MAT137 Lecture 26 — Local Extrema

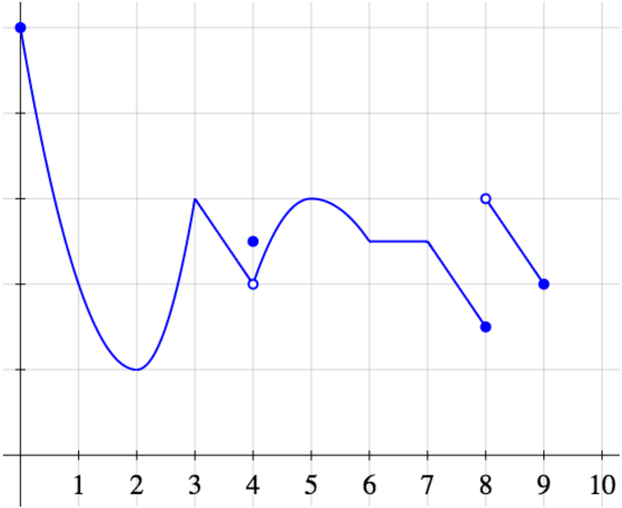
Warmup:
Write down, in set-builder notation, the domain of tan, the tangent function.

Before next class:

- **Watch videos 5.5, 5.6**

Definition of local extremum

Find local and global extrema of the function with this graph:



Where is the maximum?

We know the following about the function h:

- The domain of h is $(-4, 4)$.
- h is continuous on its domain.
- h is differentiable on its domain, except at 0.
- $h'(x) = 0 \quad \Longleftrightarrow \quad x = -1$ or 1.

What can you conclude about the maximum of h?

Where is the maximum?

We know the following about the function h:

- The domain of h is $(-4, 4)$.
- h is continuous on its domain.
- h is differentiable on its domain, except at 0.
- $h'(x) = 0 \quad \iff \quad x = -1$ or 1.

What can you conclude about the maximum of h?

(A) h has a maximum at $x = -1$, or 1.

(B) h has a maximum at $x = -1, 0$, or 1.

(C) h has a maximum at $x = -4, -1, 0, 1$, or 4.

(D) None of the above.

Fractional exponents

Let $g(x) = x^{2/3}(x-1)^3$.

Find local and global extrema of g on $[-1, 2]$.

Trig extrema

Let $f(x) = \dfrac{\sin x}{3 + \cos x}$.

Find the maximum and minimum of f.

MAT137 Lecture 27 — Rolle's Theorem

Warmup:
Write down, in set-builder notation, the domain of tan, the tangent function.

Before next class:

- **Watch videos 5.7, 5.8, 5.9**

Domain of tan

Which of these correctly describe the domain of tan?

(A) $\left\{x \in \mathbb{R} : x \neq \frac{\pi k}{2},\ k \in \{\text{odd integers}\}\right\}$

(B) $\left\{x \in \mathbb{R} : x \neq \frac{\pi k}{2},\ \forall k \in \{\text{odd integers}\}\right\}$

(C) $\left\{x \in \mathbb{R} : \forall k \in \{\text{odd integers}\}, x \neq \frac{\pi k}{2}\right\}$

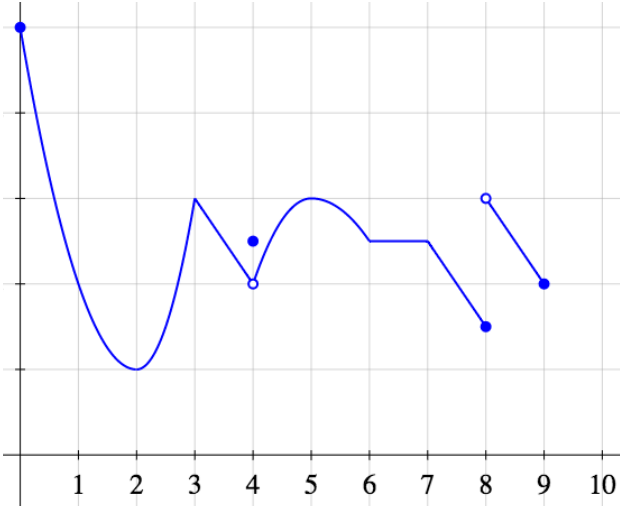
(D) $\left\{x \in \mathbb{R} : \exists k \in \{\text{odd integers}\}, x \neq \frac{\pi k}{2}\right\}$

(E) $\left\{x \in \mathbb{R} : \forall k \in \mathbb{Z}, k \text{ is odd and } x \neq \frac{\pi k}{2}\right\}$

(F) $\left\{x \in \mathbb{R} : \forall k \in \mathbb{Z}, k \text{ is odd } \implies x \neq \frac{\pi k}{2}\right\}$

Definition of local extremum

Find local and global extrema of the function with this graph:



How many zeroes?

Let

$$f(x) = e^x - \sin x + x^2 + 10x$$

How many zeroes does f have?

The second Theorem of Rolle

Complete the statement for this theorem and prove it.

Rolle's Theorem 2

Let $a < b$. Let f be a function defined on $[a, b]$.
IF

- Some conditions on continuity and derivatives *[state these conditions precisely]*
- $f(a) = f'(a) = 0$
- $f(b) = 0$

THEN $\exists c \in (a, b)$ such that $f''(c) = 0$.

Hint: Apply the 1st Rolle's Theorem to f, then do something else.

