Tangent line to a line?

What is the equation of the line tangent to the graph of $y = x$ at the point with x–coordinate 7?

1. $y = x + 7$
2. $y = x$
3. $y = 7$
4. $x = 7$
5. There is no tangent line at that point.
6. There is more than one tangent line at that point.

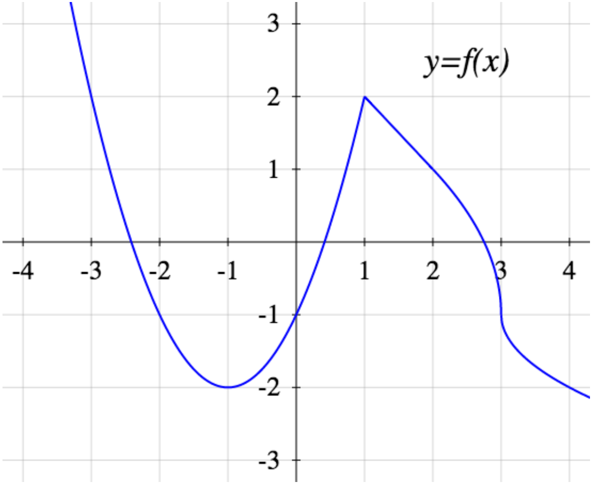
Prove these statements are false with counterexamples

Let C be a curve. Let P be a point in C.

1. The line tangent to C at P intersects C at only one point: P.
2. If a line intersects C only at P, then that line must be the tangent line to C at P.
3. The tangent line to C at P intersects C at P and "does not cross" C at P. (This means that, near P, it stays on one side of C.)
4. If a line intersects C at P and "does not cross" C at P, then it is the tangent line to C at P.

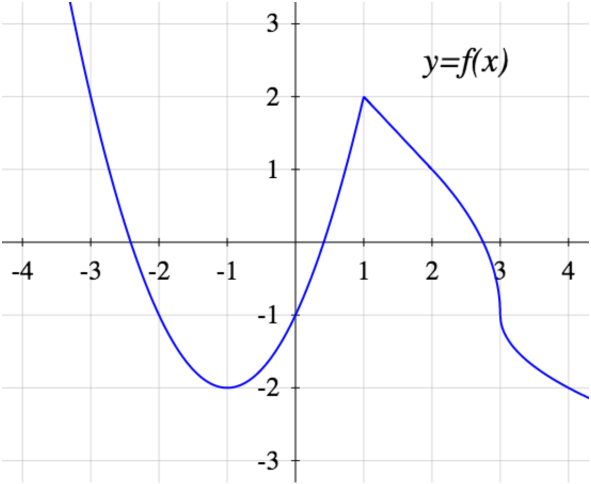
Tangent line from a graph

This is the graph of the function f. Write the equation of the line tangent to it at the point with x–coordinate -2.

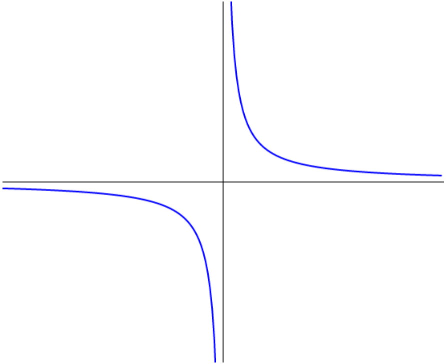


Derivative from a graph

This is the graph of the function f.
Sketch the graph of its derivative f'.



1. Sketch the graph of a continuous function with domain $\mathbb{R}$, whose derivative has the graph below.
2. Sketch the graph of a non-continuous function whose derivative has the graph below.



Estimations - 1

Let f be a continuous function with domain $\mathbb{R}$.

1. We know $f(4) = 3$ and $f(4.2) = 2.2$.
   Based only on this, give your best estimate for $f(4.1)$.

2. We know $f(4) = 3$ and $f'(4) = 0.6$.
   Based only on this, give your best estimate for $f(4.1)$.

3. We know $f(4) = 3$ and $f(4.1) = 4$.
   Based only on this, give your best estimate for $f'(4)$.

Estimations – 2

Without using a calculator, estimate $\sqrt[20]{1.01}$ as well as you can.

Hint: You know the value of $f(x) = \sqrt[20]{x}$ and its derivative at one point very close to 1.01. Use the tangent line at that point as an approximation.

