

Warm up

Calculate

$$\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$$

Hint: Use the substitution $u = \sqrt{x}$.

Computation practice: integration by substitution

Use substitutions to compute:

1. $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$

2. $\int e^x \cos(e^x) dx$

3. $\int \cot x dx$

4. $\int x^2 \sqrt{x+1} dx$

Computation practice: integration by substitution

Use substitutions to compute:

1. $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$

5. $\int \frac{e^{2x}}{\sqrt{e^x + 1}} dx$

2. $\int e^x \cos(e^x) dx$

6. $\int \frac{(\ln \ln x)^2}{x \ln x} dx$

3. $\int \cot x dx$

7. $\int x e^{-x^2} dx$

4. $\int x^2 \sqrt{x+1} dx$

8. $\int e^{-x^2} dx$

Definite integral via substitution

This final answer is right, but the write-up is WRONG. Why?

Calculate $I = \int_0^2 \sqrt{x^3 + 1} x^2 dx$

Wrong answer

Substitution: $u = x^3 + 1$, $du = 3x^2 dx$.

$$\begin{aligned} I &= \frac{1}{3} \int_0^2 \sqrt{x^3 + 1} (3x^2 dx) &= \frac{1}{3} \int_0^2 u^{1/2} du \\ &= \frac{1}{3} \frac{2}{3} u^{3/2} \Big|_0^2 &= \frac{1}{9} (x^3 + 1)^{2/3} \Big|_0^2 \\ &= \frac{2}{9} (2^3 + 1)^{3/2} - \frac{2}{9} (0 + 1)^{3/2} &= \frac{52}{9} \end{aligned}$$

Integral of products of sin and cos

We want to compute

$$I = \int \sin^3 x \cos^2 x \, dx$$

1. Attempt the substitution $u = \sin x$
2. Attempt the substitution $u = \cos x$
3. One worked better than the other. Which one? Why?
Finish the problem.

Integral of products of sin and cos

We want to compute

$$I = \int \sin^3 x \cos^2 x \, dx$$

1. Attempt the substitution $u = \sin x$
2. Attempt the substitution $u = \cos x$
3. One worked better than the other. Which one? Why? Finish the problem.
4. Assume we want to compute

$$\int \sin^n x \cos^m x \, dx$$

When will the substitution $u = \sin x$ be helpful?

When will the substitution $u = \cos x$ be helpful?

Odd functions

Theorem

Let f be a continuous function. Let $a > 0$. IF f is odd, THEN

$$\int_{-a}^a f(x) dx = 0$$

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Let f be a continuous function. Let $a > 0$. IF f is odd, THEN

$$\int_{-a}^a f(x) dx = 0$$

1. Write down the definition of “odd function”.
2. Draw a picture to interpret the theorem geometrically.
3. Prove the theorem!
Hint: Write the integral as sum of two pieces. Use a substitution to show that one of the two pieces equals minus the other.

Computation practice: Integration by parts

Use integration by parts (possibly in combination with other methods) to compute:

1. $\int x e^{-2x} dx$

2. $\int x^2 \sin x dx$

3. $\int \ln x dx$

4. $\int \sin \sqrt{x} dx$

Computation practice: Integration by parts

Use integration by parts (possibly in combination with other methods) to compute:

1. $\int x e^{-2x} dx$

5. $\int x \arctan x dx$

2. $\int x^2 \sin x dx$

6. $\int x^2 \arcsin x dx$

3. $\int \ln x dx$

7. $\int e^{\cos x} \sin^3 x dx$

4. $\int \sin \sqrt{x} dx$

8. $\int e^{ax} \sin(bx) dx$

Compute

- $\int_1^e (\ln x)^3 dx$

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- $\int_1^e (\ln x)^{10} dx$

