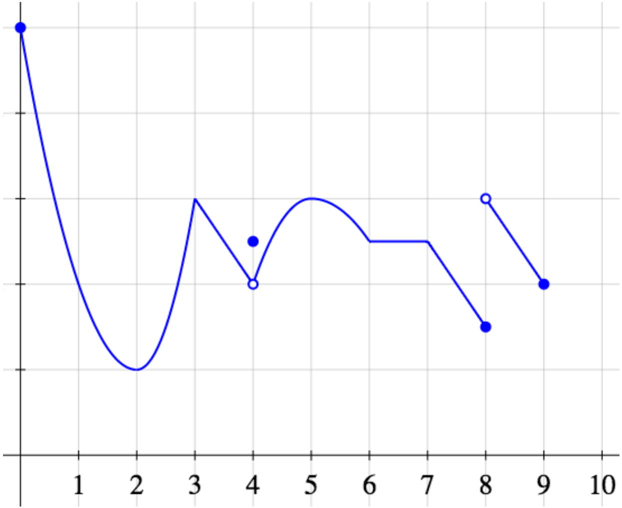
Definition of local extremum

Find local and global extrema of the function with this graph:



Where is the maximum?

We know the following about the function h:

- The domain of h is $(-4, 4)$.
- h is continuous on its domain.
- h is differentiable on its domain, except at 0.
- $h'(x) = 0 \quad \iff \quad x = -1$ or 1.

What can you conclude about the maximum of h?

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- $h'(x) = 0 \qquad \Longleftrightarrow \qquad x = -1$ or 1.

What can you conclude about the maximum of h?

1. h has a maximum at $x = -1$, or 1.
2. h has a maximum at $x = -1, 0$, or 1.
3. h has a maximum at $x = -4, -1, 0, 1$, or 4.
4. None of the above.

What can you conclude?

We know the following about the function f.

- f has domain $\mathbb{R}$.
- f is continuous
- $f(0) = 0$
- For every $x \in \mathbb{R}$, $f(x) \geq x$.

What can you conclude about $f'(0)$? Prove it.

Hint: Sketch the graph of f. Looking at the graph, make a conjecture.
To prove it, imitate the proof of the Local EVT from Video 5.3.

Fractional exponents

Let $g(x) = x^{2/3}(x-1)^3$.

Find local and global extrema of g on $[-1, 2]$.

Trig extrema

Let $f(x) = \dfrac{\sin x}{3 + \cos x}$.

Find the maximum and minimum of f.

How many zeroes?

Let

$$f(x) = e^x - \sin x + x^2 + 10x$$

How many zeroes does f have?

Zeroes of the derivative

Sketch the graph of a function f that is differentiable on $\mathbb{R}$ and such that

1. f has exactly 3 zeroes and f' has exactly 2 zeroes.
2. f has exactly 3 zeroes and f' has exactly 3 zeroes.
3. f has exactly 3 zeroes and f' has exactly 1 zero.
4. f has exactly 3 zeroes and f' has infinitely many zeroes.

Zeroes of a polynomial

You probably learned in high school that a polynomial of degree n has at most n real zeroes. Now you can prove it!

Hint: Use induction. If you are having trouble, try the case $n = 3$ first.

The second Theorem of Rolle

Complete statement for this theorem and prove it.

Rolle's Theorem 2

Let $a < b$. Let f be a function defined on $[a, b]$.
IF

- (Some conditions on continuity and derivatives)
- $f(a) = f'(a) = 0$
- $f(b) = 0$

THEN $\exists c \in (a, b)$ such that $f''(c) = 0$.

Hint: Apply the 1st Rolle's Theorem to f, then do something else.

The N-th Theorem of Rolle

Complete the statement for this theorem and prove it.

Rolle's Theorem N

Let N be a positive integer.
Let $a < b$. Let f be a function defined on $[a, b]$.
IF

- (Some conditions on continuity and derivatives)
- (Some conditions at a)
- $f(b) = 0$

THEN $\exists c \in (a, b)$ such that $f^{(N)}(c) = 0$.

A new theorem

We want to prove this theorem:

Theorem 1

Let f be a differentiable function on an open interval I.
IF $\forall x \in I, f'(x) \neq 0$
THEN f is one-to-one on I.

