

**Before next class:**

- **Watch videos 12.7, 12.8**

## Recall the definitions

- (A) **Type-1 improper integrals.** Let  $f$  be a bounded, continuous function on  $[c, \infty)$ . How do we define the improper integral

$$\int_c^{\infty} f(x) dx ?$$

- (B) **Type-2 improper integrals.** Let  $f$  be a continuous function on  $(a, b]$ , possibly with  $x = a$  as a vertical asymptote. How do we define the improper integral

$$\int_a^b f(x) dx ?$$

## Computation

Calculate, using the definition of improper integral

$$\int_1^{\infty} \frac{1}{x^2 + x} dx$$

*Hint:*  $\frac{1}{x^2 + x} = \frac{(x + 1) - (x)}{x(x + 1)}$

## Computation?

Consider the following argument. We know

$$\frac{1}{x^2 + x} = \frac{(x + 1) - (x)}{x(x + 1)} = \frac{1}{x} - \frac{1}{x + 1}$$

So

$$\int_1^{\infty} \frac{1}{x^2 + x} dx = \int_1^{\infty} \frac{1}{x} dx - \int_1^{\infty} \frac{1}{x + 1} dx$$

Since  $\int_1^{\infty} \frac{1}{x} dx = \int_1^{\infty} \frac{1}{x+1} dx = \infty$ , we see

$$\int_1^{\infty} \frac{1}{x^2 + x} dx = \infty - \infty,$$

and so it doesn't exist.

# The most important improper integrals

Use the definition of improper integral to determine for which values of  $p \in \mathbb{R}$  each of the following improper integrals converges.

$$(A) \int_1^{\infty} \frac{1}{x^p} dx$$

$$(B) \int_0^1 \frac{1}{x^p} dx$$

$$(C) \int_0^{\infty} \frac{1}{x^p} dx$$

## Positive functions

- Let  $f$  be continuous on  $[a, \infty)$ . Let  $A = \int_a^\infty f(x) dx$

Then  $A$  may be  $\left\{ \begin{array}{l} \text{convergent (a number)} \\ \text{divergent} \left\{ \begin{array}{l} \text{to } \infty \\ \text{to } -\infty \\ \text{"oscillating"} \end{array} \right. \end{array} \right.$

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- Assume  $\exists M \geq a$ , s.t.  $\forall x \geq M, f(x) \geq 0$ .

Which of the four options are still possible?



**Before next class:**

- **Watch videos 12.9, 12.10**

## Quick review

For which values of  $p \in \mathbb{R}$  is each of the following improper integrals convergent?

$$(A) \int_1^{\infty} \frac{1}{x^p} dx$$

$$(B) \int_0^1 \frac{1}{x^p} dx$$

$$(C) \int_0^{\infty} \frac{1}{x^p} dx$$

## A simple BCT application

We want to determine whether  $\int_1^{\infty} \frac{1}{x + e^x} dx$  is convergent or divergent.

We can try at least two comparisons:

(A) Compare  $\frac{1}{x}$  and  $\frac{1}{x + e^x}$ .

(B) Compare  $\frac{1}{e^x}$  and  $\frac{1}{x + e^x}$ .

Try both. What can you conclude from each one of them?

# True or False - Comparisons

Let  $a \in \mathbb{R}$ .

Let  $f$  and  $g$  be continuous functions on  $[a, \infty)$ .

Assume that  $\forall x \geq a, \quad 0 \leq f(x) \leq g(x)$ .

Which statements are correct?

(A) IF  $\int_a^\infty f(x)dx$  is convergent, THEN  $\int_a^\infty g(x)dx$  is convergent.

(B) IF  $\int_a^\infty f(x)dx = \infty$ , THEN  $\int_a^\infty g(x)dx = \infty$ .

(C) IF  $\int_a^\infty g(x)dx$  is convergent, THEN  $\int_a^\infty f(x)dx$  is convergent.

(D) IF  $\int_a^\infty g(x)dx = \infty$ , THEN  $\int_a^\infty f(x)dx = \infty$ .

# True or False - Comparisons II

Let  $a \in \mathbb{R}$ .

Let  $f$  and  $g$  be continuous functions on  $[a, \infty)$ .

Assume that  $\boxed{\forall x \geq a, \quad f(x) \leq g(x)}$ .

Which statements are correct?

(A) IF  $\int_a^\infty f(x)dx$  is convergent, THEN  $\int_a^\infty g(x)dx$  is convergent.

(B) IF  $\int_a^\infty f(x)dx = \infty$ , THEN  $\int_a^\infty g(x)dx = \infty$ .

(C) IF  $\int_a^\infty g(x)dx$  is convergent, THEN  $\int_a^\infty f(x)dx$  is convergent.

(D) IF  $\int_a^\infty g(x)dx = \infty$ , THEN  $\int_a^\infty f(x)dx = \infty$ .

