A telescopic series

I want to calculate the value of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$.

1. Find a formula for the k-th partial sum $S_k = \displaystyle\sum_{n=1}^{k} \frac{1}{n^2+2n}$.

   Hint: $\displaystyle \frac{1}{n^2+2n} = \frac{A}{n} + \frac{B}{n+2}$

2. Using the definition of series, compute the value of

$$\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$$

What is wrong with this calculation? Fix it

Claim: $$\sum_{n=2}^{\infty} \ln \frac{n}{n+1} = \ln 2$$

"Proof"

$$\begin{aligned}
\sum_{n=2}^{\infty} \ln \frac{n}{n+1} &= \sum_{n=2}^{\infty} [\ln n - \ln(n+1)] \\
&= \sum_{n=2}^{\infty} \ln(n) - \sum_{n=2}^{\infty} \ln(n+1) \\
&= (\ln 2 + \ln 3 + \ln 4 + \dots) - (\ln 3 + \ln 4 + \dots) \\
&= \ln 2
\end{aligned}$$

Trig series: convergent or divergent?

1. $\displaystyle\sum_{n=0}^{\infty} \sin(n\pi)$

2. $\displaystyle\sum_{n=0}^{\infty} \cos(n\pi)$

Help me write the next assignment

In the next assignment I want to give you a series and ask you to calculate its value from the definition. I want the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ to be

$$\forall n \geq 1, \; S_n = n^2$$

What series should I ask you to calculate?

What can you conclude?

Assume $\forall n \in \mathbb{N},\ a_n > 0$. Consider the series $\displaystyle\sum_{n=0}^{\infty} a_n$.

Let $\{S_n\}_{n=0}^{\infty}$ be its sequence of partial sums.

In each of the following cases, what can you conclude about the *series*? Is it convergent, divergent, or we do not know?

1. $\forall n \in \mathbb{N}, \quad \exists M \in \mathbb{R}$ s.t. $\quad S_n \leq M$.
2. $\exists M \in \mathbb{R}$ s.t. $\quad \forall n \in \mathbb{N}, \quad S_n \leq M$.
3. $\exists M > 0$ s.t. $\quad \forall n \in \mathbb{N}, \quad a_n \leq M$.
4. $\exists M > 0$ s.t. $\quad \forall n \in \mathbb{N}, \quad a_n \geq M$.

Harmonic series

For each $n > 0$ we define

$$r_n = \text{ smallest power of 2 that is greater than or equal to } n$$

Consider the series $S = \sum_{n=1}^{\infty} \frac{1}{r_n}$

1. Compute r_1 through r_8
2. Compute the partial sums S_1, S_2, S_4, S_8 for the series S.
3. Calculate $S = \sum_{n=1}^{\infty} \frac{1}{r_n}$.
4. Calculate $H = \sum_{n=1}^{\infty} \frac{1}{n}$. *Hint:* "Compare" H and S.

True or False – Definition of series

Let $\sum_{n=0}^{\infty} a_n$ be a series. Let $\{S_n\}_{n=0}^{\infty}$ be its partial-sum sequence.

1. IF the series $\sum_{n=0}^{\infty} a_n$ is convergent,

   THEN the sequence $\{S_n\}_{n=0}^{\infty}$ is bounded.

2. IF the series $\sum_{n=0}^{\infty} a_n$ is convergent,

   THEN the sequence $\{S_n\}_{n=0}^{\infty}$ is eventually monotonic.

3. IF the sequence $\{S_n\}_{n=0}^{\infty}$ is bounded and eventually monotonic,

   THEN the series $\sum_{n=0}^{\infty} a_n$ is convergent.

True or False – Definition of series

Let $\sum_{n=0}^{\infty} a_n$ be a series. Let $\{S_n\}_{n=0}^{\infty}$ be its partial-sum sequence.

4. IF $\forall n > 0,\ a_n > 0$,
   THEN the sequence $\{S_n\}_{n=0}^{\infty}$ is increasing.

5. IF the sequence $\{S_n\}_{n=0}^{\infty}$ is increasing,
   THEN $\forall n > 0,\ a_n > 0$.

6. IF $\forall n > 0,\ a_n \geq 0$,
   THEN the sequence $\{S_n\}_{n=0}^{\infty}$ is non-decreasing.

7. IF the sequence $\{S_n\}_{n=0}^{\infty}$ is non-decreasing,
   THEN $\forall n > 0,\ a_n \geq 0$

Rapid questions: geometric series

Convergent or divergent?

