

Warmup:

You want to show “ $\exists n \in \mathbb{N}$ s.t. $n^2 = 4$ ”. Which proofs below are correct/incorrect?

(A) Let $n = 2$. Then,
 $n \in \mathbb{N}$ and $n^2 = 4$.

(B) Take $n \in \mathbb{N}$. Let
 $n = 2$. Then $n^2 = 4$.

(C) Let $n \in \mathbb{N}$. Assume
 $n = 2$. Then $n^2 = 4$.

(D) Take $n = 2$. Then
 $n \in \mathbb{N}$ and $n^2 = 4$.

Before next class:

- Watch videos 2.1, 2.2, 2.3

Variations on induction

Let S_n be a statement depending on a positive integer n .

What conclusions can you draw in each of the following cases? (I.e., for which n do you know that S_n is true?)

(A) We have proven:

- S_3



$$\forall n \geq 1, S_n \implies S_{n+1}$$

(B) We have proven:

- S_1



$$\forall n \geq 3, S_n \implies S_{n+1}$$

(C) We have proven:

- S_1



$$\forall n \geq 1, S_n \implies S_{n+3}$$

(D) We have proven:

- S_1



$$\forall n \geq 1, S_{n+1} \implies S_n$$

Variations on induction 2

We want to prove

$$\forall n \geq 1, S_n$$

So far we have proven

- S_1
- $\forall n \geq 1, S_n \implies S_{n+3}.$

What else do we need to do?

What is wrong with this proof by induction?

Theorem

$\forall N \geq 1$, every set of N students in MAT137 will get the same grade.

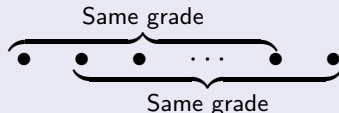
What is wrong with this proof by induction?

Theorem

$\forall N \geq 1$, every set of N students in MAT137 will get the same grade.

Proof.

- **Base case.** It is clearly true for $N = 1$.
- **Induction step.**
Assume it is true for N . I'll show it is true for $N + 1$.
Take a set of $N + 1$ students. By induction hypothesis:
 - The first N students get the same grade.
 - The last N students get the same grade.



Hence the $N + 1$ students all get the same grade.



What is wrong with this proof by induction?

For every $N \geq 1$, let

$S_N =$ “every set of N students in MAT137
will get the same grade”

What did we actually prove in the previous page?

- S_1 ?
- $\forall N \geq 1, S_N \implies S_{N+1}$?

Properties of absolute value

Let $a, b \in \mathbb{R}$. Are the following conclusions correct?

(A) $|ab| = |a||b|$

(B) $|a + b| = |a| + |b|$

If any of the conclusions is wrong, fix it.

Properties of inequalities

Let $a, b, c \in \mathbb{R}$.

Assume $a < b$. Are the following conclusions correct?

(A) $a + c < b + c$

(D) $a^2 < b^2$

(B) $a - c < b - c$

(E) $\frac{1}{a} < \frac{1}{b}$

(C) $ac < bc$

(F) $\sin a < \sin b$

If any of the conclusions is wrong, fix it.

Sets described by distance

Let $a \in \mathbb{R}$. Let $\delta > 0$.

Describe the following sets using interval notation.

(A) $A = \{x \in \mathbb{R} : |x| < \delta\}$

(B) $B = \{x \in \mathbb{R} : |x| > \delta\}$

(C) $C = \{x \in \mathbb{R} : |x - a| < \delta\}$

(D) $D = \{x \in \mathbb{R} : 0 < |x - a| < \delta\}$

Implications

Find *all* positive values of X , Y , and Z which make the following implications true.

