MAT137 Lecture 17 — Definition of Derivative

Before next class:

- **Watch videos 3.4, 3.5, 3.8**

Tangent line to a line?

What is the equation of the line tangent to the graph of $y = x$ at the point with x–coordinate 7?

(A) $y = x + 7$
(B) $y = x$
(C) $y = 7$
(D) $x = 7$
(E) There is no tangent line at that point.
(F) There is more than one tangent line at that point.

Prove these statements are false with counterexamples

Let C be a curve. Let P be a point in C.

(A) The line tangent to C at P intersects C at only one point: P.

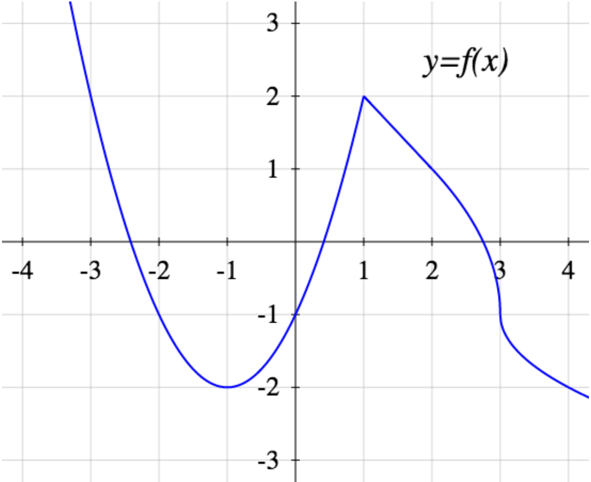
(B) If a line intersects C only at P, then that line must be the tangent line to C at P.

(C) The tangent line to C at P intersects C at P and "does not cross" C at P. (This means that, near P, it stays on one side of C.)

(D) If a line intersects C at P and "does not cross" C at P, then it is the tangent line to C at P.

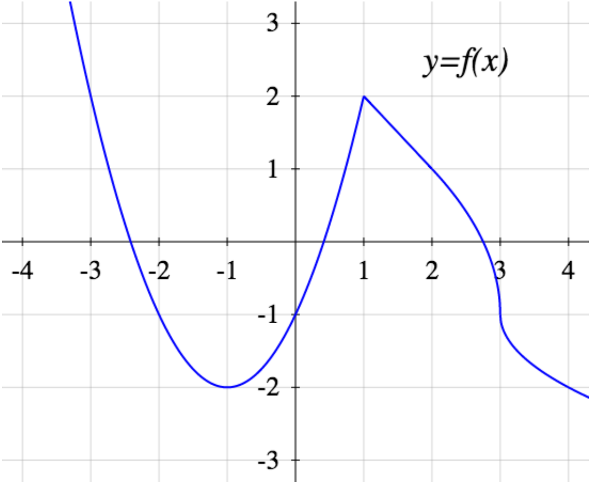
Tangent line from a graph

This is the graph of the function f. Write the (approximate) equation of the line tangent to it at the point with x–coordinate -2.



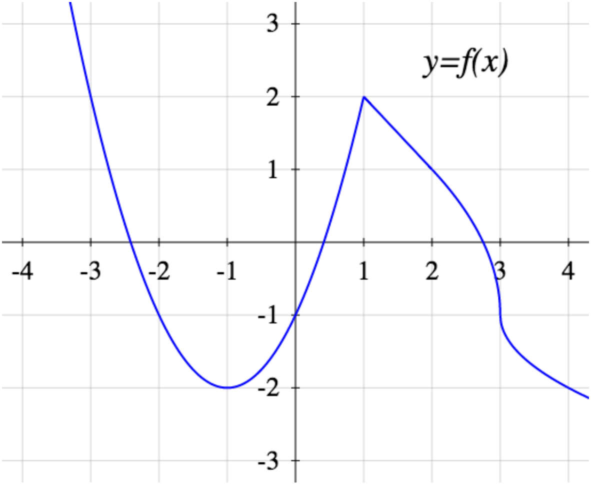
Tangent line from a graph

This is the graph of the function f. Write the (approximate) equation of the line tangent to it at the point with x–coordinate -1.



Derivative from a graph

This is the graph of the function f.
Sketch the graph of its derivative f'.



