Warmup:

You want to show "$\exists n \in \mathbb{N}$ s.t. $n^2 = 4$". Which proofs below are correct/incorrect?

(A) Let $n = 2$. Then, $n \in \mathbb{N}$ and $n^2 = 4$.

(B) Take $n \in \mathbb{N}$. Let $n = 2$. Then $n^2 = 4$.

(C) Let $n \in \mathbb{N}$. Assume $n = 2$. Then $n^2 = 4$.

(D) Take $n = 2$. Then $n \in \mathbb{N}$ and $n^2 = 4$.

Before next class:

- **Watch videos 2.1, 2.2, 2.3**

Variations on induction

Let S_n be a statement depending on a positive integer n.

What conclusions can you draw in each of the following cases? (I.e., for which n do you know that S_n is true?)

(A) We have proven:

- S_3
- $$\forall n \geq 1,\ S_n \implies S_{n+1}$$

(B) We have proven:

- S_1
- $$\forall n \geq 3,\ S_n \implies S_{n+1}$$

(C) We have proven:

- S_1
- $$\forall n \geq 1,\ S_n \implies S_{n+3}$$

(D) We have proven:

- S_1
- $$\forall n \geq 1,\ S_{n+1} \implies S_n$$

Variations on induction 2

We want to prove

$$\forall n \geq 1,\ S_n$$

So far we have proven

- S_1
- $\forall n \geq 1,\ S_n \implies S_{n+3}$.

What else do we need to do?

What is wrong with this proof by induction?

Theorem

$\forall N \geq 1$, *every set of* N *students in MAT137 will get the same grade.*

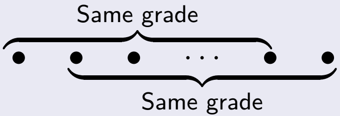
What is wrong with this proof by induction?

Theorem

$\forall N \geq 1$, *every set of N students in MAT137 will get the same grade.*

Proof.

- **Base case.** It is clearly true for $N = 1$.
- **Induction step.**
  Assume it is true for N. I'll show it is true for $N + 1$.
  Take a set of $N + 1$ students. By induction hypothesis:
  - The first N students get the same grade.
  - The last N students get the same grade.



Hence the $N + 1$ students all get the same grade.

$\square$

What is wrong with this proof by induction?

For every $N \geq 1$, let

$$S_N = \text{ “every set of } N \text{ students in MAT137} \\ \text{will get the same grade”}$$

What did we actually prove in the previous page?

- S_1 ?
- $\forall N \geq 1, \quad S_N \implies S_{N+1}$?

Properties of absolute value

Let $a, b \in \mathbb{R}$. Are the following conclusions correct?

(A) $|ab| = |a||b|$
(B) $|a + b| = |a| + |b|$

If any of the conclusions is wrong, fix it.

Properties of inequalities

Let $a, b, c \in \mathbb{R}$.
Assume $a < b$. Are the following conclusions correct?

(A) $a + c < b + c$

(B) $a - c < b - c$

(C) $ac < bc$

(D) $a^2 < b^2$

(E) $\frac{1}{a} < \frac{1}{b}$

(F) $\sin a < \sin b$

If any of the conclusions is wrong, fix it.

Sets described by distance

Let $a \in \mathbb{R}$. Let $\delta > 0$.

Describe the following sets using interval notation.

(A) $A = \{x \in \mathbb{R} \ : \ |x| < \delta\}$

(B) $B = \{x \in \mathbb{R} \ : \ |x| > \delta\}$

(C) $C = \{x \in \mathbb{R} \ : \ |x - a| < \delta\}$

(D) $D = \{x \in \mathbb{R} \ : \ 0 < |x - a| < \delta\}$

Implications

Find *all* positive values of X, Y, and Z which make the following implications true.

(A) $|t-3|<1 \implies |2t-6|<X$

(B) $|t-3|<Y \implies |2t-6|<1$

(C) $|t-3|<1 \implies |t+5|<Z$

Warmup:

Let $a, b, c \in \mathbb{R}$. Determine whether the following statements are true or false.