$$(A) \quad |t - 3| < 1 \implies |2t - 6| < X$$

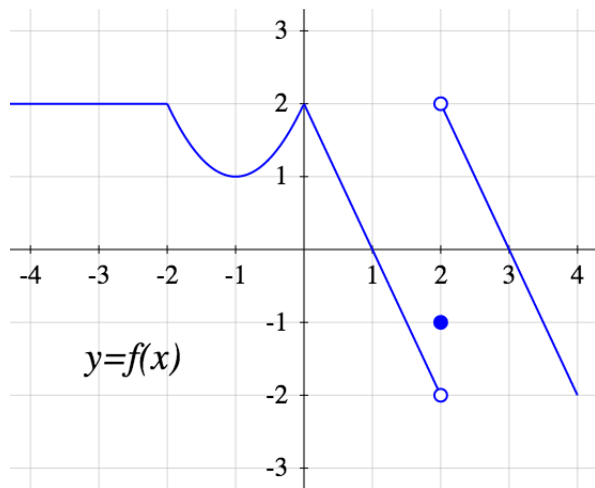
$$(B) \quad |t - 3| < Y \implies |2t - 6| < 1$$

$$(C) \quad |t - 3| < 1 \implies |t + 5| < Z$$

Before next class:

- **Watch videos 2.5, 2.6**

Limits from a graph



Find the value of

(A) $\lim_{x \rightarrow 2} f(x)$

(B) $\lim_{x \rightarrow 0} f(f(x))$

Floor

Given a real number x , we defined the *floor of x* , denoted by $\lfloor x \rfloor$, as the largest integer smaller than or equal to x . For example:

$$\lfloor \pi \rfloor = 3, \quad \lfloor 7 \rfloor = 7, \quad \lfloor -0.5 \rfloor = -1.$$

Sketch the graph of $y = \lfloor x \rfloor$. Then compute:

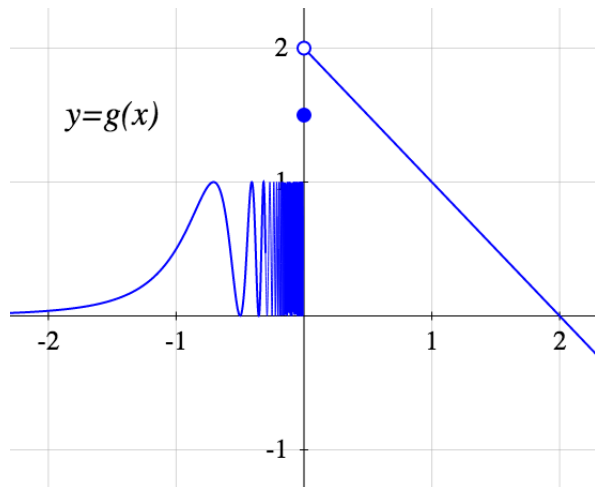
(A) $\lim_{x \rightarrow 0^+} \lfloor x \rfloor$

(C) $\lim_{x \rightarrow 0} \lfloor x \rfloor$

(B) $\lim_{x \rightarrow 0^-} \lfloor x \rfloor$

(D) $\lim_{x \rightarrow 0} \lfloor x^2 \rfloor$

More limits from a graph



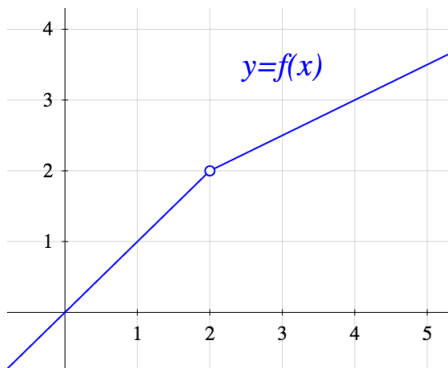
Find the value of

- (A) $\lim_{x \rightarrow 0^+} g(x)$
- (B) $\lim_{x \rightarrow 0^+} \lfloor g(x) \rfloor$
- (C) $\lim_{x \rightarrow 0^+} g(\lfloor x \rfloor)$
- (D) $\lim_{x \rightarrow 0^-} g(x)$
- (E) $\lim_{x \rightarrow 0^-} \lfloor g(x) \rfloor$
- (F) $\lim_{x \rightarrow 0^-} \lfloor \frac{g(x)}{2} \rfloor$
- (G) $\lim_{x \rightarrow 0^-} g(\lfloor x \rfloor)$

Before next class:

- **Watch videos 2.7, 2.8, 2.9**

δ from a graph



- (A) Find one value of $\delta > 0$ s.t. $0 < |x - 2| < \delta \implies |f(x) - 2| < 0.5$
- (B) Find *all* values of $\delta > 0$ s.t. $0 < |x - 2| < \delta \implies |f(x) - 2| < 0.5$

Write down the formal definition of

$$\lim_{x \rightarrow a} f(x) = L.$$

Side limits

Recall

Let $L, a \in \mathbb{R}$.

