Warm up

Write a formula for the general term of these sequences

1. $\{a_n\}_{n=0}^{\infty} = \{\,1,\ 4,\ 9,\ 16,\ 25,\ \ldots\,\}$

2. $\{b_n\}_{n=1}^{\infty} = \{\,1,\ -2,\ 4,\ -8,\ 16,\ -32,\ \ldots\,\}$

3. $\{c_n\}_{n=1}^{\infty} = \left\{\,\frac{2}{1!},\ \frac{3}{2!},\ \frac{4}{3!},\ \frac{5}{4!},\ \ldots\,\right\}$

4. $\{d_n\}_{n=1}^{\infty} = \{\,1,\ 4,\ 7,\ 10,\ 13,\ \ldots\,\}$

Sequences vs functions – convergence

For any function f with domain $[0, \infty)$,
we define a sequence as $a_n = f(n)$.
Let $L \in \mathbb{R}$. Which of these implications is true?

1. IF $\lim_{x \to \infty} f(x) = L$, THEN $\lim_{n \to \infty} a_n = L$.

2. IF $\lim_{n \to \infty} a_n = L$, THEN $\lim_{x \to \infty} f(x) = L$.

3. IF $\lim_{n \to \infty} a_n = L$, THEN $\lim_{n \to \infty} a_{n+1} = L$.

Definition of limit of a sequence

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$.
Which statements are equivalent to "$\{a_n\}_{n=0}^{\infty} \longrightarrow L$"?

1. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
2. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n > n_0 \implies |L - a_n| < \varepsilon.$
3. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{R},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
4. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{R}, \quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
5. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| \leq \varepsilon.$

6. $\forall \varepsilon \in (0, 1),\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
7. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < \dfrac{1}{\varepsilon}.$
8. $\forall k \in \mathbb{Z}^+,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < k.$
9. $\forall k \in \mathbb{Z}^+,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < \dfrac{1}{k}.$

Definition of limit of a sequence (continued)

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$.
Which statements are equivalent to "$\{a_n\}_{n=0}^{\infty} \longrightarrow L$"?

10. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains all the elements of the sequence, except the first few.
11. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains infinitely many of the elements of the sequence.
12. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains *almost all* the elements of the sequence.
13. $\forall \varepsilon > 0$, the interval $[L - \varepsilon, L + \varepsilon]$ contains *almost all* the elements of the sequence.
14. Every interval that contains L must contain *almost all* all the elements of the sequence.
15. Every open interval that contains L must contain *almost all* all the elements of the sequence.

Notation: "*almost all*" = "all, except finitely many"

Convergence and divergence

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence.
Write the formal definition of the following concepts:

1. $\{a_n\}_{n=0}^{\infty}$ is convergent.
2. $\{a_n\}_{n=0}^{\infty}$ is divergent.
3. $\{a_n\}_{n=0}^{\infty}$ is divergent to ∞.

Proof from the definition of limit

Prove, directly from the definition of limit, that

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1.$$

Suggestion:

1. Write down the definition of what you want to show.
2. Use itto decide the structure of the proof.
3. Do some rough work if necessary.
4. Write down the formal proof.

Sequences vs functions – monotonicity and boundness

For any function f with domain $[0, \infty)$,
we define a sequence as $a_n = f(n)$.
Which of these implications is true?

1. IF f is increasing, THEN $\{a_n\}_{n=0}^{\infty}$ is increasing.
2. IF $\{a_n\}_{n=0}^{\infty}$ is increasing, THEN f is increasing.
3. IF f is bounded, THEN $\{a_n\}_{n=0}^{\infty}$ is bounded.
4. IF $\{a_n\}_{n=0}^{\infty}$ is bounded, THEN f is bounded.

Examples

Construct 8 examples of sequences.
If any of them is impossible, cite a theorem to justify it.

		convergent	divergent
monotonic	bounded		
	unbounded		
not monotonic	bounded		
	unbounded		

A sequence defined by recurrence

Consider the sequence $\{R_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} & R_0 = 1 \\ \forall n \in \mathbb{N}, & R_{n+1} = \dfrac{R_n + 2}{R_n + 3} \end{cases}$$

Compute R_1, R_2, R_3.

Is this proof correct?

Let $\{R_n\}_{n=0}^{\infty}$ be the sequence in the previous slide.

Claim:

$\{R_n\}_{n=0}^{\infty} \longrightarrow -1 + \sqrt{3}$.

Is this proof correct?

Let $\{R_n\}_{n=0}^{\infty}$ be the sequence in the previous slide.

Claim:

$\{R_n\}_{n=0}^{\infty} \longrightarrow -1+\sqrt{3}$.

Proof.

