. For example, $A_{i,k} = \frac{i}{k+i^2}$.
- Decide what to write instead of each "$\boxed{?}$" so that the following identity is true:

$$\sum_{i=1}^{N} \sum_{k=1}^{i} A_{i,k} = \sum_{k=\boxed{?}}^{\boxed{?}} \sum_{i=\boxed{?}}^{\boxed{?}} A_{i,k}$$

Warm up: suprema and infima

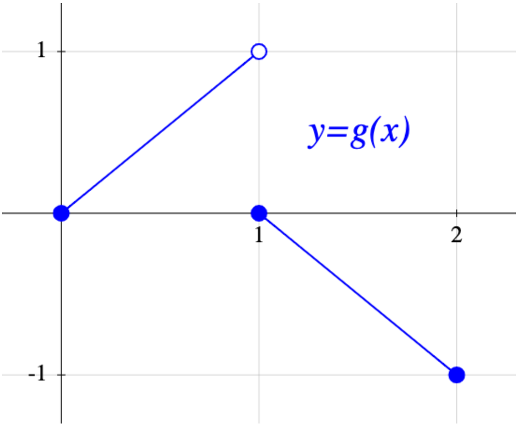
Find the supremum, infimum, maximum, and minimum of the following sets (if they exist):

1. $A = [-1, 5)$
2. $B = (-\infty, 6] \cup (8, 9)$
3. $C = \{2, 3, 4\}$
4. $D = \left\{ \frac{1}{n} \ : \ n \in \mathbb{Z}, \ n > 0 \right\}$
5. $E = \left\{ \frac{(-1)^n}{n} \ : \ n \in \mathbb{Z}, \ n > 0 \right\}$
6. $F = \{2^n \ : \ n \in \mathbb{Z}\}$

Suprema from a graph

Calculate, for the function g on the interval $[0.5, 1.5]$:

1. supremum 2. infimum 3. maximum 4. minimum



Trig suprema

Let $f(x) = \sin x$.
Find four open intervals I_1, I_2, I_3, I_4 such that

1. f has a supremum and an infimum on I_1.
2. f has a supremum and no infimum on I_2.
3. f has a maximum and a minimum on I_3.
4. f has a maximum and no minimum on I_4.

Empty set

1. Does $\emptyset$ have an upper bound ?
2. Does $\emptyset$ have a supremum?
3. Does $\emptyset$ have a maximum?
4. Is $\emptyset$ bounded above?

Equivalent definitions of supremum

Assume S is an upper bound of the set A.
Which of the following is equivalent to "S is the supremum of A"?

1. If R is an upper bound of A, then $S \leq R$.
2. $\forall R \geq S$, R is an upper bound of A.
3. $\forall R \leq S$, R is not an upper bound of A.
4. $\forall R < S$, R is not an upper bound of A.
5. $\forall R < S$, $\exists x \in A$ such that $R < x$.
6. $\forall R < S$, $\exists x \in A$ such that $R \leq x$.
7. $\forall R < S$, $\exists x \in A$ such that $R < x \leq S$.
8. $\forall R < S$, $\exists x \in A$ such that $R < x < S$.
9. $\forall \varepsilon > 0$, $\exists x \in A$ such that $S - \varepsilon < x$.
10. $\forall \varepsilon > 0$, $\exists x \in A$ such that $S - \varepsilon < x \leq S$.

Fix these FALSE statements

1. Let f and g be bounded functions on $[a, b]$. Then

$$\begin{matrix}\text{sup of } (f+g) \\ \text{on } [a,b]\end{matrix} \quad = \quad \begin{matrix}\text{sup of } f \\ \text{on } [a,b]\end{matrix} \quad + \quad \begin{matrix}\text{sup of } g \\ \text{on } [a,b]\end{matrix}$$

2. Let $a < b < c$. Let f be a bounded function on $[a, c]$. Then

$$\begin{matrix}\text{sup of } f \\ \text{on } [a,c]\end{matrix} \quad = \quad \begin{matrix}\text{sup of } f \\ \text{on } [a,b]\end{matrix} \quad + \quad \begin{matrix}\text{sup of } f \\ \text{on } [b,c]\end{matrix}$$

3. Let f be a bounded function on $[a, b]$. Let $c \in \mathbb{R}$. Then:

$$\begin{matrix}\text{sup of } (cf) \\ \text{on } [a,b]\end{matrix} \quad = \quad c \left(\begin{matrix}\text{sup of } f \\ \text{on } [a,b]\end{matrix} \right)$$

True or False - Suprema and infima

Let $A, B, C \subseteq \mathbb{R}$. Assume $C \subseteq A$. Which statements are true?
If possible, fix the false statements

1. IF A is bounded above, THEN C is bounded above.
2. IF C is bounded below, THEN A is bounded below.
3. IF A and C are bounded above, THEN $\sup C \leq \sup A$.
4. IF A and C are bounded below, THEN $\inf C \leq \inf A$.
5. IF A and B are bounded, $\sup B \leq \sup A$, and $\inf A \leq \inf B$, THEN $B \subseteq A$.
6. IF A and B are bounded above, THEN $\sup(A \cup B) = \max\{\sup A, \sup B\}$.
7. IF A and B are bounded above, THEN $\sup(A \cap B) = \min\{\sup A, \sup B\}$.