Let f be a function defined at least on an interval around a , except possibly at a .

$$\lim_{x \rightarrow a} f(x) = L$$

means

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

Write, instead, the formal definition of

$$\lim_{x \rightarrow a^+} f(x) = L, \quad \text{and} \quad \lim_{x \rightarrow a^-} f(x) = L.$$

Definition

Let $a \in \mathbb{R}$.

Let f be a function defined at least on an interval around a , except possibly at a .

Write a formal definition for

$$\lim_{x \rightarrow a} f(x) = \infty.$$

Which one(s) is the definition of $\lim_{x \rightarrow a} f(x) = \infty$?

(A) $\forall M \in \mathbb{R}, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$

(B) $\forall M \in \mathbb{Z}, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$

(C) $\forall M > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$

(D) $\forall M > 5, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$

Limit at a point

If a function f is not defined at $x = a$, then

- (A) $\lim_{x \rightarrow a} f(x)$ cannot exist
- (B) $\lim_{x \rightarrow a} f(x)$ could be 0
- (C) $\lim_{x \rightarrow a} f(x)$ must approach ∞
- (D) none of the above.

Evaluating Limits

- You're trying to guess $\lim_{x \rightarrow 0} f(x)$.
- You plug in $x = 0.1, 0.01, 0.001, \dots$ and get $f(x) = 0$ for all these values.
- In fact, you're told that for all $n = 1, 2, \dots$,
$$f\left(\frac{1}{10^n}\right) = 0.$$
- Can you conclude that $\lim_{x \rightarrow 0} f(x) = 0$?

Exponential limits

Compute:

$$\lim_{t \rightarrow 0^+} e^{1/t}, \quad \lim_{t \rightarrow 0^-} e^{1/t}.$$

Suggestion: Sketch the graph of $y = e^x$ first.

Consider the function

$$h(x) = \frac{(x-1)(2+x)}{x^2(x-1)(2-x)}.$$

- Find all real values a for which $h(a)$ is undefined.
- For each such value of a , compute $\lim_{x \rightarrow a^+} h(x)$ and $\lim_{x \rightarrow a^-} h(x)$.
- Based on your answer, and nothing else, try to sketch the graph of h .

Related implications

Let $a \in \mathbb{R}$. Let f be a function. Assume we know

$$0 < |x - a| < 0.1 \quad \implies \quad f(x) > 100$$

(A) Which values of $M \in \mathbb{R}$ satisfy ... ?

$$0 < |x - a| < 0.1 \quad \implies \quad f(x) > M$$

Related implications

Let $a \in \mathbb{R}$. Let f be a function. Assume we know

$$0 < |x - a| < 0.1 \quad \implies \quad f(x) > 100$$

(A) Which values of $M \in \mathbb{R}$ satisfy ... ?

$$0 < |x - a| < 0.1 \quad \implies \quad f(x) > M$$

(B) Which values of $\delta > 0$ satisfy ... ?

$$0 < |x - a| < \delta \quad \implies \quad f(x) > 100$$

Strict or non-strict inequality?

Let f be a function with domain \mathbb{R} . One of these statements implies the other. Which one?

(A) $\forall M \in \mathbb{R}, \exists N \in \mathbb{R} \text{ s.t. } x > N \implies f(x) > M$

(B) $\forall M \in \mathbb{R}, \exists N \in \mathbb{R} \text{ s.t. } x > N \implies f(x) \geq M$

Negation of conditionals

Write the negation of these statements:

- (A) If Justin Trudeau has a brother, then he also has a sister.
- (B) If a student in this class has a brother, then they also have a sister.

Let f be a function with domain \mathbb{R} . Write the negation of the statement:

$$\text{IF } 2 < x < 4, \quad \text{THEN } 1 < f(x) < 3.$$

Write down the formal definition of the following statements:

(A) $\lim_{x \rightarrow a} f(x) = L$

(B) $\lim_{x \rightarrow a} f(x)$ exists

(C) $\lim_{x \rightarrow a} f(x)$ does not exist

Preparation: choosing deltas

(A) Find one value of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 1.$$

(B) Find *all* values of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 1.$$

(C) Find *all* values of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 0.1.$$

(D) Let us fix $\varepsilon > 0$. Find *all* values of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < \varepsilon.$$

What is wrong with this “proof”?