# True or False - Comparisons III

Let  $a \in \mathbb{R}$ .

Let  $f$  and  $g$  be continuous functions on  $[a, \infty)$ .

Assume that  $\exists M \geq a$  s.t.  $\forall x \geq M, \quad 0 \leq f(x) \leq g(x)$ .

Which statements are correct?

(A) IF  $\int_a^\infty f(x)dx$  is convergent, THEN  $\int_a^\infty g(x)dx$  is convergent.

(B) IF  $\int_a^\infty f(x)dx = \infty$ , THEN  $\int_a^\infty g(x)dx = \infty$ .

(C) IF  $\int_a^\infty g(x)dx$  is convergent, THEN  $\int_a^\infty f(x)dx$  is convergent.

(D) IF  $\int_a^\infty g(x)dx = \infty$ , THEN  $\int_a^\infty f(x)dx = \infty$ .

# BCT calculations

Use BCT to determine whether each of the following is convergent or divergent

$$(A) \int_1^{\infty} \frac{1 + \cos^2 x}{x^{2/3}} dx$$

$$(B) \int_1^{\infty} \frac{1 + \cos^2 x}{x^{4/3}} dx$$

$$(C) \int_0^{\infty} \frac{\arctan x^2}{1 + e^x} dx$$

## BCT calculations

Use BCT to determine whether each of the following is convergent or divergent

$$(A) \int_1^{\infty} \frac{1 + \cos^2 x}{x^{2/3}} dx$$

$$(D) \int_0^{\infty} e^{-x^2} dx$$

$$(B) \int_1^{\infty} \frac{1 + \cos^2 x}{x^{4/3}} dx$$

$$(E) \int_2^{\infty} \frac{(\ln x)^{10}}{x^2} dx$$

$$(C) \int_0^{\infty} \frac{\arctan x^2}{1 + e^x} dx$$



**Before next class:**

- **Watch videos 13.2, 13.3, 13.4**

## Rapid questions: convergent or divergent?

$$(A) \int_1^{\infty} \frac{1}{x^2} dx \quad (D) \int_0^1 \frac{1}{x^2} dx \quad (G) \int_1^{\infty} \frac{3}{x^2} dx$$

$$(B) \int_1^{\infty} \frac{1}{\sqrt{x}} dx \quad (E) \int_0^1 \frac{1}{\sqrt{x}} dx \quad (H) \int_1^{\infty} \frac{1}{x^2 + 3} dx$$

$$(C) \int_1^{\infty} \frac{1}{x} dx \quad (F) \int_0^1 \frac{1}{x} dx \quad (I) \int_1^{\infty} \left( \frac{1}{x^2} + 3 \right) dx$$

## A “simple” integral

What is  $\int_{-1}^1 \frac{1}{x} dx$  ?

## A “simple” integral

What is  $\int_{-1}^1 \frac{1}{x} dx$  ?

(A)  $\int_{-1}^1 \frac{1}{x} dx = (\ln |x|) \Big|_{-1}^1 = \ln |1| - \ln |-1| = 0$

(B)  $\int_{-1}^1 \frac{1}{x} dx = 0$  because  $f(x) = \frac{1}{x}$  is an odd function.

(C)  $\int_{-1}^1 \frac{1}{x} dx$  is divergent.

## Slow questions: convergent or divergent?

$$(A) \int_1^{\infty} \frac{x^3 + 2x + 7}{x^5 + 11x^4 + 1} dx$$

$$(E) \int_0^1 \frac{\sin x}{x^{3/2}} dx$$

$$(B) \int_1^{\infty} \frac{1}{\sqrt{x^2 + x + 1}} dx$$

$$(F) \int_0^{\infty} e^{-x^2} dx$$

$$(C) \int_0^1 \frac{3 \cos x}{x + \sqrt{x}} dx$$

$$(G) \int_2^{\infty} \frac{(\ln x)^{10}}{x^2} dx$$

$$(D) \int_0^1 \sqrt{\cot x} dx$$

## What is wrong with this computation?

$$\begin{aligned}\int_{-1}^1 \frac{1}{x} dx &= \lim_{\varepsilon \rightarrow 0^+} \left[ \int_{-1}^{-\varepsilon} \frac{1}{x} dx + \int_{\varepsilon}^1 \frac{1}{x} dx \right] \\&= \lim_{\varepsilon \rightarrow 0^+} \left[ \ln |x| \Big|_{-1}^{-\varepsilon} + \ln |x| \Big|_{\varepsilon}^1 \right] \\&= \lim_{\varepsilon \rightarrow 0^+} [\ln |-\varepsilon| - \ln |\varepsilon|] \\&= \lim_{\varepsilon \rightarrow 0^+} [0] = 0\end{aligned}$$