Compute

- $\int_1^e (\ln x)^3 dx$

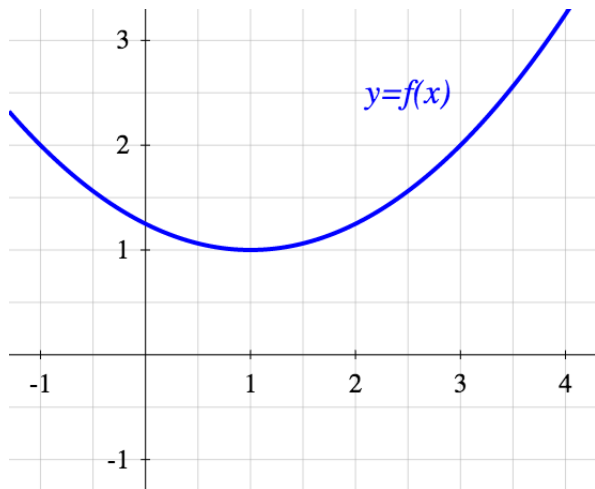
- $\int_1^e (\ln x)^{10} dx$

There is a more efficient approach. Call

$$I_n = \int_1^e (\ln x)^n dx$$

Use integration by parts on I_n . You will get an equation with I_n and I_{n-1} . Now solve the previous questions.

Integrals from a graph



Estimate:

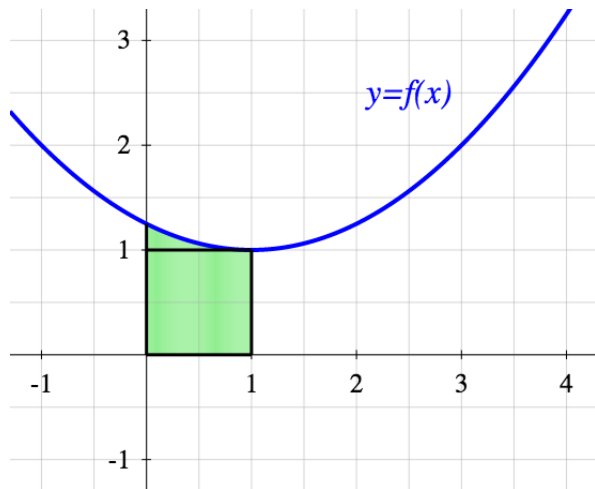
1. $\int_0^1 f(x) dx$

2. $\int_0^1 f'(x) dx$

3. $\int_0^3 x f'(x) dx$

4. $\int_0^1 f(3x) dx$

Integrals from a graph



Estimate:

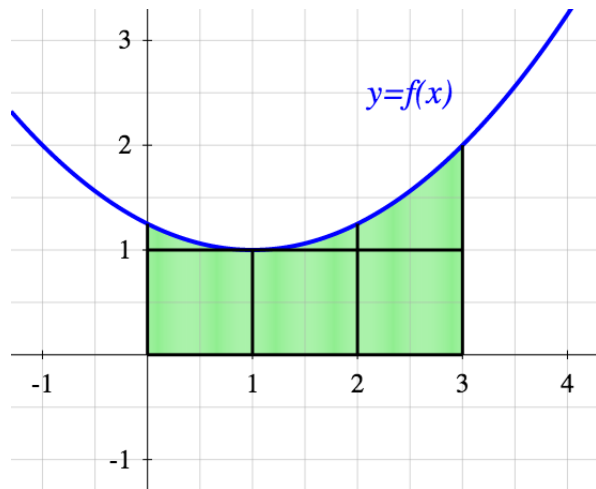
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Integrals from a graph



Estimate:

1. $\int_0^1 f(x) dx$

2. $\int_0^1 f'(x) dx$

3. $\int_0^3 x f'(x) dx$

4. $\int_0^1 f(3x) dx$

The error function

The following function is tabulated.

$$E(x) = \int_0^x e^{-t^2} dt.$$

Write the following quantities in terms of E :

1. $\int_1^2 e^{-t^2} dt$

2. $\int_0^x t^2 e^{-t^2} dt$

3. $\int_0^x e^{-2t^2} dt$

The error function

The following function is tabulated.

$$E(x) = \int_0^x e^{-t^2} dt.$$

Write the following quantities in terms of E :

