Initial Value Problem

Find a function f such that

- For every $x \in \mathbb{R}$, $f''(x) = \sin x + x^2$,
- $f'(0) = 5$,
- $f(0) = 7$.

The most misunderstood antiderivative

1. Find the *domain* and the derivative of $F_1(x) = \ln x$
2. Find the *domain* and the derivative of $F_2(x) = \ln(-x)$
3. Find the *domain* and the derivative of $F_3(x) = \ln|x|$
   Suggestion: Break the domain into two pieces.

The most misunderstood antiderivative

1. Find the *domain* and the derivative of $F_1(x) = \ln x$
2. Find the *domain* and the derivative of $F_2(x) = \ln(-x)$
3. Find the *domain* and the derivative of $F_3(x) = \ln|x|$
   Suggestion: Break the domain into two pieces.
4. Based on your answers, what is $\displaystyle\int \frac{1}{x}\,dx$?

The most misunderstood antiderivative

1. Find the *domain* and the derivative of $F_1(x) = \ln x$
2. Find the *domain* and the derivative of $F_2(x) = \ln(-x)$
3. Find the *domain* and the derivative of $F_3(x) = \ln|x|$
   Suggestion: Break the domain into two pieces.
4. Based on your answers, what is $\displaystyle\int \frac{1}{x}\,dx$?
5. Find the *domain* and the derivative of $F_4(x) = \ln|2x|$
   Why doesn't this contradict your answer to 4 ?

Compute these antiderivatives by guess 'n check

1. $\int x^5 dx$
2. $\int \left(3x^8 - 18x^5 + 1\right) dx$
3. $\int \sqrt[3]{x}\, dx$
4. $\int \frac{1}{x^9}\, dx$
5. $\int \sqrt{x}\left(x^2 + 5\right) dx$
6. $\int \frac{1}{e^{2x}}\, dx$
7. $\int \sin(3x)\, dx$
8. $\int \cos(3x + 2)\, dx$
9. $\int \sec^2 x\, dx$
10. $\int \sec x \tan x\, dx$
11. $\int \frac{1}{x}\, dx$
12. $\int \frac{1}{x + 3}\, dx$

Integration by parts 1

1. $\frac{d}{dx}[x \sin x] =$
2. $\frac{d}{dx}[\cos x] =$

Use the previous answers to calculate

3. $\int x \cos x \, dx =$

Integration by parts 2

1. $\dfrac{d}{dx}\left[xe^x\right] =$

2. ???

3. $\displaystyle\int xe^x \, dx =$

Integration by parts 3

1. ???
2. ???
3. $\displaystyle\int xe^{-x}\,dx =$

Integration by parts 4

1. $\dfrac{d}{dx}\left[x^2e^x\right] =$
2. $\dfrac{d}{dx}[xe^x] =$
3. ???
4. $\displaystyle\int x^2e^x\,dx =$

Trig-exp antiderivatives

1. $\dfrac{d}{dx}\left[e^x \sin x\right] =$
2. $\dfrac{d}{dx}\left[e^x \cos x\right] =$

Use the previous answers to calculate:

3. $\displaystyle\int e^x \sin x \, dx =$
4. $\displaystyle\int e^x \cos x \, dx =$

A challenge for guess-and-check ninjas

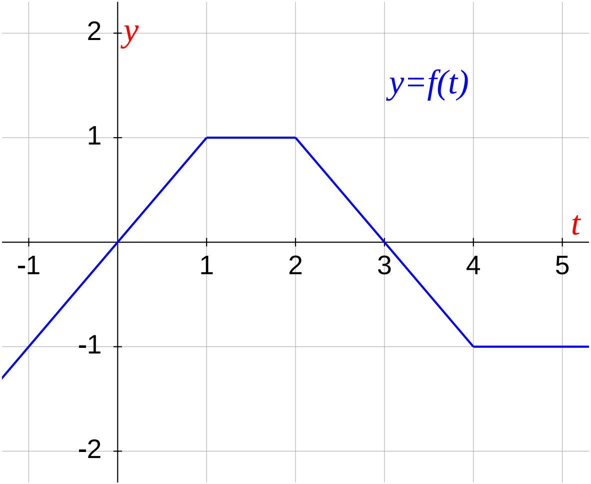
$$\int x\, e^x \cos x \, dx \;=\; ???$$

Functions defined by integrals

Which ones of these are valid ways to define functions?