Estimations – 3

1. We know $f(0) = 2, \quad f'(0) = 3, \quad g(0) = 7, \quad g'(0) = 5.$

   Compute $\displaystyle \lim_{x \to 0} \frac{f(x)}{g(x)}.$

2. We know $f(0) = 0, \quad f'(0) = 3, \quad g(0) = 0, \quad g'(0) = 5.$

   - When x is close to 0, give estimates for $f(x)$ and $g(x)$ using the tangent lines at 0.
   - Use those estimates to compute $\displaystyle \lim_{x \to 0} \frac{f(x)}{g(x)}.$

Derivatives from the definition

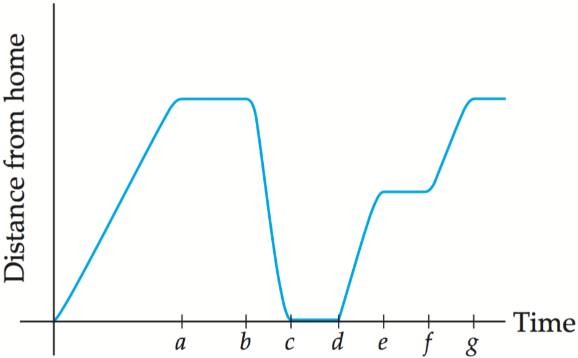
Let

$$g(x) = \frac{2}{\sqrt{x}}$$

Calculate $g'(4)$ directly from the definition of derivative as a limit.

Bella

The graph below describes Bella's distance from home one morning as she drives drive between her home and school.
Describe a possible scenario for her travels that morning.
Then sketch the corresponding graph of his velocity.



Edward and Jacob

Jacob walked at 5 km/h for 20 minutes and then sprinted at 15 km/h for 8 minutes.

1. How fast would Edward have to walk or run to go the same distance as Jacob did in the same time while moving at a constant speed?
2. Sketch a graph of Jacob's and Edward's positions over time on the same set of axes.

Computations: Basic differentiation rules

Compute the derivative of the following functions:

1. $f(x) = x^{100} - 3x^9 - 2$
2. $f(x) = \sqrt[3]{x} + 6$
3. $f(x) = \dfrac{4}{x^4}$
4. $f(x) = \sqrt{x}\,(1 + 2x)$
5. $f(x) = \dfrac{x^6 + 1}{x^3}$
6. $f(x) = \dfrac{x^2 - 2}{x^2 + 2}$

Computations: Chain rule

Compute the derivative of

1. $f(x) = \left(2x^2 + x + 1\right)^8$
2. $f(x) = \dfrac{1}{\left(x + \sqrt{x^2 + x}\right)^{137}}$

A long chain

The function below has 137 square roots:

$$f(x) = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots + \sqrt{x + \sqrt{x + 1}}}}}}$$

Find the equation of the line tangent to the graph of f at the point with x-coordinate 0.

Computations: Trig derivatives

Compute the derivatives of the following functions:

1. $f(x) = \tan(3x^2 + 1)$

2. $f(x) = (\cos x)(\sin 2x)(\tan 3x)$

3. $f(x) = \cos(\sin(\tan x))$

4. $f(x) = \cos\left(3x + \sqrt{1 + \sin^2 x^2}\right)$

Differentiable functions

Let $a \in \mathbb{R}$.
Let f be a function with domain $\mathbb{R}$.
Assume f is differentiable everywhere.
What can we conclude?

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. f is continuous at a.
4. $f'(a)$ exists.
5. $\lim_{x \to a} f'(x)$ exists.
6. f' is continuous at a.

True or False - Differentiability vs Continuity

Let f be a function with domain $\mathbb{R}$. Let $c \in \mathbb{R}$.
Which of these implications are true?

1. IF f is continuous at c, THEN f is differentiable at c
2. IF f is differentiable at c, THEN f is continuous at c
3. IF f is differentiable at c, THEN f' is continuous at c
4. IF f' is continuous at c, THEN f is continuous at c
5. IF f is differentiable at c, THEN f is continuous at and near c.
6. IF f is continuous at and near c, THEN f is differentiable at c.

True or False - Differentiability and Operations

Let f be a function with domain $\mathbb{R}$. Let $c \in \mathbb{R}$.
Let $g(x) = f(x)^2$. Which of these implications are true?

1. IF f is differentiable at c, THEN $f + f'$ is continuous at c
2. IF f is differentiable at c, THEN $3f$ is differentiable at c.
3. IF f is differentiable at c, THEN g is differentiable at c.
4. IF g is differentiable at c, THEN f is differentiable at c.
5. IF f is differentiable at c, THEN $1/f$ is differentiable at c.