Prove that

$$\lim_{x \rightarrow 3} (5x + 1) = 16$$

“Proof:”

Let $\varepsilon > 0$.

WTS $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$0 < |x - 3| < \delta \implies |(5x + 1) - (16)| < \varepsilon$$

$$|(5x + 1) - (16)| < \varepsilon \iff |5x + 15| < \varepsilon$$

$$\iff 5|x + 3| < \varepsilon \implies \delta = \frac{\varepsilon}{3}$$



Your first $\varepsilon - \delta$ proof

Goal

We want to prove that

$$\lim_{x \rightarrow 3} (5x + 1) = 16 \quad (1)$$

directly from the definition.

Your first $\varepsilon - \delta$ proof

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We want to prove that

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- (A) Write down the formal definition of the statement (1).

Your first $\varepsilon - \delta$ proof

Goal

We want to prove that

$$\lim_{x \rightarrow 3} (5x + 1) = 16 \quad (1)$$

directly from the definition.

- (A) Write down the formal definition of the statement (1).
- (B) Write down what the structure of the formal proof should be, without filling the details.

Your first $\varepsilon - \delta$ proof

Goal

We want to prove that

$$\lim_{x \rightarrow 3} (5x + 1) = 16 \quad (1)$$

directly from the definition.

- (A) Write down the formal definition of the statement (1).
- (B) Write down what the structure of the formal proof should be, without filling the details.
- (C) Write down a complete formal proof.

A harder proof

Goal

We want to prove that

$$\lim_{x \rightarrow 0} (x^3 + x^2) = 0 \quad (2)$$

directly from the definition.

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We want to prove that

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- (A) Write down the formal definition of the statement (2).

A harder proof

Goal

We want to prove that

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directly from the definition.

- (A) Write down the formal definition of the statement (2).
- (B) Write down what the structure of the formal proof should be, without filling the details.

A harder proof

Goal

We want to prove that

$$\lim_{x \rightarrow 0} (x^3 + x^2) = 0 \quad (2)$$

directly from the definition.

- (A) Write down the formal definition of the statement (2).
- (B) Write down what the structure of the formal proof should be, without filling the details.
- (C) Rough work: What is δ ?

A harder proof

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We want to prove that

$$\lim_{x \rightarrow 0} (x^3 + x^2) = 0 \quad (2)$$

directly from the definition.

- (A) Write down the formal definition of the statement (2).
- (B) Write down what the structure of the formal proof should be, without filling the details.
- (C) Rough work: What is δ ?
- (D) Write down a complete formal proof.

Is this proof correct?

Claim:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x| < \delta \implies |x^3 + x^2| < \varepsilon.$$

Proof:

- Let $\varepsilon > 0$.
- Take $\delta = \sqrt{\frac{\varepsilon}{|x+1|}}$.
- Let $x \in \mathbb{R}$. Assume $0 < |x| < \delta$. Then

$$|x^3 + x^2| = x^2|x+1| < \delta^2|x+1| = \frac{\varepsilon}{|x+1|}|x+1| = \varepsilon.$$

- I have proven that $|x^3 + x^2| < \varepsilon$. □

Choosing deltas again

Let us fix numbers $A, \varepsilon > 0$. Find:

(A) a value of $\delta > 0$ s.t. $|x| < \delta \implies |Ax^2| < \varepsilon$

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- (C) a value of $\delta > 0$ s.t. $|x| < \delta \implies |x + 1| < 10$

Choosing deltas again

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(D) *all* values of $\delta > 0$ s.t. $|x| < \delta \implies |x + 1| < 10$

(E) a value of $\delta > 0$ s.t. $|x| < \delta \implies \begin{cases} |Ax^2| < \varepsilon \\ |x + 1| < 10 \end{cases}$

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(E) a value of $\delta > 0$ s.t. $|x| < \delta \implies \begin{cases} |Ax^2| < \varepsilon \\ |x + 1| < 10 \end{cases}$

(F) a value of $\delta > 0$ s.t. $|x| < \delta \implies |(x + 1)x^2| < \varepsilon$

Indeterminate form

Let $a \in \mathbb{R}$. Let f and g be positive functions defined near a , except maybe at a .