1. $F(x) = \int_0^x \frac{t}{1+t^8}\, dt$
2. $F(x) = \int_0^x \frac{x}{1+x^8}\, dx$
3. $F(x) = \int_0^x \frac{x}{1+t^8}\, dt$
4. $F(x) = \int_0^{x^2} \frac{t}{1+t^8}\, dt$
5. $F(x) = \int_{\sin x}^{e^x} \frac{t}{1+t^8}\, dt$
6. $F(x) = \int_0^3 \frac{t}{1+x^2+t^8}\, dt$
7. $F(x) = x \int_{\sin x}^{e^x} \frac{t}{1+x^2+t^8}\, dt$
8. $F(x) = t \int_{\sin x}^{e^x} \frac{t}{1+x^2+t^8}\, dt$

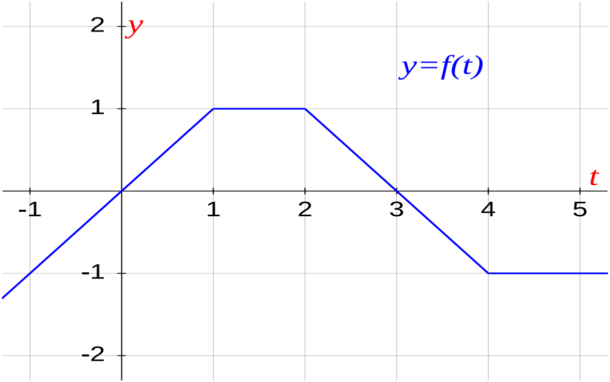
Towards FTC



Compute:

1. $\int_0^1 f(t)dt$
2. $\int_0^2 f(t)dt$
3. $\int_0^3 f(t)dt$
4. $\int_0^4 f(t)dt$
5. $\int_0^5 f(t)dt$

Towards FTC (continued)



Call $F(x) = \int_0^x f(t)dt$. This is a new function.

- Sketch the graph of $y = F(x)$.
- Using the graph you just sketched, sketch the graph of $y = F'(x)$.

Filling the tank

A tank is being filled with water. At time t water flows into the tank at a rate of

$$A\,e^{-bt}\arctan(ct)$$

litres per second, where A, b, and c are constants. The amount of water in the tank at time $t = 0s$ is V_0. Write an expression for the amount of water V in the tank at time t.

True or False?

1. If f is continuous on the interval $[a, b]$, then

$$\frac{d}{dx}\left(\int_a^b f(t)dt\right) = f(x).$$

2. If f is differentiable, then

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = \int_a^x f'(t)\,dt.$$

More True or False

Let f and g be differentiable functions with domain $\mathbb{R}$.
Assume that $f'(x) = g(x)$ for all x.
Which of the following statements must be true?

1. $f(x) = \int_0^x g(t)dt$.
2. If $f(0) = 0$, then $f(x) = \int_0^x g(t)dt$.
3. If $g(0) = 0$, then $f(x) = \int_0^x g(t)dt$.
4. There exists $C \in \mathbb{R}$ such that $f(x) = C + \int_0^x g(t)dt$.
5. There exists $C \in \mathbb{R}$ such that $f(x) = C + \int_1^x g(t)dt$.

True, False, or Shrug?

We want to find a function H with domain $\mathbb{R}$ such that $H(1) = -2$ and such that $H'(x) = e^{\sin x}$ for all x. Decide whether each of the following statements is true, false, or we do not have enough information to decide.

