

Initial Value Problem

Find a function f such that

- For every $x \in \mathbb{R}$, $f''(x) = \sin x + x^2$,
- $f'(0) = 5$,
- $f(0) = 7$.

The most misunderstood antiderivative

1. Find the *domain* and the derivative of $F_1(x) = \ln x$
2. Find the *domain* and the derivative of $F_2(x) = \ln(-x)$
3. Find the *domain* and the derivative of $F_3(x) = \ln |x|$

Suggestion: Break the domain into two pieces.

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4. Based on your answers, what is $\int \frac{1}{x} dx$?

5. Find the *domain* and the derivative of $F_4(x) = \ln |2x|$
Why doesn't this contradict your answer to 4?

Compute these antiderivatives by guess 'n check

1. $\int x^5 dx$

2. $\int (3x^8 - 18x^5 + 1) dx$

3. $\int \sqrt[3]{x} dx$

4. $\int \frac{1}{x^9} dx$

5. $\int \sqrt{x} (x^2 + 5) dx$

6. $\int \frac{1}{e^{2x}} dx$

7. $\int \sin(3x) dx$

8. $\int \cos(3x + 2) dx$

9. $\int \sec^2 x dx$

10. $\int \sec x \tan x dx$

11. $\int \frac{1}{x} dx$

12. $\int \frac{1}{x+3} dx$

Integration by parts 1

$$1. \frac{d}{dx} [x \sin x] =$$

$$2. \frac{d}{dx} [\cos x] =$$

Use the previous answers to calculate

$$3. \int x \cos x \, dx =$$

Integration by parts 2

1. $\frac{d}{dx} [xe^x] =$

2. ???

3. $\int xe^x dx =$

Integration by parts 3

1. ???

2. ???

3. $\int x e^{-x} dx =$

Integration by parts 4

1. $\frac{d}{dx} [x^2 e^x] =$

2. $\frac{d}{dx} [x e^x] =$

3. ???

4. $\int x^2 e^x dx =$

Trig-exp antiderivatives

$$1. \frac{d}{dx} [e^x \sin x] =$$

$$2. \frac{d}{dx} [e^x \cos x] =$$

Use the previous answers to calculate:

$$3. \int e^x \sin x \, dx =$$

$$4. \int e^x \cos x \, dx =$$

A challenge for guess-and-check ninjas

$$\int x e^x \cos x \, dx = ???$$

Functions defined by integrals

Which ones of these are valid ways to define functions?

1. $F(x) = \int_0^x \frac{t}{1+t^8} dt$

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4. $F(x) = \int_0^{x^2} \frac{t}{1+t^8} dt$

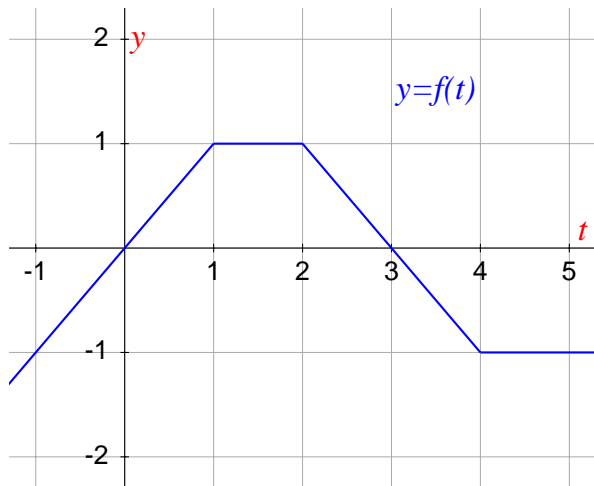
5. $F(x) = \int_{\sin x}^{e^x} \frac{t}{1+t^8} dt$

6. $F(x) = \int_0^3 \frac{t}{1+x^2+t^8} dt$

7. $F(x) = x \int_{\sin x}^{e^x} \frac{t}{1+x^2+t^8} dt$

8. $F(x) = t \int_{\sin x}^{e^x} \frac{t}{1+x^2+t^8} dt$

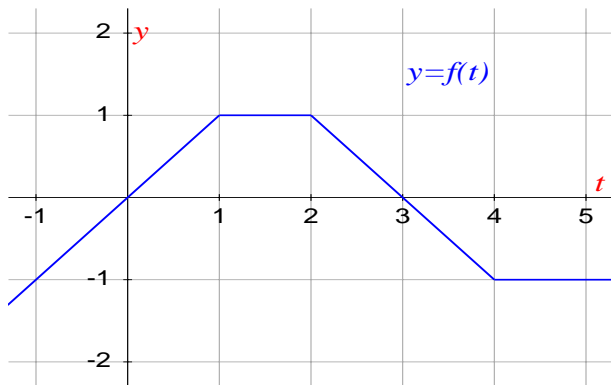
Towards FTC



Compute:

1. $\int_0^1 f(t) dt$
2. $\int_0^2 f(t) dt$
3. $\int_0^3 f(t) dt$
4. $\int_0^4 f(t) dt$
5. $\int_0^5 f(t) dt$

Towards FTC (continued)



Call $F(x) = \int_0^x f(t)dt$. This is a new function.

- Sketch the graph of $y = F(x)$.
- Using the graph you just sketched, sketch the graph of $y = F'(x)$.

