Warmup:

What are the following sets?
(A) $[2, 4] \cup (2, 5)$
(B) $[2, 4] \cap (2, 5)$
(C) $[\pi, e]$
(D) $[0, 0]$
(E) $(0, 0)$

Before next class:
- **Watch videos 1.4, 1.5, 1.6**

What are the following sets?

Write explicitly or with interval notation

(A) $A = \{x \in \mathbb{Z} : x^2 < 6\}$

(B) $B = \{x \in \mathbb{N} : x^2 < 6\}$

(C) $C = \{x \in \mathbb{R} : x^2 < 6\}$

What are the following sets?

(A) $A = \{x \in \mathbb{R} : \forall y \in [0, 1], \ x < y\}$

(B) $B = \{x \in \mathbb{R} : \exists y \in [0, 1] \text{ s.t. } x < y\}$

(C) $C = \{x \in [0, 1] : \forall y \in [0, 1], x < y\}$

(D) $D = \{x \in [0, 1] : \exists y \in [0, 1] \text{ s.t. } x < y\}$

(E) $E = \{x \in [0, 1] : \exists y \in \mathbb{R} \text{ s.t. } x < y\}$

(F) $F = \{x \in [0, 1] : y \in \mathbb{R}, \ x < y\}$

An irrational number is a number that is real but not rational.

B is the set of positive, rational numbers and negative, irrational numbers.

Write a definition for B using only mathematical notation.
(You may use the words "and", "or", and "such that".)

Homework review:

If possible, write the following sets with interval notation

(A) $A = \{x \in \mathbb{R} : \forall y \in [0,1], x < y\}$

(B) $B = \{x \in \mathbb{R} : \exists y \in [0,1] \text{ s.t. } x < y\}$

(C) $C = \{x \in [0,1] : \forall y \in [0,1], x < y\}$

(D) $D = \{x \in [0,1] : \exists y \in [0,1] \text{ s.t. } x < y\}$

(E) $E = \{x \in [0,1] : \exists y \in \mathbb{R} \text{ s.t. } x < y\}$

(F) $F = \{x \in [0,1] : y \in \mathbb{R}, x < y\}$

Before next class:
- **Watch videos 1.7, 1.8, 1.9**

Let
$$H = \{ \text{ humans } \}$$

Which statements are True/False?

(A) $\forall x \in H$, $\exists y \in H$ such that y gave birth to x

(B) $\exists x \in H$, such that $\forall y \in H$, y gave birth to x

Which of these is a correct description of the set E of even integers?

(A) $E = \{n \in \mathbb{Z} : \forall a \in \mathbb{Z}, n = 2a\}$

(B) $E = \{n \in \mathbb{Z} : \exists a \in \mathbb{Z} \text{ s.t. } n = 2a\}$

Write the negation of these statements as simply as possible:

(A) My favourite integer number is greater than 7.

(B) I know at least five students at U of T who have a cellphone.

(C) There is a country in the European Union with fewer than 1000 inhabitants.

(D) All of my friends like apples.

(E) I like apples and oranges.

Negation of $\boxed{\cdots}$ $=$ $\boxed{\cdots}$ is false.

Functions and quantifiers

Let f be a function with domain $\mathbb{R}$. Rewrite the following statements using $\forall$ or $\exists$:

(A) The graph of f intercepts the x-axis.

(B) f is the zero function.

(C) f is not the zero function.

(D) f never vanishes.

(E) The equation $f(x) = 0$ has a solution.

(F) The equation $f(x) = 0$ has no solutions.

(G) f takes both positive and negative values.

(H) f is never negative.

Before next class:
- **Watch videos 1.10, 1.11, 1.12, 1.13**

Let $x \in \mathbb{R}$.
(A) $x > 0 \implies x \geq 0$
(B) $x \geq 0 \implies x > 0$

Let $x \in \mathbb{R}$.

(A) $x > 0 \implies x \geq 0$

(B) $x \geq 0 \implies x > 0$

(C) IF $2 > 3$ THEN Jason is in love.

Rewrite the following statement as an "IF, …THEN" statement and as an implication (using $\implies$).

- **A teacher promises to give a cookie to any student who gets 100 on the test.**

Rewrite the following statement as an "IF, …THEN" statement and as an implication (using $\implies$).

- **A teacher promises to give a cookie to any student who gets 100 on the test.**

- No student gets 100.
- The teacher doesn't give out any cookies.

- Did the teacher lie?

Which of the following statements are equivalent to the statement *"Every Canadian man likes hockey"*?

