Lake ripple

We drop a pebble into a lake. It produces a circular ripple. When the radius is 2 meters and is increasing at a rate of $10cm/s$, at what rate is the area increasing?

Sliding ladder

A ten-meter long ladder is leaning against a vertical wall and sliding. The top end of the ladder is 8 meters high and sliding down at a rate of 1 meter per second. At which rate is the bottom end sliding?

Math party

The MAT137 TAs wanted to rent a disco ball for their upcoming party. However, since they are poor, they could only afford a flashlight. At the party, one TA is designated the “human disco ball”. The TA stands in the center of the room pointing the flashlight horizontally and spins at 3 revolutions per second. (Yes, they are that fast. Ask your TA to demonstratel if you don’t believe me!) The room is square with side length 8 meters. At which speed is the light from the flashlight moving across the wall when it is 3 meters away from a corner?

Sleepy ants

Two ants are taking a nap. The first one is resting at the tip of the minute hand of a cuckoo clock, which is 25 cm long. The second one is resting at the tip of the hour hand, which is half the length. At what rate is the distance between the two ants changing at 3:30?

The kite

Mary Poppins is flying a kite. The kite is 21 meters above the ground and it is being blown horizontally by the wind at 2 m/s. Mary's hands are 1 meter above the ground. Right now 30 meters of string are out. At what rate is the string being released from Mary's hands?

Coffee

A coffee filter is shaped like an inverted cone. It has a radius at the top of $4cm$ and it is $6cm$ in height. Coffee flows out of at the bottom at a rate of $2cm^3/s$. If the filter begins completely filled, how fast is the coffee level decreasing after 30 seconds?

The classic farmer problem

A farmer has 300m of fencing and wants to fence off a rectangular field and add an extra fence that divides the rectangular area in two equal parts down the middle. What is the largest area that the field can have?

Distance

Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

A matter of perspective

A painting in an art gallery has height h and is hung so that its lower edge is a distance a above your eye. How far from the wall should you stand to get the best view?

Airplane

The cost of fuel per hour for a certain airplane is proportional to the square of its speed and is \$1200 per hour for a speed of $600 km/h$. After every 5,000 hours flown, the aircraft must undergo an \$8 million dollar safety inspection. What speed should the airplane fly at in order to achieve the lowest cost per kilometre?

Fire

You hear a scream. You turn around and you see Alfonso is on fire. Literally! Luckily, you are next to a straight river. Alfonso is 10 meters away from the river and you are 5 meters away from the point P on the river closest to Alfonso. You are carrying an empty bucket. You can run twice as fast with an empty bucket as you can run with a full bucket. How far from the point P should you fill your bucket in order to get to Alfonso with a bucket full of water as fast as possible?

Dominion

Dominion is a board game where, among other things, players buy cards worth victory points. The player with the most victory points wins.

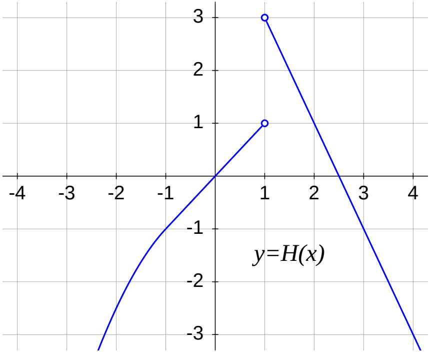
It is your last turn and you can only buy “duchies” and “dukes”. A duchy is worth 3 victory points. A duke is worth as many victory points as duchies you have. Each duchy costs 3 coins, and each duke costs 3 coins. You have not bough any duke or duchy yet.

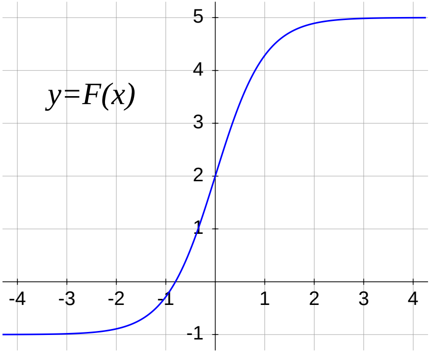
If you have N coins, how many dukes and how many duchies should you buy?

