

## Warmup:

You want to show “ $\exists n \in \mathbb{N}$  s.t.  $n^2 = 4$ ”. Which proofs below are correct/incorrect?

(A) Let  $n = 2$ . Then,  
 $n \in \mathbb{N}$  and  $n^2 = 4$ .

(B) Let  $n \in \mathbb{N}$ . Take  
 $n = 2$ . Then  $n^2 = 4$ .

(C) Let  $n \in \mathbb{N}$ . Assume  
 $n = 2$ . Then  $n^2 = 4$ .

(D) Take  $n = 2$ . Then  
 $n \in \mathbb{N}$  and  $n^2 = 4$ .

**Before next class:**

- Watch videos 2.1, 2.2, 2.3

# Variations on induction

Let  $S_n$  be a statement depending on a positive integer  $n$ .

What conclusions can you draw in each of the following cases? (I.e., for which  $n$  do you know that  $S_n$  is true?)

(A) We have proven:

- $S_3$



$$\forall n \geq 1, S_n \implies S_{n+1}$$

(B) We have proven:

- $S_1$



$$\forall n \geq 3, S_n \implies S_{n+1}$$

(C) We have proven:

- $S_1$



$$\forall n \geq 1, S_n \implies S_{n+3}$$

(D) We have proven:

- $S_1$



$$\forall n \geq 1, S_{n+1} \implies S_n$$

## Variations on induction 2

We want to prove

$$\forall n \geq 1, S_n$$

So far we have proven

- $S_1$
- $\forall n \geq 1, S_n \implies S_{n+3}.$

What else do we need to do?

# What is wrong with this proof by induction?

## Theorem

$\forall N \geq 1$ , every set of  $N$  students in MAT137 will get the same grade.

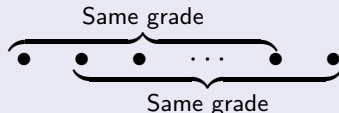
# What is wrong with this proof by induction?

## Theorem

$\forall N \geq 1$ , every set of  $N$  students in MAT137 will get the same grade.

## Proof.

- **Base case.** It is clearly true for  $N = 1$ .
- **Induction step.**  
Assume it is true for  $N$ . I'll show it is true for  $N + 1$ .  
Take a set of  $N + 1$  students. By induction hypothesis:
  - The first  $N$  students get the same grade.
  - The last  $N$  students get the same grade.



Hence the  $N + 1$  students all get the same grade.



# What is wrong with this proof by induction?

For every  $N \geq 1$ , let

$S_N =$  “every set of  $N$  students in MAT137  
will get the same grade”

What did we actually prove in the previous page?

- $S_1$  ?
- $\forall N \geq 1, S_N \implies S_{N+1}$  ?

## Properties of absolute value

Let  $a, b \in \mathbb{R}$ . Are the following conclusions correct?

(A)  $|ab| = |a||b|$

(B)  $|a + b| = |a| + |b|$

If any of the conclusions is wrong, fix it.

# Properties of inequalities

Let  $a, b, c \in \mathbb{R}$ .

Assume  $a < b$ . Are the following conclusions correct?

(A)  $a + c < b + c$

(D)  $a^2 < b^2$

(B)  $a - c < b - c$

(E)  $\frac{1}{a} < \frac{1}{b}$

(C)  $ac < bc$

(F)  $\sin a < \sin b$

If any of the conclusions is wrong, fix it.



## Sets described by distance

Let  $a \in \mathbb{R}$ . Let  $\delta > 0$ .

Describe the following sets using interval notation.

(A)  $A = \{x \in \mathbb{R} : |x| < \delta\}$

(B)  $B = \{x \in \mathbb{R} : |x| > \delta\}$

(C)  $C = \{x \in \mathbb{R} : |x - a| < \delta\}$

(D)  $D = \{x \in \mathbb{R} : 0 < |x - a| < \delta\}$

# Implications

Find *all* positive values of  $X$ ,  $Y$ , and  $Z$  which make the following implications true.

$$(A) \quad |t - 3| < 1 \implies |2t - 6| < X$$

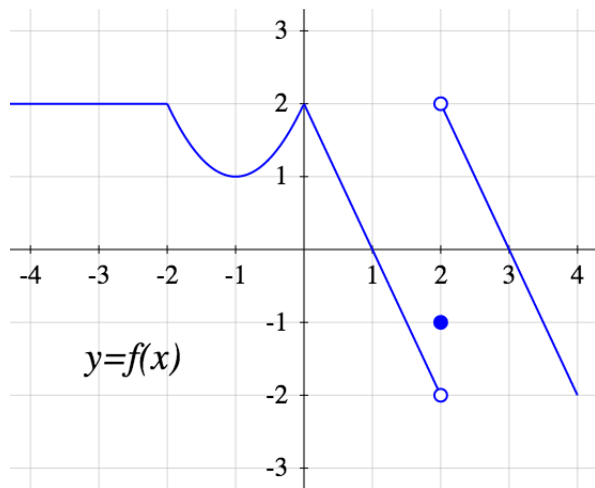
$$(B) \quad |t - 3| < Y \implies |2t - 6| < 1$$

$$(C) \quad |t - 3| < 1 \implies |t + 5| < Z$$

**Before next class:**

- **Watch videos 2.5, 2.6**

# Limits from a graph



Find the value of

(A)  $\lim_{x \rightarrow 2} f(x)$

(B)  $\lim_{x \rightarrow 0} f(f(x))$

# Floor

Given a real number  $x$ , we defined the *floor of  $x$* , denoted by  $\lfloor x \rfloor$ , as the largest integer smaller than or equal to  $x$ . For example:

$$\lfloor \pi \rfloor = 3, \quad \lfloor 7 \rfloor = 7, \quad \lfloor -0.5 \rfloor = -1.$$

