

Before next class:

- **Watch videos 7.3, 7.4**

Warm-up: sums

Compute

$$(A) \sum_{i=2}^4 (2i + 1)$$

$$(B) \sum_{i=2}^4 2i + 1$$

$$(C) \sum_{j=2}^4 (2i + 1)$$

Write these sums with Σ notation

$$(A) \quad 1^5 + 2^5 + 3^5 + 4^5 + \dots + 100^5$$

$$(B) \quad \frac{2}{4^2} + \frac{2}{5^2} + \frac{2}{6^2} + \frac{2}{7^2} + \dots + \frac{2}{N^2}$$

$$(C) \quad \cos 0 - \cos 1 + \cos 2 - \cos 3 + \dots \pm \cos(N + 1)$$

$$(D) \quad \frac{1}{0!} + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots + \frac{1}{(2N)!}$$

$$(E) \quad \frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots + \frac{1}{81!}$$

$$(F) \quad \frac{2x^3}{4!} + \frac{3x^4}{5!} + \frac{4x^5}{6!} + \dots + \frac{999x^{1000}}{1001!}$$

Telescopic sum

- Calculate the exact value of

$$\sum_{i=1}^{137} \left[\frac{1}{i} - \frac{1}{i+1} \right]$$

Hint: Write down the first few terms.

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Hint: Write down the first few terms.

- Calculate the exact value of

$$\sum_{i=1}^{10,000} \frac{1}{i(i+1)}$$

Double sums

Compute:

$$(A) \sum_{i=1}^N \sum_{k=1}^N 1$$

$$(C) \sum_{i=1}^N \sum_{k=1}^i i$$

$$(E) \sum_{i=1}^N \sum_{k=1}^i (ik)$$

$$(B) \sum_{i=1}^N \sum_{k=1}^i 1$$

$$(D) \sum_{i=1}^N \sum_{k=1}^i k$$

Useful formulas:

$$\sum_{j=1}^N j = \frac{N(N+1)}{2}, \quad \sum_{j=1}^N j^2 = \frac{N(N+1)(2N+1)}{6}, \quad \sum_{j=1}^N j^3 = \frac{N^2(N+1)^2}{4}$$

Before next class:

- **Watch videos 7.5, 7.6**

Warm up: suprema and infima

Find the supremum, infimum, maximum, and minimum of the following sets (if they exist):

(A) $A = [-1, 5)$

(B) $B = (-\infty, 6] \cup (8, 9)$

(C) $C = \{2, 3, 4\}$

(D) $D = \left\{ \frac{1}{n} : n \in \mathbb{Z}, n > 0 \right\}$

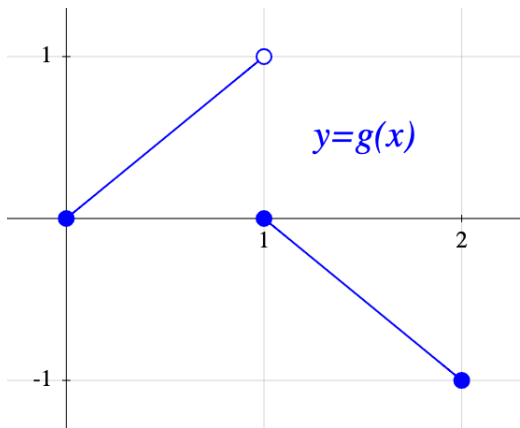
(E) $E = \left\{ \frac{(-1)^n}{n} : n \in \mathbb{Z}, n > 0 \right\}$

(F) $F = \{2^n : n \in \mathbb{Z}\}$

Suprema from a graph

Calculate, for the function g on the interval $[0.5, 1.5]$:

(A) supremum (B) infimum (C) maximum (D) minimum



Empty set

- (A) Does \emptyset have an upper bound?
- (B) Does \emptyset have a supremum?
- (C) Does \emptyset have a maximum?
- (D) Is \emptyset bounded above?

Equivalent definitions of supremum

Assume S is an upper bound of the set A .

Which of the following is equivalent to “ S is the supremum of A ”?

(A) If R is an upper bound of A , then $S \leq R$.

(B) $\forall R \geq S$, R is an upper bound of A .

(C) $\forall R \leq S$, R is not an upper bound of A .

(D) $\forall R < S$, R is not an upper bound of A .

(E) $\forall R < S$, $\exists x \in A$ such that $R < x$.

(F) $\forall R < S$, $\exists x \in A$ such that $R \leq x$.

(G) $\forall R < S$, $\exists x \in A$ such that $R < x \leq S$.

(H) $\forall R < S$, $\exists x \in A$ such that $R < x < S$.

(I) $\forall \varepsilon > 0$, $\exists x \in A$ such that $S - \varepsilon < x$.

(J) $\forall \varepsilon > 0$, $\exists x \in A$ such that $S - \varepsilon < x \leq S$.