Limits from graphs

Compute:

1. $\displaystyle\lim_{x\to 0} \frac{H(x)}{H(2+3x)-1}$
2. $\displaystyle\lim_{x\to 2} \frac{F^{-1}(x)}{x-2}$





Polynomial vs Exponential

1. Use L'Hôpital Rule to compute

$$\lim_{x \to \infty} \frac{x^7 + 5x^3 + 2}{e^x}$$

Polynomial vs Exponential

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$$\lim_{x \to \infty} \frac{x^7 + 5x^3 + 2}{e^x}$$

2. Make a conjecture for the value of

$$\lim_{x \to \infty} \frac{x^N}{e^x}$$

   where N is a positive integer. Prove it by induction.

Computations

Calculate:

1. $\displaystyle\lim_{x\to 2} \frac{x^2+2x-6}{x^2+3x-10}$
2. $\displaystyle\lim_{x\to 0} \frac{e^{2x^2}-\cos x}{x\sin x}$
3. $\displaystyle\lim_{x\to\infty} x^3 e^{-x}$
4. $\displaystyle\lim_{x\to\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}}$
5. $\displaystyle\lim_{x\to 0} x\sin\frac{2}{x}$
6. $\displaystyle\lim_{x\to\infty} x\sin\frac{2}{x}$
7. $\displaystyle\lim_{x\to\infty} x\cos\frac{2}{x}$
8. $\displaystyle\lim_{x\to 1} \left[(\ln x)\tan\frac{\pi x}{2}\right]$

Infinity minus infinity

Calculate:

1. $\lim_{x\to 0}\left[\frac{\csc x}{x} - \frac{\cot x}{x}\right]$
2. $\lim_{x\to\infty}\left[\ln(x+2) - \ln(3x+4)\right]$
3. $\lim_{x\to 1}\left[\frac{2}{x^2-1} - \frac{1}{x-1}\right]$
4. $\lim_{x\to-\infty}\left[\sqrt{x^2+3x} - \sqrt{x^2-3x}\right]$

Exponential indeterminate forms

Calculate:

1. $\lim_{x\to 0} [1 + 2\sin(3x)]^{4\cot(5x)}$
2. $\lim_{x\to\infty} \left(\frac{x+2}{x-2}\right)^{3x}$
3. $\lim_{x\to 0^+} x^x$
4. $\lim_{x\to \frac{\pi}{2}^-} (\tan x)^{\cos x}$
5. $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

Backwards L'Hôpital

Construct a polynomial P such that

$$\lim_{x \to 1} \frac{P(x)}{e^x - e \cdot x} = \frac{1}{e}$$

Come to the dark side

Help us write a difficult question for Test 3! We will ask you to compute a limit like this

$$\lim_{x \to 0} \frac{e^x + e^{-x} - 2\cos x + bx^N}{x^6}$$

where b is a real number and N is a natural number that we have not chosen yet.

We do not want the answer to be 0 or ∞ or $-\infty$ or "DNE", because you could guess that randomly.

What values of b and N should we choose? What will the value of the limit be?

Indeterminate?

Which of the following are indeterminate forms for limits? If any of them isn't, then what is the value of such limit?

1. $\frac{0}{0}$
2. $\frac{0}{\infty}$
3. $\frac{0}{1}$
4. $\frac{\infty}{0}$
5. $\frac{\infty}{\infty}$
6. $\frac{1}{\infty}$
7. $0 \cdot \infty$
8. $\infty \cdot \infty$
9. $\sqrt{\infty}$
10. $\infty - \infty$
11. 1^{∞}
12. $1^{-\infty}$
13. 0^{0}
14. 0^{∞}
15. $0^{-\infty}$
16. ∞^{0}
17. ∞^{∞}
18. $\infty^{-\infty}$

Proving something is an indeterminate form

1. Prove that $\forall c \in \mathbb{R}$, $\exists a \in \mathbb{R}$ and functions f and g such that

$$\lim_{x \to a} f(x) = 0, \quad \lim_{x \to a} g(x) = 0, \quad \lim_{x \to a} \frac{f(x)}{g(x)} = c$$

This is how you show that $\frac{0}{0}$ is an indeterminate form.