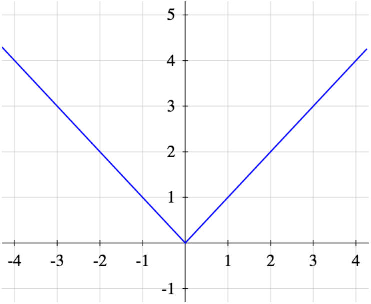
Vertical things

- Construct a function f that has a vertical asymptote at $x = 2$.
- Construct a function g that has a vertical tangent line at $x = 2$.

Absolute value and tangent lines

At (0,0) the graph of $y = |x|$...

1. ... has one tangent line: $y = 0$
2. ... has one tangent line: $x = 0$
3. ... has two tangent lines $y = x$ and $y = -x$
4. ... has no tangent line



Absolute value and derivatives - 1

Let $h(x) = x|x|$. What is $h'(0)$?

1. It is 0.
2. It doesn't exist because $|x|$ is not differentiable at 0.
3. It doesn't exist because the right- and left-limits, when computing the derivative, are different.
4. It doesn't exist because it has a corner.
5. It doesn't exist for a different reason.

Absolute value and derivatives - 2

True or False?

For all $n \in \mathbb{Z}$ and all x, $\frac{d}{dx}|x|^n = nx|x|^{n-2}$.

Write a proof for the quotient rule for derivatives

Theorem

- Let $a \in \mathbb{R}$.
- Let f and g be functions defined at and near a. Assume $g(x) \neq 0$ for x close to a.
- We define the function h by $h(x) = \dfrac{f(x)}{g(x)}$.

IF f and g are differentiable at a,
THEN h is differentiable at a, and

$$h'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Write a proof directly from the definition of derivative.
Hint: Imitate the proof of the product rule in Video 3.6.

Check your proof

1. Did you use the *definition* of derivative?
2. Are there words or only equations?
3. Does every step follow logically?
4. Did you only assume things you could assume?

5. Did you assume at some point that a function was differentiable? If so, did you justify it?
6. Did you assume at some point that a function was continuous? If so, did you justify it?

If you answered “no” to Q6, you probably missed something important.

Critique this proof

$$
\begin{aligned}
h'(a) &= \lim_{x\to a} \frac{h(x)-h(a)}{x-a} = \lim_{x\to a} \frac{\dfrac{f(x)}{g(x)} - \dfrac{f(a)}{g(a)}}{x-a} \\
&= \lim_{x\to a} \frac{f(x)g(a) - f(a)g(x)}{g(x)g(a)\,(x-a)} \\
&= \lim_{x\to a} \frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(x)}{g(x)g(a)\,(x-a)} \\
&= \lim_{x\to a} \left\{ \left[\frac{f(x)-f(a)}{x-a} g(a) - f(a) \frac{g(x)-g(a)}{x-a} \right] \frac{1}{g(x)g(a)} \right\} \\
&= \left[f'(a)g(a) - f(a)g'(a) \right] \frac{1}{g(a)g(a)}
\end{aligned}
$$

Higher order derivatives

Let $g(x) = \dfrac{1}{x^3}$.

- Calculate the first few derivatives.
- Make a conjecture for a formula for the n-th derivative $g^{(n)}(x)$.
- Prove it by induction.

Nixon

Richard Nixon, during the 1972 US Presidential campaign, (paraphrased):

> *Inflation is increasing, but the rate of increase of inflation is decreasing.*

Let

- C = cost of life
- t = time

What did Nixon say in terms of derivatives?

Quick composition

Let f and g be differentiable functions and let $h = f \circ g$. What is $h'(2)$?

1. $f'(2) \circ g'(2)$
2. $f'(2)g'(2)$
3. $f'(g(2))g'(2)$
4. $f'(g(x))g'(2)$

True or False - Differentiability and Composition

Let f and g be functions with domain $\mathbb{R}$. Let $c \in \mathbb{R}$.
Assume f and g are differentiable at c.
What can we conclude?

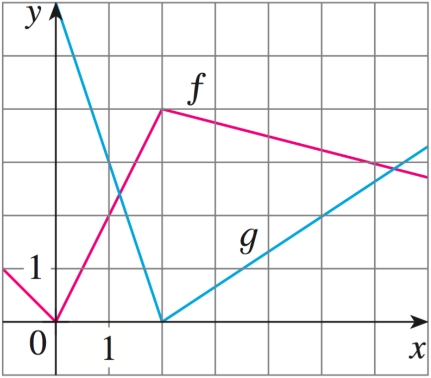
1. $f \circ g$ is differentiable at c.
2. $f \circ f$ is differentiable at c.
3. $f \circ \sin$ is differentiable at c.
4. $\sin \circ f$ is differentiable at c.

