Warmup:

What are the following sets?

(A) $[2, 4] \cup (2, 5)$

(B) $[2, 4] \cap (2, 5)$

(C) $[\pi, e]$

(D) $[0, 0]$

(E) $(0, 0)$

Before next class:

- **Watch videos 1.4, 1.5, 1.6**

Similar sets

What are the following sets?

Write explicitly or with interval notation

(A) $A = \{x \in \mathbb{Z} : x^2 < 6\}$

(B) $B = \{x \in \mathbb{N} : x^2 < 6\}$

(C) $C = \{x \in \mathbb{R} : x^2 < 6\}$

Sets and quantifiers

What are the following sets?

(A) $A = \{x \in \mathbb{R} : \forall y \in [0,1],\ x < y\}$

(B) $B = \{x \in \mathbb{R} : \exists y \in [0,1] \text{ s.t. } x < y\}$

(C) $C = \{x \in [0,1] : \forall y \in [0,1], x < y\}$

(D) $D = \{x \in [0,1] : \exists y \in [0,1] \text{ s.t. } x < y\}$

(E) $E = \{x \in [0,1] : \exists y \in \mathbb{R} \text{ s.t. } x < y\}$

(F) $F = \{x \in [0,1] : y \in \mathbb{R},\ x < y\}$

Describing a new set

An irrational number is a number that is real but not rational.

B is the set of positive, rational numbers and negative, irrational numbers.

Write a definition for B using only mathematical notation.
(You may use the words “and”, “or”, and “such that”.)

MAT137 Lecture 3 — Negation

Homework review:

If possible, write the following sets with interval notation

(A) $A = \{x \in \mathbb{R} \; : \; \forall y \in [0, 1], \; x < y\}$

(B) $B = \{x \in \mathbb{R} \; : \; \exists y \in [0, 1] \text{ s.t. } x < y\}$

(C) $C = \{x \in [0, 1] \; : \; \forall y \in [0, 1], x < y\}$

(D) $D = \{x \in [0, 1] \; : \; \exists y \in [0, 1] \text{ s.t. } x < y\}$

(E) $E = \{x \in [0, 1] \; : \; \exists y \in \mathbb{R} \text{ s.t. } x < y\}$

(F) $F = \{x \in [0, 1] \; : \; y \in \mathbb{R}, \; x < y\}$

Before next class:

- **Watch videos 1.7, 1.8, 1.9**

Mother

Let

$$H = \{ \text{ humans } \}$$

Which statements are True/False?

(A) $\forall x \in H, \exists y \in H$ such that y gave birth to x

(B) $\exists x \in H,$ such that $\forall y \in H,$ y gave birth to x

Even numbers

Which of these is a correct description of the set E of even integers?

(A) $E = \{n \in \mathbb{Z} \ : \ \forall a \in \mathbb{Z},\ n = 2a\}$

(B) $E = \{n \in \mathbb{Z} \ : \ \exists a \in \mathbb{Z} \text{ s.t. } n = 2a\}$

Negation 1

Write the negation of these statements as simply as possible:

(A) My favourite integer number is greater than 7.

(B) I know at least five students at U of T who have a cellphone.

(C) There is a country in the European Union with fewer than 1000 inhabitants.

(D) All of my friends like apples.

(E) I like apples and oranges.

$$\text{Negation of } \boxed{\cdots} \quad = \quad \boxed{\cdots} \text{ is false.}$$

Functions and quantifiers

Let f be a function with domain $\mathbb{R}$. Rewrite the following statements using $\forall$ or $\exists$:

(A) The graph of f intercepts the x-axis.

(B) f is the zero function.

(C) f is not the zero function.

(D) f never vanishes.

(E) The equation $f(x) = 0$ has a solution.

(F) The equation $f(x) = 0$ has no solutions.

(G) f takes both positive and negative values.

(H) f is never negative.

Before next class:

- **Watch videos 1.10, 1.11, 1.12, 1.13**

Conditionals - True or False?

Let $x \in \mathbb{R}$.

(A) $x > 0 \quad \Longrightarrow \quad x \geq 0$

(B) $x \geq 0 \quad \Longrightarrow \quad x > 0$

Conditionals - True or False?

Let $x \in \mathbb{R}$.