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Let f be a differentiable function on an open interval I.
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1. Transform $[P \implies Q]$ into $[(\text{not } Q) \implies (\text{not } P)]$.
   You get an equivalent Theorem (call it "Theorem 2").
   We are going to prove Theorem 2 instead.
2. Write the definition of "f is not one-to-one on I". You will need it.
3. Recall the statement of Rolle's Theorem. You will need it.
4. Do some rough work if needed.
5. Write a complete proof for Theorem 2.

A variant

Complete this variation on Theorem 2.
Use the weakest conditions you can to make it true.

Theorem 3

Let $a < b$. Let f be a function defined on $[a, b]$.
IF

- (Some conditions on continuity and differentiability)
- f is not one-to-one on $[a, b]$

THEN $\exists c \in (a, b)$ such that $f'(c) = 0$.

Why the three hypotheses are necessary

You have proven

Theorem 3

Let $a < b$. Let f be a function defined on $[a, b]$.
IF

1. f is continuous on $[a, b]$
2. f is differentiable on (a, b)
3. f is not one-to-one on $[a, b]$

THEN $\exists c \in (a, b)$ such that $f'(c) = 0$.

Give three examples to justify that each of the three hypotheses are necessary for the theorem to be true. (Graphs of the examples are enough).

MVT – True or False?

True or False

Consider $f(x) = |x|$ on the interval $[-\frac{1}{2}, 2]$.
There exists c in $(-\frac{1}{2}, 2)$ such that

$$f'(c) = \frac{f(2) - f(-\frac{1}{2})}{2 - (-\frac{1}{2})}$$

Car race - 1

A driver competes in a race.

Use MVT to prove that at some point during the race the instantaneous velocity of the driver is exactly equal to the average velocity of the driver during the race.

Car race - 2

Two drivers start a race at the same moment and finish in a tie.

Can you conclude that there was a time in the race (not counting the starting time) when the two drivers had exactly the same speed?

Car race - Is this proof correct?

Claim

IF two drivers start a race at the same moment and finish in a tie, THEN at some point in the race (not counting the starting time) they had exactly the same speed.

Proof?

- Let $f(t)$ and $g(t)$ be the positions of the two cars at time t.
- Assume the race happens in the interval $[t_1, t_2]$. By hypothesis:

$$f(t_1) = g(t_1), \qquad f(t_2) = g(t_2).$$

- Using MVT, there exists $c \in (t_1, t_2)$ such that

$$f'(c) = \frac{f(t_2) - f(t_1)}{t_2 - t_1}, \quad g'(c) = \frac{g(t_2) - g(t_1)}{t_2 - t_1}.$$

- Then $f'(c) = g'(c)$. $\square$

Car race - resolution

Two drivers start a race at the same moment and finish in a tie.

Prove that at some point during the race (not counting the starting time) the two drivers had exactly the same speed.

Speeding ticket!

On a toll road Barney takes a time stamped toll-card from the starting booth and drives directly to the end of the toll section.

After paying the required toll, Barney is surprised to receive a speeding ticket along with the toll receipt.

Which of the following are true?

1. The booth attendant does not have enough information to prove that Barney was speeding.
2. The booth attendant can prove that Barney was speeding during his trip.
3. Barney's ticket is for a lower speed than his actual maximum speed.

Proving difficult identities

Prove that, for every $x \geq 0$,

$$2 \arctan \sqrt{x} - \arcsin \frac{x-1}{x+1} = \frac{\pi}{2}$$

Hint: Derivatives.

Critique this "proof"

- $$\left[2\arctan\sqrt{x} - \arcsin\frac{x-1}{x+1}\right] = \left[\frac{\pi}{2}\right]$$
- $$\frac{d}{dx}\left[2\arctan\sqrt{x} - \arcsin\frac{x-1}{x+1}\right] = \frac{d}{dx}\left[\frac{\pi}{2}\right]$$
- $$\frac{2}{1+\left(\sqrt{x}\right)^2}\cdot\frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}\cdot\frac{(x+1)-(x-1)}{(x+1)^2} = 0$$
- $$\frac{1}{(1+x)\sqrt{x}} - \frac{1}{\sqrt{\frac{4x}{(x+1)^2}}}\cdot\frac{2}{(x+1)^2} = 0$$
- $0 = 0$
- So $2\arctan\sqrt{x} - \arcsin\dfrac{x-1}{x+1}$ is constant.
- Evaluate at $x = 0$ to find the value of the constant.
- $2\arctan 0 \; - \; \arcsin(-1) \; = \; 0 - (-\pi/2) \; = \; \pi/2$
- Therefore, $\quad 2\arctan\sqrt{x} - \arcsin\dfrac{x-1}{x+1} = \dfrac{\pi}{2}$