1. The function $H(x) = \int_0^x e^{\sin t} dt$ is a solution.
2. The function $H(x) = \int_2^x e^{\sin t} dt$ is a solution.
3. $\forall C \in \mathbb{R}$, the function $H(x) = \int_0^x e^{\sin t} dt + C$ is a solution.
4. $\exists C \in \mathbb{R}$ s.t. the function $H(x) = \int_0^x e^{\sin t} dt + C$ is a solution.
5. The function $H(x) = \int_1^x e^{\sin t} dt - 2$ is a solution.
6. There is more than one solution.

Examples of FTC-1

Compute the derivative of the following functions

1. $F_1(x) = \int_0^1 e^{-t^2} dt.$
2. $F_2(x) = \int_0^x e^{-\sin t} dt.$
3. $F_3(x) = \int_1^{x^2} \frac{\sin t}{t^2} dt.$
4. $F_4(x) = \int_x^7 \sin^3\left(\sqrt{t}\right) dt.$
5. $F_5(x) = \int_{2x}^{x^2} \frac{1}{1+t^3} dt.$

A generalized version of FTC-1

Let f, u, v be differentiable functions with domain $\mathbb{R}$.
Let us call

$$F(x) = \int_{u(x)}^{v(x)} f(t)dt$$

Find a formula for

$$F'(x)$$

in terms of f, u, v, f', u', v'.

An integral equation

Assume f is a continuous function that satisfies, for every $x \in \mathbb{R}$:

$$\int_0^x e^t f(t) = \frac{\sin x}{x^2 + 1}$$

Find an explicit expression for $f(x)$.

Compute these definite integrals

1. $\displaystyle\int_1^2 x^3 dx$
2. $\displaystyle\int_0^1 \left[e^x + e^{-x} - \cos(2x)\right] dx$
3. $\displaystyle\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}} dx$
4. $\displaystyle\int_{\pi/4}^{\pi/3} \sec^2 x \, dx$
5. $\displaystyle\int_1^2 \left[\frac{d}{dx}\left(\frac{\sin^2 x}{1+\arctan^2 x + e^{-x^2}}\right)\right] dx$

Find the error

$$\int_{-1}^{1} \frac{1}{x^4} dx \;=\; \left.\frac{-1}{3x^3}\right|_{-1}^{1} = \frac{-2}{3}$$

However, x^4 is always positive, so the integral should be positive.

Areas

Calculate the area of the bounded region...

1. ... between the x-axis and $y = 4x - x^2$.
2. ... between $y = \cos x$, the x-axis, from $x = 0$ to $x = \pi$.
3. ... between $y = x^2 + 3$ and $y = 3x + 1$.
4. ... between $y = 1$, the y-axis, and $y = \ln(x + 1)$.

Minimizing area

For each $a > 0$ consider the function

$$f_a(x) = 1 + a - ax^2$$

Find the value of a that minimizes the area of the region bounded by the graph of f_a and the x-axis. ▸ desmos

Symmetry

Calculate the value of these integrals *without computing any antiderivative*.

1. $\displaystyle\int_{-2}^{2} \sin x^3 dx$ 2. $\displaystyle\int_{0}^{\pi} \cos^2 x\, dx$ 3. $\displaystyle\int_{-1}^{1} \arccos x\, dx$

Hint: Sketch the graphs (use desmos) and use symmetry to compute the integral.
Once you guess the symmetry of the graph, try to write it algebraically.

▸ 1 ▸ 2 ▸ 3

Average Velocity

You are traveling.
Your position at time t is $s(t)$.
Your velocity at time t is $v(t)$.
The function v is continuous on an interval $[a, b]$.
Which of the following represent your average velocity on $[a, b]$?

1. $\dfrac{s(b) - s(a)}{b - a}$
2. $\dfrac{1}{b - a} \displaystyle\int_a^b v(t)dt$
3. $v(c)$ for at least one c between a and b

The Mean Value Theorem for integrals is back

Prove the following theorem.

Theorem

Let $a < b$. Let f be a continuous function on $[a, b]$.
There exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t)dt$$

The Mean Value Theorem for integrals is back

Prove the following theorem.

Theorem

Let $a < b$. Let f be a continuous function on $[a, b]$.
There exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t)dt$$

Hint: Use MVT for the function $F(x) = \displaystyle\int_a^x f(t)dt$.