(A) $a > b \implies ac > bc$

(B) $a > b \implies \text{not } (ac > bc)$

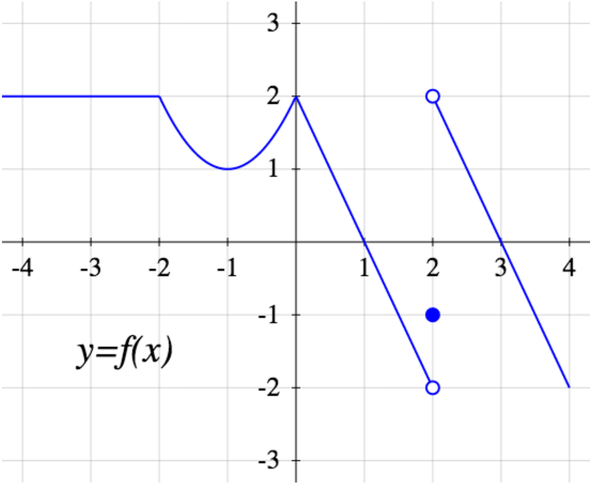
(C) $\text{not } (a > b) \implies ac > bc$

(D) $\text{not } \left(a > b \implies ac > bc \right)$

Before next class:

- **Watch videos 2.5, 2.6**

Limits from a graph



Find the value of

(A) $\lim_{x \to 2} f(x)$

(B) $\lim_{x \to 0} f(f(x))$

Floor

Given a real number x, we defined the *floor of* x, denoted by $\lfloor x \rfloor$, as the largest integer smaller than or equal to x. For example:

$$\lfloor \pi \rfloor = 3, \qquad \lfloor 7 \rfloor = 7, \qquad \lfloor -0.5 \rfloor = -1.$$

Sketch the graph of $y = \lfloor x \rfloor$. Then compute:

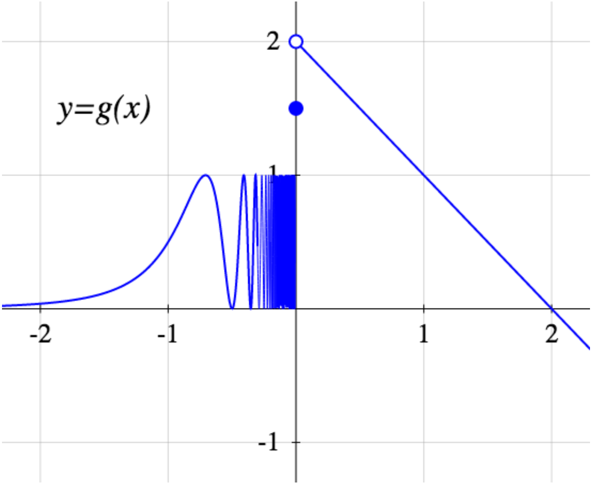
(A) $\lim_{x \to 0^+} \lfloor x \rfloor$

(B) $\lim_{x \to 0^-} \lfloor x \rfloor$

(C) $\lim_{x \to 0} \lfloor x \rfloor$

(D) $\lim_{x \to 0} \lfloor x^2 \rfloor$

More limits from a graph



Find the value of

(A) $\lim_{x\to 0^+} g(x)$

(B) $\lim_{x\to 0^+} \lfloor g(x) \rfloor$

(C) $\lim_{x\to 0^+} g(\lfloor x \rfloor)$

(D) $\lim_{x\to 0^-} g(x)$

(E) $\lim_{x\to 0^-} \lfloor g(x) \rfloor$

(F) $\lim_{x\to 0^-} \lfloor \frac{g(x)}{2} \rfloor$

(G) $\lim_{x\to 0^-} g(\lfloor x \rfloor)$

Warmup:
Compute the follwoing limits

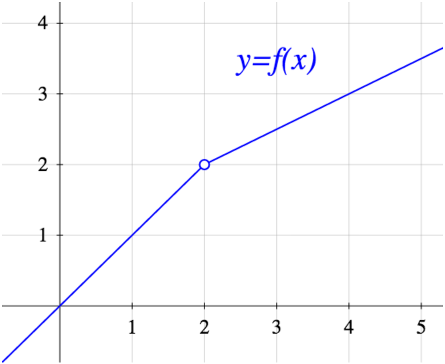
(A) $\lim_{x \to 0} \lfloor x \rfloor$

(B) $\lim_{x \to 0} \lfloor x^2 \rfloor$

Before next class:

- **Watch videos 2.7, 2.8, 2.9**

δ from a graph



(A) Find one value of $\delta > 0$ s.t. $0 < |x - 2| < \delta \implies |f(x) - 2| < 0.5$

(B) Find *all* values of $\delta > 0$ s.t. $0 < |x - 2| < \delta \implies |f(x) - 2| < 0.5$

Warm-up

Write down the formal definition of

$$\lim_{x \to a} f(x) = L.$$

Side limits

Recall

Let $L, a \in \mathbb{R}$.
Let f be a function defined at least on an interval around a, except possibly at a.

$$\lim_{x \to a} f(x) = L$$

means

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t.} \qquad 0 < |x-a| < \delta \implies |f(x)-L| < \varepsilon.$$

Write, instead, the formal definition of

$$\lim_{x \to a^+} f(x) = L, \qquad \text{and} \qquad \lim_{x \to a^-} f(x) = L.$$

Infinite limits

Definition

Let $a \in \mathbb{R}$.
Let f be a function defined at least on an interval around a, except possibly at a.
Write a formal definition for

$$\lim_{x \to a} f(x) = \infty.$$

Infinite limits 2

Which statements are equivalent to the definition of $\lim_{x \to a} f(x) = \infty$?

(A) $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies |f(x) - \infty| < \varepsilon$

(B) $\forall M \in \mathbb{R}, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$

(C) $\forall \delta > 0, \exists M > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$

(D) $\forall M > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$

(E) $\forall M > 5, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$

(F) $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > \varepsilon$

Warmup:

- Write the moral definition of $\lim_{x \to a} f(x) = \infty$
- Write the formal definition of $\lim_{x \to a} f(x) = \infty$

Before next class:

- **Watch videos 2.10, 2.11**

Evaluating Limits

- You're trying to guess $\lim_{x \to 0} f(x)$.
- You plug in $x = 0.1, 0.01, 0.001, \ldots$ and get $f(x) = 0$ for all these values.
- In fact, you're told that for all $n = 1, 2, \ldots,$ $f\left(\frac{1}{10^n}\right) = 0.$
- Can you conclude that $\lim_{x \to 0} f(x) = 0$?

Proving a Limit

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x$. We want to prove

$$\lim_{x \to 3} f(x) = 6.$$

Find all values of $X, Y, Z, W > 0$ that make the following statements true

(A) $0 < |t - 3| < X \implies |f(t) - 6| < 1$

(B) $0 < |t - 3| < Y \implies |f(t) - 6| < 1/10$

(C) $0 < |t - 3| < Z \implies |f(t) - 6| < 1/100$

Proving a Limit

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x$. We want to prove

$$\lim_{x \to 3} f(x) = 6.$$

Find all values of $X, Y, Z, W > 0$ that make the following statements true

(A) $0 < |t - 3| < X \implies |f(t) - 6| < 1$

(B) $0 < |t - 3| < Y \implies |f(t) - 6| < 1/10$

(C) $0 < |t - 3| < Z \implies |f(t) - 6| < 1/100$

(D) $0 < |t - 3| < W \implies |f(t) - 6| < \varepsilon$
(assuming $\varepsilon > 0$ is fixed)

Proving a Limit 2

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x$. We want to prove

$$\lim_{x \to 3} f(x) = 6.$$

Write out

- The definition of $\lim_{x \to 3} f(x) = 6$
- The structure of a proof that $\lim_{x \to 3} f(x) = 6$

Proving a Limit 2

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x$. We want to prove

$$\lim_{x \to 3} f(x) = 6.$$

Write out

- The definition of $\lim_{x \to 3} f(x) = 6$
- The structure of a proof that $\lim_{x \to 3} f(x) = 6$

- Prove that $\lim_{x \to 3} f(x) = 6$

Disproving a Limit

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x$. We want to prove

$$\lim_{x \to 3} f(x) \neq 7.$$

Write out

- The definition of $\lim\limits_{x \to 3} f(x) \neq 7$
- The structure of a proof that $\lim\limits_{x \to 3} f(x) \neq 7$

Disproving a Limit

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x$. We want to prove

$$\lim_{x \to 3} f(x) \neq 7.$$

Write out

- The definition of $\lim_{x \to 3} f(x) \neq 7$
- The structure of a proof that $\lim_{x \to 3} f(x) \neq 7$

- Prove that $\lim_{x \to 3} f(x) \neq 7$

Before next class:

- **Watch videos 2.12, 2.13**

What is wrong with this “proof”?