Sketch the graph of  $y = \lfloor x \rfloor$ . Then compute:

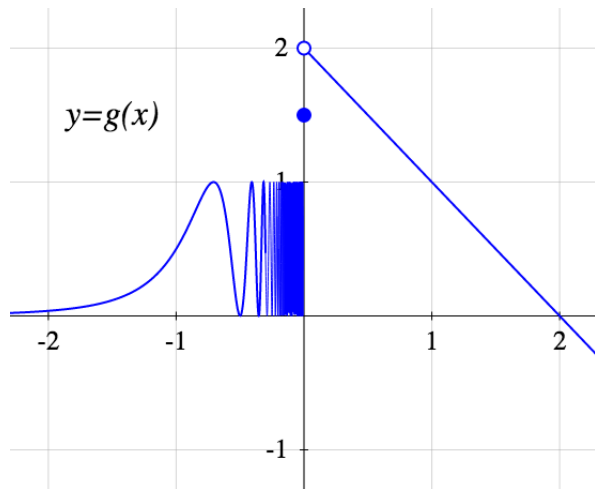
(A)  $\lim_{x \rightarrow 0^+} \lfloor x \rfloor$

(C)  $\lim_{x \rightarrow 0} \lfloor x \rfloor$

(B)  $\lim_{x \rightarrow 0^-} \lfloor x \rfloor$

(D)  $\lim_{x \rightarrow 0} \lfloor x^2 \rfloor$

## More limits from a graph



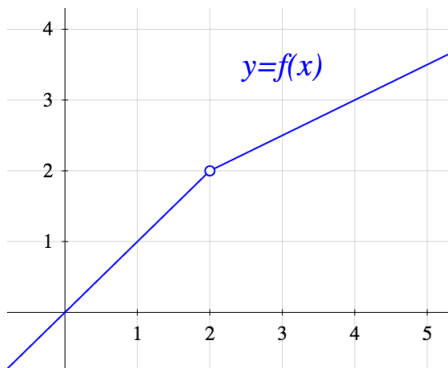
Find the value of

- (A)  $\lim_{x \rightarrow 0^+} g(x)$
- (B)  $\lim_{x \rightarrow 0^+} \lfloor g(x) \rfloor$
- (C)  $\lim_{x \rightarrow 0^+} g(\lfloor x \rfloor)$
- (D)  $\lim_{x \rightarrow 0^-} g(x)$
- (E)  $\lim_{x \rightarrow 0^-} \lfloor g(x) \rfloor$
- (F)  $\lim_{x \rightarrow 0^-} \lfloor \frac{g(x)}{2} \rfloor$
- (G)  $\lim_{x \rightarrow 0^-} g(\lfloor x \rfloor)$

**Before next class:**

- **Watch videos 2.7, 2.8, 2.9**

## $\delta$ from a graph



- (A) Find one value of  $\delta > 0$  s.t.  $0 < |x - 2| < \delta \implies |f(x) - 2| < 0.5$
- (B) Find *all* values of  $\delta > 0$  s.t.  $0 < |x - 2| < \delta \implies |f(x) - 2| < 0.5$



Write down the formal definition of

$$\lim_{x \rightarrow a} f(x) = L.$$

## Side limits

### Recall

Let  $L, a \in \mathbb{R}$ .

Let  $f$  be a function defined at least on an interval around  $a$ , except possibly at  $a$ .

$$\lim_{x \rightarrow a} f(x) = L$$

means

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

Write, instead, the formal definition of

$$\lim_{x \rightarrow a^+} f(x) = L, \quad \text{and} \quad \lim_{x \rightarrow a^-} f(x) = L.$$

## Definition

Let  $a \in \mathbb{R}$ .

Let  $f$  be a function defined at least on an interval around  $a$ , except possibly at  $a$ .

Write a formal definition for

$$\lim_{x \rightarrow a} f(x) = \infty.$$

Which one(s) is the definition of  $\lim_{x \rightarrow a} f(x) = \infty$  ?

(A)  $\forall M \in \mathbb{R}, \exists \delta > 0$  s.t.  $0 < |x - a| < \delta \implies f(x) > M$

(B)  $\forall M \in \mathbb{Z}, \exists \delta > 0$  s.t.  $0 < |x - a| < \delta \implies f(x) > M$

(C)  $\forall M > 0, \exists \delta > 0$  s.t.  $0 < |x - a| < \delta \implies f(x) > M$

(D)  $\forall M > 5, \exists \delta > 0$  s.t.  $0 < |x - a| < \delta \implies f(x) > M$

## Limit at a point

If a function  $f$  is not defined at  $x = a$ , then

- (A)  $\lim_{x \rightarrow a} f(x)$  cannot exist
- (B)  $\lim_{x \rightarrow a} f(x)$  could be 0
- (C)  $\lim_{x \rightarrow a} f(x)$  must approach  $\infty$
- (D) none of the above.

# Evaluating Limits

- You're trying to guess  $\lim_{x \rightarrow 0} f(x)$ .
- You plug in  $x = 0.1, 0.01, 0.001, \dots$  and get  $f(x) = 0$  for all these values.
- In fact, you're told that for all  $n = 1, 2, \dots$ ,  
$$f\left(\frac{1}{10^n}\right) = 0.$$
- Can you conclude that  $\lim_{x \rightarrow 0} f(x) = 0$ ?

# Exponential limits

Compute:

$$\lim_{t \rightarrow 0^+} e^{1/t}, \quad \lim_{t \rightarrow 0^-} e^{1/t}.$$

Suggestion: Sketch the graph of  $y = e^x$  first.

