MAT137 Lecture 7 — Absolute Values

Warmup:

You want to show "$\exists n \in \mathbb{N}$ s.t. $n^2 = 4$". Which proofs below are correct/incorrect?

(A) Let $n = 2$. Then, $n \in \mathbb{N}$ and $n^2 = 4$.

(B) Let $n \in \mathbb{N}$. Take $n = 2$. Then $n^2 = 4$.

(C) Let $n \in \mathbb{N}$. Assume $n = 2$. Then $n^2 = 4$.

(D) Take $n = 4$. Then $n \in \mathbb{N}$ and $n^2 = 4$.

Before next class:

- **Watch videos 2.1, 2.2, 2.3**

Variations on induction

Let S_n be a statement depending on a positive integer n.

What conclusions can you draw in each of the following cases? (I.e., for which n do you know that S_n is true?)

(A) We have proven:

- S_3
- $$\forall n \geq 1,\ S_n \implies S_{n+1}$$

(B) We have proven:

- S_1
- $$\forall n \geq 3,\ S_n \implies S_{n+1}$$

(C) We have proven:

- S_1
- $$\forall n \geq 1,\ S_n \implies S_{n+3}$$

(D) We have proven:

- S_1
- $$\forall n \geq 1,\ S_{n+1} \implies S_n$$

Variations on induction 2

We want to prove

$$\forall n \geq 1, \; S_n$$

So far we have proven

- S_1
- $\forall n \geq 1, \; S_n \implies S_{n+3}$.

What else do we need to do?

What is wrong with this proof by induction?

Theorem

$\forall N \geq 1$, *every set of* N *students in MAT137 will get the same grade.*

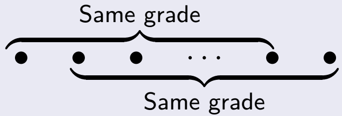
What is wrong with this proof by induction?

Theorem

$\forall N \geq 1$, every set of N students in MAT137 will get the same grade.

Proof.

- **Base case.** It is clearly true for $N = 1$.
- **Induction step.**
  Assume it is true for N. I'll show it is true for $N + 1$.
  Take a set of $N + 1$ students. By induction hypothesis:
  - The first N students get the same grade.
  - The last N students get the same grade.



Hence the $N + 1$ students all get the same grade.

$\square$

What is wrong with this proof by induction?

For every $N \geq 1$, let

$$S_N = \text{ “every set of } N \text{ students in MAT137} \\ \text{will get the same grade”}$$

What did we actually prove in the previous page?

- S_1 ?
- $\forall N \geq 1, \quad S_N \implies S_{N+1}$?

Properties of absolute value

Let $a, b \in \mathbb{R}$. Are the following conclusions correct?

(A) $|ab| = |a||b|$
(B) $|a + b| = |a| + |b|$

If any of the conclusions is wrong, fix it.

Properties of inequalities

Let $a, b, c \in \mathbb{R}$.
Assume $a < b$. Are the following conclusions correct?

(A) $a + c < b + c$

(B) $a - c < b - c$

(C) $ac < bc$

(D) $a^2 < b^2$

(E) $\frac{1}{a} < \frac{1}{b}$

(F) $\sin a < \sin b$

If any of the conclusions is wrong, fix it.

Sets described by distance

Let $a \in \mathbb{R}$. Let $\delta > 0$.

Describe the following sets using interval notation.

(A) $A = \{x \in \mathbb{R} \ : \ |x| < \delta\}$

(B) $B = \{x \in \mathbb{R} \ : \ |x| > \delta\}$

(C) $C = \{x \in \mathbb{R} \ : \ |x - a| < \delta\}$

(D) $D = \{x \in \mathbb{R} \ : \ 0 < |x - a| < \delta\}$

Implications

Find *all* positive values of X, Y, and Z which make the following implications true.

(A) $|t-3|<1 \implies |2t-6|<X$

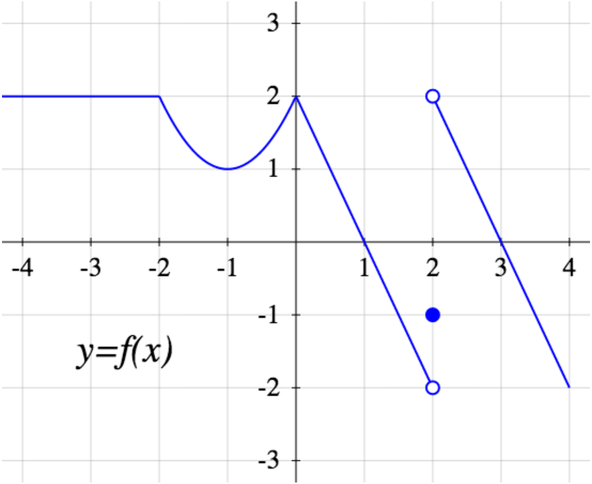
(B) $|t-3|<Y \implies |2t-6|<1$

(C) $|t-3|<1 \implies |t+5|<Z$

Before next class:

- **Watch videos 2.5, 2.6**

Limits from a graph

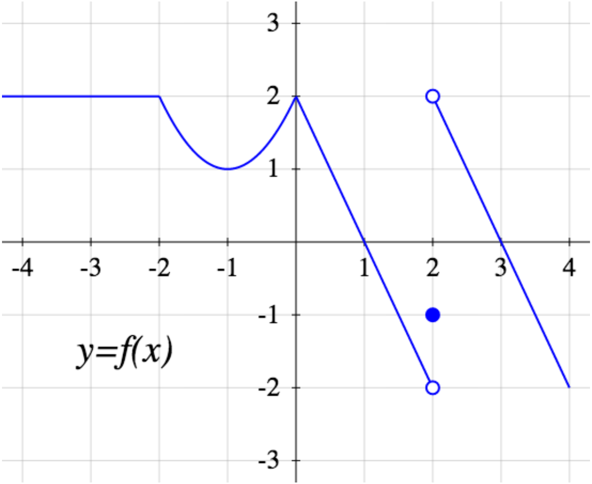


Find the value of

(A) $\lim_{x \to 2} f(x)$

(B) $\lim_{x \to 0} f(f(x))$

Limits from a graph



Find the value of

(A) $\lim_{x \to 2} f(x)$

(B) $\lim_{x \to 0} f(f(x))$

(C) $\lim_{x \to 2} [f(x)]^2$

(D) $\lim_{x \to 0} f(2\cos x)$

Floor

Given a real number x, we defined the *floor of* x, denoted by $\lfloor x \rfloor$, as the largest integer smaller than or equal to x. For example:

$$\lfloor \pi \rfloor = 3, \qquad \lfloor 7 \rfloor = 7, \qquad \lfloor -0.5 \rfloor = -1.$$

Sketch the graph of $y = \lfloor x \rfloor$. Then compute:

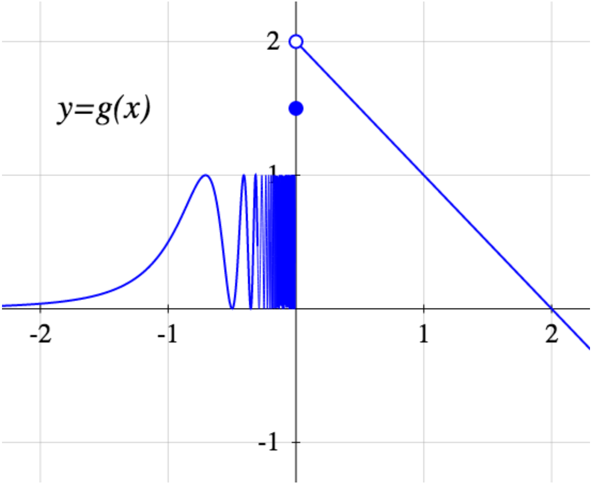
(A) $\lim_{x \to 0^+} \lfloor x \rfloor$

(B) $\lim_{x \to 0^-} \lfloor x \rfloor$

(C) $\lim_{x \to 0} \lfloor x \rfloor$

(D) $\lim_{x \to 0} \lfloor x^2 \rfloor$

More limits from a graph



Find the value of

(A) $\lim_{x\to 0^+} g(x)$

(B) $\lim_{x\to 0^+} \lfloor g(x) \rfloor$

(C) $\lim_{x\to 0^+} g(\lfloor x \rfloor)$

(D) $\lim_{x\to 0^-} g(x)$

(E) $\lim_{x\to 0^-} \lfloor g(x) \rfloor$

(F) $\lim_{x\to 0^-} \lfloor \frac{g(x)}{2} \rfloor$

(G) $\lim_{x\to 0^-} g(\lfloor x \rfloor)$

Limit at a point

If a function f is not defined at $x = a$, then

(A) $\lim_{x \to a} f(x)$ cannot exist

(B) $\lim_{x \to a} f(x)$ could be 0

(C) $\lim_{x \to a} f(x)$ must approach ∞

(D) none of the above.

- You're trying to guess $\lim_{x \to 0} f(x)$.
- You plug in $x = 0.1, 0.01, 0.001, \ldots$ and get $f(x) = 0$ for all these values.
- In fact, you're told that for all $n = 1, 2, \ldots,$
$$f\left(\frac{1}{10^n}\right) = 0.$$
- Can you conclude that $\lim_{x \to 0} f(x) = 0$?

