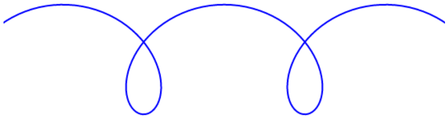
MAT137 Lecture 22 — Inverse Functions

Before next class:

- **Watch videos 4.3, 4.4**

Worm up

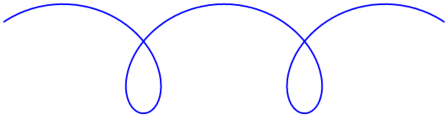
A worm is crawling accross the table. The path of the worm looks something like this:



True or False?

The position of the worm is a function.

Worm function



A worm is crawling accross the table.
For any time t, let $f(t)$ be the position of the worm.
This defines a function f.

(A) What is the domain of f?

(B) What is the codomain of f?

(C) What is the range of f?

Function, number, or nonsense?

Let f, g be functions. Let x be a number. Classify each expression as a **function**, **number**, or **nonsense**.

(A) $f(x)$

(B) $f \circ g$

(C) $f \circ (g(x))$

(D) $(f \circ g)(x)$

(E) $f(x) \circ g(x)$

(F) $f(x)g(x)$

(G) $f(g(x))$

(H) $f(g)$

(I) $f(g)(x)$

(J) $f(g(x)f(x))$

(K) e^x

(L) $\ln x$

(M) $\ln$

(N) $\sin \circ e^x$

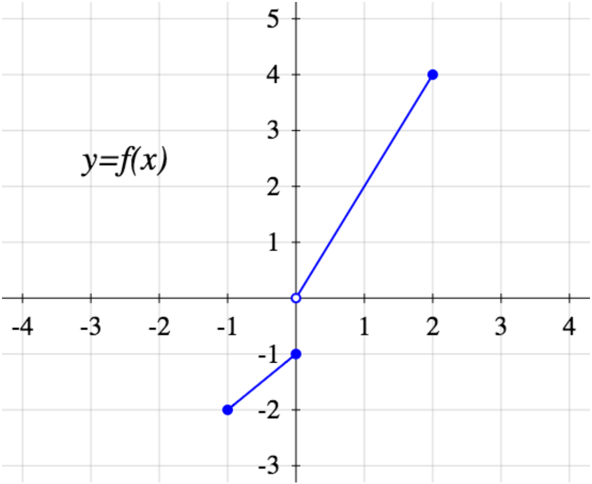
(O) $\sin \circ \ln$

(P) $(\ln \circ \sin)(e^x)$

(Q) $e^x \circ \sin$

(R) $\sin^2$

Inverse function from a graph



Calculate:

(A) $f(2)$

(B) $f(0)$

(C) $f^{-1}(2)$

(D) $f^{-1}(0)$

(E) $f^{-1}(-1)$

Absolute value and inverses

Let

$$h(x) = x|x| + 1$$

(A) Calculate $h^{-1}(-8)$.

Absolute value and inverses

Let

$$h(x) = x|x| + 1$$

(A) Calculate $h^{-1}(-8)$.

(B) Sketch the graph of h.

(C) Find an equation for h^{-1}.

(D) Sketch the graph of h^{-1}.

(E) Verify that

- for every $t \in \boxed{???}$, $\quad h(h^{-1}(t)) = t$.
- for every $t \in \boxed{???}$, $\quad h^{-1}(h(t)) = t$.

Composition and inverses

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$.

Let f and g be functions. Assume they each have an inverse.

Is $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$?

- If YES, prove it.
- If NO, fix the statement.

Composition and inverses

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- If YES, prove it.
- If NO, fix the statement.

If you do not know how to start, experiment with the functions

$$f(x) = x + 1, \qquad g(x) = 2x.$$

Before next class:

- **Watch videos 4.5, 4.7, 4.8, 4.9**

Fill in the Blank

Given that f is an invertible function, fill in the blanks.

(A) If $f(-1) = 0$, then $f^{-1}(0) =$ ——.

(B) If $f^{-1}(2) = 1$, then $f(1) =$ ——.

(C) If $(2, 3)$ is on the graph of f, then —— is on the graph of f^{-1}.

(D) If —— is on the graph of f, then $(-2, 4)$ is on the graph of f^{-1}.

Where is the error?

- We know that $\boxed{(f^{-1})' = \frac{1}{f'}}$
- Let $f(x) = x^2$, restricted to the domain $x \in (0, \infty)$

$$f'(x) = 2x \quad \text{and} \quad \boxed{f'(4) = 8}$$

- Then $f^{-1}(x) = \sqrt{x}$

$$(f^{-1})'(x) = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \boxed{(f^{-1})'(4) = \frac{1}{4}}$$

- So $\boxed{(f^{-1})'(4) \neq \frac{1}{f'(4)}}$

Derivatives of the inverse function

Let f be a one-to-one function.
Let $a, b \in \mathbb{R}$ such that $b = f(a)$.