Consider the function

$$h(x) = \frac{(x-1)(2+x)}{x^2(x-1)(2-x)}.$$

- Find all real values  $a$  for which  $h(a)$  is undefined.
- For each such value of  $a$ , compute  $\lim_{x \rightarrow a^+} h(x)$  and  $\lim_{x \rightarrow a^-} h(x)$ .
- Based on your answer, and nothing else, try to sketch the graph of  $h$ .



## Related implications

Let  $a \in \mathbb{R}$ . Let  $f$  be a function. Assume we know

$$0 < |x - a| < 0.1 \quad \implies \quad f(x) > 100$$

(A) Which values of  $M \in \mathbb{R}$  satisfy ... ?

$$0 < |x - a| < 0.1 \quad \implies \quad f(x) > M$$

## Related implications

Let  $a \in \mathbb{R}$ . Let  $f$  be a function. Assume we know

$$0 < |x - a| < 0.1 \quad \implies \quad f(x) > 100$$

(A) Which values of  $M \in \mathbb{R}$  satisfy ... ?

$$0 < |x - a| < 0.1 \quad \implies \quad f(x) > M$$

(B) Which values of  $\delta > 0$  satisfy ... ?

$$0 < |x - a| < \delta \quad \implies \quad f(x) > 100$$

## Strict or non-strict inequality?

Let  $f$  be a function with domain  $\mathbb{R}$ . One of these statements implies the other. Which one?

(A)  $\forall M \in \mathbb{R}, \exists N \in \mathbb{R} \text{ s.t. } x > N \implies f(x) > M$

(B)  $\forall M \in \mathbb{R}, \exists N \in \mathbb{R} \text{ s.t. } x > N \implies f(x) \geq M$

## Negation of conditionals

Write the negation of these statements:

- (A) If Justin Trudeau has a brother, then he also has a sister.
- (B) If a student in this class has a brother, then they also have a sister.

Let  $f$  be a function with domain  $\mathbb{R}$ . Write the negation of the statement:

$$\text{IF } 2 < x < 4, \quad \text{THEN } 1 < f(x) < 3.$$

Write down the formal definition of the following statements:

(A)  $\lim_{x \rightarrow a} f(x) = L$

(B)  $\lim_{x \rightarrow a} f(x)$  exists

(C)  $\lim_{x \rightarrow a} f(x)$  does not exist

## Preparation: choosing deltas

(A) Find one value of  $\delta > 0$  such that

$$|x - 3| < \delta \implies |5x - 15| < 1.$$

(B) Find *all* values of  $\delta > 0$  such that

$$|x - 3| < \delta \implies |5x - 15| < 1.$$

(C) Find *all* values of  $\delta > 0$  such that

$$|x - 3| < \delta \implies |5x - 15| < 0.1.$$

(D) Let us fix  $\varepsilon > 0$ . Find *all* values of  $\delta > 0$  such that

$$|x - 3| < \delta \implies |5x - 15| < \varepsilon.$$

# What is wrong with this “proof”?

Prove that

$$\lim_{x \rightarrow 3} (5x + 1) = 16$$

“Proof:”

Let  $\varepsilon > 0$ .

WTS  $\forall \varepsilon > 0, \exists \delta > 0$  s.t.

$$0 < |x - 3| < \delta \implies |(5x + 1) - (16)| < \varepsilon$$

$$|(5x + 1) - (16)| < \varepsilon \iff |5x + 15| < \varepsilon$$

$$\iff 5|x + 3| < \varepsilon \implies \delta = \frac{\varepsilon}{3}$$





# Your first $\varepsilon - \delta$ proof

## Goal

We want to prove that

$$\lim_{x \rightarrow 3} (5x + 1) = 16 \quad (1)$$

directly from the definition.

## Your first $\varepsilon - \delta$ proof

### Goal

We want to prove that

$$\lim_{x \rightarrow 3} (5x + 1) = 16 \quad (1)$$

directly from the definition.

- (A) Write down the formal definition of the statement (1).

## Your first $\varepsilon - \delta$ proof

### Goal

We want to prove that

$$\lim_{x \rightarrow 3} (5x + 1) = 16 \quad (1)$$

directly from the definition.

- (A) Write down the formal definition of the statement (1).
- (B) Write down what the structure of the formal proof should be, without filling the details.

## Your first $\varepsilon - \delta$ proof

### Goal

We want to prove that

$$\lim_{x \rightarrow 3} (5x + 1) = 16 \quad (1)$$

directly from the definition.

- (A) Write down the formal definition of the statement (1).
- (B) Write down what the structure of the formal proof should be, without filling the details.
- (C) Write down a complete formal proof.

## A harder proof

### Goal

We want to prove that

$$\lim_{x \rightarrow 0} (x^3 + x^2) = 0 \quad (2)$$

directly from the definition.

## A harder proof

### Goal

We want to prove that

$$\lim_{x \rightarrow 0} (x^3 + x^2) = 0 \quad (2)$$

directly from the definition.

- (A) Write down the formal definition of the statement (2).

## A harder proof

### Goal

We want to prove that

$$\lim_{x \rightarrow 0} (x^3 + x^2) = 0 \quad (2)$$

directly from the definition.

- (A) Write down the formal definition of the statement (2).
- (B) Write down what the structure of the formal proof should be, without filling the details.

### Goal

We want to prove that

$$\lim_{x \rightarrow 0} (x^3 + x^2) = 0 \quad (2)$$

directly from the definition.

- (A) Write down the formal definition of the statement (2).
- (B) Write down what the structure of the formal proof should be, without filling the details.
- (C) Rough work: What is  $\delta$ ?



## A harder proof

### Goal

We want to prove that

$$\lim_{x \rightarrow 0} (x^3 + x^2) = 0 \quad (2)$$

directly from the definition.