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n/2}}$

4. $\displaystyle\sum_{n=5}^{\infty} \frac{3^n}{2^{2n+1}}$

5. $\displaystyle\sum_{n=3}^{\infty} \frac{3^n}{1000 \cdot 2^{n+2}}$

6. $\displaystyle\sum_{n=0}^{\infty} (-1)^n$

Geometric series

Calculate the value of the following series:

1. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots$

2. $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \dots$

3. $\frac{3}{2} - \frac{9}{4} + \frac{27}{8} - \frac{81}{16} + \dots$

4. $1 + \frac{1}{2^{0.5}} + \frac{1}{2} + \frac{1}{2^{1.5}} + \frac{1}{2^2} + \frac{1}{2^{2.5}} + \dots$

5. $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{2^{2n+1}}$

6. $\sum_{n=k}^{\infty} x^n$

Is $0.999999\ldots = 1$?

Is $0.999999\ldots = 1$?

1. Write the number $\quad 0.9999999\ldots \quad$ as a series
   Hint: $427 = 400 + 20 + 7$.

2. Compute the first few partial sums

3. Add up the series.
   Hint: it is geometric.

Decimal expansions of rational numbers

We can interpret any finite decimal expansion as a finite sum. For example:

$$2.13096 = 2 + \frac{1}{10} + \frac{3}{10^2} + \frac{0}{10^3} + \frac{9}{10^4} + \frac{6}{10^5}$$

Similarly, we can interpret any infinite decimal expansion as an infinite series.

Interpret the following numbers as series, and add up the series to calculate their value as fractions:

1. $0.99999\ldots$
2. $0.11111\ldots$
3. $0.252525\ldots$
4. $0.3121212\ldots$

Hint: Use geometric series

Functions as series

You know that when $|x| < 1$:

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Find similar ways to write the following functions as series:

1. $g(x) = \dfrac{1}{1+x}$
2. $h(x) = \dfrac{1}{1-x^2}$
3. $A(x) = \dfrac{1}{2-x}$

Functions as series

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2. $h(x) = \dfrac{1}{1-x^2}$
3. $A(x) = \dfrac{1}{2-x}$
4. $G(x) = \ln(1+x)$

Hint: For the last one, compute G'.

Challenge

We want to calculate the value of

$$A = \sum_{n=0}^{\infty} \frac{1}{2^n}, \qquad B = \sum_{n=1}^{\infty} \frac{n}{2^n}, \qquad C = \sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

Let $f(x) = \dfrac{1}{1-x}$.

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1. Recall that $f(x) = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. Use it to compute A.
2. Pretend you can take derivatives of series the way you take them of finite sums. Write $f'(x)$ as a series.
3. Use it to compute B.
4. Do something similar to compute C.

Challenge - 2

We want to calculate the value of

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,3^n}$$

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1. Compute $\sum_{n=0}^{\infty}(-1)^n x^{2n}$
2. Compute $\frac{d}{dx}[\arctan x]$
3. Pretend you can take derivatives and antiderivatives of series the way you can take them of finite sums. Which series adds up to $\arctan x$?
4. Now calculate the value of the original series.

Examples

1. A series $\sum_{n=0}^{\infty} a_n$ may be

$$\begin{cases} \text{convergent (a number)} \\ \text{divergent} \begin{cases} \text{to } \infty \\ \text{to } -\infty \\ \text{``oscillating''} \end{cases} \end{cases}$$

Give one example of each of the four results.

Examples

1. A series $\sum_{n=0}^{\infty} a_n$ may be

$$\begin{cases} \text{convergent (a number)} \\ \text{divergent} \begin{cases} \text{to } \infty \\ \text{to } -\infty \\ \text{“oscillating”} \end{cases} \end{cases}$$

   Give one example of each of the four results.

2. Now assume $\forall n \in \mathbb{N},\ a_n \geq 0$.
   Which of the four outcomes is still possible?

True or False – The tail of a series

1. IF the series $\displaystyle\sum_{n=0}^{\infty} a_n$ converges,

   THEN the series $\displaystyle\sum_{n=7}^{\infty} a_n$ converges

2. IF the series $\displaystyle\sum_{n=7}^{\infty} a_n$ converges,

   THEN the series $\displaystyle\sum_{n=0}^{\infty} a_n$ converges

3. IF the series $\displaystyle\sum_{n=0}^{\infty} a_n$ converges,

   THEN the series $\displaystyle\sum_{n=7}^{\infty} a_n$ converges to a smaller number.

True or False – The Necessary Condition

1. IF $\lim_{n\to\infty} a_n = 0$, THEN $\sum_n^{\infty} a_n$ is convergent.

2. IF $\lim_{n\to\infty} a_n \neq 0$, THEN $\sum_n^{\infty} a_n$ is divergent.

3. IF $\sum_n^{\infty} a_n$ is convergent THEN $\lim_{n\to\infty} a_n = 0$.

4. IF $\sum_n^{\infty} a_n$ is divergent THEN $\lim_{n\to\infty} a_n \neq 0$.

True or False – Harder questions

1. IF $\sum_{n=0}^{\infty} a_n$ is convergent, THEN $\lim_{k\to\infty}\left[\sum_{n=k}^{\infty} a_n\right] = 0.$

2. IF $\lim_{k\to\infty}\left[\sum_{n=k}^{\infty} a_n\right] = 0,$ THEN $\sum_{n=0}^{\infty} a_n$ is convergent.

3. IF $\sum_{n=1}^{\infty} a_{2n}$ and $\sum_{n=1}^{\infty} a_{2n+1}$ are convergent,

   THEN $\sum_{n=1}^{\infty} a_n$ is convergent.