Warm up

1. Let f be a function defined on an interval I.
   Write the definition of "f is increasing on I".
2. Write the statement of the Mean Value Theorem.

Positive derivative implies increasing

Use the MVT to prove

Theorem

Let $a < b$. Let f be a differentiable function on (a, b).

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN f is increasing on (a, b).

Positive derivative implies increasing

Use the MVT to prove

Theorem

Let $a < b$. Let f be a differentiable function on (a, b).

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN f is increasing on (a, b).

1. Recall the definition of what you are trying to prove.
2. **From that definition, figure out the structure of the proof.**
3. If you have used a theorem, did you verify the hypotheses?
4. Are there words in your proof, or just equations?

What is wrong with this proof?

Theorem

Let $a < b$. Let f be a differentiable function on (a, b).

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN f is increasing on (a, b).

Proof.

- From the MVT, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$
- We know $b - a > 0$ and $f'(c) > 0$
- Therefore $f(b) - f(a) > 0$. Thus $f(b) > f(a)$.
- f is increasing.

$\square$

Your first integration

Find all functions f such that, for all $x \in \mathbb{R}$:
$f''(x) = x + \sin x$.

Intervals of monotonicity

Let $g(x) = x^3(x^2 - 4)^{1/3}$.

Find out on which intervals this function is increasing or decreasing.
Using that information, sketch its graph.

To save time, here is the first derivative:

$$g'(x) = \frac{x^2(11x^2 - 36)}{3(x^2 - 4)^{2/3}}$$

True or False – Monotonicity and local extrema

Let I be an interval. Let f be a function defined on I. Let $c \in I$. Which implications are true?

1. IF f is increasing on I, THEN $\forall x \in I,\ f'(x) > 0$.
2. IF $\forall x \in I,\ f'(x) > 0$, THEN f is increasing on I.
3. IF f has a local extremum at c, THEN $f'(c) = 0$.
4. IF $f'(c) = 0$, THEN f has a local extremum at c.
5. IF f has local extremum at c, THEN f has an extremum at c
6. IF f has an extremum at c, THEN f has local extremum at c

Inequalities

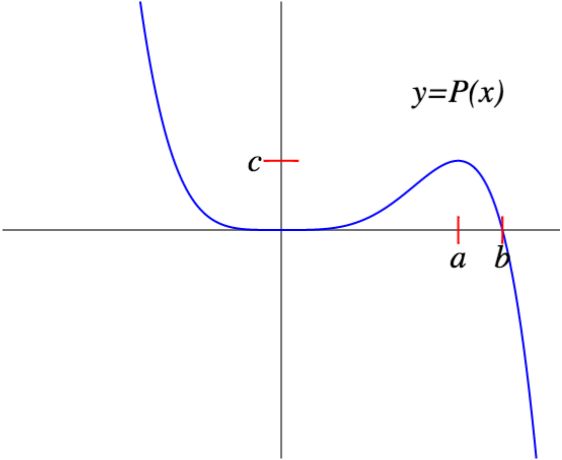
Prove that, for every $x \in \mathbb{R}$

$$e^x \geq 1 + x$$

Hint: Where is the function $f(x) = e^x - 1 - x$ increasing or decreasing? What is its minimum?

Backwards graphing

Below is the graph of a polynomial P. Notice that it is not at scale. The coordinates in the graph are $a = 24$, $b = 25$, and $c = 1$. Find the equation of P.



A sneaky function

Construct a function f satisfying all the following properties:

- Domain $f = \mathbb{R}$
- f is continuous
- $f'(0) = 0$
- f does not have a local extremum at 0.
- There isn't an interval centered at 0 on which f is increasing.
- There isn't an interval centered at 0 on which f is decreasing.