Exponential limits

Compute:

$$\lim_{t \to 0^+} e^{1/t}, \qquad \lim_{t \to 0^-} e^{1/t}.$$

Suggestion: Sketch the graph of $y = e^x$ first.

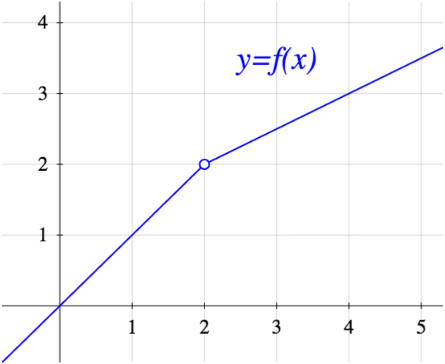
Rational limits

Consider the function

$$h(x) = \frac{(x-1)(2+x)}{x^2(x-1)(2-x)}.$$

- Find all real values a for which $h(a)$ is undefined.
- For each such value of a, compute $\lim_{x \to a^+} h(x)$ and $\lim_{x \to a^-} h(x)$.
- Based on your answer, and nothing else, try to sketch the graph of h.

δ from a graph



(A) Find one value of $\delta > 0$ s.t. $0 < |x - 2| < \delta \implies |f(x) - 2| < 0.5$

(B) Find *all* values of $\delta > 0$ s.t. $0 < |x - 2| < \delta \implies |f(x) - 2| < 0.5$

Warm-up

Write down the formal definition of

$$\lim_{x \to a} f(x) = L.$$

Side limits

Recall

Let $L, a \in \mathbb{R}$.
Let f be a function defined at least on an interval around a, except possibly at a.

$$\lim_{x \to a} f(x) = L$$

means

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \quad 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

Write, instead, the formal definition of

$$\lim_{x \to a^+} f(x) = L, \quad \text{and} \quad \lim_{x \to a^-} f(x) = L.$$

Infinite limits

Definition

Let $a \in \mathbb{R}$.
Let f be a function defined at least on an interval around a, except possibly at a.
Write a formal definition for

$$\lim_{x \to a} f(x) = \infty.$$

Infinite limits - 2

Which one(s) is the definition of $\lim_{x \to a} f(x) = \infty$?

(A)
$$\forall M \in \mathbb{R},\ \exists \delta > 0 \ \text{ s.t. } \ 0 < |x - a| < \delta \implies f(x) > M$$

(B)
$$\forall M \in \mathbb{Z},\ \exists \delta > 0 \ \text{ s.t. } \ 0 < |x - a| < \delta \implies f(x) > M$$

(C)
$$\forall M > 0,\ \exists \delta > 0 \ \text{ s.t. } \ 0 < |x - a| < \delta \implies f(x) > M$$

(D)
$$\forall M > 5,\ \exists \delta > 0 \ \text{ s.t. } \ 0 < |x - a| < \delta \implies f(x) > M$$

Related implications

Let $a \in \mathbb{R}$. Let f be a function. Assume we know

$$0 < |x - a| < 0.1 \quad \implies \quad f(x) > 100$$

(A) Which values of $M \in \mathbb{R}$ satisfy ... ?

$$0 < |x - a| < 0.1 \quad \implies \quad f(x) > M$$

Related implications

Let $a \in \mathbb{R}$. Let f be a function. Assume we know

$$0 < |x - a| < 0.1 \quad \Longrightarrow \quad f(x) > 100$$

(A) Which values of $M \in \mathbb{R}$ satisfy ... ?

$$0 < |x - a| < 0.1 \quad \Longrightarrow \quad f(x) > M$$

(B) Which values of $\delta > 0$ satisfy ... ?

$$0 < |x - a| < \delta \quad \Longrightarrow \quad f(x) > 100$$

Strict or non-strict inequality?

Let f be a function with domain $\mathbb{R}$. One of these statements implies the other. Which one?

(A) $\forall M \in \mathbb{R}, \exists N \in \mathbb{R}$ s.t. $x > N \implies f(x) > M$

(B) $\forall M \in \mathbb{R}, \exists N \in \mathbb{R}$ s.t. $x > N \implies f(x) \geq M$

Negation of conditionals

Write the negation of these statements:

(A) If Justin Trudeau has a brother, then he also has a sister.

(B) If a student in this class has a brother, then they also have a sister.