Before next class:

- **Watch videos 7.7, 7.8, 7.11**

Equivalent or not

Let $A \subseteq \mathbb{R}$ and let $s \in \mathbb{R}$.

(A) $\exists x \in A$ such that $2 < x$

(B) $\exists x \in A$ such that $2 \leq x$

1. Does **(A)** imply **(B)**?
2. Does **(B)** imply **(A)**?

(C) $\forall r < s, \exists x \in A$ such that $r < x$.

(D) $\forall r < s, \exists x \in A$ such that $r \leq x$.

3. Does **(C)** imply **(D)**?
4. Does **(D)** imply **(C)**?

Warm up: partitions

Which ones are partitions of $[0, 2]$?

- (A) $[0, 2]$
- (B) $\{0.5, 1, 1.5\}$
- (C) $\{0, 2\}$
- (D) $\{1, 2\}$
- (E) $\{0, e, 2\}$
- (F) $\{0, 1.5, 1.6, 1.7, 1.8, 1.9, 2\}$
- (G) $\left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\} \cup \{2\}$

Warm up: lower and upper sums

Let $f(x) = \sin x$.

Consider the partition $P = \{0, 1, 3\}$ of the interval $[0, 3]$.

Calculate $L_P(f)$ and $U_P(f)$.

Equations for lower and upper sums

Let f be a **decreasing**, bounded function on $[a, b]$.

Let $P = \{x_0, x_1, \dots, x_N\}$ be a partition of $[a, b]$

Which ones are a valid equation for $L_P(f)$? For $U_P(f)$?

$$\begin{array}{lll} \text{(A)} \sum_{i=0}^N f(x_i) \Delta x_i & \text{(C)} \sum_{i=0}^{N-1} f(x_i) \Delta x_i & \text{(E)} \sum_{i=1}^N f(x_{i-1}) \Delta x_i \\ \text{(B)} \sum_{i=1}^N f(x_i) \Delta x_i & \text{(D)} \sum_{i=1}^N f(x_{i+1}) \Delta x_i & \text{(F)} \sum_{i=0}^{N-1} f(x_i) \Delta x_{i+1} \end{array}$$

Recall: $\Delta x_i = x_i - x_{i-1}$.

Joining partitions

Assume

$$L_P(f) = 2, \quad U_P(f) = 6$$

$$L_Q(f) = 3, \quad U_Q(f) = 8$$

(A) Is $P \subseteq Q$?

(B) Is $Q \subseteq P$?

(C) What can you say about $L_{P \cup Q}(f)$ and $U_{P \cup Q}(f)$?

A tricky question

Let f be a bounded function on $[a, b]$. Which statement is true?

(A) There exists a partition P of $[a, b]$ such that

$$\underline{I}_a^b(f) = L_P(f) \quad \text{and} \quad \overline{I}_a^b(f) = U_P(f).$$

(B) There exist partitions P and Q of $[a, b]$ such that

$$\underline{I}_a^b(f) = L_P(f) \quad \text{and} \quad \overline{I}_a^b(f) = U_Q(f).$$

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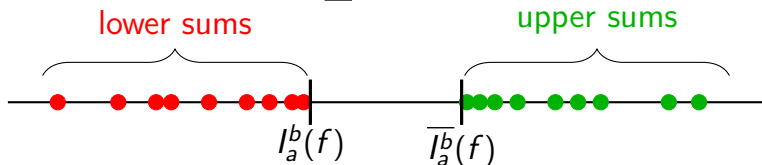
$$\underline{I}_a^b(f) = L_P(f) \quad \text{and} \quad \overline{I}_a^b(f) = U_P(f).$$

(B) There exist partitions P and Q of $[a, b]$ such that

$$\underline{I}_a^b(f) = L_P(f) \quad \text{and} \quad \overline{I}_a^b(f) = U_Q(f).$$

(C) There exists a partition P of $[a, b]$ such that

$$\underline{I}_a^b(f) = L_P(f).$$



MAT137 Lecture 40 — Examples and Properties of the Integral

Before next class:

- **Watch videos 7.9, 7.10**

Properties of the integral

Assume we know the following

$$\int_0^2 f(x) dx = 3, \quad \int_0^4 f(x) dx = 9, \quad \int_0^4 g(x) dx = 2.$$

Compute:

(A) $\int_0^2 f(t) dt$

(B) $\int_0^2 f(x) dx$

(C) $\int_0^2 f(t) dx$

(D) $\int_2^0 f(x) dx$

(E) $\int_2^4 f(x) dx$

(F) $\int_{-2}^0 f(x) dx$

(G) $\int_0^4 [f(x) - 2g(x)] dx$

Example: a non-continuous function

Consider the function $f(x) = \begin{cases} 0 & x = 0 \\ 5 & 0 < x \leq 1 \end{cases}$, defined on $[0, 1]$.

