Warm-up

What are the following sets?

1. $[2,4] \cup (2,5)$
2. $[2,4] \cap (2,5)$
3. $[\pi, e]$
4. $[0,0]$
5. $(0,0)$

Similar sets

What are the following sets?

1. $A = \{x \in \mathbb{Z} : x^2 < 6\}$
2. $B = \{x \in \mathbb{N} : x^2 < 6\}$
3. $C = \{x \in \mathbb{R} : x^2 < 6\}$

Describing a new set

An irrational number is a number that is real but not rational.

B is the set of positive, rational numbers and negative, irrational numbers.

Write a definition for B using only mathematical notation.
(You may use the words “and”, “or”, and “such that”.)

Sets and quantifiers

What are the following sets?

1. $A = \{x \in \mathbb{R} : \forall y \in [0,1], x < y\}$
2. $B = \{x \in \mathbb{R} : \exists y \in [0,1] \text{ s.t. } x < y\}$
3. $C = \{x \in [0,1] : \forall y \in [0,1], x < y\}$
4. $D = \{x \in [0,1] : \exists y \in [0,1] \text{ s.t. } x < y\}$
5. $E = \{x \in [0,1] : \exists y \in \mathbb{R} \text{ s.t. } x < y\}$
6. $F = \{x \in [0,1] : y \in \mathbb{R}, x < y\}$

Functions and quantifiers

Let f be a function with domain $\mathbb{R}$. Rewrite the following statements using $\forall$ or $\exists$:

1. The graph of f intercepts the x-axis.
2. f is the zero function.
3. f is not the zero function.
4. f never vanishes.
5. The equation $f(x) = 0$ has a solution.
6. The equation $f(x) = 0$ has no solutions.
7. f takes both positive and negative values.
8. f is never negative.

Negation 1

Write the negation of these statements as simply as possible:

1. My favourite integer number is greater than 7.
2. I know at least five students at U of T who have a cellphone.
3. There is a country in the European Union with fewer than 1000 inhabitants.
4. All of my friends like apples.
5. I like apples and oranges.

$$\text{Negation of } \boxed{\cdots} = \boxed{\cdots} \text{ is false.}$$

Negation 2

Write the negation of this statement without using any negative words ("no", "not", "none", etc.):

> *"Every page in this book contains at least one word whose first and last letters both come alphabetically before M."*

Negation 3

Negate the following statement without using any negative words ("no", "not", "none", etc.):

> *"I have a friend all of whose former boyfriends had at least two siblings with exactly three different vowels in their name."*

Symmetric difference

Given two sets A and B, we define

- $A \setminus B = \{x \in A \ : \ x \notin B\}$
- $A \triangle B = (A \setminus B) \cup (B \setminus A)$

Let

- $C_1 = \{$ students under 18 $\}$
- $C_2 = \{$ students born in Ontario $\}$

What is the set $C_1 \triangle C_2$?

Symmetric difference - 2

Given two sets A and B, we define

- $A \setminus B = \{x \in A \ : \ x \notin B\}$
- $A \triangle B = (A \setminus B) \cup (B \setminus A)$

Is the following equality

$$(A \triangle B) \triangle C \ = \ A \triangle (B \triangle C)$$

true for all sets A, B, and C?

Even numbers

Write a description of the set E of even integers using set-building notation.

Even numbers

Which of these is a correct description of the set E of even integers?

1. $E = \{n \in \mathbb{Z} \ : \ \forall a \in \mathbb{Z},\, n = 2a\}$
2. $E = \{n \in \mathbb{Z} \ : \ \exists a \in \mathbb{Z} \text{ s.t. } n = 2a\}$

Even numbers

Which of these is a correct description of the set E of even integers?

1. $E = \{n \in \mathbb{Z} \ : \ \forall a \in \mathbb{Z}, \ n = 2a\}$
2. $E = \{n \in \mathbb{Z} \ : \ \exists a \in \mathbb{Z} \text{ s.t. } n = 2a\}$

Which of these statements is true?

3. $\forall a \in \mathbb{Z}$, the number $n = 2a$ is even.
4. $\exists a \in \mathbb{Z}$ s.t. the number $n = 2a$ is even.

Mother

Let

$$H = \{ \text{ humans } \}$$

True or False?

1. $\forall x \in H,\ \exists y \in H$ such that y gave birth to x
2. $\exists y \in H$ such that $\forall x \in H,\ y$ gave birth to x

Elephants

True or False?

1. There is a pink elephant in this room.
2. All elephants in this room are pink.

Indecisive function

Construct a function f that satisfies all of the following properties at once:

- The domain of f is $\mathbb{R}$.
- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that

$$x < y \text{ and } f(x) < f(y)$$

- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that

$$x < y \text{ and } f(x) > f(y)$$

Conditionals - True or False?

Let $x \in \mathbb{R}$.

1. $x > 0 \quad \Longrightarrow \quad x \geq 0$
2. $x \geq 0 \quad \Longrightarrow \quad x > 0$

Conditionals - True or False?