(A) If a man is Canadian, then he likes hockey.

(B) If a man likes hockey, then he is Canadian.

(C) If a man does not like hockey, then he is not Canadian.

(D) If a man is not Canadian, then he likes hockey.

(E) Non-Canadian men do not like hockey.

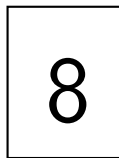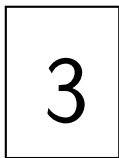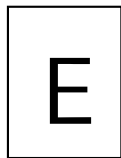(F) If a Canadian does not like hockey, then they are not a man.

Write the negation of these statements:

(A) If Justin Trudeau has a brother, then he also has a sister.

(B) If a student in this class has a brother, then they also have a sister.

Cards

Take a look at the following cards.

| E | 3 | P | 8 |

Each card has a letter on one side and a number on the other, and I tell you:

> "**If** a card has a vowel on one side,
> **then** it has an odd number on the other side."

Which cards do you need to turn over in order to verify whether I am telling the truth or not?

Four cards lie on the table in front of you. You know that each card has a letter on one side and a number on the other.

Negate the following statement:

"**If** a card has a vowel on one side,
then it has an odd number on the other side."

Write the negation of this statement without using any negative words ("no", "not", "none", etc.):

"Every page in this book contains at least one word whose first and last letters both come alphabetically before M."

Negate the following statement without using any
negative words ("no", "not", "none", etc.):

> *"I have a friend all of whose former boyfriends
> had at least two siblings with exactly three dif-
> ferent vowels in their name."*

Warmup:
The law says "**if you check out a library book, you must return it**".

- I did not check out any library books.
- I did not return any library books.

Did I violate the law?

Before next class:
- **Watch videos 1.14, 1.15**

True or False?

(A) There is a pink elephant in this room.

(B) All elephants in this room are pink.

Let f be a function with domain D.

f is *one-to-one* means that ...
- ... different inputs (x) ...
- ... must produce different outputs ($f(x)$).

Write a formal definition of "one-to-one".

Definition: Let f be a function with domain D.
f is one-to-one means …

(A) $f(x_1) \neq f(x_2)$

(B) $\exists x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$

(C) $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$

(D) $\forall x_1, x_2 \in D,\ x_1 \neq x_2,\ f(x_1) \neq f(x_2)$

(E) $\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$

(F) $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2) \implies x_1 \neq x_2$

(G) $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

Let f be a function with domain D.

What does each of the following mean?

(A) $f(x_1) \neq f(x_2)$

(B) $\exists x_1, x_2 \in D, \ f(x_1) \neq f(x_2)$

(C) $\forall x_1, x_2 \in D, \ f(x_1) \neq f(x_2)$

(D) $\forall x_1, x_2 \in D, \ x_1 \neq x_2, \ f(x_1) \neq f(x_2)$

(E) $\forall x_1, x_2 \in D, \ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$

(F) $\forall x_1, x_2 \in D, \ f(x_1) \neq f(x_2) \implies x_1 \neq x_2$

(G) $\forall x_1, x_2 \in D, \ f(x_1) = f(x_2) \implies x_1 = x_2$

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $\forall x_1, x_2 \in D, \ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
- OR, equivalently, $\forall x_1, x_2 \in D, \ f(x_1) = f(x_2) \implies x_1 = x_2$

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $\qquad \forall x_1, x_2 \in D, \ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
- OR, equivalently, $\forall x_1, x_2 \in D, \ f(x_1) = f(x_2) \implies x_1 = x_2$

Suppose I give you a specific function f and I ask you to prove it is one-to-one.

Proving a function is one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $\forall x_1, x_2 \in D, \ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
- OR, equivalently, $\forall x_1, x_2 \in D, \ f(x_1) = f(x_2) \implies x_1 = x_2$

Suppose I give you a specific function f and I ask you to prove it is one-to-one.

- **For the first definition**, write the structure of your proof (How do you begin? What do you assume? What do you conclude?).
- **For the second definition**, write the structure of your proof if you use the second definition.

Proving a function is one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $\forall x_1, x_2 \in D, \; x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
- OR, equivalently, $\forall x_1, x_2 \in D, \; f(x_1) = f(x_2) \implies x_1 = x_2$

Suppose I give you a specific function f and I ask you to prove it is one-to-one.

- **For the first definition**, write the structure of your proof (How do you begin? What do you assume? What do you conclude?).
- **For the second definition**, write the structure of your proof if you use the second definition.

Exercise

Prove that $f(x) = 3x + 2$, with domain $\mathbb{R}$, is one-to-one.