- (A) Write down the formal definition of the statement (2).
- (B) Write down what the structure of the formal proof should be, without filling the details.
- (C) Rough work: What is  $\delta$ ?
- (D) Write down a complete formal proof.

# Is this proof correct?

## Claim:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x| < \delta \implies |x^3 + x^2| < \varepsilon.$$

## Proof:

- Let  $\varepsilon > 0$ .
- Take  $\delta = \sqrt{\frac{\varepsilon}{|x+1|}}$ .
- Let  $x \in \mathbb{R}$ . Assume  $0 < |x| < \delta$ . Then

$$|x^3 + x^2| = x^2|x+1| < \delta^2|x+1| = \frac{\varepsilon}{|x+1|}|x+1| = \varepsilon.$$

- I have proven that  $|x^3 + x^2| < \varepsilon$ . □

## Choosing deltas again

Let us fix numbers  $A, \varepsilon > 0$ . Find:

(A) a value of  $\delta > 0$  s.t.  $|x| < \delta \implies |Ax^2| < \varepsilon$

## Choosing deltas again

Let us fix numbers  $A, \varepsilon > 0$ . Find:

(A) a value of  $\delta > 0$  s.t.  $|x| < \delta \implies |Ax^2| < \varepsilon$

(B) *all* values of  $\delta > 0$  s.t.  $|x| < \delta \implies |Ax^2| < \varepsilon$

## Choosing deltas again

Let us fix numbers  $A, \varepsilon > 0$ . Find:

- (A) a value of  $\delta > 0$  s.t.  $|x| < \delta \implies |Ax^2| < \varepsilon$
- (B) *all* values of  $\delta > 0$  s.t.  $|x| < \delta \implies |Ax^2| < \varepsilon$
- (C) a value of  $\delta > 0$  s.t.  $|x| < \delta \implies |x + 1| < 10$

## Choosing deltas again

Let us fix numbers  $A, \varepsilon > 0$ . Find:

- (A) a value of  $\delta > 0$  s.t.  $|x| < \delta \implies |Ax^2| < \varepsilon$
- (B) *all* values of  $\delta > 0$  s.t.  $|x| < \delta \implies |Ax^2| < \varepsilon$
- (C) a value of  $\delta > 0$  s.t.  $|x| < \delta \implies |x + 1| < 10$
- (D) *all* values of  $\delta > 0$  s.t.  $|x| < \delta \implies |x + 1| < 10$

## Choosing deltas again

Let us fix numbers  $A, \varepsilon > 0$ . Find:

(A) a value of  $\delta > 0$  s.t.  $|x| < \delta \implies |Ax^2| < \varepsilon$

(B) *all* values of  $\delta > 0$  s.t.  $|x| < \delta \implies |Ax^2| < \varepsilon$

(C) a value of  $\delta > 0$  s.t.  $|x| < \delta \implies |x + 1| < 10$

(D) *all* values of  $\delta > 0$  s.t.  $|x| < \delta \implies |x + 1| < 10$

(E) a value of  $\delta > 0$  s.t.  $|x| < \delta \implies \begin{cases} |Ax^2| < \varepsilon \\ |x + 1| < 10 \end{cases}$

## Choosing deltas again

Let us fix numbers  $A, \varepsilon > 0$ . Find:

(A) a value of  $\delta > 0$  s.t.  $|x| < \delta \implies |Ax^2| < \varepsilon$

(B) *all* values of  $\delta > 0$  s.t.  $|x| < \delta \implies |Ax^2| < \varepsilon$

(C) a value of  $\delta > 0$  s.t.  $|x| < \delta \implies |x + 1| < 10$

(D) *all* values of  $\delta > 0$  s.t.  $|x| < \delta \implies |x + 1| < 10$

(E) a value of  $\delta > 0$  s.t.  $|x| < \delta \implies \begin{cases} |Ax^2| < \varepsilon \\ |x + 1| < 10 \end{cases}$

(F) a value of  $\delta > 0$  s.t.  $|x| < \delta \implies |(x + 1)x^2| < \varepsilon$



## Indeterminate form

Let  $a \in \mathbb{R}$ . Let  $f$  and  $g$  be positive functions defined near  $a$ , except maybe at  $a$ .

Assume  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ .

What can we conclude about  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  ?

- (A) The limit is 1. exist.
- (B) The limit is 0.
- (C) The limit is  $\infty$ .
- (D) The limit does not
- (E) We do not have enough information to decide.

## A theorem about limits

Let  $f$  be a function with domain  $\mathbb{R}$  such that

$$\lim_{x \rightarrow 0} f(x) = 3$$

Prove that

$$\lim_{x \rightarrow 0} [5f(2x)] = 15$$

directly from the definition of limit. Do not use any of the limit laws.

## A theorem about limits

Let  $f$  be a function with domain  $\mathbb{R}$  such that

$$\lim_{x \rightarrow 0} f(x) = 3$$

Prove that

$$\lim_{x \rightarrow 0} [5f(2x)] = 15$$

directly from the definition of limit. Do not use any of the limit laws.

- (A) Write down the formal definition of the statement you want to prove.

## A theorem about limits

Let  $f$  be a function with domain  $\mathbb{R}$  such that

$$\lim_{x \rightarrow 0} f(x) = 3$$

Prove that

$$\lim_{x \rightarrow 0} [5f(2x)] = 15$$

directly from the definition of limit. Do not use any of the limit laws.

- (A) Write down the formal definition of the statement you want to prove.
- (B) Write down what the structure of the formal proof should be, without filling the details.

## A theorem about limits

Let  $f$  be a function with domain  $\mathbb{R}$  such that

$$\lim_{x \rightarrow 0} f(x) = 3$$

Prove that

$$\lim_{x \rightarrow 0} [5f(2x)] = 15$$

directly from the definition of limit. Do not use any of the limit laws.