Assume $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$.

What can we conclude about $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$?

- (A) The limit is 1. exist.
- (B) The limit is 0.
- (C) The limit is ∞ .
- (D) The limit does not
- (E) We do not have enough information to decide.

A theorem about limits

Let f be a function with domain \mathbb{R} such that

$$\lim_{x \rightarrow 0} f(x) = 3$$

Prove that

$$\lim_{x \rightarrow 0} [5f(2x)] = 15$$

directly from the definition of limit. Do not use any of the limit laws.

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- (A) Write down the formal definition of the statement you want to prove.

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- (C) Rough work.

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- (A) Write down the formal definition of the statement you want to prove.
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- (C) Rough work.
- (D) Write down a complete proof.

Proof feedback

- (A) Is the structure of the proof correct?
(First fix ε , then choose δ , then ...)
- (B) Did you say exactly what δ is?
- (C) Is the proof self-contained?
(I do not need to read the rough work)
- (D) Are all variables defined? In the right order?
- (E) Do all steps follow logically from what comes before?
Do you start from what you know and prove what you have to prove?
- (F) Are you proving your conclusion or assuming it?

A new squeeze

This is the Squeeze Theorem, as you know it:

The (classical) Squeeze Theorem

Let $a, L \in \mathbb{R}$.

Let f , g , and h be functions defined near a , except possibly at a .

- IF
- For x close to a but not a , $h(x) \leq g(x) \leq f(x)$
 - $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} h(x) = L$

THEN

- $\lim_{x \rightarrow a} g(x) = L$

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THEN

- $\lim_{x \rightarrow a} g(x) = L$

Come up with a new version of the theorem about limits being infinity. (The conclusion should be $\lim_{x \rightarrow a} g(x) = \infty$.)

Hint: Draw a picture for the classical Squeeze Theorem. Then draw a picture for the new theorem.

A new squeeze

The (new) Squeeze Theorem

Let $a \in \mathbb{R}$.

Let g and h be functions defined near a , except possibly at a .

- IF
- For x close to a but not a , $h(x) \leq g(x)$
 - $\lim_{x \rightarrow a} h(x) = \infty$

- THEN
- $\lim_{x \rightarrow a} g(x) = \infty$

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(A) Replace the first hypothesis with a more precise mathematical statement.

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THEN • $\lim_{x \rightarrow a} g(x) = \infty$

- (A) Replace the first hypothesis with a more precise mathematical statement.
- (B) Write down the definition of what you want to prove.

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- (A) Replace the first hypothesis with a more precise mathematical statement.
- (B) Write down the definition of what you want to prove.
- (C) Write down the structure of the formal proof.

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- (D) Rough work

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- (A) Replace the first hypothesis with a more precise mathematical statement.
- (B) Write down the definition of what you want to prove.
- (C) Write down the structure of the formal proof.
- (D) Rough work
- (E) Write down a complete, formal proof.

True or False?

Is this theorem true?

Claim

Let $a \in \mathbb{R}$.

Let f and g be functions defined near a .

- IF $\lim_{x \rightarrow a} f(x) = 0$,
- THEN $\lim_{x \rightarrow a} [f(x)g(x)] = 0$.

A new theorem about products

Theorem

Let $a \in \mathbb{R}$. Let f and g be functions with domain \mathbb{R} , except possibly a . Assume

- $\lim_{x \rightarrow a} f(x) = 0$, and
- g is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN $\lim_{x \rightarrow a} [f(x)g(x)] = 0$

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- $\lim_{x \rightarrow a} f(x) = 0$, and
- g is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN $\lim_{x \rightarrow a} [f(x)g(x)] = 0$

- (A) Write down the formal definition of what you want to prove.
- (B) Write down what the structure of the formal proof.

A new theorem about products

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- (A) Write down the formal definition of what you want to prove.
- (B) Write down what the structure of the formal proof.
- (C) Rough work.
- (D) Write down a complete formal proof.