1. $\int_1^2 e^{-t^2} dt$

4. $\int_0^1 e^{-t^2+6t} dt$

2. $\int_0^x t^2 e^{-t^2} dt$

5. $\int_{x_1}^{x_2} e^{-\frac{(t-\mu)^2}{\sigma^2}} dt$

3. $\int_0^x e^{-2t^2} dt$

6. $\int_1^2 \frac{e^{-t}}{\sqrt{t}} dt$

Exp-trig antiderivative

We want to compute

$$I = \int e^{ax} \sin(bx) dx$$

- Try once integration by parts choosing $u = e^{ax}$. Stop.
- Go back to I . Now try integration by parts once choosing $u = \sin(bx)$ instead. Stop.
- Look at what you did. Think.

Practice: Integrals with trigonometric functions

Compute the following antiderivatives. (Once you get them to a form from where you see a path to finish them, even if long, you may stop.)

1. $\int \sin^{10} x \cos x \, dx$

4. $\int \cos^2 x \, dx$

2. $\int \sin^{10} x \cos^7 x \, dx$

5. $\int \cos^4 x \, dx$

3. $\int e^{\cos x} \cos x \sin^3 x \, dx$

6. $\int \csc x \, dx$

Useful trig identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

Integral of products of secant and tangent

To integrate

$$\int \sec^n x \tan^m x \, dx$$

- If , then use the substitution $u = \tan x$.
- If , then use the substitution $u = \sec x$.

Hint: You will need

- $\frac{d}{dx} [\tan x] = \dots$
- $\frac{d}{dx} [\sec x] = \dots$
- The trig identity involving sec and tan

A reduction formula

Let $I_n = \int_0^{2\pi} \sin^n x \, dx$.

1. Compute I_0 and I_1 .
2. Write an equation for I_n in terms of I_{n-2} . This is called a reduction formula.

Hint: Starting with I_n , use integration by parts once. Then use $\sin^2 x + \cos^2 x = 1$ to rewrite the new integral in terms of I_n and I_{n-2} .

3. Write a formula for I_n for all natural numbers n .

A different kind of substitution

Calculate

$$\int_0^1 \sqrt{1-x^2} \, dx$$

using the substitution

$$\begin{cases} x = \sin \theta \\ dx = ?? \end{cases}$$

Rational integrals

1. Calculate $\int \frac{1}{x+a} dx$

2. Reduce to common denominator $\frac{2}{x} - \frac{3}{x+3}$

3. Calculate $\int \frac{-x+6}{x^2+3x} dx$

4. Calculate $\int \frac{1}{x^2+3x} dx$

5. Calculate $\int \frac{1}{x^3-x} dx$

Repeated factors

1. Calculate $\int \frac{1}{(x+1)^n} dx$ for $n > 1$

2. Calculate $\int \frac{(x+1) - 1}{(x+1)^2} dx$

3. Calculate $\int \frac{2x+6}{(x+1)^2} dx$

4. Calculate $\int \frac{x^2}{(x+1)^3} dx$

Irreducible quadratics

1. Calculate $\int \frac{1}{x^2 + 1} dx$ and $\int \frac{x}{x^2 + 1} dx$.

Hint: These two are very short.

2. Calculate $\int \frac{2x + 3}{x^2 + 1} dx$

3. Calculate $\int \frac{x^2}{x^2 + 1} dx$

4. Calculate $\int \frac{x}{x^2 + x + 1} dx$

Hint: Complete the square in the denominator and use a substitution to transform into one of the previous ones.

Repeated quadratics

1. Calculate

$$\frac{d}{dx} [\arctan x], \quad \frac{d}{dx} \left[\frac{x}{1+x^2} \right].$$

2. Use the previous answer to calculate

$$\int \frac{1}{(1+x^2)^2} dx$$

The integral of secant

Compute

$$\int \sec x \, dx$$

using the substitution $u = \sin x$.

Messier rational functions

1. How could we compute an integral of the form

$$\int \frac{\text{polynomial}}{(x+1)^3(x+2)} dx ?$$

2. How could we compute an integral of the form

$$\int \frac{\text{polynomial}}{x^4(x+1)^3(x+2)(x^2+1)(x^2+4)} dx ?$$