Derivatives from the definition

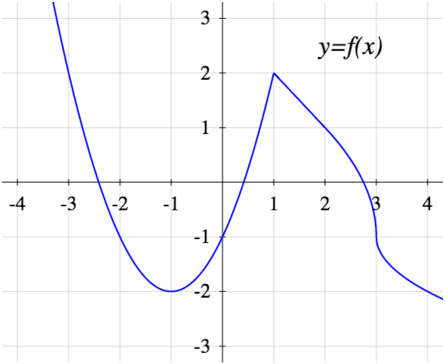
Let

$$g(x) = \frac{2}{\sqrt{x}}$$

Calculate $g'(4)$ directly from the definition of derivative as a limit.

Warmup:

Sketch $y = f'(x)$.



Before next class:

- **Watch videos 3.6, 3.7, 3.9**

Differentiable functions

Let $a \in \mathbb{R}$.
Let f be a function with domain $\mathbb{R}$.
Assume f is differentiable everywhere.
What can we conclude?

(A) $f(a)$ is defined.

(B) $\lim_{x \to a} f(x)$ exists.

(C) f is continuous at a.

(D) $f'(a)$ exists.

(E) $\lim_{x \to a} f'(x)$ exists.

(F) f' is continuous at a.

Compute the derivative of the following functions:

(A) $f(x) = x^{100} - 3x^9 - 2$

(B) $f(x) = \sqrt[3]{x} + 6$

(C) $f(x) = \dfrac{4}{x^4}$

(D) $f(x) = \sqrt{x}\,(1 + 2x)$

(E) $f(x) = \dfrac{x^6 + 1}{x^3}$

(F) $f(x) = \dfrac{x^2 - 2}{x^2 + 2}$

Higher order derivatives

Let $g(x) = \dfrac{1}{x^3}$.

- Calculate the first few derivatives.
- Make a conjecture for a formula for the n-th derivative $g^{(n)}(x)$.
- Prove it by induction.

Estimations - 1

Let f be a continuous function with domain $\mathbb{R}$.

(A) We know $f(4) = 3$ and $f(4.2) = 2.2$.
Based only on this, give your best estimate for $f(4.1)$.

(B) We know $f(4) = 3$ and $f'(4) = 0.6$.
Based only on this, give your best estimate for $f(4.1)$.

(C) We know $f(4) = 3$ and $f(4.1) = 4$.
Based only on this, give your best estimate for $f'(4)$.

Estimations – 2

Without using a calculator, estimate $\sqrt[20]{1.01}$ as well as you can.

Hint: You know the value of $f(x) = \sqrt[20]{x}$ and its derivative at one point very close to 1.01. Use the tangent line at that point as an approximation.

Estimations – 3

(A) We know
$f(0) = 2, \quad f'(0) = 3, \quad g(0) = 7, \quad g'(0) = 5.$

Compute $\displaystyle \lim_{x \to 0} \frac{f(x)}{g(x)}$.

(B) We know
$f(0) = 0, \quad f'(0) = 3, \quad g(0) = 0, \quad g'(0) = 5.$

- When x is close to 0, give estimates for $f(x)$ and $g(x)$ using the tangent lines at 0.
- Use those estimates to compute $\displaystyle \lim_{x \to 0} \frac{f(x)}{g(x)}$.

Before next class:

- **Watch videos 3.10, 3.11**

True or False - Differentiability vs Continuity

Let f be a function with domain $\mathbb{R}$. Let $c \in \mathbb{R}$.
Which of these implications are true?

(A) IF f is continuous at c, THEN f is differentiable at c

(B) IF f is differentiable at c, THEN f is continuous at c

(C) IF f is differentiable at c, THEN f' is continuous at c

(D) IF f' is continuous at c, THEN f is continuous at c

(E) IF f is differentiable at c, THEN f is continuous at and near c.

(F) IF f is continuous at and near c, THEN f is differentiable at c.

True or False - Differentiability and Operations

Let f be a function with domain $\mathbb{R}$. Let $c \in \mathbb{R}$.
Let $g(x) = f(x)^2$. Which of these implications are true?

(A) IF f is differentiable at c, THEN $f + f'$ is continuous at c

(B) IF f is differentiable at c, THEN $3f$ is differentiable at c.