Prove that

$$\lim_{x \to 3} (5x + 1) = 16$$

“Proof:”

Let $\varepsilon > 0$.
WTS $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$0 < |x - 3| < \delta \implies |(5x + 1) - (16)| < \varepsilon$$

$$\begin{aligned} |(5x+1) - (16)| < \varepsilon &\iff |5x + 15| < \varepsilon \\ &\iff 5|x+3| < \varepsilon \implies \delta = \frac{\varepsilon}{3} \end{aligned}$$

$\square$

Indeterminate form

Let $a \in \mathbb{R}$. Let f and g be positive functions defined near a, except maybe at a.

Assume $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$.

What can we conclude about $\lim_{x \to a} \dfrac{f(x)}{g(x)}$?

(A) The limit is 1.
(B) The limit is 0.
(C) The limit is ∞.
(D) Limit does not exist.
(E) We do not have enough information to decide.

True or False?

Is this claim true?

Claim

Let $a \in \mathbb{R}$.

Let f and g be functions defined near a.

- IF $\lim_{x \to a} f(x) = 0$,
- THEN $\lim_{x \to a} [f(x)g(x)] = 0$.

A new theorem about products

Theorem

Let $a \in \mathbb{R}$. Let f and g be functions with domain $\mathbb{R}$, except possibly a. Assume

- $\lim_{x \to a} f(x) = 0$, and
- g is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN $\lim_{x \to a} [f(x)g(x)] = 0$

A new theorem about products

Theorem

Let $a \in \mathbb{R}$. Let f and g be functions with domain $\mathbb{R}$, except possibly a. Assume

- $\lim_{x \to a} f(x) = 0$, and
- g is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN $\lim_{x \to a} [f(x)g(x)] = 0$

(A) Write down the formal definition of what you want to prove.

A new theorem about products

Theorem

Let $a \in \mathbb{R}$. Let f and g be functions with domain $\mathbb{R}$, except possibly a. Assume

- $\lim_{x \to a} f(x) = 0$, and
- g is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN $\lim_{x \to a} [f(x)g(x)] = 0$

(A) Write down the formal definition of what you want to prove.

(B) Write down what the structure of the formal proof.

A new theorem about products

Theorem

Let $a \in \mathbb{R}$. Let f and g be functions with domain $\mathbb{R}$, except possibly a. Assume

- $\lim_{x \to a} f(x) = 0$, and
- g is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN $\lim_{x \to a} [f(x)g(x)] = 0$

(A) Write down the formal definition of what you want to prove.

(B) Write down what the structure of the formal proof.

(C) Rough work.

A new theorem about products

Theorem

Let $a \in \mathbb{R}$. Let f and g be functions with domain $\mathbb{R}$, except possibly a. Assume

- $\lim_{x \to a} f(x) = 0$, and
- g is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN $\lim_{x \to a} [f(x)g(x)] = 0$

(A) Write down the formal definition of what you want to prove.
(B) Write down what the structure of the formal proof.
(C) Rough work.
(D) Write down a complete formal proof.

Critique this "proof" – #1

- WTS $\lim_{x \to a} [f(x)g(x)] = 0$:
  $\forall \varepsilon > 0, \exists \delta > 0 \quad \text{s.t.} \quad 0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon.$
- We know $\lim_{x \to a} f(x) = 0$
  $\forall \varepsilon_1 > 0, \exists \delta_1 > 0 \quad \text{s.t.} \quad 0 < |x - a| < \delta_1 \implies |f(x)| < \varepsilon_1.$
- We know $\exists M > 0 \quad \text{s.t.} \quad \forall x \neq 0, \ |g(x)| \leq M.$
- $|f(x)g(x)| = |f(x)||g(x)| < \varepsilon_1 M$
- $\varepsilon = \varepsilon_1 M \implies \varepsilon_1 = \dfrac{\varepsilon}{M}$
- Take $\delta = \delta_1$

Before next class:

- **Watch videos 2.14, 2.15**

Limits involving $\sin(1/x)$ Part I

$\lim_{x \to 0} \sin(1/x)$

(A) DNE because the function values oscillate around 0

(B) DNE because $1/0$ is undefined

(C) DNE because no matter how close x gets to 0, there are x's near 0 for which $\sin(1/x) = 1$, and some for which $\sin(1/x) = -1$

