

**Before next class:**

- **Watch videos 13.5, 13.6, 13.7**

## Rapid questions: improper integrals

Convergent or divergent?

$$(A) \int_1^{\infty} \frac{1}{x^2} dx$$

$$(B) \int_1^{\infty} \frac{1}{x} dx$$

$$(C) \int_1^{\infty} \frac{1}{\sqrt{x}} dx$$

## Rapid questions: improper integrals

Convergent or divergent?

(A)  $\int_1^{\infty} \frac{1}{x^2} dx$

(D)  $\int_1^{\infty} \frac{x+1}{x^3+2} dx$

(B)  $\int_1^{\infty} \frac{1}{x} dx$

(E)  $\int_1^{\infty} \frac{\sqrt{x^2+5}}{x^2+6} dx$

(C)  $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$

(F)  $\int_1^{\infty} \frac{x^2+3}{\sqrt{x^5+2}} dx$

## A telescopic series

I want to calculate the value of the series  $\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n}$ .

(A) Find a formula for the  $k$ -th partial sum  $S_k = \sum_{n=1}^k \frac{1}{n^2 + 2n}$ .

*Hint:*  $\frac{1}{n^2 + 2n} = \frac{A}{n} + \frac{B}{n+2}$

(B) Using the definition of series, compute the value of

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n}$$

# What is wrong with this calculation? Fix it

**Claim:** 
$$\sum_{n=2}^{\infty} \ln \frac{n}{n+1} = \ln 2$$

“Proof”

$$\begin{aligned} \sum_{n=2}^{\infty} \ln \frac{n}{n+1} &= \sum_{n=2}^{\infty} [\ln n - \ln(n+1)] \\ &= \sum_{n=2}^{\infty} \ln(n) - \sum_{n=2}^{\infty} \ln(n+1) \\ &= (\ln 2 + \ln 3 + \ln 4 + \dots) - (\ln 3 + \ln 4 + \dots) \\ &= \ln 2 \end{aligned}$$

# True or False – The tail of a series

(A) IF the series  $\sum_{n=0}^{\infty} a_n$  converges,

THEN the series  $\sum_{n=7}^{\infty} a_n$  converges

(B) IF the series  $\sum_{n=7}^{\infty} a_n$  converges,

THEN the series  $\sum_{n=0}^{\infty} a_n$  converges

(C) IF the series  $\sum_{n=0}^{\infty} a_n$  converges,

THEN the series  $\sum_{n=7}^{\infty} a_n$  converges to a smaller number.

**Before next class:**

- **Watch videos 13.8, 13.9**

# Geometric series

Calculate the value of the following series:

$$(A) \quad 1 + \frac{13}{+} \frac{19}{+} \frac{1}{27} + \frac{1}{81} + \dots$$

$$(B) \quad \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \dots$$

$$(C) \quad \frac{3}{2} - \frac{9}{4} + \frac{27}{8} - \frac{81}{16} + \dots$$

$$(D) \quad 1 + \frac{1}{2^{0.5}} + \frac{1}{2} + \frac{1}{2^{1.5}} + \frac{1}{2^2} + \frac{1}{2^{2.5}} + \dots$$

$$(E) \quad \sum_{n=1}^{\infty} (-1)^n \frac{3^n}{2^{2n+1}}$$

$$(F) \quad \sum_{n=k}^{\infty} x^n$$



Is  $0.999999\dots = 1$ ?

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(A) Write the number  $0.999999 \dots$  as a series  
*Hint:*  $427 = 400 + 20 + 7$ .

(B) Compute the first few partial sums

(C) Add up the series.  
*Hint:* it is geometric.

# Examples

(A) A series  $\sum_{n=0}^{\infty} a_n$  may be

$$\left\{ \begin{array}{l} \text{convergent (a number)} \\ \text{divergent} \left\{ \begin{array}{l} \text{to } \infty \\ \text{to } -\infty \\ \text{"oscillating"} \end{array} \right. \end{array} \right.$$

Give one example of each of the four results.

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Give one example of each of the four results.

(B) Now assume  $\forall n \in \mathbb{N}, a_n \geq 0$ .  
Which of the four outcomes is still possible?

## True or False – Definition of series

Let  $\sum_{n=0}^{\infty} a_n$  be a series. Let  $\{S_n\}_{n=0}^{\infty}$  be its partial-sum sequence.

(A) IF the series  $\sum_{n=0}^{\infty} a_n$  is convergent,

THEN the sequence  $\{S_n\}_{n=0}^{\infty}$  is bounded.

(B) IF the series  $\sum_{n=0}^{\infty} a_n$  is convergent,

THEN the sequence  $\{S_n\}_{n=0}^{\infty}$  is eventually monotonic.