More negation

Let f be a function with domain $\mathbb{R}$. Write the negation of the statement:

$$\text{IF } \ 2 < x < 4, \qquad \text{THEN } \ 1 < f(x) < 3.$$

Existence

Write down the formal definition of the following statements:

(A) $\lim_{x \to a} f(x) = L$

(B) $\lim_{x \to a} f(x)$ exists

(C) $\lim_{x \to a} f(x)$ does not exist

Preparation: choosing deltas

(A) Find one value of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 1.$$

(B) Find *all* values of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 1.$$

(C) Find *all* values of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 0.1.$$

(D) Let us fix $\varepsilon > 0$. Find *all* values of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < \varepsilon.$$

What is wrong with this "proof"?

Prove that

$$\lim_{x \to 3} (5x + 1) = 16$$

"Proof:"

Let $\varepsilon > 0$.
WTS $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$0 < |x - 3| < \delta \implies |(5x + 1) - (16)| < \varepsilon$$

$$|(5x+1) - (16)| < \varepsilon \iff |5x + 15| < \varepsilon$$

$$\iff 5|x + 3| < \varepsilon \implies \delta = \frac{\varepsilon}{3}$$

$\square$

Your first $\varepsilon - \delta$ proof

Goal

We want to prove that

$$\lim_{x \to 3} (5x + 1) = 16 \tag{1}$$

directly from the definition.

Your first $\varepsilon - \delta$ proof

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We want to prove that

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directly from the definition.

(A) Write down the formal definition of the statement (1).

Your first $\varepsilon - \delta$ proof

Goal

We want to prove that

$$\lim_{x \to 3} (5x + 1) = 16 \tag{1}$$

directly from the definition.

(A) Write down the formal definition of the statement (1).

(B) Write down what the structure of the formal proof should be, without filling the details.

Your first $\varepsilon - \delta$ proof

Goal

We want to prove that

$$\lim_{x \to 3} (5x + 1) = 16 \tag{1}$$

directly from the definition.

(A) Write down the formal definition of the statement (1).

(B) Write down what the structure of the formal proof should be, without filling the details.

(C) Write down a complete formal proof.

A harder proof

Goal

We want to prove that

$$\lim_{x \to 0} \left(x^3 + x^2\right) = 0 \tag{2}$$

directly from the definition.

A harder proof

Goal

We want to prove that

$$\lim_{x \to 0} \left(x^3 + x^2\right) = 0 \tag{2}$$

directly from the definition.

(A) Write down the formal definition of the statement (2).

A harder proof

Goal

We want to prove that

$$\lim_{x \to 0} \left(x^3 + x^2 \right) = 0 \tag{2}$$

directly from the definition.

(A) Write down the formal definition of the statement (2).

(B) Write down what the structure of the formal proof should be, without filling the details.

A harder proof

Goal

We want to prove that

$$\lim_{x \to 0} \left(x^3 + x^2\right) = 0 \tag{2}$$

directly from the definition.

(A) Write down the formal definition of the statement (2).

(B) Write down what the structure of the formal proof should be, without filling the details.

(C) Rough work: What is δ?

A harder proof

Goal

We want to prove that

$$\lim_{x \to 0} \left(x^3 + x^2\right) = 0 \tag{2}$$

directly from the definition.

(A) Write down the formal definition of the statement (2).
(B) Write down what the structure of the formal proof should be, without filling the details.
(C) Rough work: What is δ?
(D) Write down a complete formal proof.

Is this proof correct?

Claim:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \quad 0 < |x| < \delta \implies |x^3 + x^2| < \varepsilon.$$

Proof:

- Let $\varepsilon > 0$.
- Take $\delta = \sqrt{\dfrac{\varepsilon}{|x+1|}}$.
- Let $x \in \mathbb{R}$. Assume $0 < |x| < \delta$. Then

$$|x^3+x^2| = x^2|x+1| < \delta^2|x+1| = \frac{\varepsilon}{|x+1|}|x+1| = \varepsilon.$$

- I have proven that $|x^3 + x^2| < \varepsilon$. $\square$

Choosing deltas again

Let us fix numbers $A, \varepsilon > 0$. Find:

(A) a value of $\delta > 0$ s.t. $|x| < \delta \implies |Ax^2| < \varepsilon$

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(C) a value of $\delta > 0$ s.t. $|x| < \delta \implies |x + 1| < 10$

Choosing deltas again

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(C) a value of $\delta > 0$ s.t. $|x| < \delta \implies |x + 1| < 10$

(D) *all* values of $\delta > 0$ s.t. $|x| < \delta \implies |x + 1| < 10$

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(C) a value of $\delta > 0$ s.t. $|x| < \delta \implies |x + 1| < 10$