Warmup:

For each of the following statements, list **all** conditions under which it is false.

- If a student gets a 100 on the test, I will give them a cookie.
- If I check out a library book, I will return it.
- If there is an elephant in this room, it is pink.

Before next class:

- **Watch videos 2.4**

A teacher promises: **If a student gets a 100 on the test, then I will give them a cookie.**

- No student gets 100.
- The teacher doesn't give out any cookies.

Did the teacher lie?

The law says, "**If you check out a library book, you must return it**".

- I did not check out any library books.
- I did not return any library books.

Did I violate the law?

Consider the statement "**if this room contains an elephant, that elephant is pink**".

The facts are:

- This room contains no elephants.
- If there were an elephant in this room, it would come from the zoo and would be a grey elephant.

Is the statement true or false?

Consider the statements

(A) If I have a favorite number, then it is an even number.

(B) If I have a favorite number, then it is an odd number.

Can both statements be true at the same time?

Proving a function is NOT one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $\qquad\qquad\qquad \forall x_1, x_2 \in D, \; x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
- OR, equivalently, $\forall x_1, x_2 \in D, \; f(x_1) = f(x_2) \implies x_1 = x_2$

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $$\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$
- OR, equivalently, $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

Suppose I give you a specific function f and I ask you to prove it is **not** one-to-one.

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $$\forall x_1, x_2 \in D, \ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$
- OR, equivalently, $\forall x_1, x_2 \in D, \ f(x_1) = f(x_2) \implies x_1 = x_2$

Suppose I give you a specific function f and I ask you to prove it is **not** one-to-one. You need to prove f satisfies the *negation* of the definition.

- Write the negation of the first definition.
- Write the negation of the second definition.
- Write the structure of your proof.

Proving a function is NOT one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $\forall x_1, x_2 \in D, \ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
- OR, equivalently, $\forall x_1, x_2 \in D, \ f(x_1) = f(x_2) \implies x_1 = x_2$

Suppose I give you a specific function f and I ask you to prove it is **not** one-to-one. You need to prove f satisfies the *negation* of the definition.

- Write the negation of the first definition.
- Write the negation of the second definition.
- Write the structure of your proof.

Exercise

Prove that $f(x) = x^2$, with domain $\mathbb{R}$, is not one-to-one.

Variations on induction

Let S_n be a statement depending on a positive integer n.

What conclusions can you draw in each of the following cases? (I.e., for which n do you know that S_n is true?)

(A) We have proven:
- S_3
-
  $\forall n \geq 1,\ S_n \implies S_{n+1}$

(B) We have proven:
- S_1
-
  $\forall n \geq 3,\ S_n \implies S_{n+1}$

(C) We have proven:
- S_1
-
  $\forall n \geq 1,\ S_n \implies S_{n+3}$

(D) We have proven:
- S_1
-
  $\forall n \geq 1,\ S_{n+1} \implies S_n$

We want to prove
$$\forall n \geq 1, \ S_n$$

So far we have proven

- S_1
- $\forall n \geq 1, \ S_n \implies S_{n+3}.$

What else do we need to do?

What is wrong with this proof by induction?

Theorem

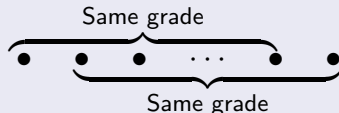$\forall N \geq 1$, *every set of N students in MAT137 will get the same grade.*

What is wrong with this proof by induction?

Theorem

$\forall N \geq 1$, every set of N students in MAT137 will get the same grade.

Proof.

- **Base case.** It is clearly true for $N = 1$.
- **Induction step.**
  Assume it is true for N. I'll show it is true for $N + 1$.
  Take a set of $N + 1$ students. By induction hypothesis:
  - The first N students get the same grade.
  - The last N students get the same grade.

Same grade

Same grade

  Hence the $N + 1$ students all get the same grade.

$\square$

For every $N \geq 1$, let

$$S_N = \text{``every set of } N \text{ students in MAT137}$$
$$\text{will get the same grade''}$$

What did we actually prove in the previous page?

- S_1 ?
- $\forall N \geq 1, \quad S_N \implies S_{N+1}$?

Theorem

The sum of two odd numbers is even.

Proof.

3 is odd.
5 is odd.
$3 + 5 = 8$ is even. $\square$

Theorem

The sum of two odd numbers is even.

Proof.

The sum of two odd numbers is always even.

even + even = even

even + odd = odd

odd + even = odd

odd + odd = even. □