- (A) Write down the formal definition of the statement you want to prove.
- (B) Write down what the structure of the formal proof should be, without filling the details.
- (C) Rough work.

## A theorem about limits

Let  $f$  be a function with domain  $\mathbb{R}$  such that

$$\lim_{x \rightarrow 0} f(x) = 3$$

Prove that

$$\lim_{x \rightarrow 0} [5f(2x)] = 15$$

directly from the definition of limit. Do not use any of the limit laws.

- (A) Write down the formal definition of the statement you want to prove.
- (B) Write down what the structure of the formal proof should be, without filling the details.
- (C) Rough work.
- (D) Write down a complete proof.

## Proof feedback

- (A) Is the structure of the proof correct?  
(First fix  $\varepsilon$ , then choose  $\delta$ , then ...)
- (B) Did you say exactly what  $\delta$  is?
- (C) Is the proof self-contained?  
(I do not need to read the rough work)
- (D) Are all variables defined? In the right order?
- (E) Do all steps follow logically from what comes before?  
Do you start from what you know and prove what you have to prove?
- (F) Are you proving your conclusion or assuming it?

# A new squeeze

This is the Squeeze Theorem, as you know it:

## The (classical) Squeeze Theorem

Let  $a, L \in \mathbb{R}$ .

Let  $f$ ,  $g$ , and  $h$  be functions defined near  $a$ , except possibly at  $a$ .

- IF
- For  $x$  close to  $a$  but not  $a$ ,  $h(x) \leq g(x) \leq f(x)$
  - $\lim_{x \rightarrow a} f(x) = L$     and     $\lim_{x \rightarrow a} h(x) = L$

THEN

- $\lim_{x \rightarrow a} g(x) = L$



# A new squeeze

This is the Squeeze Theorem, as you know it:

## The (classical) Squeeze Theorem

Let  $a, L \in \mathbb{R}$ .

Let  $f$ ,  $g$ , and  $h$  be functions defined near  $a$ , except possibly at  $a$ .

- IF
- For  $x$  close to  $a$  but not  $a$ ,  $h(x) \leq g(x) \leq f(x)$
  - $\lim_{x \rightarrow a} f(x) = L$  and  $\lim_{x \rightarrow a} h(x) = L$

THEN

- $\lim_{x \rightarrow a} g(x) = L$

Come up with a new version of the theorem about limits being infinity. (The conclusion should be  $\lim_{x \rightarrow a} g(x) = \infty$ .)

*Hint:* Draw a picture for the classical Squeeze Theorem. Then draw a picture for the new theorem.

# A new squeeze

## The (new) Squeeze Theorem

Let  $a \in \mathbb{R}$ .

Let  $g$  and  $h$  be functions defined near  $a$ , except possibly at  $a$ .

- IF
- For  $x$  close to  $a$  but not  $a$ ,  $h(x) \leq g(x)$
  - $\lim_{x \rightarrow a} h(x) = \infty$

THEN

- $\lim_{x \rightarrow a} g(x) = \infty$

# A new squeeze

## The (new) Squeeze Theorem

Let  $a \in \mathbb{R}$ .

Let  $g$  and  $h$  be functions defined near  $a$ , except possibly at  $a$ .

IF      • For  $x$  close to  $a$  but not  $a$ ,  $h(x) \leq g(x)$   
         •  $\lim_{x \rightarrow a} h(x) = \infty$

THEN   •  $\lim_{x \rightarrow a} g(x) = \infty$

(A) Replace the first hypothesis with a more precise mathematical statement.

## The (new) Squeeze Theorem

Let  $a \in \mathbb{R}$ .

Let  $g$  and  $h$  be functions defined near  $a$ , except possibly at  $a$ .

IF      • For  $x$  close to  $a$  but not  $a$ ,  $h(x) \leq g(x)$   
         •  $\lim_{x \rightarrow a} h(x) = \infty$

THEN   •  $\lim_{x \rightarrow a} g(x) = \infty$

- (A) Replace the first hypothesis with a more precise mathematical statement.
- (B) Write down the definition of what you want to prove.

## The (new) Squeeze Theorem

Let  $a \in \mathbb{R}$ .

Let  $g$  and  $h$  be functions defined near  $a$ , except possibly at  $a$ .

IF      • For  $x$  close to  $a$  but not  $a$ ,  $h(x) \leq g(x)$   
         •  $\lim_{x \rightarrow a} h(x) = \infty$

THEN   •  $\lim_{x \rightarrow a} g(x) = \infty$

- (A) Replace the first hypothesis with a more precise mathematical statement.
- (B) Write down the definition of what you want to prove.
- (C) Write down the structure of the formal proof.

## The (new) Squeeze Theorem

Let  $a \in \mathbb{R}$ .

Let  $g$  and  $h$  be functions defined near  $a$ , except possibly at  $a$ .

IF      • For  $x$  close to  $a$  but not  $a$ ,  $h(x) \leq g(x)$   
         •  $\lim_{x \rightarrow a} h(x) = \infty$

THEN   •  $\lim_{x \rightarrow a} g(x) = \infty$

- (A) Replace the first hypothesis with a more precise mathematical statement.
- (B) Write down the definition of what you want to prove.
- (C) Write down the structure of the formal proof.
- (D) Rough work

## The (new) Squeeze Theorem

Let  $a \in \mathbb{R}$ .

Let  $g$  and  $h$  be functions defined near  $a$ , except possibly at  $a$ .

IF      • For  $x$  close to  $a$  but not  $a$ ,  $h(x) \leq g(x)$   
         •  $\lim_{x \rightarrow a} h(x) = \infty$

THEN   •  $\lim_{x \rightarrow a} g(x) = \infty$

- (A) Replace the first hypothesis with a more precise mathematical statement.
- (B) Write down the definition of what you want to prove.
- (C) Write down the structure of the formal proof.
- (D) Rough work
- (E) Write down a complete, formal proof.