(C) IF f is differentiable at c, THEN g is differentiable at c.

(D) IF g is differentiable at c, THEN f is differentiable at c.

(E) IF f is differentiable at c, THEN $1/f$ is differentiable at c.

Absolute value and tangent lines

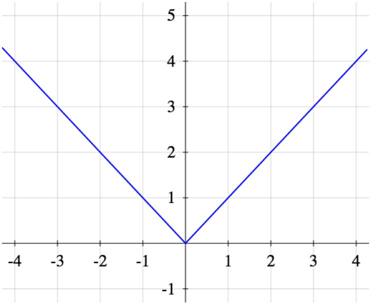
At (0,0) the graph of $y = |x|$...

(A) ... has one tangent line: $y = 0$

(B) ... has one tangent line: $x = 0$

(C) ... has two tangent lines $y = x$ and $y = -x$

(D) ... has no tangent line



Absolute value and derivatives

Let $h(x) = x|x|$. What is $h'(0)$?

(A) It is 0.

(B) It doesn't exist because $|x|$ is not differentiable at 0.

(C) It doesn't exist because the right- and left-limits, when computing the derivative, are different.

(D) It doesn't exist because it has a corner.

(E) It doesn't exist for a different reason.

Write a proof for the quotient rule for derivatives

Theorem

- Let $a \in \mathbb{R}$.
- Let f and g be functions defined at and near a. Assume $g(x) \neq 0$ for x close to a.
- We define the function h by $h(x) = \dfrac{f(x)}{g(x)}$.

IF f and g are differentiable at a,
THEN h is differentiable at a, and

$$h'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Write a proof directly from the definition of derivative.
Hint: Imitate the proof of the product rule in Video 3.6.

Check your proof

(A) Did you use the *definition* of derivative?
(B) Are there words or only equations?
(C) Does every step follow logically?
(D) Did you only assume things you could assume?

(E) Did you assume at some point that a function was differentiable? If so, did you justify it?
(F) Did you assume at some point that a function was continuous? If so, did you justify it?

If you answered "no" to Q6, you probably missed something important.

Critique this proof

$$
\begin{aligned}
h'(a) &= \lim_{x\to a} \frac{h(x)-h(a)}{x-a} = \lim_{x\to a} \frac{\dfrac{f(x)}{g(x)} - \dfrac{f(a)}{g(a)}}{x-a} \\
&= \lim_{x\to a} \frac{f(x)g(a)-f(a)g(x)}{g(x)g(a)\,(x-a)} \\
&= \lim_{x\to a} \frac{f(x)g(a)-f(a)g(a)+f(a)g(a)-f(a)g(x)}{g(x)g(a)\,(x-a)} \\
&= \lim_{x\to a} \left\{ \left[\frac{f(x)-f(a)}{x-a} g(a) - f(a)\frac{g(x)-g(a)}{x-a} \right] \frac{1}{g(x)g(a)} \right\} \\
&= \left[f'(a)g(a) - f(a)g'(a) \right] \frac{1}{g(a)g(a)}
\end{aligned}
$$

Before next class:

- **Watch videos 3.12, 3.13**

Quick composition

Let f and g be differentiable functions and let $h = f \circ g$. What is $h'(2)$?

(A) $f'(2) \circ g'(2)$

(B) $f'(2)g'(2)$

(C) $f'(g(2))g'(2)$

(D) $f'(g(x))g'(2)$

True or False - Differentiability and Composition

Let f and g be functions with domain $\mathbb{R}$. Let $c \in \mathbb{R}$.
Assume f and g are differentiable at c.
What can we conclude?

(A) $f \circ g$ is differentiable at c.

(B) $f \circ f$ is differentiable at c.

(C) $f \circ \sin$ is differentiable at c.

(D) $\sin \circ f$ is differentiable at c.