Let $x \in \mathbb{R}$.

1. $x > 0 \quad \Longrightarrow \quad x \geq 0$
2. $x \geq 0 \quad \Longrightarrow \quad x > 0$

3. IF $2 > 3$ THEN Alfonso is in love.

Negation of conditionals

Write the negation of these statements:

1. If Justin Trudeau has a brother, then he also has a sister.
2. If a student in this class has a brother, then they also have a sister.

Cards

Take a look at the following cards.

E	3	P	8

Each card has a letter on one side and a number on the other, and I tell you:

> "If *a card has a vowel on one side,*
> then *it has an odd number on the other side.*"

Which cards do you need to turn over in order to verify whether I am telling the truth or not?

Cards - 2

Four cards lie on the table in front of you. You know that each card has a letter on one side and a number on the other.

Negate the following statement:

"If *a card has a vowel on one side*,
then *it has an odd number on the other side.*"

Hockey

Which of the following statements are equivalent to the statement *"Every Canadian man likes hockey"*?

1. If a man is Canadian, then he likes hockey.
2. If a man likes hockey, then he is Canadian.
3. If a man does not like hockey, then he is not Canadian.
4. If a man is not Canadian, then he likes hockey.
5. Non-Canadian men do not like hockey.
6. If a Canadian does not like hockey, then she is not a man.

Graphs

Draw the graph of a function f with domain $\mathbb{R}$ that satisfies:

$$\text{If } 2 < x < 4 \text{ then } 1 < f(x) < 2.$$

Draw the graph of a function g with domain $\mathbb{R}$ that satisfies:

$$2 < x < 4 \text{ if and only if } 1 < g(x) < 2.$$

One-to-one functions

Let f be a function with domain D.

f is *one-to-one* means that ...

- ... different inputs (x) ...
- ... must produce different outputs ($f(x)$).

One-to-one functions

Let f be a function with domain D.

f is *one-to-one* means that ...

- ... different inputs (x) ...
- ... must produce different outputs ($f(x)$).

Write a formal definition of "one-to-one".

One-to-one functions

Definition: Let f be a function with domain D.
f is one-to-one means ...

1. $f(x_1) \neq f(x_2)$
2. $\exists x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$
3. $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$
4. $\forall x_1, x_2 \in D,\ x_1 \neq x_2,\ f(x_1) \neq f(x_2)$
5. $\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
6. $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2) \implies x_1 \neq x_2$
7. $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

One-to-one functions

Let f be a function with domain D.

What does each of the following mean?

1. $f(x_1) \neq f(x_2)$
2. $\exists x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$
3. $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$
4. $\forall x_1, x_2 \in D,\ x_1 \neq x_2,\ f(x_1) \neq f(x_2)$
5. $\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
6. $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2) \implies x_1 \neq x_2$
7. $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

Proving a function is one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $$\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$
- OR, equivalently, $$\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$$

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Suppose I give you a specific function f and I ask you to prove it is one-to-one.

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Suppose I give you a specific function f and I ask you to prove it is one-to-one.

- Write the structure of your proof (how do you begin? what do you assume? what do you conclude?) if you use the first definition.
- Write the structure of your proof if you use the second definition.

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- Write the structure of your proof if you use the second definition.

Exercise

Prove that $f(x) = 3x + 2$, with domain $\mathbb{R}$, is one-to-one.

Proving a function is NOT one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $$\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$
- OR, equivalently, $$\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$$

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Let f be a function with domain D.
We say f is one-to-one when

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- OR, equivalently, $$\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$$

Suppose I give you a specific function f and I ask you to prove it is not one-to-one.

Proving a function is NOT one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
- OR, equivalently, $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

Suppose I give you a specific function f and I ask you to prove it is not one-to-one. You need to prove f satisfies the *negation* of the definition.

- Write the negation of the first definition.
- Write the negation of the second definition.
- Write the structure of your proof.

Proving a function is NOT one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $$\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$
- OR, equivalently, $$\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$$

Suppose I give you a specific function f and I ask you to prove it is not one-to-one. You need to prove f satisfies the *negation* of the definition.

- Write the negation of the first definition.
- Write the negation of the second definition.
- Write the structure of your proof.

Exercise

Prove that $f(x) = x^2$, with domain $\mathbb{R}$, is not one-to-one.

Proving a theorem

Theorem

Let f be a function with domain D.

- IF f is increasing on D
- THEN f is one-to-one on D

Proving a theorem

Theorem

Let f be a function with domain D.

- IF f is increasing on D
- THEN f is one-to-one on D

1. Remind yourself of the precise definition of "increasing" and "one-to-one".

Proving a theorem

Theorem

Let f be a function with domain D.

- IF f is increasing on D
- THEN f is one-to-one on D

1. Remind yourself of the precise definition of "increasing" and "one-to-one".
2. To prove the theorem, what will you assume? what do you want to show?

Proving a theorem

Theorem

Let f be a function with domain D.