## True or False?

Is this theorem true?

### Claim

Let  $a \in \mathbb{R}$ .

Let  $f$  and  $g$  be functions defined near  $a$ .

- IF  $\lim_{x \rightarrow a} f(x) = 0$ ,
- THEN  $\lim_{x \rightarrow a} [f(x)g(x)] = 0$ .



# A new theorem about products

## Theorem

Let  $a \in \mathbb{R}$ . Let  $f$  and  $g$  be functions with domain  $\mathbb{R}$ , except possibly  $a$ . Assume

- $\lim_{x \rightarrow a} f(x) = 0$ , and
- $g$  is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN  $\lim_{x \rightarrow a} [f(x)g(x)] = 0$

# A new theorem about products

## Theorem

Let  $a \in \mathbb{R}$ . Let  $f$  and  $g$  be functions with domain  $\mathbb{R}$ , except possibly  $a$ . Assume

- $\lim_{x \rightarrow a} f(x) = 0$ , and
- $g$  is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN  $\lim_{x \rightarrow a} [f(x)g(x)] = 0$

- (A) Write down the formal definition of what you want to prove.

# A new theorem about products

## Theorem

Let  $a \in \mathbb{R}$ . Let  $f$  and  $g$  be functions with domain  $\mathbb{R}$ , except possibly  $a$ . Assume

- $\lim_{x \rightarrow a} f(x) = 0$ , and
- $g$  is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN  $\lim_{x \rightarrow a} [f(x)g(x)] = 0$

- (A) Write down the formal definition of what you want to prove.
- (B) Write down what the structure of the formal proof.

# A new theorem about products

## Theorem

Let  $a \in \mathbb{R}$ . Let  $f$  and  $g$  be functions with domain  $\mathbb{R}$ , except possibly  $a$ . Assume

- $\lim_{x \rightarrow a} f(x) = 0$ , and
- $g$  is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN  $\lim_{x \rightarrow a} [f(x)g(x)] = 0$

- (A) Write down the formal definition of what you want to prove.
- (B) Write down what the structure of the formal proof.
- (C) Rough work.

# A new theorem about products

## Theorem

Let  $a \in \mathbb{R}$ . Let  $f$  and  $g$  be functions with domain  $\mathbb{R}$ , except possibly  $a$ . Assume

- $\lim_{x \rightarrow a} f(x) = 0$ , and
- $g$  is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN  $\lim_{x \rightarrow a} [f(x)g(x)] = 0$

- (A) Write down the formal definition of what you want to prove.
- (B) Write down what the structure of the formal proof.
- (C) Rough work.
- (D) Write down a complete formal proof.

# Critique this “proof” – #1

- WTS  $\lim_{x \rightarrow a} [f(x)g(x)] = 0$ :  
 $\forall \varepsilon > 0, \exists \delta > 0 \quad \text{s.t.} \quad 0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon.$
- We know  $\lim_{x \rightarrow a} f(x) = 0$   
 $\forall \varepsilon_1 > 0, \exists \delta_1 > 0 \quad \text{s.t.} \quad 0 < |x - a| < \delta_1 \implies |f(x)| < \varepsilon_1.$
- We know  $\exists M > 0 \quad \text{s.t.} \quad \forall x \neq 0, |g(x)| \leq M.$
- $|f(x)g(x)| = |f(x)||g(x)| < \varepsilon_1 M$
- $\varepsilon = \varepsilon_1 M \implies \varepsilon_1 = \frac{\varepsilon}{M}$
- Take  $\delta = \delta_1$

## Critique this “proof” – #2

- WTS  $\lim_{x \rightarrow a} [f(x)g(x)] = 0$ . By definition, WTS:  
$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon$$
- Let  $\varepsilon > 0$ .
- Use the value  $\frac{\varepsilon}{M}$  as “epsilon” in the definition of  
$$\lim_{x \rightarrow a} f(x) = 0$$
  
$$\exists \delta_1 \in \mathbb{R} \text{ s.t. } 0 < |x - a| < \delta_1 \implies |f(x)| < \frac{\varepsilon}{M}.$$
- Take  $\delta = \delta_1$ .
- Let  $x \in \mathbb{R}$ . Assume  $0 < |x - a| < \delta$
- Since  $\exists M > 0$  s.t.  $\forall x \neq 0, |g(x)| \leq M$   
$$|f(x)g(x)| < \frac{\varepsilon}{M} \cdot M = \varepsilon.$$

## Critique this “proof” – #3

- Since  $g$  is bounded,  $\exists M > 0$  s.t.  $\forall x \neq 0, |g(x)| \leq M$
- Since  $\lim_{x \rightarrow a} f(x) = 0$ , there exists  $\delta_1 > 0$  s.t.  
if  $0 < |x - a| < \delta_1$ , then  $|f(x) - 0| = |f(x)| < \varepsilon_1 = \frac{\varepsilon}{M}$ .
- $|f(x)g(x)| = |f(x)| \cdot |g(x)| \leq |f(x)| \cdot M < \varepsilon_1 \cdot M = \frac{\varepsilon}{M} \cdot M = \varepsilon$
- In summary, by setting  $\delta = \min\{\delta_1\}$ , we find that  
if  $0 < |x - a| < \delta$ , then  $|f(x) \cdot g(x)| < \varepsilon$ .