(D) *all* values of $\delta > 0$ s.t. $|x| < \delta \implies |x + 1| < 10$

(E) a value of $\delta > 0$ s.t. $|x| < \delta \implies \begin{cases} |Ax^2| < \varepsilon \\ |x + 1| < 10 \end{cases}$

Choosing deltas again

Let us fix numbers $A, \varepsilon > 0$. Find:

(A) a value of $\delta > 0$ s.t. $|x| < \delta \implies |Ax^2| < \varepsilon$

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(E) a value of $\delta > 0$ s.t. $|x| < \delta \implies \begin{cases} |Ax^2| < \varepsilon \\ |x+1| < 10 \end{cases}$

(F) a value of $\delta > 0$ s.t. $|x| < \delta \implies |(x+1)x^2| < \varepsilon$

Indeterminate form

Let $a \in \mathbb{R}$. Let f and g be positive functions defined near a, except maybe at a.

Assume $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$.

What can we conclude about $\lim_{x \to a} \frac{f(x)}{g(x)}$?

(A) The limit is 1.
(B) The limit is 0.
(C) The limit is ∞.
(D) The limit does not exist.
(E) We do not have enough information to decide.

A theorem about limits

Let f be a function with domain $\mathbb{R}$ such that

$$\lim_{x \to 0} f(x) = 3$$

Prove that

$$\lim_{x \to 0} [5f(2x)] = 15$$

directly from the definition of limit. Do not use any of the limit laws.

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(A) Write down the formal definition of the statement you want to prove.

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(C) Rough work.

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directly from the definition of limit. Do not use any of the limit laws.

(A) Write down the formal definition of the statement you want to prove.

(B) Write down what the structure of the formal proof should be, without filling the details.

(C) Rough work.

(D) Write down a complete proof.

Proof feedback

(A) Is the structure of the proof correct?
(First fix ε, then choose δ, then ...)

(B) Did you say exactly what δ is?

(C) Is the proof self-contained?
(I do not need to read the rough work)

(D) Are all variables defined? In the right order?

(E) Do all steps follow logically from what comes before?
Do you start from what you know and prove what you have to prove?

(F) Are you proving your conclusion or assuming it?

A new squeeze

This is the Squeeze Theorem, as you know it:

The (classical) Squeeze Theorem

Let $a, L \in \mathbb{R}$.
Let f, g, and h be functions defined near a, except possibly at a.

IF
- For x close to a but not a, $\quad h(x) \leq g(x) \leq f(x)$
- $\lim_{x \to a} f(x) = L \quad$ and $\quad \lim_{x \to a} h(x) = L$

THEN
- $\lim_{x \to a} g(x) = L$

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- $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} h(x) = L$

THEN
- $\lim_{x \to a} g(x) = L$

Come up with a new version of the theorem about limits being infinity. (The conclusion should be $\lim_{x \to a} g(x) = \infty$.)

Hint: Draw a picture for the classical Squeeze Theorem. Then draw a picture for the new theorem.

A new squeeze

The (new) Squeeze Theorem

Let $a \in \mathbb{R}$.
Let g and h be functions defined near a, except possibly at a.

IF
- For x close to a but not a, $h(x) \leq g(x)$
- $\lim_{x \to a} h(x) = \infty$

THEN
- $\lim_{x \to a} g(x) = \infty$

A new squeeze

The (new) Squeeze Theorem

Let $a \in \mathbb{R}$.
Let g and h be functions defined near a, except possibly at a.

IF
- For x close to a but not a, $\quad h(x) \leq g(x)$
- $\lim\limits_{x \to a} h(x) = \infty$

THEN
- $\lim\limits_{x \to a} g(x) = \infty$

(A) Replace the first hypothesis with a more precise mathematical statement.

A new squeeze

The (new) Squeeze Theorem

Let $a \in \mathbb{R}$.
Let g and h be functions defined near a, except possibly at a.

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- For x close to a but not a, $\quad h(x) \leq g(x)$
- $\lim_{x \to a} h(x) = \infty$

THEN
- $\lim_{x \to a} g(x) = \infty$

(A) Replace the first hypothesis with a more precise mathematical statement.

(B) Write down the definition of what you want to prove.

A new squeeze

The (new) Squeeze Theorem

Let $a \in \mathbb{R}$.
Let g and h be functions defined near a, except possibly at a.

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- For x close to a but not a, $\quad h(x) \leq g(x)$
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THEN
- $\lim_{x \to a} g(x) = \infty$

(A) Replace the first hypothesis with a more precise mathematical statement.
(B) Write down the definition of what you want to prove.
(C) Write down the structure of the formal proof.

A new squeeze

The (new) Squeeze Theorem

Let $a \in \mathbb{R}$.
Let g and h be functions defined near a, except possibly at a.

