MAT137 Lecture 55 — Improper Integrals

Before next class:

- **Watch videos 12.7, 12.8**

Recall the definitions

(A) **Type-1 improper integrals.** Let f be a bounded, continuous function on $[c, \infty)$. How do we define the improper integral

$$\int_c^\infty f(x)dx\,?$$

(B) **Type-2 improper integrals.** Let f be a continuous function on $(a, b]$, possibly with $x = a$ as a vertical asymptote. How do we define the improper integral

$$\int_a^b f(x)dx\,?$$

Computation

Calculate, using the definition of improper integral

$$\int_1^{\infty} \frac{1}{x^2+x} dx$$

Hint: $\dfrac{1}{x^2+x} = \dfrac{(x+1)-(x)}{x(x+1)}$

The most important improper integrals

Use the definition of improper integral to determine for which values of $p \in \mathbb{R}$ each of the following improper integrals converges.

(A) $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$

(B) $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$

(C) $\displaystyle\int_0^\infty \frac{1}{x^p}\,dx$

Positive functions

- Let f be continuous on $[a, \infty)$. Let $A = \displaystyle\int_a^\infty f(x)dx$

  $$\text{Then } A \text{ may be } \begin{cases} \text{convergent (a number)} \\ \text{divergent} \begin{cases} \text{to } \infty \\ \text{to } -\infty \\ \text{``oscillating''} \end{cases} \end{cases}$$

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- Assume $\forall x \geq a, f(x) \geq 0$.

  $\boxed{\text{Which of the four options are still possible?}}$

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- Assume $\forall x \geq a, f(x) \geq 0$.

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- Assume $\exists M \geq a,$ s.t. $\forall x \geq M, f(x) \geq 0$.

  $\boxed{\text{Which of the four options are still possible?}}$

Before next class:

- **Watch videos 12.9, 12.10**

Quick review

For which values of $p \in \mathbb{R}$ is each of the following improper integrals convergent?

(A) $\displaystyle\int_1^{\infty} \frac{1}{x^p}\, dx$

(B) $\displaystyle\int_0^{1} \frac{1}{x^p}\, dx$

(C) $\displaystyle\int_0^{\infty} \frac{1}{x^p}\, dx$

A simple BCT application

We want to determine whether $\displaystyle\int_1^{\infty} \frac{1}{x+e^x}\,dx$ is convergent or divergent.

We can try at least two comparisons:

(A) Compare $\dfrac{1}{x}$ and $\dfrac{1}{x+e^x}$.

(B) Compare $\dfrac{1}{e^x}$ and $\dfrac{1}{x+e^x}$.

Try both. What can you conclude from each one of them?

True or False - Comparisons

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, \infty)$.
Assume that $\boxed{\forall x \geq a, \quad 0 \leq f(x) \leq g(x)}$.
What can we conclude?

(A) IF $\int_a^\infty f(x)dx$ is convergent, THEN $\int_a^\infty g(x)dx$ is convergent.

(B) IF $\int_a^\infty f(x)dx = \infty$, THEN $\int_a^\infty g(x)dx = \infty$.

(C) IF $\int_a^\infty g(x)dx$ is convergent, THEN $\int_a^\infty f(x)dx$ is convergent.

(D) IF $\int_a^\infty g(x)dx = \infty$, THEN $\int_a^\infty f(x)dx = \infty$.

True or False - Comparisons II

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, \infty)$.

Assume that $\boxed{\forall x \geq a, \quad f(x) \leq g(x)}$.

What can we conclude?

(A) IF $\displaystyle\int_a^\infty f(x)dx$ is convergent, THEN $\displaystyle\int_a^\infty g(x)dx$ is convergent.

(B) IF $\displaystyle\int_a^\infty f(x)dx = \infty$, THEN $\displaystyle\int_a^\infty g(x)dx = \infty$.

(C) IF $\displaystyle\int_a^\infty g(x)dx$ is convergent, THEN $\displaystyle\int_a^\infty f(x)dx$ is convergent.

(D) IF $\displaystyle\int_a^\infty g(x)dx = \infty$, THEN $\displaystyle\int_a^\infty f(x)dx = \infty$.