Filling the tank

A tank is being filled with water. At time t water flows into the tank at a rate of

$$A e^{-bt} \arctan(ct)$$

litres per second, where A , b , and c are constants. The amount of water in the tank at time $t = 0$ s is V_0 . Write an expression for the amount of water V in the tank at time t .

True or False?

1. If f is continuous on the interval $[a, b]$, then

$$\frac{d}{dx} \left(\int_a^b f(t) dt \right) = f(x).$$

2. If f is differentiable, then

$$\frac{d}{dx} \left(\int_a^x f(t) dt \right) = \int_a^x f'(t) dt.$$

More True or False

Let f and g be differentiable functions with domain \mathbb{R} .

Assume that $f'(x) = g(x)$ for all x .

Which of the following statements must be true?

1. $f(x) = \int_0^x g(t)dt.$

2. If $f(0) = 0$, then $f(x) = \int_0^x g(t)dt.$

3. If $g(0) = 0$, then $f(x) = \int_0^x g(t)dt.$

4. There exists $C \in \mathbb{R}$ such that $f(x) = C + \int_0^x g(t)dt.$

5. There exists $C \in \mathbb{R}$ such that $f(x) = C + \int_1^x g(t)dt.$

True, False, or Shrug?

We want to find a function H with domain \mathbb{R} such that $H(1) = -2$ and such that $H'(x) = e^{\sin x}$ for all x . Decide whether each of the following statements is true, false, or we do not have enough information to decide.

1. The function $H(x) = \int_0^x e^{\sin t} dt$ is a solution.
2. The function $H(x) = \int_2^x e^{\sin t} dt$ is a solution.
3. $\forall C \in \mathbb{R}$, the function $H(x) = \int_0^x e^{\sin t} dt + C$ is a solution.
4. $\exists C \in \mathbb{R}$ s.t. the function $H(x) = \int_0^x e^{\sin t} dt + C$ is a solution.
5. The function $H(x) = \int_1^x e^{\sin t} dt - 2$ is a solution.
6. There is more than one solution.

Examples of FTC-1

Compute the derivative of the following functions

$$1. F_1(x) = \int_0^1 e^{-t^2} dt.$$

$$2. F_2(x) = \int_0^x e^{-\sin t} dt.$$

$$3. F_3(x) = \int_1^{x^2} \frac{\sin t}{t^2} dt.$$

$$4. F_4(x) = \int_x^7 \sin^3(\sqrt{t}) dt.$$

$$5. F_5(x) = \int_{2x}^{x^2} \frac{1}{1+t^3} dt.$$

A generalized version of FTC-1

Let f , u , v be differentiable functions with domain \mathbb{R} .

Let us call

$$F(x) = \int_{u(x)}^{v(x)} f(t) dt$$

Find a formula for

$$F'(x)$$

in terms of f , u , v , f' , u' , v' .

An integral equation

Assume f is a continuous function that satisfies, for every $x \in \mathbb{R}$:

$$\int_0^x e^t f(t) dt = \frac{\sin x}{x^2 + 1}$$

Find an explicit expression for $f(x)$.

Compute these definite integrals

1. $\int_1^2 x^3 dx$

2. $\int_0^1 [e^x + e^{-x} - \cos(2x)] dx$

3. $\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}} dx$

4. $\int_{\pi/4}^{\pi/3} \sec^2 x dx$

5. $\int_1^2 \left[\frac{d}{dx} \left(\frac{\sin^2 x}{1 + \arctan^2 x + e^{-x^2}} \right) \right] dx$

Find the error

$$\int_{-1}^1 \frac{1}{x^4} dx = \left. \frac{-1}{3x^3} \right|_{-1}^1 = \frac{-2}{3}$$

However, x^4 is always positive, so the integral should be positive.

Calculate the area of the bounded region...

1. ... between the x -axis and $y = 4x - x^2$.
2. ... between $y = \cos x$, the x -axis, from $x = 0$ to $x = \pi$.
3. ... between $y = x^2 + 3$ and $y = 3x + 1$.
4. ... between $y = 1$, the y -axis, and $y = \ln(x + 1)$.

Minimizing area

For each $a > 0$ consider the function

$$f_a(x) = 1 + a - ax^2$$

Find the value of a that minimizes the area of the region bounded by the graph of f_a and the x -axis.

► [desmos](#)

Symmetry

Calculate the value of these integrals *without computing any antiderivative*.

1. $\int_{-2}^2 \sin x^3 dx$ 2. $\int_0^{\pi} \cos^2 x dx$ 3. $\int_{-1}^1 \arccos x dx$

Hint: Sketch the graphs (use desmos) and use symmetry to compute the integral.

Once you guess the symmetry of the graph, try to write it algebraically.

Average Velocity

You are traveling.

Your position at time t is $s(t)$.

Your velocity at time t is $v(t)$.

The function v is continuous on an interval $[a, b]$.

Which of the following represent your average velocity on $[a, b]$?

1. $\frac{s(b) - s(a)}{b - a}$

2. $\frac{1}{b - a} \int_a^b v(t) dt$

3. $v(c)$ for at least one c between a and b

The Mean Value Theorem for integrals is back

Prove the following theorem.

Theorem

Let $a < b$. Let f be a continuous function on $[a, b]$.

There exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t) dt$$

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Hint: Use MVT for the function $F(x) = \int_a^x f(t) dt$.