Computations: Chain rule

Compute the derivative of

(A) $f(x) = \left(2x^2 + x + 1\right)^8$

(B) $f(x) = \dfrac{1}{\left(x + \sqrt{x^2 + x}\right)^{137}}$

MAT137 Lecture 21 — Trig Derivatives and Implicit Differentiation

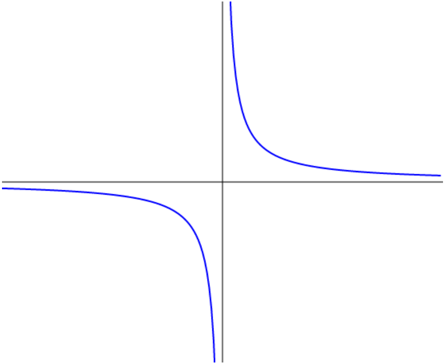
Before next class:

- **Watch videos 4.1, 4.2**

From the derivative to the function

(A) Sketch the graph of a continuous function with domain $\mathbb{R}$, whose derivative has the graph below.

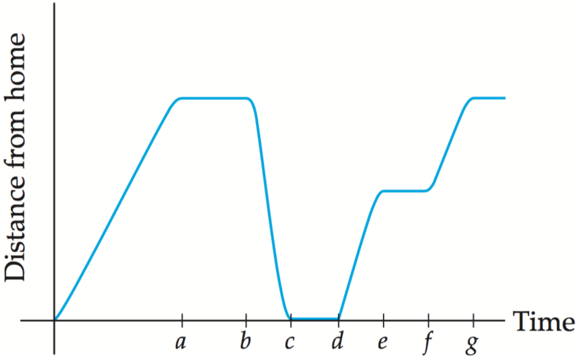
(B) Sketch the graph of a non-continuous function whose derivative has the graph below.



Bella

The graph below describes Bella's distance from home one morning as she drives drive between her home and school.

Describe a possible scenario for her travels that morning. Then sketch the corresponding graph of his velocity.



Jacob walked at 5 km/h for 20 minutes and then sprinted at 15 km/h for 8 minutes.

(A) How fast would Edward have to walk or run to go the same distance as Jacob did in the same time while moving at a constant speed?

(B) Sketch a graph of Jacob's and Edward's positions over time on the same set of axes.

A long chain

The function below has 137 square roots:

$$f(x) = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots + \sqrt{x + \sqrt{x + 1}}}}}}$$

Find the equation of the line tangent to the graph of f at the point with x-coordinate 0.

Computations: Trig derivatives

Compute the derivatives of the following functions:

(A) $f(x) = \tan(3x^2 + 1)$

(B) $f(x) = (\cos x)(\sin 2x)(\tan 3x)$

(C) $f(x) = \cos(\sin(\tan x))$

(D) $f(x) = \cos\left(3x + \sqrt{1 + \sin^2 x^2}\right)$

Vertical things

- Construct a function f that has a vertical asymptote at $x = 2$.
- Construct a function g that has a vertical tangent line at $x = 2$.

Absolute value and derivatives - 2

True or False?

For all $n \in \mathbb{Z}$ and all x, $\frac{d}{dx}|x|^n = nx|x|^{n-2}$.

Nixon

Richard Nixon, during the 1972 US Presidential campaign, (paraphrased):

> *Inflation is increasing, but the rate of increase of inflation is decreasing.*

Let

- C = cost of life
- t = time

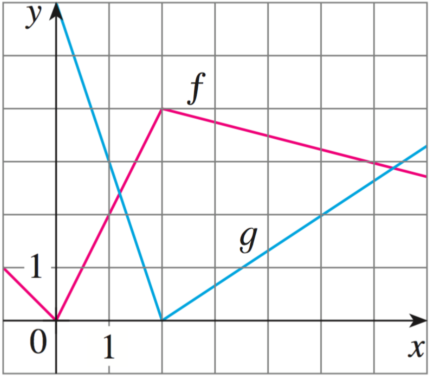
What did Nixon say in terms of derivatives?

Chain rule from a graph

If f and g are the functions whose graphs are shown.
Let $u(x) = f(g(x))$ and $v(x) = g(f(x))$.

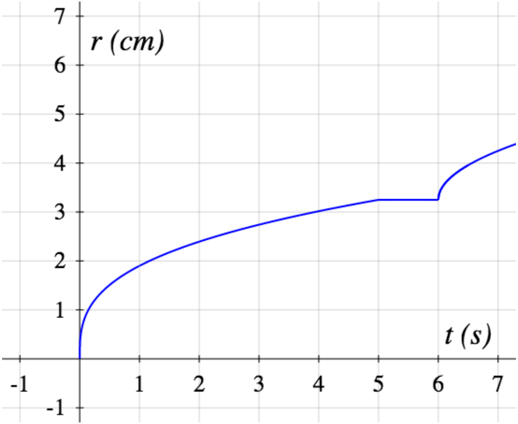
Find each derivative, if it exists.
If it does not exist, explain why.