(A) Obtain a formula for $\left(f^{-1}\right)'(b)$ in terms of $f'(a)$.

Hint: This was done in Video 4.4
Take $\dfrac{d}{dy}$ of both sides of $\quad f(f^{-1}((y)) = y$.

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Take $\dfrac{d}{dy}$ of both sides of $\quad f(f^{-1}((y)) = y$.

(B) Obtain a formula for $\left(f^{-1}\right)''(b)$ in terms of $f'(a)$ and $f''(a)$.

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Hint: This was done in Video 4.4
Take $\dfrac{d}{dy}$ of both sides of $\quad f(f^{-1}((y)) = y$.

(B) Obtain a formula for $\left(f^{-1}\right)''(b)$ in terms of $f'(a)$ and $f''(a)$.

(C) *Challenge:* Obtain a formula for $\left(f^{-1}\right)'''(b)$ in terms of $f'(a)$, $f''(a)$, and $f'''(a)$.

Composition of one-to-one functions

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$. Prove the following theorem.

Theorem A

Let f and g be functions.
IF f and g are one-to-one,
THEN $f \circ g$ is one-to-one.

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Theorem A

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Suggestion:

(A) Write the definition of what you want to prove.
(B) Figure out the formal structure of the proof.
(C) Complete the proof (use the hypotheses!)

Before next class:

- **Watch videos 4.12, 4.13, 4.14**

Computations - Exponentials and logarithms

Compute the derivative of the following functions:

(A) $f(x) = e^{\sin x + \cos x} \ln x$

(B) $f(x) = \pi^{\tan x}$

(C) $f(x) = \ln\left[e^x + \ln \ln \ln x\right]$

(D) $f(x) = \log_{10}(2x + 3)$

Logarithm and Absolute Value

The function F is defined by the equation

$$F(x) = \ln |x|.$$

What is its derivative?

(A) $F'(x) = \dfrac{1}{x}$

(B) $F'(x) = \dfrac{1}{|x|}$

(C) F is not differentiable

Logarithmic differentiation

Calculate the derivative of

$$g(x) = x^{\tan x}.$$

More logarithmic differentiation

Calculate the derivative of

$$f(x) = (\sin x)^{\cos x} + (\cos x)^{\sin x}\,.$$

More logarithmic differentiation

Calculate the derivative of

$$f(x) = (\sin x)^{\cos x} + (\cos x)^{\sin x}.$$

What is wrong with this answer?

$$\begin{aligned}
\ln f(x) =& (\cos x)\ln(\sin x) + (\sin x)(\ln\cos x) \\
\frac{d}{dx}[\ln f(x)] =& \frac{d}{dx}[(\cos x)\ln(\sin x)] + \frac{d}{dx}[(\sin x)(\ln\cos x)] \\
\frac{f'(x)}{f(x)} =& -(\sin x)\ln(\sin x) + (\cos x)\frac{\cos x}{\sin x} \\
& + (\cos x)\ln(\cos x) + (\sin x)\frac{-\sin x}{\cos x}
\end{aligned}$$

$$f'(x) = f(x)\left[-(\sin x)\ln(\sin x) + (\cos x)\ln(\cos x) + \frac{\cos^2 x}{\sin x} - \frac{\sin^2 x}{\cos x}\right]$$

Cue Boss Music

Calculate the derivative of

$$h(x) = \sqrt[3]{\frac{\left(\sin^6 x\right)\sqrt{x^7 + 6x + 2}}{3^x\left(x^{10} + 2x\right)^{10}}}$$

Before next class:

- **Watch videos 5.2, 5.3, 5.4**

Definition of arctan

(A) Sketch the graph of tan.

(B) Prove that tan is not one-to-one.

(C) Select the largest interval containing 0 such that the restriction of tan to it is one-to-one. We define arctan as the inverse of this restriction. Let $x, y \in \mathbb{R}$

$$\arctan y = x \qquad \iff \qquad ???$$

(D) What is the domain of arctan? What is the range of arctan?
Sketch the graph of arctan.