(A) $x > 0 \quad \Longrightarrow \quad x \geq 0$

(B) $x \geq 0 \quad \Longrightarrow \quad x > 0$

(C) IF $2 > 3$ THEN Jason is in love.

Cookies

Rewrite the following statement as an "IF, ...THEN" statement and as an implication (using $\implies$).

- **A teacher promises to give a cookie to any student who gets 100 on the test.**

Cookies

Rewrite the following statement as an "IF, ...THEN" statement and as an implication (using $\implies$).

- **A teacher promises to give a cookie to any student who gets 100 on the test.**

- No student gets 100.
- The teacher doesn't give out any cookies.

- Did the teacher lie?

Hockey

Which of the following statements are equivalent to the statement *"Every Canadian man likes hockey"*?

(A) If a man is Canadian, then he likes hockey.

(B) If a man likes hockey, then he is Canadian.

(C) If a man does not like hockey, then he is not Canadian.

(D) If a man is not Canadian, then he likes hockey.

(E) Non-Canadian men do not like hockey.

(F) If a Canadian does not like hockey, then they are not a man.

Negation of conditionals

Write the negation of these statements:

(A) If Justin Trudeau has a brother, then he also has a sister.

(B) If a student in this class has a brother, then they also have a sister.

Cards

Take a look at the following cards.

E	3	P	8

Each card has a letter on one side and a number on the other, and I tell you:

> *"**If** a card has a vowel on one side,*
> ***then** it has an odd number on the other side."*

Which cards do you need to turn over in order to verify whether I am telling the truth or not?

Four cards lie on the table in front of you. You know that each card has a letter on one side and a number on the other.

Negate the following statement:

*"**If** a card has a vowel on one side,*
***then** it has an odd number on the other side."*

Negation 2

Write the negation of this statement without using any negative words (“no”, “not”, “none”, etc.):

> *“Every page in this book contains at least one word whose first and last letters both come alphabetically before M.”*

Negation 3

Negate the following statement without using any negative words (“no”, “not”, “none”, etc.):

> *“I have a friend all of whose former boyfriends had at least two siblings with exactly three different vowels in their name.”*

Warmup:
The law says "**if you check out a library book, you must return it**".

- I did not check out any library books.
- I did not return any library books.

Did I violate the law?

Before next class:

- **Watch videos 1.14, 1.15**

Elephants

True or False?

(A) There is a pink elephant in this room.

(B) All elephants in this room are pink.

One-to-one functions

Let f be a function with domain D.

f is *one-to-one* means that ...

- ... different inputs (x) ...
- ... must produce different outputs ($f(x)$).

Write a formal definition of “one-to-one”.

One-to-one functions

Definition: Let f be a function with domain D.
f is one-to-one means ...

(A) $f(x_1) \neq f(x_2)$
(B) $\exists x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$
(C) $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$
(D) $\forall x_1, x_2 \in D,\ x_1 \neq x_2,\ f(x_1) \neq f(x_2)$
(E) $\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
(F) $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2) \implies x_1 \neq x_2$
(G) $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

One-to-one functions

Let f be a function with domain D.

What does each of the following mean?

(A) $f(x_1) \neq f(x_2)$

(B) $\exists x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$

(C) $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$

(D) $\forall x_1, x_2 \in D,\ x_1 \neq x_2,\ f(x_1) \neq f(x_2)$

(E) $\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$

(F) $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2) \implies x_1 \neq x_2$

(G) $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

Proving a function is one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
- OR, equivalently, $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

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Suppose I give you a specific function f and I ask you to prove it is one-to-one.

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Suppose I give you a specific function f and I ask you to prove it is one-to-one.

- **For the first definition**, write the structure of your proof (How do you begin? What do you assume? What do you conclude?).
- **For the second definition**, write the structure of your proof if you use the second definition.

Proving a function is one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $\forall x_1, x_2 \in D, \; x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
- OR, equivalently, $\forall x_1, x_2 \in D, \; f(x_1) = f(x_2) \implies x_1 = x_2$

Suppose I give you a specific function f and I ask you to prove it is one-to-one.

- **For the first definition**, write the structure of your proof (How do you begin? What do you assume? What do you conclude?).
- **For the second definition**, write the structure of your proof if you use the second definition.

Exercise

Prove that $f(x) = 3x + 2$, with domain $\mathbb{R}$, is one-to-one.