Warm up: partitions

Which ones are partitions of $[0, 2]$?

1. $[0, 2]$
2. $\{0.5,\ 1,\ 1.5\}$
3. $\{0,\ 2\}$
4. $\{1,\ 2\}$
5. $\{0,\ e,\ 2\}$
6. $\{0,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2\}$
7. $\left\{\dfrac{n}{n+1}\ :\ n \in \mathbb{N}\right\} \cup \{2\}$

Partitions of different intervals

Let $a < b < c$. Which of these statements are true?
If any are false, fix them.

1. IF P and Q are partitions of $[a, b]$,
   THEN $P \cup Q$ is a partition of $[a, b]$.
2. IF P and Q are partitions of $[a, b]$,
   THEN $P \cap Q$ is a partition of $[a, b]$.
3. IF P is a partition of $[a, b]$ and Q is a partition of $[b, c]$
   THEN $P \cup Q$ is a partition of $[a, c]$
4. IF P is a partition of $[a, c]$,
   THEN $P \cap [a, b]$ is a partition of $[a, b]$

Warm up: lower and upper sums

Let $f(x) = \sin x$.

Consider the partition $P = \{0, 1, 3\}$ of the interval $[0, 3]$.

Calculate $L_P(f)$ and $U_P(f)$.

Equations for lower and upper sums

Let f be a **decreasing**, bounded function on $[a, b]$.
Let $P = \{x_0, x_1, \ldots, x_N\}$ be a partition of $[a, b]$

Which ones are a valid equation for $L_P(f)$? For $U_P(f)$?

1. $\displaystyle\sum_{i=0}^{N} f(x_i)\,\Delta x_i$
2. $\displaystyle\sum_{i=1}^{N} f(x_i)\,\Delta x_i$
3. $\displaystyle\sum_{i=0}^{N-1} f(x_i)\,\Delta x_i$
4. $\displaystyle\sum_{i=1}^{N} f(x_{i+1})\,\Delta x_i$
5. $\displaystyle\sum_{i=1}^{N} f(x_{i-1})\,\Delta x_i$
6. $\displaystyle\sum_{i=0}^{N-1} f(x_i)\,\Delta x_{i+1}$

Recall: $\Delta x_i = x_i - x_{i-1}$.

Easier than it looks

Let f be a bounded function on $[a, b]$.
Assume f is not constant.
Prove that there exists a partition P of $[a, b]$ such that

$$L_P(f) \neq U_P(f).$$

Joining partitions

Assume

$$L_P(f) = 2, \quad U_P(f) = 6$$
$$L_Q(f) = 3, \quad U_Q(f) = 8$$

1. Is $P \subseteq Q$?
2. Is $Q \subseteq P$?
3. What can you say about $L_{P \cup Q}(f)$ and $U_{P \cup Q}(f)$?

A tricky question

Let f be a bounded function on $[a, b]$. Which statement is true?

1. There exists a partition P of $[a, b]$ such that

$$\underline{I_a^b}(f) = L_P(f) \qquad \text{and} \qquad \overline{I_a^b}(f) = U_P(f).$$

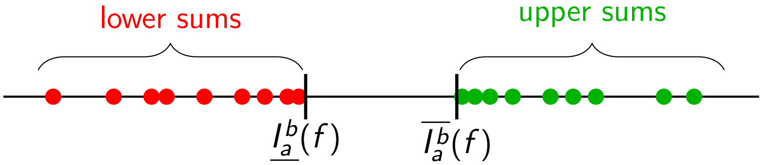
2. There exist partitions P and Q of $[a, b]$ such that

$$\underline{I_a^b}(f) = L_P(f) \qquad \text{and} \qquad \overline{I_a^b}(f) = U_Q(f).$$

A tricky question

Let f be a bounded function on $[a, b]$. Which statement is true?