Critique this “proof” – #1

- WTS $\lim_{x \rightarrow a} [f(x)g(x)] = 0$:
 $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon$.
- We know $\lim_{x \rightarrow a} f(x) = 0$
 $\forall \varepsilon_1 > 0, \exists \delta_1 > 0$ s.t. $0 < |x - a| < \delta_1 \implies |f(x)| < \varepsilon_1$.
- We know $\exists M > 0$ s.t. $\forall x \neq 0, |g(x)| \leq M$.
- $|f(x)g(x)| = |f(x)||g(x)| < \varepsilon_1 M$
- $\varepsilon = \varepsilon_1 M \implies \varepsilon_1 = \frac{\varepsilon}{M}$
- Take $\delta = \delta_1$

Critique this “proof” – #2

- WTS $\lim_{x \rightarrow a} [f(x)g(x)] = 0$. By definition, WTS:
$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon$$
- Let $\varepsilon > 0$.
- Use the value $\frac{\varepsilon}{M}$ as “epsilon” in the definition of
$$\lim_{x \rightarrow a} f(x) = 0$$

$$\exists \delta_1 \in \mathbb{R} \text{ s.t. } 0 < |x - a| < \delta_1 \implies |f(x)| < \frac{\varepsilon}{M}.$$
- Take $\delta = \delta_1$.
- Let $x \in \mathbb{R}$. Assume $0 < |x - a| < \delta$
- Since $\exists M > 0$ s.t. $\forall x \neq 0, |g(x)| \leq M$
$$|f(x)g(x)| < \frac{\varepsilon}{M} \cdot M = \varepsilon.$$

Critique this “proof” – #3

- Since g is bounded, $\exists M > 0$ s.t. $\forall x \neq 0, |g(x)| \leq M$
- Since $\lim_{x \rightarrow a} f(x) = 0$, there exists $\delta_1 > 0$ s.t.
if $0 < |x - a| < \delta_1$, then $|f(x) - 0| = |f(x)| < \varepsilon_1 = \frac{\varepsilon}{M}$.
- $|f(x)g(x)| = |f(x)| \cdot |g(x)| \leq |f(x)| \cdot M < \varepsilon_1 \cdot M = \frac{\varepsilon}{M} \cdot M = \varepsilon$
- In summary, by setting $\delta = \min\{\delta_1\}$, we find that
if $0 < |x - a| < \delta$, then $|f(x) \cdot g(x)| < \varepsilon$.

Limits involving $\sin(1/x)$ Part I

The reason that $\lim_{x \rightarrow 0} \sin(1/x)$ does not exist is:

- (A) because the function values oscillate around 0
- (B) because $1/0$ is undefined
- (C) because no matter how close x gets to 0, there are x 's near 0 for which $\sin(1/x) = 1$, and some for which $\sin(1/x) = -1$
- (D) all of the above

Limits involving $\sin(1/x)$ Part II

The limit $\lim_{x \rightarrow 0} x^2 \sin(1/x)$

- (A) does not exist because the function values oscillate around 0
- (B) does not exist because $1/0$ is undefined
- (C) does not exist because no matter how close x gets to 0, there are x 's near 0 for which $\sin(1/x) = 1$, and some for which $\sin(1/x) = -1$
- (D) equals 0
- (E) equals 1

Absolute value and the Squeeze Theorem

Use the Squeeze Theorem to prove:

Theorem

IF $\lim_{x \rightarrow a} |f(x)| = 0$, THEN $\lim_{x \rightarrow a} f(x) = 0$.

Hint: Recall that $-|c| \leq c \leq |c|$ for every $c \in \mathbb{R}$.

Undefined function

Let $a \in \mathbb{R}$ and let f be a function. Assume $f(a)$ is undefined.

What can we conclude?

- (A) $\lim_{x \rightarrow a} f(x)$ exist
- (B) $\lim_{x \rightarrow a} f(x)$ doesn't exist.
- (C) No conclusion. $\lim_{x \rightarrow a} f(x)$ may or may not exist.

What else can we conclude?

- (D) f is continuous at a .
- (E) f is not continuous at a .
- (F) No conclusion. f may or may not be continuous at a .

A new function

- Let $x, y \in \mathbb{R}$. What does the following expression calculate? Prove it.

$$f(x, y) = \frac{x + y + |x - y|}{2}$$

Suggestion: If you don't know how to start, try some sample values of x and y .

A new function

- Let $x, y \in \mathbb{R}$. What does the following expression calculate? Prove it.

$$f(x, y) = \frac{x + y + |x - y|}{2}$$

Suggestion: If you don't know how to start, try some sample values of x and y .