4. IF $\sum_{n=1}^{\infty} a_n$ is convergent,

   THEN $\sum_{n=1}^{\infty} a_{2n}$ and $\sum_{n=1}^{\infty} a_{2n+1}$ are convergent.

Series are linear

Let $\sum_{n=0}^{\infty} a_n$ be a series. Let $c \in \mathbb{R}$. Prove that

- IF $\sum_{n=0}^{\infty} a_n$ is convergent.
- THEN $\sum_{n=0}^{\infty} (ca_n)$ is convergent and $\sum_{n=0}^{\infty} (ca_n) = c \left[\sum_{n=0}^{\infty} a_n \right]$.

Write a proof directly from the definition of series.

Rapid questions: improper integrals

Convergent or divergent?

1. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

2. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

3. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

Rapid questions: improper integrals

Convergent or divergent?

1. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
3. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$
4. $\displaystyle\int_1^\infty \frac{x+1}{x^3+2}\,dx$
5. $\displaystyle\int_1^\infty \frac{\sqrt{x^2+5}}{x^2+6}\,dx$
6. $\displaystyle\int_1^\infty \frac{x^2+3}{\sqrt{x^5+2}}\,dx$

For which values of $a \in \mathbb{R}$ are these series convergent?

1. $\displaystyle\sum_{n}^{\infty} \frac{1}{a^n}$
2. $\displaystyle\sum_{n}^{\infty} \frac{1}{n^a}$
3. $\displaystyle\sum_{n}^{\infty} a^n$
4. $\displaystyle\sum_{n}^{\infty} n^a$

Quick comparisons: convergent or divergent?

1. $\displaystyle\sum_{n}^{\infty} \frac{n+1}{n^2+1}$

2. $\displaystyle\sum_{n}^{\infty} \frac{n^2+3n}{n^4+5n+1}$

3. $\displaystyle\sum_{n}^{\infty} \frac{\sqrt{n}+1}{n^2+1}$

4. $\displaystyle\sum_{n}^{\infty} \frac{\sqrt[3]{n^2+1}+1}{\sqrt{n^3+n}+n+1}$

Slow comparisons: convergent or divergent?

1. $\displaystyle\sum_{n}^{\infty} \frac{2^n - 40}{3^n - 20}$

2. $\displaystyle\sum_{n}^{\infty} \frac{(\ln n)^{20}}{n^2}$

3. $\displaystyle\sum_{n}^{\infty} \sin^2 \frac{1}{n}$

4. $\displaystyle\sum_{n}^{\infty} \frac{1}{n(\ln n)^3}$

5. $\displaystyle\sum_{n}^{\infty} \frac{1}{n \ln n}$

6. $\displaystyle\sum_{n}^{\infty} e^{-n^2}$

Convergence tests: ninja level

We know

- $\forall n \in \mathbb{N},\ a_n > 0.$
- the series $\sum_{n}^{\infty} a_n$ is convergent

Determine whether the following series are convergent, divergent, or we do not have enough information to decide:

1. $\sum_{n}^{\infty} \sin a_n$
2. $\sum_{n}^{\infty} \cos a_n$
3. $\sum_{n}^{\infty} \sqrt{a_n}$
4. $\sum_{n}^{\infty} (a_n)^2$

Are all decimal expansions well-defined?

We had defined a real number as “any number with a decimal expansion”. Now we understand what it means to write a number with an infinite decimal expansion. It is a series!

$$0.a_1 a_2 a_3 a_4 a_5 \cdots = \frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{1000} + \ldots$$

for any “digits” a_1, a_2, a_3, ...

But this raises a question: are these series always convergent, no matter which infinite string of digits we choose?

Yes, they are! Prove it.
(Hint: all the terms in the series are positive.)

Rapid questions: alternating series test

Convergent or divergent?

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.5}}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3}}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sin n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.5}}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3}}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sin n}$

True or False - Odd and even partial sums

Let $\sum_{n=0}^{\infty} a_n$ be a series. Let $\{S_n\}_{n=0}^{\infty}$ be its partial-sum sequence.