1. There exists a partition P of $[a, b]$ such that
$$\underline{I_a^b}(f) = L_P(f) \qquad \text{and} \qquad \overline{I_a^b}(f) = U_P(f).$$
2. There exist partitions P and Q of $[a, b]$ such that
$$\underline{I_a^b}(f) = L_P(f) \qquad \text{and} \qquad \overline{I_a^b}(f) = U_Q(f).$$
3. There exists a partition P of $[a, b]$ such that
$$\underline{I_a^b}(f) = L_P(f).$$



An alternative definition

Let f be a bounded function on the interval $[a, b]$. Let $M \in \mathbb{R}$. Some of these four statements imply others. What implies what?

1. $\forall$ partition P of $[a, b]$, $\quad L_P(f) \leq M$,
2. $\forall \varepsilon > 0$, $\exists$ partition P of $[a, b]$ $\quad$ s.t. $\quad M - \varepsilon < L_P(f)$
3. $M \leq \underline{I}_a^b(f)$
4. $\underline{I}_a^b(f) \leq M$

An alternative definition

Let f be a bounded function on the interval $[a, b]$. Let $M \in \mathbb{R}$.
Some of these four statements imply others. What implies what?

1. $\forall$ partition P of $[a, b]$, $\quad L_P(f) \leq M$,
2. $\forall \varepsilon > 0$, $\exists$ partition P of $[a, b]$ s.t. $\quad M - \varepsilon < L_P(f)$
3. $M \leq \underline{I_a^b}(f)$
4. $\underline{I_a^b}(f) \leq M$

Based on this exercise, we could have defined $\underline{I_a^b}(f)$ as
"the only number $M \in \mathbb{R}$ satisfying these two properties: ..."

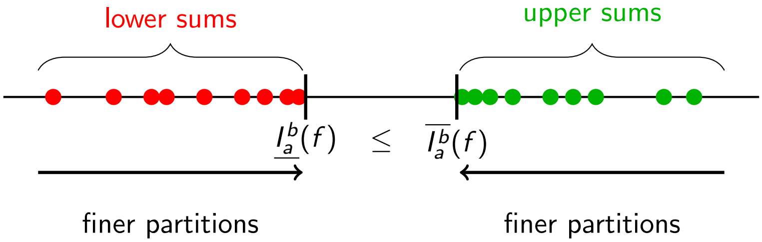
Use the same idea to write an alternative definition of $\overline{I_a^b}(f)$.

The "ε–characterization" of integrability

True or False?

Let f be a bounded function on $[a, b]$.

1. IF "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$ s.t. $U_P(f) - L_P(f) < \varepsilon$",
   THEN f is integrable on $[a, b]$
2. IF f is integrable on $[a, b]$
   THEN "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$ s.t. $U_P(f) - L_P(f) < \varepsilon$".



The "ε–characterization" of integrability - Part 1

True or False?

Let f be a bounded function on $[a, b]$.

- IF "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$ s.t. $U_P(f) - L_P(f) < \varepsilon$",
- THEN f is integrable on $[a, b]$

Hints:

1. Recall the definition of "f is integrable on $[a, b]$".
2. Let P be a partition.
   Order the numbers $U_P(f)$, $L_P(f)$, $\overline{I_a^b}(f)$, $\underline{I_a^b}(f)$.
   (Draw a picture of these numbers in the real line.)

The "ε–characterization" of integrability - Part 2

True or False?

Let f be a bounded function on $[a, b]$.

- IF f is integrable on $[a, b]$
- THEN "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$ s.t. $U_P(f) - L_P(f) < \varepsilon$".

Hints: Assume f is integrable on $[a, b]$. Let I be the integral. Fix $\varepsilon > 0$.

1. Recall the definition of "f is integrable on $[a, b]$".
2. There exist a partition P_1 s.t. $U_{P_1}(f) < I + \frac{\varepsilon}{2}$. Why?
3. There exist a partition P_2 s.t. $L_{P_2}(f) > I - \frac{\varepsilon}{2}$. Why?
4. What can you say about $U_{P_1}(f) - L_{P_2}(f)$?
5. Construct a partition P s.t. $L_{P_2}(f) \leq L_P(f) \leq U_P(f) \leq U_{P_1}(f)$.

Example 1: a constant function

Consider the function $f(x) = 2$ on $[0, 4]$.

1. Given $P = \{0, 1, e, \pi, 4\}$, compute $L_P(f)$ and $U_P(f)$.
2. Explicitly compute *all* the upper sums and *all* the lower sums.
3. Compute $\overline{I_0^4}(f)$
4. Compute $\underline{I_0^4}(f)$
5. Is f integrable on $[0, 4]$?

Example 2: a non-continuous function

Consider the function $f(x) = \begin{cases} 0 & x = 0 \\ 5 & 0 < x \leq 1 \end{cases}$, defined on $[0, 1]$.