Before next class:

- **Watch videos 5.10, 5.11, 5.12**

True or False—Local Extrema Again

Let I be an open interval. Let f be a differentiable function defined on I. Let $c \in I$.
Which implications are true?

(A) IF f has local extreme at c , THEN f has an extreme at c

(B) IF f has an extreme at c, THEN f has local extreme at c

(C) IF f has a local extreme at c, THEN $f'(c) = 0$.

(D) IF $f'(c) = 0$, THEN f has a local extreme at c.

Proving difficult identities

Prove that, for every $x \geq 0$,

$$2 \arctan \sqrt{x} - \arcsin \frac{x-1}{x+1} = \frac{\pi}{2}$$

Hint: You are trying to prove a function is constant. Use derivatives.

Critique this "proof"

- $$\left[2\arctan\sqrt{x} - \arcsin\frac{x-1}{x+1}\right] = \left[\frac{\pi}{2}\right]$$
- $$\frac{d}{dx}\left[2\arctan\sqrt{x} - \arcsin\frac{x-1}{x+1}\right] = \frac{d}{dx}\left[\frac{\pi}{2}\right]$$
- $$\frac{2}{1+\left(\sqrt{x}\right)^2}\cdot\frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}\cdot\frac{(x+1)-(x-1)}{(x+1)^2} = 0$$
- $$\frac{1}{(1+x)\sqrt{x}} - \frac{1}{\sqrt{\frac{4x}{(x+1)^2}}}\cdot\frac{2}{(x+1)^2} = 0$$
- $0 = 0$
- So $2\arctan\sqrt{x} - \arcsin\dfrac{x-1}{x+1}$ is constant.
- Evaluate at $x = 0$ to find the value of the constant.
- $2\arctan 0 \ - \ \arcsin(-1) \ = \ 0 - (-\pi/2) \ = \ \pi/2$
- Therefore, $\quad 2\arctan\sqrt{x} - \arcsin\dfrac{x-1}{x+1} = \dfrac{\pi}{2}$

Car race - 1

A driver competes in a race.

Use MVT to prove that at some point during the race the instantaneous velocity of the driver is exactly equal to the average velocity of the driver during the race.

Car race - 2

Two drivers start a race at the same moment and finish in a tie.

Can you conclude that there was a time in the race (not counting the starting time) when the two drivers had exactly the same speed?

Car race - Is this proof correct?

Claim

IF two drivers start a race at the same moment and finish in a tie, THEN at some point in the race (not counting the starting time) they had exactly the same speed.

Proof?

- Let $f(t)$ and $g(t)$ be the positions of the two cars at time t.
- Assume the race happens in the interval $[t_1, t_2]$. By hypothesis:

$$f(t_1) = g(t_1), \qquad f(t_2) = g(t_2).$$

- Using MVT, there exists $c \in (t_1, t_2)$ such that

$$f'(c) = \frac{f(t_2) - f(t_1)}{t_2 - t_1}, \quad g'(c) = \frac{g(t_2) - g(t_1)}{t_2 - t_1}.$$

- Then $f'(c) = g'(c)$. $\square$

Car race - resolution

Two drivers start a race at the same moment and finish in a tie.

Prove that at some point during the race (not counting the starting time) the two drivers had exactly the same speed.

Before next class:

- **Watch videos 6.1, 6.2**

Definition of increasing

Let f be defined by $f(x) = x^3$.
Which statements are TRUE?

(A) f is increasing on $(0, \infty)$.
(B) f is increasing on $[0, \infty)$.
(C) f is increasing on $(-\infty, 0)$.
(D) f is increasing on $(-\infty, 0]$.
(E) f is increasing on $(-\infty, 0)$ and on $(0, \infty)$.
(F) f is increasing on $(-\infty, 0]$ and on $[0, \infty)$.
(G) f is increasing on $\mathbb{R}$.
(H) f is increasing on $[1, 2]$.

True or False—Again, Again!

Let I be an open interval.
Let f be a function defined on I.
Let $c \in I$. Which implications are true?

(A) IF f is increasing on I , THEN $\forall x \in I$, $f'(x) > 0$.

(B) IF $\forall x \in I$, $f'(x) > 0$, THEN f is increasing on I.

(C) IF f has a local extreme at c, THEN $f'(c) = 0$.

(D) IF $f'(c) = 0$, THEN f has a local extreme at c.

Preparation

(A) Let f be a function defined on an interval I.
Write the definition of "f is increasing on I".

(B) Write the statement of the Mean Value Theorem

Positive derivative implies increasing

Use the MVT to prove

Theorem

Let $a < b$. Let f be a differentiable function on (a, b).

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN f is increasing on (a, b).

Positive derivative implies increasing

Use the MVT to prove

Theorem

Let $a < b$. Let f be a differentiable function on (a, b).

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN f is increasing on (a, b).

(A) Recall the definition of what you are trying to prove.

(B) **From that definition, figure out the structure of the proof.**

(C) If you have used a theorem, did you verify the hypotheses?

(D) Are there words in your proof, or just equations?

What is wrong with this proof?

Theorem

Let $a < b$. Let f be a differentiable function on (a, b).

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN f is increasing on (a, b).

Proof.

- From the MVT, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$
- We know $b - a > 0$ and $f'(c) > 0$
- Therefore $f(b) - f(a) > 0$. Thus $f(b) > f(a)$.
- f is increasing.

$\square$

Inequalities

Prove that, for every $x \in \mathbb{R}$

$$e^x \geq 1 + x$$

Hint: Where is the function $f(x) = e^x - 1 - x$ increasing or decreasing? What is its minimum?