## Limits involving $\sin(1/x)$ Part I

The reason that  $\lim_{x \rightarrow 0} \sin(1/x)$  does not exist is:

- (A) because the function values oscillate around 0
- (B) because  $1/0$  is undefined
- (C) because no matter how close  $x$  gets to 0, there are  $x$ 's near 0 for which  $\sin(1/x) = 1$ , and some for which  $\sin(1/x) = -1$
- (D) all of the above

## Limits involving $\sin(1/x)$ Part II

The limit  $\lim_{x \rightarrow 0} x^2 \sin(1/x)$

- (A) does not exist because the function values oscillate around 0
- (B) does not exist because  $1/0$  is undefined
- (C) does not exist because no matter how close  $x$  gets to 0, there are  $x$ 's near 0 for which  $\sin(1/x) = 1$ , and some for which  $\sin(1/x) = -1$
- (D) equals 0
- (E) equals 1

# Absolute value and the Squeeze Theorem

Use the Squeeze Theorem to prove:

## Theorem

*IF  $\lim_{x \rightarrow a} |f(x)| = 0$ , THEN  $\lim_{x \rightarrow a} f(x) = 0$ .*

*Hint:* Recall that  $-|c| \leq c \leq |c|$  for every  $c \in \mathbb{R}$ .

## Undefined function

Let  $a \in \mathbb{R}$  and let  $f$  be a function. Assume  $f(a)$  is undefined.

What can we conclude?

- (A)  $\lim_{x \rightarrow a} f(x)$  exist
- (B)  $\lim_{x \rightarrow a} f(x)$  doesn't exist.
- (C) No conclusion.  $\lim_{x \rightarrow a} f(x)$  may or may not exist.

What else can we conclude?

- (D)  $f$  is continuous at  $a$ .
- (E)  $f$  is not continuous at  $a$ .
- (F) No conclusion.  $f$  may or may not be continuous at  $a$ .

## A new function

- Let  $x, y \in \mathbb{R}$ . What does the following expression calculate? Prove it.

$$f(x, y) = \frac{x + y + |x - y|}{2}$$

*Suggestion:* If you don't know how to start, try some sample values of  $x$  and  $y$ .

## A new function

- Let  $x, y \in \mathbb{R}$ . What does the following expression calculate? Prove it.

$$f(x, y) = \frac{x + y + |x - y|}{2}$$

*Suggestion:* If you don't know how to start, try some sample values of  $x$  and  $y$ .

- Write a similar expression to compute  $\min\{x, y\}$ .

## More continuous functions

We want to prove the following theorem

### Theorem

IF  $f$  and  $g$  are continuous functions  
THEN  $h(x) = \max\{f(x), g(x)\}$  is also a continuous function.

You are allowed to use all results that we already know.  
What is the fastest way to prove this?

*Hint:* There is a way to prove this quickly without writing any epsilons.

## True or False? – Discontinuities

- (A) IF  $f$  and  $g$  have removable discontinuities at  $a$   
THEN  $f + g$  has a removable discontinuity at  $a$
  
- (B) IF  $f$  and  $g$  have non-removable discontinuities at  $a$   
THEN  $f + g$  has a non-removable discontinuity at  $a$



Which one is the correct claim?

Claim 1?

(Assuming these limits exist)

$$\lim_{x \rightarrow a} g(f(x)) = g\left(\lim_{x \rightarrow a} f(x)\right)$$

Claim 2?

IF      (A)  $\lim_{x \rightarrow a} f(x) = L$ ,    and    (B)  $\lim_{t \rightarrow L} g(t) = M$   
THEN    (C)  $\lim_{x \rightarrow a} g(f(x)) = M$

## A difficult example

Construct a pair of functions  $f$  and  $g$  such that

$$\lim_{x \rightarrow 0} f(x) = 1$$

$$\lim_{t \rightarrow 1} g(t) = 2$$

$$\lim_{x \rightarrow 0} g(f(x)) = 42$$

## Transforming limits

The only thing we know about the function  $g$  is that

$$\lim_{x \rightarrow 0} \frac{g(x)}{x^2} = 2.$$

Use it to compute the following limits:

$$(A) \quad \lim_{x \rightarrow 0} \frac{g(x)}{x}$$

$$(B) \quad \lim_{x \rightarrow 0} \frac{g(x)}{x^4}$$

$$(C) \quad \lim_{x \rightarrow 0} \frac{g(3x)}{x^2}$$

# Limits at infinity

Compute:

$$\begin{array}{ll} \text{(A)} \quad \lim_{x \rightarrow \infty} (x^7 - 2x^5 + 11) & \text{(D)} \quad \lim_{x \rightarrow \infty} \frac{x^2 + 2x + 3}{3x^2 + 4x + 5} \\ \text{(B)} \quad \lim_{x \rightarrow \infty} \left( x^2 - \sqrt{x^5 + 1} \right) & \\ \text{(C)} \quad \lim_{x \rightarrow \infty} \frac{x^2 + 11}{x + 1} & \text{(E)} \quad \lim_{x \rightarrow \infty} \frac{x^3 + \sqrt{2x^6 + 1}}{2x^3 + \sqrt{x^5 + 1}} \end{array}$$