(C) IF the sequence  $\{S_n\}_{n=0}^{\infty}$  is bounded and eventually monotonic,

THEN the series  $\sum_{n=0}^{\infty} a_n$  is convergent.

## True or False – Definition of series

Let  $\sum_{n=0}^{\infty} a_n$  be a series. Let  $\{S_n\}_{n=0}^{\infty}$  be its partial-sum sequence.

(D) IF  $\forall n > 0, a_n > 0$ ,

THEN the sequence  $\{S_n\}_{n=0}^{\infty}$  is increasing.

(E) IF the sequence  $\{S_n\}_{n=0}^{\infty}$  is increasing,

THEN  $\forall n > 0, a_n > 0$ .

(F) IF  $\forall n > 0, a_n \geq 0$ ,

THEN the sequence  $\{S_n\}_{n=0}^{\infty}$  is non-decreasing.

(G) IF the sequence  $\{S_n\}_{n=0}^{\infty}$  is non-decreasing,

THEN  $\forall n > 0, a_n \geq 0$

**Before next class:**

- **Watch videos 13.10, 13.12**

## Rapid questions: geometric series

Convergent or divergent?

$$(A) \sum_{n=0}^{\infty} \frac{1}{2^n}$$

$$(D) \sum_{n=5}^{\infty} \frac{3^n}{2^{2n+1}}$$

$$(B) \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$$

$$(E) \sum_{n=3}^{\infty} \frac{3^n}{1000 \cdot 2^{n+2}}$$

$$(C) \sum_{n=1}^{\infty} \frac{1}{2^{n/2}}$$

$$(F) \sum_{n=0}^{\infty} (-1)^n$$



## True or False – The Necessary Condition

(A) IF  $\lim_{n \rightarrow \infty} a_n = 0$ , THEN  $\sum_n^{\infty} a_n$  is convergent.

(B) IF  $\lim_{n \rightarrow \infty} a_n \neq 0$ , THEN  $\sum_n^{\infty} a_n$  is divergent.

(C) IF  $\sum_n^{\infty} a_n$  is convergent THEN  $\lim_{n \rightarrow \infty} a_n = 0$ .

(D) IF  $\sum_n^{\infty} a_n$  is divergent THEN  $\lim_{n \rightarrow \infty} a_n \neq 0$ .

# What can you conclude?

Assume  $\forall n \in \mathbb{N}$ ,  $a_n > 0$ . Consider the series  $\sum_{n=0}^{\infty} a_n$ .

Let  $\{S_n\}_{n=0}^{\infty}$  be its sequence of partial sums.

In each of the following cases, what can you conclude about the *series*? Is it convergent, divergent, or we do not know?

(A)  $\forall n \in \mathbb{N}$ ,  $\exists M \in \mathbb{R}$  s.t.  $S_n \leq M$ .

(B)  $\exists M \in \mathbb{R}$  s.t.  $\forall n \in \mathbb{N}$ ,  $S_n \leq M$ .

(C)  $\exists M > 0$  s.t.  $\forall n \in \mathbb{N}$ ,  $a_n \leq M$ .

(D)  $\exists M > 0$  s.t.  $\forall n \in \mathbb{N}$ ,  $a_n \geq M$ .

# Functions as series

You know that when  $|x| < 1$ :

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Find similar ways to write the following functions as series:

(A)  $g(x) = \frac{1}{1+x}$

(C)  $A(x) = \frac{1}{2-x}$

(B)  $h(x) = \frac{1}{1-x^2}$

## Functions as series

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(C)  $A(x) = \frac{1}{2-x}$

(B)  $h(x) = \frac{1}{1-x^2}$

(D)  $G(x) = \ln(1+x)$

*Hint:* For the last one, compute  $G'$ .

# MAT137 Lecture 61 — Integral test and comparison tests

**Before next class:**

- **Watch videos 13.13**

For which values of  $a \in \mathbb{R}$  are these series convergent?

$$(A) \sum_n^{\infty} \frac{1}{a^n}$$

$$(C) \sum_n^{\infty} a^n$$

$$(B) \sum_n^{\infty} \frac{1}{n^a}$$

$$(D) \sum_n^{\infty} n^a$$

## Quick comparisons: convergent or divergent?

$$(A) \sum_n^{\infty} \frac{n+1}{n^2+1}$$

$$(C) \sum_n^{\infty} \frac{\sqrt{n}+1}{n^2+1}$$

$$(B) \sum_n^{\infty} \frac{n^2+3n}{n^4+5n+1}$$

$$(D) \sum_n^{\infty} \frac{\sqrt[3]{n^2+1}+1}{\sqrt{n^3+n}+n+1}$$

## Slow comparisons: convergent or divergent?

$$(A) \sum_n^{\infty} \frac{2^n - 40}{3^n - 20}$$

$$(D) \sum_n^{\infty} \frac{1}{n (\ln n)^3}$$

$$(B) \sum_n^{\infty} \frac{(\ln n)^{20}}{n^2}$$

$$(E) \sum_n^{\infty} \frac{1}{n \ln n}$$

$$(C) \sum_n^{\infty} \sin^2 \frac{1}{n}$$

$$(F) \sum_n^{\infty} e^{-n^2}$$



**Before next class:**

- **Watch videos 13.15**

## Rapid questions: alternating series test

Convergent or divergent?