(A) $u'(1)$
(B) $v'(1)$

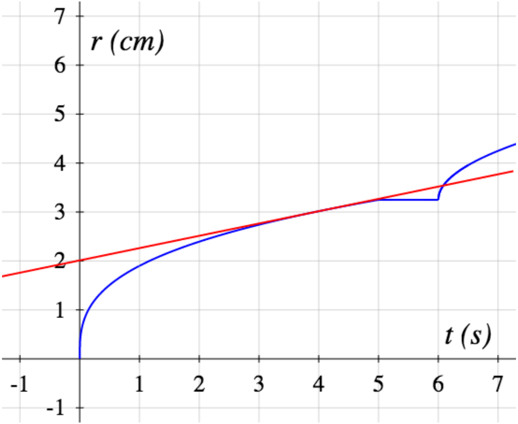


Balloon

I am inflating a spherical balloon. Below is the graph of the radius r (in cm) as a function of time t (in s). At what rate is the volume of the balloon increasing at time $4s$?



I am inflating a spherical balloon. Below is the graph of the radius r (in cm) as a function of time t (in s). At what rate is the volume of the balloon increasing at time $4s$?



An alternative proof of the quotient rule

Assume we have already proven the product rule, the power rule, and the chain rule.

Obtain a formula for the derivative of $h(x) = \dfrac{f(x)}{g(x)}$.

Hint: $\dfrac{f(x)}{g(x)} = f(x) \cdot g(x)^{-1}$

Derivatives of $(f \circ g)$

Assume f and g are functions that have all their derivatives. Find formulas for

(A) $(f \circ g)'(x)$

(B) $(f \circ g)''(x)$

(C) $(f \circ g)'''(x)$

in terms of the values of f, g and their derivatives.

Hint: The first one is simply the chain rule.

Derivatives of $(f \circ g)$

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(A) $(f \circ g)'(x)$
(B) $(f \circ g)''(x)$
(C) $(f \circ g)'''(x)$

in terms of the values of f, g and their derivatives.

Hint: The first one is simply the chain rule.

Challenge: Find a formula for $(f \circ g)^{(n)}(x)$
(This is beyond the scope of this course).

Derivative of cos

Let $g(x) = \cos x$.

Obtain and prove a formula for its derivative directly from the definition of derivative as a limit.

Hint: Imitate the derivation in Video 3.12.
If you need a trig identity that you do not know, google it or ask another student.

Derivatives of the other trig functions

Use the basic differentiation rules, as well as

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x,$$

to quickly obtain and prove formulas for the derivatives of tan, cot, sec, and csc.

Product of trig functions

Let $f(x) = \sin x \cos x$. What is its derivative $f'(x)$?

(A) $1 - 2\sin^2(x)$

(B) $2\cos^2(x) - 1$

(C) $\cos 2x$

(D) all of the above

(E) none of the above

A pesky function

Let $h(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

(A) Calculate $h'(x)$ for any $x \neq 0$.

(B) Using the definition of derivative, calculate $h'(0)$.

(C) Calculate $\lim_{x \to 0} h'(x)$

Hint: Questions 2 and 3 have different answers.

A pesky function

Let $h(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

(A) Calculate $h'(x)$ for any $x \neq 0$.

(B) Using the definition of derivative, calculate $h'(0)$.

(C) Calculate $\lim_{x \to 0} h'(x)$

Hint: Questions 2 and 3 have different answers.

(D) Is h continuous at 0?

(E) Is h differentiable at 0?

(F) Is h' continuous at 0?

Implicit differentiation

The equation

$$\sin(x+y) + xy^2 = 0$$

defines a function $y = h(x)$ near $(0,0)$. graph
Using implicit differentiation, compute

(A) $h(0)$ (B) $h'(0)$ (C) $h''(0)$ (D) $h'''(0)$