IF
- For x close to a but not a, $h(x) \leq g(x)$
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THEN
- $\lim_{x \to a} g(x) = \infty$

(A) Replace the first hypothesis with a more precise mathematical statement.
(B) Write down the definition of what you want to prove.
(C) Write down the structure of the formal proof.
(D) Rough work

A new squeeze

The (new) Squeeze Theorem

Let $a \in \mathbb{R}$.
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THEN
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(A) Replace the first hypothesis with a more precise mathematical statement.
(B) Write down the definition of what you want to prove.
(C) Write down the structure of the formal proof.
(D) Rough work
(E) Write down a complete, formal proof.

True or False?

Is this theorem true?

Claim

Let $a \in \mathbb{R}$.

Let f and g be functions defined near a.

- IF $\lim_{x \to a} f(x) = 0$,
- THEN $\lim_{x \to a} [f(x)g(x)] = 0$.

A new theorem about products

Theorem

Let $a \in \mathbb{R}$. Let f and g be functions with domain $\mathbb{R}$, except possibly a. Assume

- $\lim_{x \to a} f(x) = 0$, and
- g is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN $\lim_{x \to a} [f(x)g(x)] = 0$

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(A) Write down the formal definition of what you want to prove.
(B) Write down what the structure of the formal proof.
(C) Rough work.
(D) Write down a complete formal proof.

Critique this "proof" – #1

- WTS $\lim_{x \to a} [f(x)g(x)] = 0$:
  $\forall \varepsilon > 0, \exists \delta > 0 \quad \text{s.t.} \quad 0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon.$
- We know $\lim_{x \to a} f(x) = 0$
  $\forall \varepsilon_1 > 0, \exists \delta_1 > 0 \quad \text{s.t.} \quad 0 < |x - a| < \delta_1 \implies |f(x)| < \varepsilon_1.$
- We know $\exists M > 0 \quad \text{s.t.} \quad \forall x \neq 0, \ |g(x)| \leq M.$
- $|f(x)g(x)| = |f(x)||g(x)| < \varepsilon_1 M$
- $\varepsilon = \varepsilon_1 M \implies \varepsilon_1 = \dfrac{\varepsilon}{M}$
- Take $\delta = \delta_1$

Critique this "proof" – #2

- WTS $\lim_{x \to a} [f(x)g(x)] = 0$. By definition, WTS:
$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon$$
- Let $\varepsilon > 0$.
- Use the value $\frac{\varepsilon}{M}$ as "epsilon" in the definition of $\lim_{x \to a} f(x) = 0$
$$\exists \delta_1 \in \mathbb{R} \text{ s.t. } 0 < |x - a| < \delta_1 \implies |f(x)| < \frac{\varepsilon}{M}.$$
- Take $\delta = \delta_1$.
- Let $x \in \mathbb{R}$. Assume $0 < |x - a| < \delta$
- Since $\exists M > 0$ s.t. $\forall x \neq 0, |g(x)| \leq M$
$$|f(x)g(x)| < \frac{\varepsilon}{M} \cdot M = \varepsilon.$$

Critique this "proof" – #3

- Since g is bounded, $\exists M > 0 \quad \text{s.t.} \quad \forall x \neq 0,\ |g(x)| \leq M$

- Since $\lim_{x \to a} f(x) = 0$, there exists $\delta_1 > 0$ s.t.

  if $0 < |x - a| < \delta_1, \quad \text{then} \quad |f(x) - 0| = |f(x)| < \varepsilon_1 = \dfrac{\varepsilon}{M}$.

- $|f(x)g(x)| = |f(x)| \cdot |g(x)| \leq |f(x)| \cdot M < \varepsilon_1 \cdot M = \dfrac{\varepsilon}{M} \cdot M = \varepsilon$

- In summary, by setting $\delta = \min\{\delta_1\}$, we find that

  if $0 < |x - a| < \delta, \quad \text{then} \quad |f(x) \cdot g(x)| < \varepsilon$.

Limits involving $\sin(1/x)$ Part I

The reason that $\lim_{x \to 0} \sin(1/x)$ does not exist is:

(A) because the function values oscillate around 0

(B) because $1/0$ is undefined

(C) because no matter how close x gets to 0, there are x's near 0 for which $\sin(1/x) = 1$, and some for which $\sin(1/x) = -1$

(D) all of the above

Limits involving $\sin(1/x)$ Part II

The limit $\lim_{x \to 0} x^2 \sin(1/x)$

(A) does not exist because the function values oscillate around 0

(B) does not exist because $1/0$ is undefined

(C) does not exist because no matter how close x gets to 0, there are x's near 0 for which $\sin(1/x) = 1$, and some for which $\sin(1/x) = -1$

(D) equals 0

(E) equals 1

Absolute value and the Squeeze Theorem

Use the Squeeze Theorem to prove:

Theorem

IF $\lim_{x \to a} |f(x)| = 0$, *THEN* $\lim_{x \to a} f(x) = 0$.