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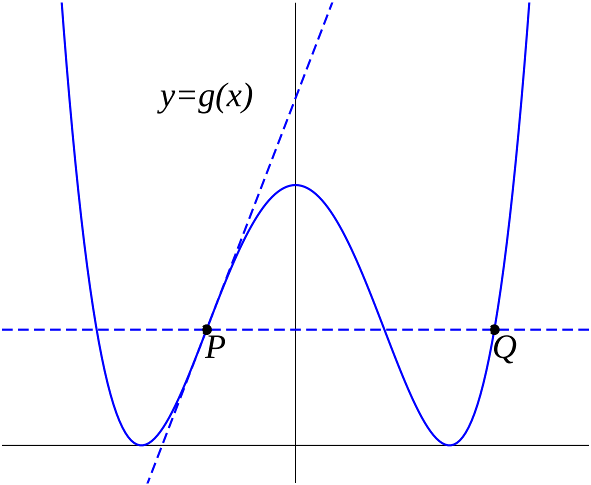
2. Prove the same way that $\frac{\infty}{\infty}$, $0 \cdot \infty$, and $\infty - \infty$ are also indeterminate forms.

3. Prove that 1^{∞}, 0^0, and ∞^0 are indeterminate forms.
   (You will only get $c \geq 0$ this time)

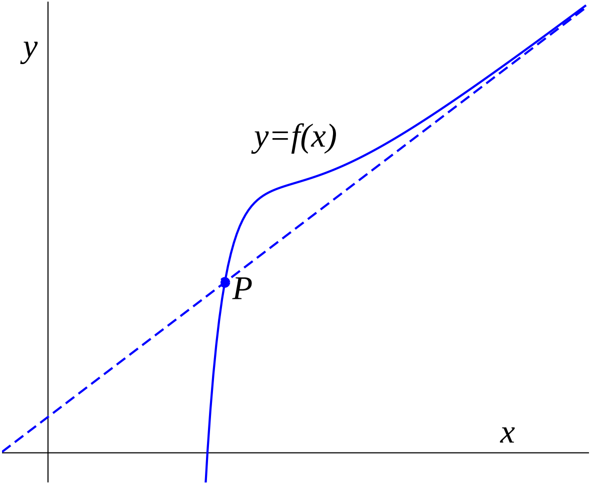
Find the coordinates of P and Q

$$g(x) = x^4 - 6x^2 + 9$$



Find the coordinates of P

$$f(x) = 3x + 4 + \frac{2x - 10}{x^2}$$



True or False – Concavity and inflection points

Let f be a differentiable function with domain $\mathbb{R}$.
Let $c \in \mathbb{R}$. Let I be an interval. Which implications are true?

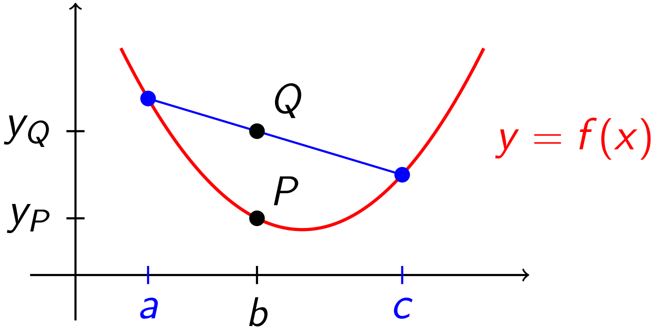
1. IF f is concave up on I, THEN $\forall x \in I$, $f''(x) > 0$.
2. IF $\forall x \in I$, $f''(x) > 0$, THEN f is concave up on I.
3. IF f is concave up on I THEN f' is increasing on I.
4. IF f' is increasing on I, THEN f is concave up on I.

5. IF f has an I.P. at c, THEN $f''(c) = 0$.
6. IF $f''(c) = 0$, THEN f has an I.P. at c.
7. IF f has an I.P. at c, THEN f' has a local extremum at c
8. IF f' has a local extremum at c, THEN f has an I.P. at c.