# Trig computations

Using that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ , compute the following limits:

$$(A) \lim_{x \rightarrow 2} \frac{\sin x}{x}$$

$$(B) \lim_{x \rightarrow 0} \frac{\sin(5x)}{x}$$

$$(C) \lim_{x \rightarrow 0} \frac{\tan^2(2x^2)}{x^4}$$

$$(D) \lim_{x \rightarrow 0} \frac{\sin e^x}{e^x}$$

$$(E) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$$

$$(F) \lim_{x \rightarrow 0} \frac{\tan^{10}(2x^{20})}{\sin^{200}(3x)}$$

# Trig computations

Using that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ , compute the following limits:

$$(A) \lim_{x \rightarrow 2} \frac{\sin x}{x}$$

$$(B) \lim_{x \rightarrow 0} \frac{\sin(5x)}{x}$$

$$(C) \lim_{x \rightarrow 0} \frac{\tan^2(2x^2)}{x^4}$$

$$(G) \lim_{x \rightarrow 0} [(\sin x) (\cos(2x)) (\tan(3x)) (\sec(4x)) (\csc(5x)) (\cot(6x))]$$

$$(D) \lim_{x \rightarrow 0} \frac{\sin e^x}{e^x}$$

$$(E) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$$

$$(F) \lim_{x \rightarrow 0} \frac{\tan^{10}(2x^{20})}{\sin^{200}(3x)}$$

## Plus or minus infinity?

Compute:

$$(A) \lim_{x \rightarrow -3^+} \frac{x^2 - 9}{3 - 2x - x^2} \quad (B) \lim_{x \rightarrow 1^+} \frac{x^2 - 9}{3 - 2x - x^2}$$

## A harder limit

Calculate

$$\lim_{x \rightarrow 2} \frac{[\sqrt{2+x} - 2] [\sqrt{3+x} - 3]}{\sqrt{x-1} - 1}$$



# Which solution is right?

Compute  $L = \lim_{x \rightarrow -\infty} \left[ x - \sqrt{x^2 + x} \right]$ .

## • Solution 1

$$\begin{aligned} L &= \lim_{x \rightarrow -\infty} \frac{\left[ x - \sqrt{x^2 + x} \right] \left[ x + \sqrt{x^2 + x} \right]}{\left[ x + \sqrt{x^2 + x} \right]} = \lim_{x \rightarrow -\infty} \frac{x^2 - (x^2 + x)}{\left[ x + \sqrt{x^2 + x} \right]} \\ &= \lim_{x \rightarrow -\infty} \frac{-x}{x \left[ 1 + \sqrt{1 + \frac{1}{x}} \right]} = \lim_{x \rightarrow -\infty} \frac{-1}{\left[ 1 + \sqrt{1 + \frac{1}{x}} \right]} = \frac{-1}{2} \end{aligned}$$

## • Solution 2

$$L = \lim_{x \rightarrow -\infty} \left[ x - \sqrt{x^2 + x} \right] = (-\infty) - \infty = -\infty$$

## Can we conclude this?

- Consider the function  $f(x) = \frac{4}{x}$ .
- We have  $f(-1) = -4 < 0$  and  $f(1) = 4 > 0$ .
- Use IVT.  
Can we conclude  $f(c) = 0$  for some  $c \in (-1, 1)$ ?

## Existence of solutions

Prove that the equation

$$x^4 - 2x = 100$$

has at least two solutions.

## Can this be proven? (Use IVT)

- (A) Prove that there exists a time of the day when the hour hand and the minute hand of a clock form an angle of exactly 23 degrees.

## Can this be proven? (Use IVT)

- (A) Prove that there exists a time of the day when the hour hand and the minute hand of a clock form an angle of exactly 23 degrees.
- (B) During a Raptors basketball game, at half time the Raptors have 52 points. Prove that at some point they had exactly 26 points.

## Can this be proven? (Use IVT)

- (A) Prove that there exists a time of the day when the hour hand and the minute hand of a clock form an angle of exactly 23 degrees.
- (B) During a Raptors basketball game, at half time the Raptors have 52 points. Prove that at some point they had exactly 26 points.
- (C) Prove that at some point during Alfonso's life, his height in centimetres was exactly equal to 10 times his weight in kilograms. Some data:
  - His height at birth: 47 cm
  - His weight at birth: 3.2 kg
  - His height today: 172 cm

# Temperature

On June 09, 2016, the outside temperature in Toronto at 6 AM was  $10^{\circ}$ . At 4 PM, the temperature was  $20^{\circ}$ .

- (A) Must there have been a time between 6 AM and 4 PM when the temperature was  $15^{\circ}$ ? Explain how you know. Which assumption about temperature allows you to reach your conclusion?
- (B) Must there have been a time between 6 AM and 4 PM when the temperature was  $22^{\circ}$ ? Explain how you know.
- (C) Could there have been a time between 6 AM and 4 PM when the temperature was  $22^{\circ}$ ? Explain how you know.

In each of the following cases, does the function  $f$  have a maximum and a minimum on the interval  $I$ ?

(A)  $f(x) = x^2, \quad I = (-1, 1).$

(B)  $f(x) = \frac{(e^x + 2) \sin x}{x} - \cos x + 3, \quad I = [2, 6]$

(C)  $f(x) = \frac{(e^x + 2) \sin x}{x} - \cos x + 3, \quad I = (0, 5]$



## Definition of maximum

Let  $f$  be a function with domain  $I$ .

Which one (or ones) of the following is (or are) a definition of “ $f$  has a maximum on  $I$ ”?

(A)  $\forall x \in I, \exists C \in \mathbb{R} \text{ s.t. } f(x) \leq C$

(B)  $\exists C \in I \text{ s.t. } \forall x \in I, f(x) \leq C$

(C)  $\exists C \in \mathbb{R} \text{ s.t. } \forall x \in I, f(x) \leq C$

(D)  $\exists C \in \mathbb{R} \text{ s.t. } \forall x \in I, f(x) < C$

## More on the definition of maximum

Let  $f$  be a function with domain  $I$ .

What does each of the following mean?

- (A)  $\exists C \in \mathbb{R}$  s.t.  $\forall x \in I, f(x) \leq C$
- (B)  $\exists C \in \mathbb{R}$  s.t.  $\forall x \in I, f(x) < C$
- (C)  $\exists a \in I$  s.t.  $\forall x \in I, f(x) \leq f(a)$
- (D)  $\exists a \in I$  s.t.  $\forall x \in I, f(x) < f(a)$