- IF f is increasing on D
- THEN f is one-to-one on D

1. Remind yourself of the precise definition of "increasing" and "one-to-one".
2. To prove the theorem, what will you assume? what do you want to show?
3. Look at the part you want to show. Based on the definition, what is the structure of the proof?

Proving a theorem

Theorem

Let f be a function with domain D.

- IF f is increasing on D
- THEN f is one-to-one on D

1. Remind yourself of the precise definition of "increasing" and "one-to-one".
2. To prove the theorem, what will you assume? what do you want to show?
3. Look at the part you want to show. Based on the definition, what is the structure of the proof?
4. Complete the proof.

DISproving a theorem

FALSE Theorem

Let f be a function with domain D.

- IF f is one-to-one on D
- THEN f is increasing on D

DISproving a theorem

FALSE Theorem

Let f be a function with domain D.

- IF f is one-to-one on D
- THEN f is increasing on D

1. This theorem is false. What do you need to do to prove it is false?

DISproving a theorem

FALSE Theorem

Let f be a function with domain D.

- IF f is one-to-one on D
- THEN f is increasing on D

1. This theorem is false. What do you need to do to prove it is false?
2. Prove the theorem is false.

What is wrong with this proof? (1)

Theorem

The sum of two odd numbers is even.

What is wrong with this proof? (1)

Theorem

The sum of two odd numbers is even.

Proof.

3 is odd.

5 is odd.

$3 + 5 = 8$ is even. $\square$

What is wrong with this proof? (2)

Theorem

The sum of two odd numbers is even.

What is wrong with this proof? (2)

Theorem

The sum of two odd numbers is even.

Proof.

The sum of two odd numbers is always even.

even $+$ even $=$ even

even $+$ odd $=$ odd

odd $+$ even $=$ odd

odd $+$ odd $=$ even. $\square$

Definition of odd and even

Write a definition of “odd integer” and “even integer”.

Definition of odd and even

Write a definition of "odd integer" and "even integer".

Definition

Let $x \in \mathbb{Z}$. We say that x is odd when ...

1. $x = 2a + 1$?
2. $\forall a \in \mathbb{Z}, x = 2a + 1$?
3. $\exists a \in \mathbb{Z}$ s.t. $x = 2a + 1$?

What is wrong with this proof? (3)

Theorem

The sum of two odd numbers is always even.

What is wrong with this proof? (3)

Theorem

The sum of two odd numbers is always even.

Proof.

$x = 2a + 1$ odd

$y = 2b + 1$ odd

$x + y = 2n$ even

$2a + 1 + 2b + 1 = 2n$

$2a + 2b + 2 = 2n$

$a + b + 1 = n$ $\square$

Write a correct proof!

Theorem

The sum of two odd numbers is always even.

Variations on induction

Let S_n be a statement depending on a positive integer n.

In each of the following cases, which statements are guaranteed to be true?

Variations on induction

Let S_n be a statement depending on a positive integer n.

In each of the following cases, which statements are guaranteed to be true?

1. We have proven:
   - S_3
   - $\forall n \geq 1,\ S_n \implies S_{n+1}$
2. We have proven:
   - S_1
   - $\forall n \geq 3,\ S_n \implies S_{n+1}$
3. We have proven:
   - S_1
   - $\forall n \geq 1,\ S_n \implies S_{n+3}$
4. We have proven:
   - S_1
   - $\forall n \geq 1,\ S_{n+1} \implies S_n$

Variations on induction 2

We want to prove

$$\forall n \geq 1,\ S_n$$

So far we have proven

- S_1
- $\forall n \geq 1,\ S_n \implies S_{n+3}.$

What else do we need to do?

Variations on induction 3

We want to prove

$$\forall n \in \mathbb{Z},\ S_n$$

So far we have proven

- S_1

What else do we need to do?

What is wrong with this proof by induction?

Theorem

$\forall N \geq 1$, *every set of* N *students in MAT137 will get the same grade.*

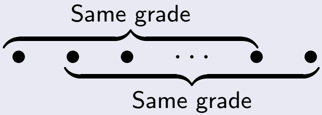
What is wrong with this proof by induction?

Theorem

$\forall N \geq 1$, every set of N students in MAT137 will get the same grade.

Proof.

- **Base case.** It is clearly true for $N = 1$.
- **Induction step.**
  Assume it is true for N. I'll show it is true for $N + 1$.
  Take a set of $N + 1$ students. By induction hypothesis:
  - The first N students get the same grade.
  - The last N students get the same grade.



Hence the $N + 1$ students all get the same grade.

$\square$

What is wrong with this proof by induction?

For every $N \geq 1$, let

$$S_N = \text{"every set of } N \text{ students in MAT137 will get the same grade"}$$

What is wrong with this proof by induction?

For every $N \geq 1$, let

$$S_N = \text{“every set of } N \text{ students in MAT137} \\ \text{will get the same grade”}$$

What did we actually prove in the previous page?

- S_1 ?
- $\forall N \geq 1, \quad S_N \implies S_{N+1}$?