- Write a similar expression to compute $\min\{x, y\}$.

More continuous functions

We want to prove the following theorem

Theorem

IF f and g are continuous functions
THEN $h(x) = \max\{f(x), g(x)\}$ is also a continuous function.

You are allowed to use all results that we already know.
What is the fastest way to prove this?

Hint: There is a way to prove this quickly without writing any epsilons.

True or False? – Discontinuities

- (A) IF f and g have removable discontinuities at a
THEN $f + g$ has a removable discontinuity at a

- (B) IF f and g have non-removable discontinuities at a
THEN $f + g$ has a non-removable discontinuity at a

Which one is the correct claim?

Claim 1?

(Assuming these limits exist)

$$\lim_{x \rightarrow a} g(f(x)) = g\left(\lim_{x \rightarrow a} f(x)\right)$$

Claim 2?

IF (A) $\lim_{x \rightarrow a} f(x) = L$, and (B) $\lim_{t \rightarrow L} g(t) = M$
THEN (C) $\lim_{x \rightarrow a} g(f(x)) = M$

A difficult example

Construct a pair of functions f and g such that

$$\lim_{x \rightarrow 0} f(x) = 1$$

$$\lim_{t \rightarrow 1} g(t) = 2$$

$$\lim_{x \rightarrow 0} g(f(x)) = 42$$

Transforming limits

The only thing we know about the function g is that

$$\lim_{x \rightarrow 0} \frac{g(x)}{x^2} = 2.$$

Use it to compute the following limits:

$$(A) \quad \lim_{x \rightarrow 0} \frac{g(x)}{x}$$

$$(B) \quad \lim_{x \rightarrow 0} \frac{g(x)}{x^4}$$

$$(C) \quad \lim_{x \rightarrow 0} \frac{g(3x)}{x^2}$$

Limits at infinity

Compute:

$$\begin{array}{ll} \text{(A)} \quad \lim_{x \rightarrow \infty} (x^7 - 2x^5 + 11) & \text{(D)} \quad \lim_{x \rightarrow \infty} \frac{x^2 + 2x + 3}{3x^2 + 4x + 5} \\ \text{(B)} \quad \lim_{x \rightarrow \infty} \left(x^2 - \sqrt{x^5 + 1} \right) & \\ \text{(C)} \quad \lim_{x \rightarrow \infty} \frac{x^2 + 11}{x + 1} & \text{(E)} \quad \lim_{x \rightarrow \infty} \frac{x^3 + \sqrt{2x^6 + 1}}{2x^3 + \sqrt{x^5 + 1}} \end{array}$$

Trig computations

Using that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, compute the following limits:

$$(A) \lim_{x \rightarrow 2} \frac{\sin x}{x}$$

$$(B) \lim_{x \rightarrow 0} \frac{\sin(5x)}{x}$$

$$(C) \lim_{x \rightarrow 0} \frac{\tan^2(2x^2)}{x^4}$$

$$(D) \lim_{x \rightarrow 0} \frac{\sin e^x}{e^x}$$

$$(E) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$$

$$(F) \lim_{x \rightarrow 0} \frac{\tan^{10}(2x^{20})}{\sin^{200}(3x)}$$

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$$(G) \lim_{x \rightarrow 0} [(\sin x) (\cos(2x)) (\tan(3x)) (\sec(4x)) (\csc(5x)) (\cot(6x))]$$

$$(D) \lim_{x \rightarrow 0} \frac{\sin e^x}{e^x}$$

$$(E) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$$

$$(F) \lim_{x \rightarrow 0} \frac{\tan^{10}(2x^{20})}{\sin^{200}(3x)}$$

Plus or minus infinity?

Compute:

$$(A) \lim_{x \rightarrow -3^+} \frac{x^2 - 9}{3 - 2x - x^2} \quad (B) \lim_{x \rightarrow 1^+} \frac{x^2 - 9}{3 - 2x - x^2}$$

A harder limit

Calculate

$$\lim_{x \rightarrow 2} \frac{[\sqrt{2+x} - 2] [\sqrt{3+x} - 3]}{\sqrt{x-1} - 1}$$

Which solution is right?