True or False - Comparisons III

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, \infty)$.
Assume that $\boxed{\exists M \geq a \text{ s.t. } \forall x \geq M, \quad 0 \leq f(x) \leq g(x)}$.
What can we conclude?

(A) IF $\displaystyle\int_a^\infty f(x)dx$ is convergent, THEN $\displaystyle\int_a^\infty g(x)dx$ is convergent.

(B) IF $\displaystyle\int_a^\infty f(x)dx = \infty$, THEN $\displaystyle\int_a^\infty g(x)dx = \infty$.

(C) IF $\displaystyle\int_a^\infty g(x)dx$ is convergent, THEN $\displaystyle\int_a^\infty f(x)dx$ is convergent.

(D) IF $\displaystyle\int_a^\infty g(x)dx = \infty$, THEN $\displaystyle\int_a^\infty f(x)dx = \infty$.

BCT calculations

Use BCT to determine whether each of the following is convergent or divergent

(A) $\displaystyle\int_1^\infty \frac{1+\cos^2 x}{x^{2/3}}\,dx$

(B) $\displaystyle\int_1^\infty \frac{1+\cos^2 x}{x^{4/3}}\,dx$

(C) $\displaystyle\int_0^\infty \frac{\arctan x^2}{1+e^x}\,dx$

BCT calculations

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(B) $\displaystyle\int_1^\infty \frac{1+\cos^2 x}{x^{4/3}}\,dx$

(C) $\displaystyle\int_0^\infty \frac{\arctan x^2}{1+e^x}\,dx$

(D) $\displaystyle\int_0^\infty e^{-x^2}\,dx$

(E) $\displaystyle\int_2^\infty \frac{(\ln x)^{10}}{x^2}\,dx$

Before next class:

- **Watch videos 13,2, 13.3, 13.4**

Rapid questions: convergent or divergent?

(A) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

(B) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

(C) $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

(D) $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$

(E) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

(F) $\displaystyle\int_0^1 \frac{1}{x}\,dx$

(G) $\displaystyle\int_1^\infty \frac{3}{x^2}\,dx$

(H) $\displaystyle\int_1^\infty \frac{1}{x^2+3}\,dx$

(I) $\displaystyle\int_1^\infty \left(\frac{1}{x^2}+3\right)dx$

A "simple" integral

What is $\displaystyle\int_{-1}^{1} \frac{1}{x}\, dx \quad ?$

A "simple" integral

What is $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx$?

(A) $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx = (\ln|x|)\Big|_{-1}^{1} = \ln|1| - \ln|-1| = 0$

(B) $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx = 0$ because $f(x) = \dfrac{1}{x}$ is an odd function.

(C) $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx$ is divergent.

Slow questions: convergent or divergent?

(A) $\displaystyle\int_1^\infty \frac{x^3+2x+7}{x^5+11x^4+1}\,dx$

(B) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2+x+1}}\,dx$

(C) $\displaystyle\int_0^1 \frac{3\cos x}{x+\sqrt{x}}\,dx$

(D) $\displaystyle\int_0^1 \sqrt{\cot x}\,dx$

(E) $\displaystyle\int_0^1 \frac{\sin x}{x^{3/2}}\,dx$

(F) $\displaystyle\int_0^\infty e^{-x^2}\,dx$

(G) $\displaystyle\int_2^\infty \frac{(\ln x)^{10}}{x^2}\,dx$

What is wrong with this computation?

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{x}\,dx &= \lim_{\varepsilon\to 0^+} \left[\int_{-1}^{-\varepsilon} \frac{1}{x}\,dx + \int_{\varepsilon}^{1} \frac{1}{x}\,dx \right] \\
&= \lim_{\varepsilon\to 0^+} \left[\ln|x| \Big|_{-1}^{-\varepsilon} + \ln|x| \Big|_{\varepsilon}^{1} \right] \\
&= \lim_{\varepsilon\to 0^+} \left[\ln|-\varepsilon| - \ln|\varepsilon| \right] \\
&= \lim_{\varepsilon\to 0^+} [0] = 0
\end{aligned}$$