(D) all of the above

Limits involving $\sin(1/x)$ Part II

$\lim_{x \to 0} x^2 \sin(1/x)$

(A) DNE because the function values oscillate around 0

(B) DNE because $1/0$ is undefined

(C) DNE because no matter how close x gets to 0, there are x's near 0 for which $\sin(1/x) = 1$, and some for which $\sin(1/x) = -1$

(D) equals 0

(E) equals 1

A new squeeze

This is the Squeeze Theorem, as you know it:

The (classical) Squeeze Theorem

Let $a, L \in \mathbb{R}$.
Let f, g, and h be functions defined near a, except possibly at a.

IF
- For x close to a but not a, $\quad h(x) \leq g(x) \leq f(x)$
- $\lim_{x \to a} f(x) = L \quad$ and $\quad \lim_{x \to a} h(x) = L$

THEN
- $\lim_{x \to a} g(x) = L$

A new squeeze

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The (classical) Squeeze Theorem

Let $a, L \in \mathbb{R}$.
Let f, g, and h be functions defined near a, except possibly at a.

IF
- For x close to a but not a, $\quad h(x) \leq g(x) \leq f(x)$
- $\lim_{x \to a} f(x) = L \quad$ and $\quad \lim_{x \to a} h(x) = L$

THEN
- $\lim_{x \to a} g(x) = L$

Come up with a new version of the theorem about limits being infinity. (The conclusion should be $\lim_{x \to a} g(x) = \infty$.)

Hint: Draw a picture for the classical Squeeze Theorem. Then draw a picture for the new theorem.

A new theorem about products

Theorem

Let $a \in \mathbb{R}$. Let f and g be functions with domain $\mathbb{R}$, except possibly a. Assume

- $\lim\limits_{x \to a} f(x) = 0$, and
- g is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN $\lim\limits_{x \to a} [f(x)g(x)] = 0$

Critique this “proof” – #1

- WTS $\lim_{x \to a} [f(x)g(x)] = 0$:
  $\forall \varepsilon > 0, \exists \delta > 0 \quad \text{s.t.} \quad 0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon.$
- We know $\lim_{x \to a} f(x) = 0$
  $\forall \varepsilon_1 > 0, \exists \delta_1 > 0 \quad \text{s.t.} \quad 0 < |x - a| < \delta_1 \implies |f(x)| < \varepsilon_1.$
- We know $\exists M > 0 \quad \text{s.t.} \quad \forall x \neq 0, \ |g(x)| \leq M.$
- $|f(x)g(x)| = |f(x)||g(x)| < \varepsilon_1 M$
- $\varepsilon = \varepsilon_1 M \implies \varepsilon_1 = \dfrac{\varepsilon}{M}$
- Take $\delta = \delta_1$

Critique this "proof" – #2

- WTS $\lim_{x \to a} [f(x)g(x)] = 0$. By definition, WTS:
$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon$$
- Let $\varepsilon > 0$.
- Use the value $\dfrac{\varepsilon}{M}$ as "epsilon" in the definition of $\lim_{x \to a} f(x) = 0$
$$\exists \delta_1 \in \mathbb{R} \text{ s.t. } 0 < |x - a| < \delta_1 \implies |f(x)| < \frac{\varepsilon}{M}.$$
- Take $\delta = \delta_1$.
- Let $x \in \mathbb{R}$. Assume $0 < |x - a| < \delta$
- Since $\exists M > 0 \text{ s.t. } \forall x \neq 0, |g(x)| \leq M$
$$|f(x)g(x)| < \frac{\varepsilon}{M} \cdot M = \varepsilon.$$

Critique this "proof" – #3

- Since g is bounded, $\exists M > 0 \quad \text{s.t.} \quad \forall x \neq 0, \ |g(x)| \leq M$

- Since $\lim_{x \to a} f(x) = 0$, there exists $\delta_1 > 0$ s.t.

  if $0 < |x - a| < \delta_1, \quad$ then $\quad |f(x) - 0| = |f(x)| < \varepsilon_1 = \dfrac{\varepsilon}{M}$.