Warmup:

For each of the following statements, list **all** conditions under which it is false.

- If a student gets a 100 on the test, I will give them a cookie.
- If I check out a library book, I will return it.
- If there is an elephant in this room, it is pink.

Before next class:

- **Watch videos 2.4**

Cookies Redux

A teacher promises to give a cookie to any student who gets 100 on the test.

- No student gets 100.
- The teacher doesn't give out any cookies.

- Did the teacher lie?

Library Redux

The law says "**if you check out a library book, you must return it**".

- I did not check out any library books.
- I did not return any library books.

Did I violate the law?

Elephant Redux

Consider the statement "**if this room contains an elephant, that elephant is pink**".

The facts are:

- This room contains no elephants.
- If there were an elephant in this room, it would come from the zoo and would be a grey elephant.

Is the statement true or false?

My Mind

Consider the statements

(A) If I have a favorite number, then it is an even number.

(B) If I have a favorite number, then it is an odd number.

Can both statements be true at the same time?

Proving a function is NOT one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $$\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$
- OR, equivalently, $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

Proving a function is NOT one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
- OR, equivalently, $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

Suppose I give you a specific function f and I ask you to prove it is not one-to-one.

Proving a function is NOT one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $$\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$
- OR, equivalently, $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

Suppose I give you a specific function f and I ask you to prove it is not one-to-one. You need to prove f satisfies the *negation* of the definition.

- Write the negation of the first definition.
- Write the negation of the second definition.
- Write the structure of your proof.

Proving a function is NOT one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $$\forall x_1, x_2 \in D, \; x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$
- OR, equivalently, $\forall x_1, x_2 \in D, \; f(x_1) = f(x_2) \implies x_1 = x_2$

Suppose I give you a specific function f and I ask you to prove it is not one-to-one. You need to prove f satisfies the *negation* of the definition.

- Write the negation of the first definition.
- Write the negation of the second definition.
- Write the structure of your proof.

Exercise

Prove that $f(x) = x^2$, with domain $\mathbb{R}$, is not one-to-one.

Variations on induction

Let S_n be a statement depending on a positive integer n.

What conclusions can you draw in each of the following cases?

(A) We have proven:

- S_3
- $$\forall n \geq 1,\ S_n \implies S_{n+1}$$

(B) We have proven:

- S_1
- $$\forall n \geq 3,\ S_n \implies S_{n+1}$$

(C) We have proven:

- S_1
- $$\forall n \geq 1,\ S_n \implies S_{n+3}$$

(D) We have proven:

- S_1
- $$\forall n \geq 1,\ S_{n+1} \implies S_n$$

Variations on induction 2

We want to prove

$$\forall n \geq 1,\ S_n$$

So far we have proven

- S_1
- $\forall n \geq 1,\ S_n \implies S_{n+3}$.

What else do we need to do?

What is wrong with this proof by induction?

Theorem

$\forall N \geq 1$, *every set of* N *students in MAT137 will get the same grade.*

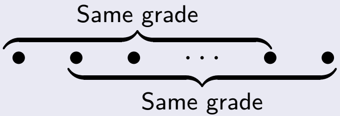
What is wrong with this proof by induction?

Theorem

$\forall N \geq 1$, *every set of* N *students in MAT137 will get the same grade.*

Proof.

- **Base case.** It is clearly true for $N = 1$.
- **Induction step.**
  Assume it is true for N. I'll show it is true for $N + 1$.
  Take a set of $N + 1$ students. By induction hypothesis:
  - The first N students get the same grade.
  - The last N students get the same grade.



Hence the $N + 1$ students all get the same grade.

$\square$

What is wrong with this proof by induction?

For every $N \geq 1$, let

$$S_N = \text{ ``every set of } N \text{ students in MAT137} \\ \text{will get the same grade''}$$

What did we actually prove in the previous page?

- S_1 ?
- $\forall N \geq 1, \quad S_N \implies S_{N+1}$?

What is wrong with this proof? (1)

Theorem

The sum of two odd numbers is even.

Proof.

3 is odd.

5 is odd.

$3 + 5 = 8$ is even. $\square$

What is wrong with this proof? (2)

Theorem

The sum of two odd numbers is even.

Proof.

The sum of two odd numbers is always even.

even + even = even

even + odd = odd

odd + even = odd

odd + odd = even. $\square$