1. Let $P = \{0, 0.2, 0.5, 0.9, 1\}$.
   Calculate $L_P(f)$ and $U_P(f)$ for this partition.
2. Fix an arbitrary partition $P = \{x_0, x_1, \ldots, x_N\}$ of $[0, 1]$.
   What is $U_P(f)$? What is $L_P(f)$? (Draw a picture!)
3. Find a partition P with exactly 3 points (2 subintervals) such that $L_P(f) = 4.99$.
4. What is the upper integral, $\overline{I_0^1}(f)$?
5. What is the lower integral, $\underline{I_0^1}(f)$?
6. Is f integrable on $[0, 1]$?

Example 3: a very non-continuous function

Consider the function f defined on $[0,1]$:

$$f(x) = \begin{cases} 1/2 & \text{if } 0 \leq x \leq 1/2 \\ 1 & \text{if } 1/2 < x \leq 1 \text{ and } x \in \mathbb{Q} \\ 0 & \text{if } 1/2 < x \leq 1 \text{ and } x \notin \mathbb{Q} \end{cases}$$

1. Draw a picture!
2. Let $P = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. Calculate $L_P(f)$ and $U_P(f)$.
3. Construct a partition P such that $L_P(f) = \dfrac{1}{4}$ and $U_P(f) = \dfrac{3}{4}$
4. What is the upper integral, $\overline{I_0^1}(f)$?
5. What is the lower integral, $\underline{I_0^1}(f)$?
6. Is f integrable on $[0,1]$?

Sum of non-integrable functions

Find bounded functions f and g on $[0, 1]$ such that

- f is non-integrable on $[0, 1]$,
- g is non-integrable on $[0, 1]$,
- $f + g$ is integrable on $[0, 1]$.

or prove this is impossible.

Properties of the integral

Assume we know the following

$$\int_0^2 f(x)dx = 3, \qquad \int_0^4 f(x)dx = 9, \qquad \int_0^4 g(x)dx = 2.$$

Compute:

1. $\int_0^2 f(t)dt$
2. $\int_0^2 f(x)dx$
3. $\int_0^2 f(t)dx$
4. $\int_2^0 f(x)dx$
5. $\int_2^4 f(x)dx$
6. $\int_{-2}^0 f(x)dx$
7. $\int_0^4 [f(x) - 2g(x)]\ dx$

The norm of a partition

1. Construct a partition P of $[0,1]$ such that $||P|| = \frac{\pi}{10}$.
2. Construct a sequence of partitions of $[0,1]$

$$P_1, P_2, P_3, \ldots$$

as simple as possible, such that $\lim_{n\to\infty} ||P_n|| = 0$.

3. Construct a *different* sequence of partitions of $[0,1]$

$$Q_1, Q_2, Q_3, \ldots$$

such that $\lim_{n\to\infty} ||Q_n|| = 0$.

Compute $\int_1^2 x^2 dx$ using Riemann sums

Let $f(x) = x^2$ on $[1, 2]$. Let P_n be the partition that breaks $[1, 2]$ into n subintervals of equal length.

1. Write a explicit formula for P_n.
2. What is Δx_i?
3. Write the Riemann sum $S^*_{P_n}(f)$ with sigma notation (choose x_i^* as the right endpoint).
4. Add the sum
5. Compute $\lim_{n\to\infty} S^*_{P_n}(f)$.
6. Repeat the last 3 questions when we choose x_i^* as the left endpoint.

Helpful identities: $\displaystyle\sum_{i=1}^{N} i = \frac{N(N+1)}{2}, \qquad \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$

Riemann sums backwards

Interpret the following limits as integrals:

1. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \sin \frac{i}{n}$
2. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{n+i}{n^2}$

Hint: Let f be a continuous function on $[0,1]$. Write a formula for $\displaystyle\int_0^1 f(x)dx$ as a limit of Riemann sums, making the simplest choices you can.

The Mean Value Theorem for integrals

Prove the following theorem.

Theorem

Let $a < b$. Let f be a continuous function on $[a, b]$.
There exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t)dt$$

The Mean Value Theorem for integrals

Prove the following theorem.

Theorem

Let $a < b$. Let f be a continuous function on $[a, b]$.
There exists $c \in [a, b]$ such that

$$f(c) \; = \; \frac{1}{b-a} \int_a^b f(t)dt$$

Hints:

1. Compute $L_P(f)$ and $U_P(f)$ for the partition $P = \{a, b\}$.
2. Use that $L_P(f) \leq \int_a^b f(t)dt \leq U_P(f)$ to prove that
$$??? \; \leq \; \frac{1}{b-a} \int_a^b f(t)dt \; \leq \; ???$$
3. Use EVT and IVT.