1. IF $\lim_{n\to\infty} S_{2n}$ exists, THEN $\sum_{n=0}^{\infty} a_n$ is convergent.
2. IF $\lim_{n\to\infty} S_{2n}$ exists and $\lim_{n\to\infty} S_{2n+1}$ exists,

   THEN $\sum_{n=0}^{\infty} a_n$ is convergent.
3. IF $\lim_{n\to\infty} S_{2n}$ exists and $\lim_{n\to\infty} a_n = 0$,

   THEN $\sum_{n=0}^{\infty} a_n$ is convergent.

An Alternating Series Test example

Verify carefully the 3 hypotheses of the Alternating Series Test for

$$\sum_{n=0}^{\infty} (-1)^n \frac{n-\pi}{e^n}$$

Can we conclude it is convergent?

Estimation

Estimate the sum

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

with an error smaller than 0.001. Write your final answer as a rational number (i.e. as a quotient of two integers).

Not exactly alternating

Are these series convergent or divergent?

$$A = 1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11} - \frac{1}{12} + \dots$$

$$B = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} - \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} - \dots$$

Suggestion: Draw the partial sums on the real line.

A counterexample to Alternating Series Test?

Construct a series of the form $\sum_{n=1}^{\infty}(-1)^n b_n$ such that

- $b_n > 0$ for all $n \geq 1$
- $\lim_{n\to\infty} b_n = 0$
- the series $\sum_{n=1}^{\infty}(-1)^n b_n$ is divergent.

Absolutely convergent or conditionally convergent?

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.5}}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1.5}}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sin n}$

True or False - Absolute Values

1. IF $\{a_n\}_{n=1}^{\infty}$ is convergent, THEN $\{\,|a_n|\,\}_{n=1}^{\infty}$ is convergent.

2. IF $\{\,|a_n|\,\}_{n=1}^{\infty}$ is convergent, THEN $\{a_n\}_{n=1}^{\infty}$ is convergent.

3. IF $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, THEN $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is convergent.

4. IF $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is convergent, THEN $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

Positive and negative terms - 1

- Let $\sum a_n$ be a series.
- Call $\sum$(P.T.) the sum of only the positive terms of the same series.
- Call $\sum$(N.T.) the sum of only the negative terms of the same series.

Positive and negative terms - 1

- Let $\sum a_n$ be a series.
- Call $\sum$(P.T.) the sum of only the positive terms of the same series.
- Call $\sum$(N.T.) the sum of only the negative terms of the same series.

IF $\sum$(P.T.) is...	AND $\sum$(N.T.) is...	THEN $\sum a_n$ may be...
CONV	CONV	
∞	CONV	
CONV	$-\infty$	
∞	$-\infty$	

Positive and negative terms - 2

- Let $\sum a_n$ be a series.
- $\sum(\text{P.T.}) \;=\;$ sum of only the positive terms of the same series.
- $\sum(\text{N.T.}) \;=\;$ sum of only the negative terms of the same series.

Positive and negative terms - 2

- Let $\sum a_n$ be a series.
- $\sum(\text{P.T.}) \quad = \quad$ sum of only the positive terms of the same series.
- $\sum(\text{N.T.}) \quad = \quad$ sum of only the negative terms of the same series.

	$\sum(\text{P.T.})$ may be...	$\sum(\text{N.T.})$ may be...
If $\sum a_n$ is CONV		
If $\sum \lvert a_n \rvert$ is CONV		
If $\sum a_n$ is ABS CONV		
If $\sum a_n$ is COND CONV		
If $\sum a_n = \infty$		
If $\sum a_n$ is DIV oscillating		

Quick review: Convergent or divergent?

1. $\sum_{n}^{\infty}(1.1)^n$

2. $\sum_{n}^{\infty}(0.9)^n$

3. $\sum_{n}^{\infty}\frac{1}{n^{1.1}}$

4. $\sum_{n}^{\infty}\frac{1}{n^{0.9}}$

5. $\sum_{n}^{\infty}\frac{(-1)^n}{\ln n}$

6. $\sum_{n}^{\infty}\frac{(-1)^n}{e^{1/n}}$

7. $\sum_{n}^{\infty}\frac{n^3+n^2+11}{n^4+2n-3}$

8. $\sum_{n}^{\infty}\frac{\sqrt{n^5+2n+16}}{n^4-11n+7}$

Ratio Test: Convergent or divergent?

Use Ratio Test to decide which series are convergent.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2 \; 3^{n+1}}$

3. $\displaystyle\sum_{n=2}^{\infty} \frac{n!}{n^n}$

4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

Root test

Here is a new convergence test

Theorem

Let $\sum_n a_n$ be a series. Assume the limit $L = \lim_{n \to \infty} \sqrt[n]{|a_n|}$ exists.

- IF $0 \leq L < 1$ THEN the series is ???
- IF $L > 1$ THEN the series is ???

Without writing an actual proof, guess the conclusion of the theorem and argue why it makes sense.

Hint: Imitate the argument on Video 13.18 for the Ratio Test. For large values of n, what is $|a_n|$ approximately?