$$(A) \sum_{n=1}^{\infty} \frac{1}{n^{0.5}}$$

$$(D) \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.5}}$$

$$(B) \sum_{n=1}^{\infty} \frac{1}{n^3}$$

$$(E) \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

$$(C) \sum_{n=1}^{\infty} \frac{1}{\sin n}$$

$$(F) \sum_{n=1}^{\infty} \frac{(-1)^n}{\sin n}$$

Estimate the sum

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

with an error smaller than 0.001. Write your final answer as a rational number (i.e. as a quotient of two integers).

# Convergence tests: ninja level

We know

- $\forall n \in \mathbb{N}, a_n > 0.$
- the series  $\sum_n^{\infty} a_n$  is convergent

Determine whether the following series are convergent, divergent, or we do not have enough information to decide:

$$(A) \sum_n^{\infty} \sin a_n$$

$$(B) \sum_n^{\infty} \cos a_n$$

$$(C) \sum_n^{\infty} \sqrt{a_n}$$

$$(D) \sum_n^{\infty} (a_n)^2$$

# MAT137 Lecture 63 — Absolute and conditional convergence

**Before next class:**

- **Watch videos 13.18, 13.19**

## True or False - Absolute Values

(A) IF  $\{a_n\}_{n=1}^{\infty}$  is convergent, THEN  $\{|a_n|\}_{n=1}^{\infty}$  is convergent.

(B) IF  $\{|a_n|\}_{n=1}^{\infty}$  is convergent, THEN  $\{a_n\}_{n=1}^{\infty}$  is convergent.

(C) IF  $\sum_{n=1}^{\infty} a_n$  is convergent, THEN  $\sum_{n=1}^{\infty} |a_n|$  is convergent.

(D) IF  $\sum_{n=1}^{\infty} |a_n|$  is convergent, THEN  $\sum_{n=1}^{\infty} a_n$  is convergent.

# Absolutely convergent or conditionally convergent?

$$(A) \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.5}}$$

$$(B) \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1.5}}$$

$$(C) \sum_{n=1}^{\infty} \frac{(-1)^n}{\sin n}$$

# Convergence tests: ninja level

We know

- $\forall n \in \mathbb{N}, a_n > 0.$
- the series  $\sum_n^{\infty} a_n$  is convergent

Determine whether the following series are convergent, divergent, or we do not have enough information to decide:

$$(A) \sum_n^{\infty} \sin a_n$$

$$(B) \sum_n^{\infty} \cos a_n$$

$$(C) \sum_n^{\infty} \sqrt{a_n}$$

$$(D) \sum_n^{\infty} (a_n)^2$$



**Before next class:**

- **Watch videos 14.1, 14.2**

## Quick review: Convergent or divergent?

$$(A) \sum_{n=1}^{\infty} (1.1)^n$$

$$(E) \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$$

$$(B) \sum_{n=1}^{\infty} (0.9)^n$$

$$(F) \sum_{n=1}^{\infty} \frac{(-1)^n}{e^{1/n}}$$

$$(C) \sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$$

$$(G) \sum_{n=1}^{\infty} \frac{n^3 + n^2 + 11}{n^4 + 2n - 3}$$

$$(D) \sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$$

$$(H) \sum_{n=1}^{\infty} \frac{\sqrt{n^5 + 2n + 16}}{n^4 - 11n + 7}$$

## Ratio Test: Convergent or divergent?

Use Ratio Test to decide which series are convergent.

$$(A) \sum_{n=1}^{\infty} \frac{3^n}{n!}$$

$$(C) \sum_{n=2}^{\infty} \frac{n!}{n^n}$$

$$(B) \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2 3^{n+1}}$$

$$(D) \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

## Challenge

We want to calculate the value of

$$A = \sum_{n=0}^{\infty} \frac{1}{2^n}, \quad B = \sum_{n=1}^{\infty} \frac{n}{2^n}, \quad C = \sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

Let  $f(x) = \frac{1}{1-x}$ .

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- (A) Recall that  $f(x) = \sum_{n=0}^{\infty} x^n$  for  $|x| < 1$ . Use it to compute  $A$ .
- (B) Pretend you can take derivatives of series the way you take them of finite sums. Write  $f'(x)$  as a series.
- (C) Use it to compute  $B$ .
- (D) Do something similar to compute  $C$ .