- Let $L = \lim_{n\to\infty} R_n$.
- $R_{n+1} = \dfrac{R_n+2}{R_n+3}$
- $\lim_{n\to\infty} R_{n+1} = \lim_{n\to\infty} \dfrac{R_n+2}{R_n+3}$
- $L = \dfrac{L+2}{L+3}$
- $L(L+3) = L+2$
- $L^2+2L-2=0$
- $L = -1 \pm \sqrt{3}$
- L must be positive, so $L = -1+\sqrt{3}$

$\square$

Another sequence defined by recurrence

Consider the sequence $\{a_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} & a_0 = 1 \\ \forall n \in \mathbb{N}, & a_{n+1} = 1 - a_n \end{cases}$$

- Use the same method as in the previous slide to compute its limit.
- **After** you have computed the limit, calculate a_2, a_3, a_4, and a_5.
- What happened?

The original sequence defined by recurrence – done right

Consider the sequence $\{R_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} & R_0 = 1 \\ \forall n \in \mathbb{N}, & R_{n+1} = \dfrac{R_n + 2}{R_n + 3} \end{cases}$$

1. Prove $\{R_n\}_{n=0}^{\infty}$ is bounded below by 0.
2. Prove $\{R_n\}_{n=0}^{\infty}$ is decreasing (use induction)
3. Prove $\{R_n\}_{n=0}^{\infty}$ is convergent (use a theorem)
4. Now the calculation in the earlier slide is correct, and we can get the value of the limit.

True or False - convergence, monotonicity, and boundedness

1. If a sequence is convergent, then it is bounded above.
2. If a sequence is bounded, then it is convergent
3. If a sequence is convergent, then it is eventually monotonic.
4. If a sequence is positive and converges to 0, then it is eventually monotonic.
5. If a sequence diverges to ∞, then it is eventually monotonic.
6. If a sequence diverges, then it is unbounded.
7. If a sequence diverges and is unbounded above, then it diverges to ∞.
8. If a sequence is eventually monotonic, then it is either convergent, divergent to ∞, or divergent to $-\infty$.

True or False - Rapid fire

1. (convergent) $\implies$ (bounded)
2. (convergent) $\implies$ (monotonic)
3. (convergent) $\implies$ (eventually monotonic)
4. (bounded) $\implies$ (convergent)
5. (monotonic) $\implies$ (convergent)
6. (bounded $+$ monotonic) $\implies$ (convergent)
7. (divergent to ∞) $\implies$ (eventually monotonic)
8. (divergent to ∞) $\implies$ (unbounded above)
9. (unbounded above) $\implies$ (divergent to ∞)

Fill in the blanks

Let $\{a_n\}$ be a decreasing, bounded sequence.
Assume $a_1 = 1$ and a_n is never 0.
Let m be the greatest lower bound of $\{a_n\}$.

For each of the statements below, find **all** the values of m that make the statement true.

1. IF $\boxed{}$ THEN $\{1/a_n\}$ is bounded
2. IF $\boxed{}$ THEN $\{1/a_n\}$ is increasing
3. IF $\boxed{}$ THEN $\{\sin a_n\}$ is bounded
4. IF $\boxed{}$ THEN $\{\sin a_n\}$ is decreasing

Proof of Theorem 3

Write a proof for the following Theorem

Theorem 3

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence.

- IF $\{a_n\}_{n=0}^{\infty}$ is increasing AND unbounded above,
- THEN $\{a_n\}_{n=0}^{\infty}$ is divergent to ∞

Proof of Theorem 3

Write a proof for the following Theorem

Theorem 3

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence.

- IF $\{a_n\}_{n=0}^{\infty}$ is increasing AND unbounded above,
- THEN $\{a_n\}_{n=0}^{\infty}$ is divergent to ∞

1. Write the definitions of "increasing", "unbounded above", and "divergent to ∞"
2. Using the definition of what you want to prove, write down the structure of the formal proof.
3. Do some rough work if necessary.
4. Write a formal proof.

Proof feedback

1. Does your proof have the correct structure?
2. Are all your variables fixed (not quantified)? In the right order? Do you know what depends on what?
3. Is the proof self-contained? Or do I need to read the rough work to understand it?
4. Does each statement follow logically from previous statements?
5. Did you explain what you were doing? Would your reader be able to follow your thought process without reading your mind?

Critique this proof - #1

- $\forall M \in \mathbb{R},\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies x_n > M$

- M is not an upper bound: $\exists n_0 \in \mathbb{N}$ s.t. $x_{n_0} > M$

- $n \geq n_0 \implies x_n \geq x_{n_0} > M$

Critique this proof - #2

- WTS $a_n \to \infty$. This means:
  $\forall M \in \mathbb{R},\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies x_n > M$
- bounded above: $\exists M \in \mathbb{R},\ \forall n \in \mathbb{N},\ x_n \leq M$
- negation: $\forall M \in \mathbb{R},\ \exists n \in \mathbb{N},\ x_n > M$
- $\forall n \in \mathbb{N}$, take $n = n_0$.