Chain rule from a graph

If f and g are the functions whose graphs are shown.
Let $u(x) = f(g(x))$ and $v(x) = g(f(x))$.

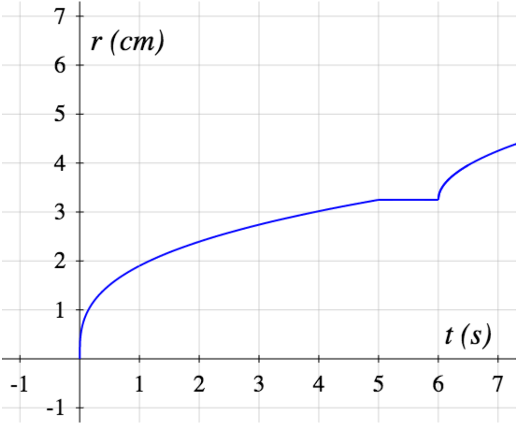
Find each derivative, if it exists.
If it does not exist, explain why.

1. $u'(1)$
2. $v'(1)$



Balloon

I am inflating a spherical balloon. Below is the graph of the radius r (in cm) as a function of time t (in s). At what rate is the volume of the balloon increasing at time $4s$?



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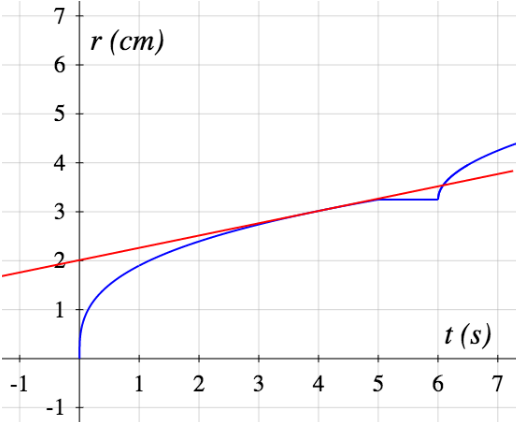

An alternative proof of the quotient rule

Assume we have already proven the product rule, the power rule, and the chain rule.

Obtain a formula for the derivative of $h(x) = \dfrac{f(x)}{g(x)}$.

Hint: $\dfrac{f(x)}{g(x)} = f(x) \cdot g(x)^{-1}$

Derivatives of $(f \circ g)$

Assume f and g are functions that have all their derivatives. Find formulas for

1. $(f \circ g)'(x)$
2. $(f \circ g)''(x)$
3. $(f \circ g)'''(x)$

in terms of the values of f, g and their derivatives.

Hint: The first one is simply the chain rule.

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in terms of the values of f, g and their derivatives.

Hint: The first one is simply the chain rule.

Challenge: Find a formula for $(f \circ g)^{(n)}(x)$
(This is beyond the scope of this course).

Derivative of cos

Let $g(x) = \cos x$.

Obtain and prove a formula for its derivative directly from the definition of derivative as a limit.

Hint: Imitate the derivation in Video 3.12.
If you need a trig identity that you do not know, google it or ask another student.

Derivatives of the other trig functions

Use the basic differentiation rules, as well as

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x,$$

to quickly obtain and prove formulas for the derivatives of tan, cot, sec, and csc.

Product of trig functions

Let $f(x) = \sin x \cos x$. What is its derivative $f'(x)$?

1. $1 - 2\sin^2(x)$
2. $2\cos^2(x) - 1$
3. $\cos 2x$
4. all of the above
5. none of the above

A pesky function

Let $h(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

1. Calculate $h'(x)$ for any $x \neq 0$.
2. Using the definition of derivative, calculate $h'(0)$.
3. Calculate $\lim_{x \to 0} h'(x)$

   Hint: Questions 2 and 3 have different answers.

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2. Using the definition of derivative, calculate $h'(0)$.
3. Calculate $\lim\limits_{x \to 0} h'(x)$

   Hint: Questions 2 and 3 have different answers.

4. Is h continuous at 0?
5. Is h differentiable at 0?
6. Is h' continuous at 0?

Implicit differentiation

The equation

$$\sin(x+y)+xy^2=0$$

defines a function $y=h(x)$ near $(0,0)$. ▸ graph
Using implicit differentiation, compute

1. $h(0)$
2. $h'(0)$
3. $h''(0)$
4. $h'''(0)$