I.P. = "inflection point"

"Secant segments are above the graph"

Let f be a function defined on an interval I.
In Video 6.13 you learned that an alternative to our definition of "f is concave up on I" is "the secant segments stay above the graph".



Rewrite this as a precise mathematical statement of the form

"$\forall a, b, c \in I, \quad a < b < c \implies$ $\boxed{\text{an inequality involving } f,\ a,\ b,\ c}$"

A polynomial from 3 points

Construct a polynomial that satisfies the following three properties at once:

1. It has an inflection point at $x = 2$
2. It has a a local extremum at $x = 1$
3. It has y-intercept at $y = 1$.

Monotonicity and concavity

Let $f(x) = xe^{-x^2/2}$.

1. Find the intervals where f is increasing or decreasing, and its local extrema.
2. Find the intervals where f is concave up or concave down, and its inflection points.
3. Calculate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.
4. Using this information, sketch the graph of f.

Fractional exponents

Let $h(x) = \dfrac{x^{2/3}}{(x-1)^{2/3}}$. Its first two derviatives are

$$h'(x) = \frac{-2}{3x^{1/3}(x-1)^{5/3}} \qquad h''(x) = \frac{2(6x-1)}{9x^{4/3}(x-1)^{8/3}}$$

1. Find all asymptotes of h
2. Study the monotonicity of h and local extrema
3. Study the concavity of h and inflection points
4. With this information, sketch the graph of h

Hyperbolic tangent

The function tanh, defined by

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

is called the "hyperbolic tangent".

1. Find its two asymptotes
2. Study its monotonicity
3. Study its concavity
4. With this information, sketch its graph.

A very hard function to graph

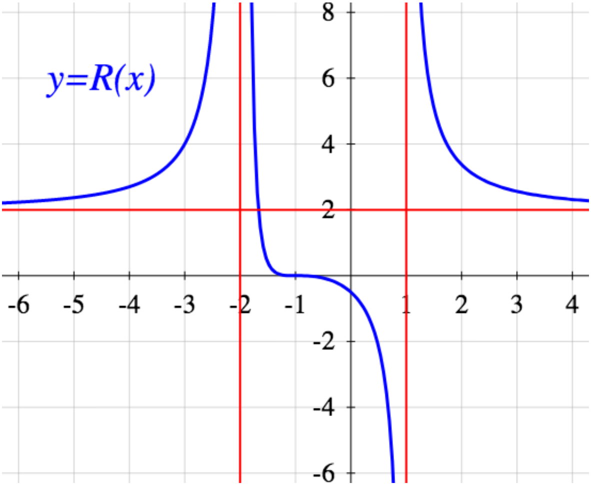
The function $G(x) = xe^{1/x}$ is deceiving. To help you out:

$$G'(x) = \frac{x-1}{x}e^{1/x}, \qquad G''(x) = \frac{e^{1/x}}{x^3}$$

1. Carefully study the behaviour as $x \to \pm\infty$.
   You should find an asymptote, but it is not easy.
2. Carefully study the behaviour as $x \to 0^+$ and $x \to 0^-$. The two are very different.
3. Use G' to study monotonocity.
4. Use G'' to study concavity.
5. Sketch the graph of G.

Backwards graphing

R is a rational function (a quotient of polynomials).
Find its equation.



Unexpected asymptotes

Find the two asymptotes of the function

$$F(x) = x + \sqrt{x^2 + 2x + 2}$$

Hint: The behaviour as $x \to \infty$ is very different from $x \to -\infty$.

graph

Unusual examples

Construct three functions f, g, and h.

1. f has domain at least $(0, \infty)$, is continuous, is always concave up, and satisfies $\lim_{x\to\infty} f(x) = -\infty$
2. g has domain $\mathbb{R}$, is continuous, has a local minimum at $x = 0$, and has an inflection point also at $x = 0$.
3. h has domain $\mathbb{R}$, is differentiable, is strictly increasing. In addition, h' is periodic with period 2, and h' is not constant.