- $|f(x)g(x)| = |f(x)| \cdot |g(x)| \leq |f(x)| \cdot M < \varepsilon_1 \cdot M = \dfrac{\varepsilon}{M} \cdot M = \varepsilon$

- In summary, by setting $\delta = \min\{\delta_1\}$, we find that

  if $0 < |x - a| < \delta, \quad$ then $\quad |f(x) \cdot g(x)| < \varepsilon$.

Before next class:

- **Watch videos 2.16, 2.17, 2.18**

Undefined function

Let $a \in \mathbb{R}$ and let f be a function. Assume $f(a)$ is undefined.

What can we conclude?

(A) $\lim_{x \to a} f(x)$ exist

(B) $\lim_{x \to a} f(x)$ doesn't exist.

(C) No conclusion. $\lim_{x \to a} f(x)$ may or may not exist.

What else can we conclude?

(D) f is continuous at a.

(E) f is not continuous at a.

(F) No conclusion. f may or may not be continuous at a.

Definition of continuity

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$.

Which statements are equivalent to "f is continuous at a"?

(A) $\lim_{x \to a} f(x)$ exists.

(B) $\lim_{x \to a} f(x)$ exists and $f(a)$ is defined.

(C) $\lim_{x \to a} f(x) = f(a)$.

(D) $\forall \varepsilon > 0,\ \exists \delta > 0,\ \text{s.t. } \forall x \in \mathbb{R},\ 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$

(E) $\forall \varepsilon > 0,\ \exists \delta > 0,\ \text{s.t. } \forall x \in \mathbb{R},\ 0 < |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$

(F) $\forall \varepsilon > 0,\ \exists \delta > 0,\ \text{s.t. } \forall x \in \mathbb{R},\ |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$

A new function

- Let $x, y \in \mathbb{R}$. What does the following expression calculate? Prove it.

$$f(x, y) = \frac{x + y + |x - y|}{2}$$

Suggestion: If you don't know how to start, try some sample values of x and y.

A new function

- Let $x, y \in \mathbb{R}$. What does the following expression calculate? Prove it.

$$f(x,y) = \frac{x + y + |x - y|}{2}$$

  Suggestion: If you don't know how to start, try some sample values of x and y.

- Write a similar expression to compute $\min\{x, y\}$.

More continuous functions

We want to prove the following theorem

Theorem

IF f and g are continuous functions
THEN $h(x) = \max\{f(x), g(x)\}$ is also a continuous function.

You are allowed to use all results that we already know.
What is the fastest way to prove this?

Hint: There is a way to prove this quickly without writing any epsilons.

Existence

Write down the formal definition of the following statements:

(A) $\lim_{x \to a} f(x) = L$

(B) $\lim_{x \to a} f(x)$ exists

(C) $\lim_{x \to a} f(x)$ does not exist

Before next class:

- **Watch videos 2.19, 2.20**

True or False? – Discontinuities

(A) IF f and g have removable discontinuities at a
THEN $f + g$ has a removable discontinuity at a

(B) IF f and g have non-removable discontinuities at a
THEN $f + g$ has a non-removable discontinuity at a

Which one is the correct claim?

Claim 1?

(Assuming these limits exist)

$$\lim_{x \to a} g(f(x)) \;=\; g\left(\lim_{x \to a} f(x)\right)$$

Claim 2?

IF (A) $\lim_{x \to a} f(x) = L$, and (B) $\lim_{t \to L} g(t) = M$

THEN (C) $\lim_{x \to a} g(f(x)) = M$

A difficult example

Construct a pair of functions f and g such that

$$\begin{aligned}
\lim_{x \to 0} f(x) &= 1 \\
\lim_{t \to 1} g(t) &= 2 \\
\lim_{x \to 0} g(f(x)) &= 42
\end{aligned}$$

Continuity and Quantifiers

Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

Which statements are equivalent to "f is continuous"?

(A) $\lim\limits_{x \to a} f(x) = f(a)$.