Hint: Recall that $-|c| \leq c \leq |c|$ for every $c \in \mathbb{R}$.

Undefined function

Let $a \in \mathbb{R}$ and let f be a function. Assume $f(a)$ is undefined.

What can we conclude?

(A) $\lim_{x \to a} f(x)$ exist

(B) $\lim_{x \to a} f(x)$ doesn't exist.

(C) No conclusion. $\lim_{x \to a} f(x)$ may or may not exist.

What else can we conclude?

(D) f is continuous at a.

(E) f is not continuous at a.

(F) No conclusion. f may or may not be continuous at a.

A new function

- Let $x, y \in \mathbb{R}$. What does the following expression calculate? Prove it.

$$f(x, y) = \frac{x + y + |x - y|}{2}$$

Suggestion: If you don't know how to start, try some sample values of x and y.

A new function

- Let $x, y \in \mathbb{R}$. What does the following expression calculate? Prove it.

$$f(x, y) = \frac{x + y + |x - y|}{2}$$

  Suggestion: If you don't know how to start, try some sample values of x and y.

- Write a similar expression to compute $\min\{x, y\}$.

More continuous functions

We want to prove the following theorem

Theorem

IF f and g are continuous functions
THEN $h(x) = \max\{f(x), g(x)\}$ is also a continuous function.

You are allowed to use all results that we already know. What is the fastest way to prove this?

Hint: There is a way to prove this quickly without writing any epsilons.

True or False? – Discontinuities

(A) IF f and g have removable discontinuities at a
THEN $f + g$ has a removable discontinuity at a

(B) IF f and g have non-removable discontinuities at a
THEN $f + g$ has a non-removable discontinuity at a

Which one is the correct claim?

Claim 1?

(Assuming these limits exist)

$$\lim_{x \to a} g(f(x)) \;=\; g\left(\lim_{x \to a} f(x)\right)$$

Claim 2?

IF (A) $\lim_{x \to a} f(x) = L$, and (B) $\lim_{t \to L} g(t) = M$

THEN (C) $\lim_{x \to a} g(f(x)) = M$

A difficult example

Construct a pair of functions f and g such that

$$\begin{aligned}\lim_{x\to 0} f(x) &= 1 \\ \lim_{t\to 1} g(t) &= 2 \\ \lim_{x\to 0} g(f(x)) &= 42\end{aligned}$$

Transforming limits

The only thing we know about the function g is that

$$\lim_{x \to 0} \frac{g(x)}{x^2} = 2.$$

Use it to compute the following limits:

(A) $\displaystyle\lim_{x \to 0} \frac{g(x)}{x}$

(B) $\displaystyle\lim_{x \to 0} \frac{g(x)}{x^4}$

(C) $\displaystyle\lim_{x \to 0} \frac{g(3x)}{x^2}$

Compute:

(A) $\lim\limits_{x\to\infty} \left(x^7 - 2x^5 + 11\right)$

(B) $\lim\limits_{x\to\infty} \left(x^2 - \sqrt{x^5+1}\right)$

(C) $\lim\limits_{x\to\infty} \dfrac{x^2+11}{x+1}$

(D) $\lim\limits_{x\to\infty} \dfrac{x^2+2x+3}{3x^2+4x+5}$

(E) $\lim\limits_{x\to\infty} \dfrac{x^3+\sqrt{2x^6+1}}{2x^3+\sqrt{x^5+1}}$

Trig computations

Using that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, compute the following limits:

(A) $\lim_{x \to 2} \frac{\sin x}{x}$

(B) $\lim_{x \to 0} \frac{\sin(5x)}{x}$

(C) $\lim_{x \to 0} \frac{\tan^2(2x^2)}{x^4}$

(D) $\lim_{x \to 0} \frac{\sin e^x}{e^x}$

(E) $\lim_{x \to 0} \frac{1 - \cos x}{x}$

(F) $\lim_{x \to 0} \frac{\tan^{10}(2x^{20})}{\sin^{200}(3x)}$

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(G) $\lim_{x \to 0} \left[(\sin x)\,(\cos(2x))\,(\tan(3x))\,(\sec(4x))\,(\csc(5x))\,(\cot(6x)) \right]$

Plus or minus infinity?