(E) Compute

(E) $\arctan(\tan(1))$

(E) $\arctan(\tan(3))$

(E) $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$

(E) $\arctan(\tan(-6))$

(E) $\tan(\arctan(0))$

(E) $\tan(\arctan(10))$

Trig-inverse-trig

Find simple expressions for these quantities and state the domain on which they are valid:

(A) $\sin(\arccos x)$

(B) $\sec(\arccos x)$

(C) $\sec(\arctan x)$

(D) $\tan(\operatorname{arcsec} x)$

Trig-inverse-trig

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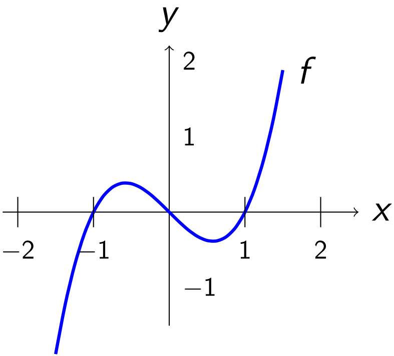
Hint: There are two standard ways to attack these problems:

- Use a trig identity
  e.g.: a trig identity relating $\sin$ and $\cos$ for (1)
- Or draw a right triangle with side lengths 1 and x
  e.g.: with an angle θ such that $\cos\theta = x$ for (1)

If you need to take a square root, you must justify which branch ($+$ or $-$) you are choosing.



(A) Find the largest interval containing 0 on which f is invertible.

(B) Find the largest interval containing 1 on which f is invertible.

Functions, inverses, and graphs

Sketch the graph of a function g satisfying all the following properties:

(A) The domain of g is $\mathbb{R}$.

(B) g is continuous everywhere except at -2.

(C) g is differentiable everywhere except at -2 and 1.

(D) g has an inverse function.

(E) $g(0) = 2$

(F) $g'(0) = 2$

(G) $\left(g^{-1}\right)'(-3) = -2$.

Functions, inverses, and graphs - 2

Draw the graph of a function f satisfying all of the following:

(A) The domain of f is $\mathbb{R}$.

(B) f is differentiable everywhere.

(C) The restriction of f to $[0, \infty)$ is one-to-one, and its INVERSE has a vertical tangent line at 2.

(D) The restriction of f to $(-\infty, 0]$ is one-to-one, and its INVERSE has derivative 2 at 2.

Composition of one-to-one functions – 2

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$. Prove the following theorem.

Theorem B

Let f and g be functions.
IF $f \circ g$ is one-to-one, THEN g is one-to-one.

Composition of one-to-one functions – 2

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Theorem B

Let f and g be functions.
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Suggestion:

(A) Transform the "$P \implies Q$" theorem into an equivalent "(not Q) $\implies$ (not P)" theorem. You will prove that one instead.

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Suggestion:

(A) Transform the "$P \implies Q$" theorem into an equivalent "(not Q) $\implies$ (not P)" theorem. You will prove that one instead.

(B) Write the definition of the hypotheses and of the conclusion.

(C) Write the proof.

Composition of one-to-one functions – 3

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$.

Prove the following claim is FALSE with a counterexample.

Claim

Let f and g be functions.
IF $f \circ g$ is one-to-one,
THEN f is one-to-one.

Increasing and one-to-one

Definition

Let f be a function with domain D. We say that f is *increasing* on D when

$$\forall x_1, x_2 \in D, \quad x_1 < x_2 \implies f(x_1) < f(x_2).$$

(A) Prove that if a function is increasing, then it is one-to-one.

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$$\forall x_1, x_2 \in D, \quad x_1 < x_2 \implies f(x_1) < f(x_2).$$

(A) Prove that if a function is increasing, then it is one-to-one.

(B) Use this to show that $g(x) = x^5 + 4x^3 + 2x + 1$ has an inverse.

(C) Find $(g^{-1})'(1)$.

A different type of logarithm

Calculate the derivative of

$$f(x) = \log_{x+1}(x^2 + 1)$$

Note: This is a new function. We have not given you a formula for it yet, That is on purpose.

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Hint: If you do not know where to start, remember the definition of logarithm:

$$\log_a b = c \iff a^c = b.$$

An Implicit Function

Find y' if $x^y = y^x$.

Derivative of arctan

Obtain (and prove) a formula for the derivative of arctan.

Hint: Call $f(t) = \arctan t$ and differentiate

$$\forall t \in \dots \qquad \tan(f(t)) = t$$

Computations - Inverse trig functions

Compute the derivatives of these functions, and simplify them as much as possible:

(A) $f(x) = \arcsin\left(x^{3/2}\right)$

(B) $f(x) = 2x^2 \arctan(x^2) - \ln(x^4 + 1)$

(A) Complete: “We define arcsec as the inverse of the restriction of sec to ...”
Hint: Sketch the graph of sec.

(B) What are the domain and range of arcsec?
Sketch its graph.

(C) Obtain (and prove) a formula for the derivative of arcsec in the same way you did for arctan.

(D) Now obtain the same formula in a different way:
use $\sec x = \dfrac{1}{\cos x}$ to write arcsec in terms of arccos.