- (A) Let $P = \{0, 0.2, 0.5, 0.9, 1\}$.
Calculate $L_P(f)$ and $U_P(f)$ for this partition.
- (B) Fix an arbitrary partition $P = \{x_0, x_1, \dots, x_N\}$ of $[0, 1]$.
What is $U_P(f)$? What is $L_P(f)$? (Draw a picture!)
- (C) Find a partition P with exactly 3 points (2 subintervals) such that $L_P(f) = 4.99$.
- (D) What is the upper integral, $\overline{I}_0^1(f)$?
- (E) What is the lower integral, $\underline{I}_0^1(f)$?
- (F) Is f integrable on $[0, 1]$?

An alternative definition

Let f be a bounded function on the interval $[a, b]$. Let $M \in \mathbb{R}$. Some of these four statements imply others. What implies what?

(A) \forall partition P of $[a, b]$, $L_P(f) \leq M$,

(B) $\forall \varepsilon > 0$, \exists partition P of $[a, b]$ s.t. $M - \varepsilon < L_P(f)$

(C) $M \leq \underline{I}_a^b(f)$

(D) $\underline{I}_a^b(f) \leq M$

An alternative definition

Let f be a bounded function on the interval $[a, b]$. Let $M \in \mathbb{R}$. Some of these four statements imply others. What implies what?

(A) \forall partition P of $[a, b]$, $L_P(f) \leq M$,

(B) $\forall \varepsilon > 0, \exists$ partition P of $[a, b]$ s.t. $M - \varepsilon < L_P(f)$

(C) $M \leq \underline{I}_a^b(f)$

(D) $\underline{I}_a^b(f) \leq M$

Based on this exercise, we could have defined $\underline{I}_a^b(f)$ as “the only number $M \in \mathbb{R}$ satisfying these two properties: ...”
Use the same idea to write an alternative definition of $\overline{I}_a^b(f)$.

Before next class:

- **Watch videos 8.1, 8.2**

The norm of a partition

- (A) Construct a partition P of $[0, 1]$ such that $\|P\| = \frac{\pi}{10}$.
- (B) Construct a sequence of partitions of $[0, 1]$

$$P_1, P_2, P_3, \dots$$

as simple as possible, such that $\lim_{n \rightarrow \infty} \|P_n\| = 0$.

- (C) Construct a *different* sequence of partitions of $[0, 1]$

$$Q_1, Q_2, Q_3, \dots$$

such that $\lim_{n \rightarrow \infty} \|Q_n\| = 0$.

Compute $\int_1^2 x^2 dx$ using Riemann sums

Let $f(x) = x^2$ on $[1, 2]$. Let P_n be the partition that breaks $[1, 2]$ into n subintervals of equal length.

- (A) Write an explicit formula for P_n .
- (B) What is Δx_i ?
- (C) Write the Riemann sum $S_{P_n}^*(f)$ with sigma notation (choose x_i^* as the right endpoint).
- (D) Add the sum
- (E) Compute $\lim_{n \rightarrow \infty} S_{P_n}^*(f)$.
- (F) Repeat the last 3 questions when we choose x_i^* as the left endpoint.

Helpful identities:
$$\sum_{i=1}^N i = \frac{N(N+1)}{2}, \quad \sum_{i=1}^N i^2 = \frac{N(N+1)(2N+1)}{6}$$

Example 3: a very non-continuous function

Consider the function f defined on $[0, 1]$:

$$f(x) = \begin{cases} 1/2 & \text{if } 0 \leq x \leq 1/2 \\ 1 & \text{if } 1/2 < x \leq 1 \text{ and } x \in \mathbb{Q} \\ 0 & \text{if } 1/2 < x \leq 1 \text{ and } x \notin \mathbb{Q} \end{cases}$$

- (A) Draw a picture!
- (B) Let $P = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. Calculate $L_P(f)$ and $U_P(f)$.
- (C) Construct a partition P such that $L_P(f) = \frac{1}{4}$ and $U_P(f) = \frac{3}{4}$
- (D) What is the upper integral, $\overline{I}_0^1(f)$?
- (E) What is the lower integral, $\underline{I}_0^1(f)$?
- (F) Is f integrable on $[0, 1]$?

Fix these FALSE statements

(A) Let f and g be bounded functions on $[a, b]$. Then

$$\begin{array}{ccccc} \text{sup of } (f + g) & = & \text{sup of } f & + & \text{sup of } g \\ \text{on } [a, b] & & \text{on } [a, b] & & \text{on } [a, b] \end{array}$$

(B) Let $a < b < c$. Let f be a bounded function on $[a, c]$.
Then

$$\begin{array}{ccccc} \text{sup of } f & = & \text{sup of } f & + & \text{sup of } f \\ \text{on } [a, c] & & \text{on } [a, b] & & \text{on } [b, c] \end{array}$$

(C) Let f be a bounded function on $[a, b]$. Let $c \in \mathbb{R}$. Then:

$$\begin{array}{ccc} \text{sup of } (cf) & = & c \left(\text{sup of } f \right) \\ \text{on } [a, b] & & \text{on } [a, b] \end{array}$$