(B) $\forall \varepsilon > 0,\ \exists \delta > 0,\ \text{s.t. } \forall x \in \mathbb{R},$
$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$

(C) $\forall a \in \mathbb{R},\ \forall \varepsilon > 0,\ \exists \delta > 0,\ \text{s.t. } \forall x \in \mathbb{R},$
$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$

(D) $\forall x \in \mathbb{R},\ \forall \varepsilon > 0,\ \exists \delta > 0,\ \text{s.t. } \forall a \in \mathbb{R},$
$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$

(E) $\forall \varepsilon > 0,\ \exists \delta > 0,\ \text{s.t. } \forall a \in \mathbb{R},\ \forall x \in \mathbb{R},$
$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$

Before next class:

- **Watch videos 2.21, 2.22**

Transforming limits

The only thing we know about the function g is that

$$\lim_{x \to 0} \frac{g(x)}{x^2} = 2.$$

Use it to compute the following limits:

(A) $\displaystyle \lim_{x \to 0} \frac{g(x)}{x}$

(B) $\displaystyle \lim_{x \to 0} \frac{g(x)}{x^4}$

(C) $\displaystyle \lim_{x \to 0} \frac{g(3x)}{x^2}$

Compute:

(A) $\lim_{x\to\infty} \left(x^7 - 2x^5 + 11\right)$

(B) $\lim_{x\to\infty} \left(x^2 - \sqrt{x^5 + 1}\right)$

(C) $\lim_{x\to\infty} \frac{x^2 + 11}{x + 1}$

(D) $\lim_{x\to\infty} \frac{x^2 + 2x + 3}{3x^2 + 4x + 5}$

(E) $\lim_{x\to 0} \frac{\sin(5x)}{x}$

(F) $\lim_{x\to 0} \frac{\tan^2(2x^2)}{x^4}$

(G) $\lim_{x\to 0} \frac{\sin e^x}{e^x}$

(H) $\lim_{x\to 0} \frac{1 - \cos x}{x}$

(I) $\lim_{x\to\infty} \frac{x^3 + \sqrt{2x^6 + 1}}{2x^3 + \sqrt{x^5 + 1}}$

Which solution is right?

Compute $L = \lim_{x\to-\infty} \left[x - \sqrt{x^2+x}\right]$.

- **Solution 1**

$$L = \lim_{x\to-\infty} \frac{\left[x - \sqrt{x^2+x}\right]\left[x + \sqrt{x^2+x}\right]}{\left[x + \sqrt{x^2+x}\right]} = \lim_{x\to-\infty} \frac{x^2 - (x^2+x)}{\left[x + \sqrt{x^2+x}\right]}$$

$$= \lim_{x\to-\infty} \frac{-x}{x\left[1 + \sqrt{1+\frac{1}{x}}\right]} = \lim_{x\to-\infty} \frac{-1}{\left[1 + \sqrt{1+\frac{1}{x}}\right]} = \frac{-1}{2}$$

- **Solution 2**

$$L = \lim_{x\to-\infty} \left[{\color{red}x} - {\color{blue}\sqrt{x^2+x}}\right] = {\color{red}(-\infty)} - {\color{blue}\infty} = -\infty$$

Before next class:

- **Watch videos 3.1, 3.2, 3.3**

Existence of solutions

Prove that the equation

$$x^4 - 2x = 100$$

has at least two solutions.

Can this be proven? (Use IVT)

(A) Prove that there exists a time of the day when the hour hand and the minute hand of a clock form an angle of exactly 23 degrees.

(B) During a Raptors basketball game, at half time the Raptors have 52 points. Prove that at some point they had exactly 26 points.

(C) Prove that at some point during Jason's life, his height in centimetres was exactly equal to 10 times his weight in kilograms. Some data:

- His height at birth: 47 cm
- His weight at birth: 3.2 kg
- His height today: 170 cm

Definition of maximum

Let f be a function with domain I.
Which one (or ones) of the following is (or are) a definition of "f has a maximum on I"?

(A) $\forall x \in I, \exists C \in \mathbb{R}$ s.t. $f(x) \leq C$

(B) $\exists C \in I$ s.t. $\forall x \in I, f(x) \leq C$

(C) $\exists C \in \mathbb{R}$ s.t. $\forall x \in I, f(x) \leq C$

(D) $\exists C \in \mathbb{R}$ s.t. $\forall x \in I, f(x) < C$

Extrema

In each of the following cases, does the function f have a maximum and a minimum on the interval A?

(A) $f(x) = x^2, \quad A = (-1, 1).$

(B) $f(x) = \dfrac{(e^x + 2)\sin x}{x} - \cos x + 3, \quad A = [2, 6]$

(C) $f(x) = \dfrac{(e^x + 2)\sin x}{x} - \cos x + 3, \quad A = (0, 5]$