Composition law

Write a proof for the following Theorem

Theorem

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$. Let f be a function.

- IF $\begin{cases} \{a_n\}_{n=0}^{\infty} \longrightarrow L \\ f \text{ is continuous at } L \end{cases}$
- THEN $\{f(a_n)\}_{n=0}^{\infty} \longrightarrow f(L)$.

Composition law

Write a proof for the following Theorem

Theorem

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$. Let f be a function.

- IF $\begin{cases} \{a_n\}_{n=0}^{\infty} \longrightarrow L \\ f \text{ is continuous at } L \end{cases}$
- THEN $\{f(a_n)\}_{n=0}^{\infty} \longrightarrow f(L)$.

1. Write the definition of your hypotheses and your conclusion.
2. Using the definition of your conclusion, figure out the structure of the proof.
3. Do some rough work if necessary.
4. Write a formal proof.

Calculations

1. $\lim_{n \to \infty} \dfrac{n! + 2e^n}{3n! + 4e^n}$

2. $\lim_{n \to \infty} \dfrac{2^n + (2n)^2}{2^{n+1} + n^2}$

3. $\lim_{n \to \infty} \dfrac{5n^5 + 5^n + 5n!}{n^n}$

True or False – The Big Theorem

Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be positive sequences.

1. IF $a_n << b_n$, THEN $\forall m \in \mathbb{N},\ a_m < b_m$.

2. IF $a_n << b_n$, THEN $\exists m \in \mathbb{N}$ s.t. $a_m < b_m$.

3. IF $a_n << b_n$, THEN $\exists n_0 \in \mathbb{N}$ s.t.
   $\forall m \in \mathbb{N},\ m \geq n_0 \implies a_m < b_m$.

True or False – The Big Theorem

Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be positive sequences.

1. IF $a_n << b_n$, THEN $\forall m \in \mathbb{N},\ a_m < b_m$.
2. IF $a_n << b_n$, THEN $\exists m \in \mathbb{N}$ s.t. $a_m < b_m$.
3. IF $a_n << b_n$, THEN $\exists n_0 \in \mathbb{N}$ s.t. $\forall m \in \mathbb{N},\ m \geq n_0 \implies a_m < b_m$.
4. IF $\forall m \in \mathbb{N},\ a_m < b_m$, THEN $a_n << b_n$.
5. IF $\exists m \in \mathbb{N}$ s.t. $a_m < b_m$, THEN $a_n << b_n$.
6. IF $\exists n_0 \in \mathbb{N}$ s.t. $\forall m \in \mathbb{N},\ m \geq n_0 \implies a_m < b_m$, THEN $a_n << b_n$.

Refining the Big Theorem - 1

1. Construct a sequence $\{u_n\}_n$ such that

$$\begin{cases} \forall a < 0, & n^a << u_n \\ \forall a \geq 0, & u_n << n^a \end{cases}$$

2. Construct a sequence $\{v_n\}_n$ such that

$$\begin{cases} \forall a \leq 0, & n^a << v_n \\ \forall a > 0, & v_n << n^a \end{cases}$$

Refining the Big Theorem - 2

1. Construct a sequence $\{u_n\}_n$ such that

$$\begin{cases} \forall a < 2, & n^a << u_n \\ \forall a \geq 2, & u_n << n^a \end{cases}$$

2. Construct a sequence $\{v_n\}_n$ such that

$$\begin{cases} \forall a \leq 2, & n^a << v_n \\ \forall a > 2, & v_n << n^a \end{cases}$$

True or False - Review

1. If $\{a_n\}_{n=0}^{\infty}$ diverges and is increasing, then $\exists n \in \mathbb{N}$ s.t. $a_n > 100$.
2. If $\lim_{n\to\infty} a_n = L$, then $\forall n \in \mathbb{N}$, $a_n < L + 1$.
3. If $\lim_{n\to\infty} a_n = L$, then $\exists n \in \mathbb{N}$ s.t. $a_n < L + 1$.
4. If $\lim_{n\to\infty} a_n = L$, then $\exists \varepsilon > 0$ s.t. $\forall n \in \mathbb{N}$, $a_n < L + \varepsilon$.
5. If $\{a_n\}_{n=0}^{\infty}$ is convergent and $b_n = a_n$ for *almost all* $n \in \mathbb{N}$, then $\{b_n\}_{n=0}^{\infty}$ is convergent.
6. If $a_n << b_n$, then $\exists n \in \mathbb{N}$ s.t. $a_n < b_n$.
7. If $a_n << b_n$, then $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ s.t. $a_n < \varepsilon b_n$.
8. If $a_n << b_n$, then $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n \geq n_0 \implies a_n < \varepsilon b_n$,