Compute:

(A) $\lim_{x \to -3^+} \dfrac{x^2 - 9}{3 - 2x - x^2}$ (B) $\lim_{x \to 1^+} \dfrac{x^2 - 9}{3 - 2x - x^2}$

A harder limit

Calculate

$$\lim_{x \to 2} \frac{\left[\sqrt{2+x} - 2\right]\left[\sqrt{3+x} - 3\right]}{\sqrt{x-1} - 1}$$

Which solution is right?

Compute $L = \lim_{x\to-\infty}\left[x - \sqrt{x^2+x}\right]$.

- **Solution 1**

$$L = \lim_{x\to-\infty} \frac{\left[x - \sqrt{x^2+x}\right]\ \left[x + \sqrt{x^2+x}\right]}{\left[x + \sqrt{x^2+x}\right]} = \lim_{x\to-\infty} \frac{x^2 - (x^2+x)}{\left[x + \sqrt{x^2+x}\right]}$$

$$= \lim_{x\to-\infty} \frac{-x}{x\left[1 + \sqrt{1+\frac{1}{x}}\right]} = \lim_{x\to-\infty} \frac{-1}{\left[1 + \sqrt{1+\frac{1}{x}}\right]} = \frac{-1}{2}$$

- **Solution 2**

$$L = \lim_{x\to-\infty}\left[{\color{red}x} - {\color{blue}\sqrt{x^2+x}}\right] = {\color{red}(-\infty)} - {\color{blue}\infty} = -\infty$$

Can we conclude this?

- Consider the function $(x) = \dfrac{4}{x}$.
- We have $f(-1) = -4 < 0$ and $f(1) = 4 > 0$.
- Use IVT.
  Can we conclude $f(c) = 0$ for some $c \in (-1, 1)$?

Existence of solutions

Prove that the equation

$$x^4 - 2x = 100$$

has at least two solutions.

Can this be proven? (Use IVT)

(A) Prove that there exists a time of the day when the hour hand and the minute hand of a clock form an angle of exactly 23 degrees.

Can this be proven? (Use IVT)

(A) Prove that there exists a time of the day when the hour hand and the minute hand of a clock form an angle of exactly 23 degrees.

(B) During a Raptors basketball game, at half time the Raptors have 52 points. Prove that at some point they had exactly 26 points.

Can this be proven? (Use IVT)

(A) Prove that there exists a time of the day when the hour hand and the minute hand of a clock form an angle of exactly 23 degrees.

(B) During a Raptors basketball game, at half time the Raptors have 52 points. Prove that at some point they had exactly 26 points.

(C) Prove that at some point during Alfonso's life, his height in centimetres was exactly equal to 10 times his weight in kilograms. Some data:

- His height at birth: 47 cm
- His weight at birth: 3.2 kg
- His height today: 172 cm

Temperature

On June 09, 2016, the outside temperature in Toronto at 6 AM was 10°. At 4 PM, the temperature was 20°.

(A) Must there have been a time between 6 AM and 4 PM when the temperature was 15°? Explain how you know. Which assumption about temperature allows you to reach your conclusion?

(B) Must there have been a time between 6 AM and 4 PM when the temperature was 22°? Explain how you know.

(C) Could there have been a time between 6 AM and 4 PM when the temperature was 22°? Explain how you know.

In each of the following cases, does the function f have a maximum and a minimum on the interval I?

(A) $f(x) = x^2, \qquad I = (-1, 1).$

(B) $f(x) = \dfrac{(e^x + 2)\sin x}{x} - \cos x + 3, \qquad I = [2, 6]$

(C) $f(x) = \dfrac{(e^x + 2)\sin x}{x} - \cos x + 3, \qquad I = (0, 5]$

Definition of maximum

Let f be a function with domain I.
Which one (or ones) of the following is (or are) a definition of "f has a maximum on I"?

(A) $\forall x \in I,\ \exists C \in \mathbb{R}$ s.t. $f(x) \leq C$

(B) $\exists C \in I$ s.t. $\forall x \in I,\ f(x) \leq C$

(C) $\exists C \in \mathbb{R}$ s.t. $\forall x \in I,\ f(x) \leq C$

(D) $\exists C \in \mathbb{R}$ s.t. $\forall x \in I,\ f(x) < C$

More on the definition of maximum

Let f be a function with domain I.
What does each of the following mean?

(A) $\exists C \in \mathbb{R}$ s.t. $\forall x \in I$, $f(x) \leq C$
(B) $\exists C \in \mathbb{R}$ s.t. $\forall x \in I$, $f(x) < C$
(C) $\exists a \in I$ s.t. $\forall x \in I$, $f(x) \leq f(a)$
(D) $\exists a \in I$ s.t. $\forall x \in I$, $f(x) < f(a)$