Compute $L = \lim_{x \rightarrow -\infty} \left[x - \sqrt{x^2 + x} \right]$.

• Solution 1

$$\begin{aligned} L &= \lim_{x \rightarrow -\infty} \frac{\left[x - \sqrt{x^2 + x} \right] \left[x + \sqrt{x^2 + x} \right]}{\left[x + \sqrt{x^2 + x} \right]} = \lim_{x \rightarrow -\infty} \frac{x^2 - (x^2 + x)}{\left[x + \sqrt{x^2 + x} \right]} \\ &= \lim_{x \rightarrow -\infty} \frac{-x}{x \left[1 + \sqrt{1 + \frac{1}{x}} \right]} = \lim_{x \rightarrow -\infty} \frac{-1}{\left[1 + \sqrt{1 + \frac{1}{x}} \right]} = \frac{-1}{2} \end{aligned}$$

• Solution 2

$$L = \lim_{x \rightarrow -\infty} \left[x - \sqrt{x^2 + x} \right] = (-\infty) - \infty = -\infty$$

Can we conclude this?

- Consider the function $f(x) = \frac{4}{x}$.
- We have $f(-1) = -4 < 0$ and $f(1) = 4 > 0$.
- Use IVT.
Can we conclude $f(c) = 0$ for some $c \in (-1, 1)$?

Existence of solutions

Prove that the equation

$$x^4 - 2x = 100$$

has at least two solutions.

Can this be proven? (Use IVT)

- (A) Prove that there exists a time of the day when the hour hand and the minute hand of a clock form an angle of exactly 23 degrees.

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- (A) Prove that there exists a time of the day when the hour hand and the minute hand of a clock form an angle of exactly 23 degrees.
- (B) During a Raptors basketball game, at half time the Raptors have 52 points. Prove that at some point they had exactly 26 points.

Can this be proven? (Use IVT)

- (A) Prove that there exists a time of the day when the hour hand and the minute hand of a clock form an angle of exactly 23 degrees.
- (B) During a Raptors basketball game, at half time the Raptors have 52 points. Prove that at some point they had exactly 26 points.
- (C) Prove that at some point during Alfonso's life, his height in centimetres was exactly equal to 10 times his weight in kilograms. Some data:
 - His height at birth: 47 cm
 - His weight at birth: 3.2 kg
 - His height today: 172 cm

Temperature

On June 09, 2016, the outside temperature in Toronto at 6 AM was 10° . At 4 PM, the temperature was 20° .

- (A) Must there have been a time between 6 AM and 4 PM when the temperature was 15° ? Explain how you know. Which assumption about temperature allows you to reach your conclusion?
- (B) Must there have been a time between 6 AM and 4 PM when the temperature was 22° ? Explain how you know.
- (C) Could there have been a time between 6 AM and 4 PM when the temperature was 22° ? Explain how you know.

In each of the following cases, does the function f have a maximum and a minimum on the interval I ?

(A) $f(x) = x^2, \quad I = (-1, 1).$

(B) $f(x) = \frac{(e^x + 2) \sin x}{x} - \cos x + 3, \quad I = [2, 6]$

(C) $f(x) = \frac{(e^x + 2) \sin x}{x} - \cos x + 3, \quad I = (0, 5]$

Definition of maximum

Let f be a function with domain I .

Which one (or ones) of the following is (or are) a definition of “ f has a maximum on I ”?

(A) $\forall x \in I, \exists C \in \mathbb{R} \text{ s.t. } f(x) \leq C$

(B) $\exists C \in I \text{ s.t. } \forall x \in I, f(x) \leq C$

(C) $\exists C \in \mathbb{R} \text{ s.t. } \forall x \in I, f(x) \leq C$

(D) $\exists C \in \mathbb{R} \text{ s.t. } \forall x \in I, f(x) < C$

More on the definition of maximum

Let f be a function with domain I .

What does each of the following mean?

- (A) $\exists C \in \mathbb{R}$ s.t. $\forall x \in I, f(x) \leq C$
- (B) $\exists C \in \mathbb{R}$ s.t. $\forall x \in I, f(x) < C$
- (C) $\exists a \in I$ s.t. $\forall x \in I, f(x) \leq f(a)$
- (D) $\exists a \in I$ s.t. $\forall x \in I, f(x) < f(a)$