

### **Warmup:**

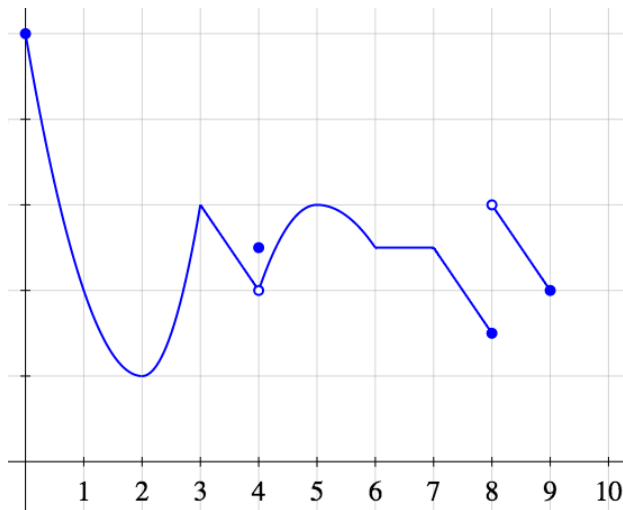
Write down, in set-builder notation, the domain of  $\tan$ , the tangent function.

### **Before next class:**

- **Watch videos 5.5, 5.6**

## Definition of local extremum

Find local and global extrema of the function with this graph:



## Where is the maximum?

We know the following about the function  $h$ :

- The domain of  $h$  is  $(-4, 4)$ .
- $h$  is continuous on its domain.
- $h$  is differentiable on its domain, except at 0.
- $h'(x) = 0 \iff x = -1 \text{ or } 1.$

What can you conclude about the maximum of  $h$ ?

## Where is the maximum?

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What can you conclude about the maximum of  $h$ ?

- (A)  $h$  has a maximum at  $x = -1$ , or 1.
- (B)  $h$  has a maximum at  $x = -1, 0$ , or 1.
- (C)  $h$  has a maximum at  $x = -4, -1, 0, 1$ , or 4.
- (D) None of the above.

## Fractional exponents

Let  $g(x) = x^{2/3}(x - 1)^3$ .

Find local and global extrema of  $g$  on  $[-1, 2]$ .

## Trig extrema

Let  $f(x) = \frac{\sin x}{3 + \cos x}$ .

Find the maximum and minimum of  $f$ .

## **Warmup:**

Write down, in set-builder notation, the domain of  $\tan$ , the tangent function.

## **Before next class:**

- **Watch videos 5.7, 5.8, 5.9**

## Domain of tan

Which of these correctly describe the domain of tan?

(A)  $\left\{ x \in \mathbb{R} : x \neq \frac{\pi k}{2}, k \in \{\text{odd integers}\} \right\}$

(B)  $\left\{ x \in \mathbb{R} : x \neq \frac{\pi k}{2}, \forall k \in \{\text{odd integers}\} \right\}$

(C)  $\left\{ x \in \mathbb{R} : \forall k \in \{\text{odd integers}\}, x \neq \frac{\pi k}{2} \right\}$

(D)  $\left\{ x \in \mathbb{R} : \exists k \in \{\text{odd integers}\}, x \neq \frac{\pi k}{2} \right\}$

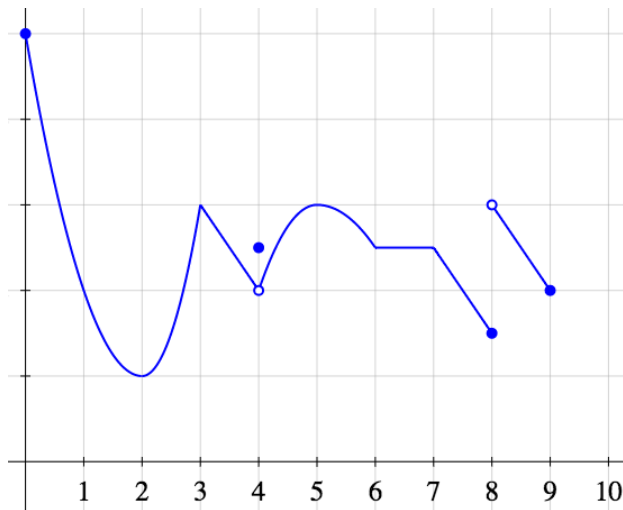
(E)  $\left\{ x \in \mathbb{R} : \forall k \in \mathbb{Z}, k \text{ is odd and } x \neq \frac{\pi k}{2} \right\}$

(F)  $\left\{ x \in \mathbb{R} : \forall k \in \mathbb{Z}, k \text{ is odd} \implies x \neq \frac{\pi k}{2} \right\}$



## Definition of local extremum

Find local and global extrema of the function with this graph:



## How many zeroes?

Let

$$f(x) = e^x - \sin x + x^2 + 10x$$

How many zeroes does  $f$  have?

## The second Theorem of Rolle

Complete the statement for this theorem and prove it.

### Rolle's Theorem 2

Let  $a < b$ . Let  $f$  be a function defined on  $[a, b]$ .

IF

- Some conditions on continuity and derivatives  
*[state these conditions precisely]*
- $f(a) = f(b) = 0$
- $f'(a) = f'(b) = 0$

THEN  $\exists c \in (a, b)$  such that  $f''(c) = 0$ .

*Hint:* Apply the 1st Rolle's Theorem to  $f'$ , then do something else.

**Before next class:**

- **Watch videos 5.10, 5.11, 5.12**

## True or False—Local Extrema Again

Let  $I$  be an open interval. Let  $f$  be a differentiable function defined on  $I$ . Let  $c \in I$ .

Which implications are true?

- (A) IF  $f$  has local extreme at  $c$ , THEN  $f$  has an extreme at  $c$
- (B) IF  $f$  has an extreme at  $c$ , THEN  $f$  has local extreme at  $c$
- (C) IF  $f$  has a local extreme at  $c$ , THEN  $f'(c) = 0$ .
- (D) IF  $f'(c) = 0$ , THEN  $f$  has a local extreme at  $c$ .

## Proving difficult identities

Prove that, for every  $x \geq 0$ ,

$$2 \arctan \sqrt{x} - \arcsin \frac{x-1}{x+1} = \frac{\pi}{2}$$

*Hint:* You are trying to prove a function is constant. Use derivatives.

## Critique this “proof”

- $\left[ 2 \arctan \sqrt{x} - \arcsin \frac{x-1}{x+1} \right] = \left[ \frac{\pi}{2} \right]$
- $\frac{d}{dx} \left[ 2 \arctan \sqrt{x} - \arcsin \frac{x-1}{x+1} \right] = \frac{d}{dx} \left[ \frac{\pi}{2} \right]$
- $\frac{2}{1 + (\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{1 - \left(\frac{x-1}{x+1}\right)^2}} \cdot \frac{(x+1) - (x-1)}{(x+1)^2} = 0$
- $\frac{1}{(1+x)\sqrt{x}} - \frac{1}{\sqrt{\frac{4x}{(x+1)^2}}} \cdot \frac{2}{(x+1)^2} = 0$
- $0 = 0$
- So  $2 \arctan \sqrt{x} - \arcsin \frac{x-1}{x+1}$  is constant.
- Evaluate at  $x = 0$  to find the value of the constant.
- $2 \arctan 0 - \arcsin(-1) = 0 - (-\pi/2) = \pi/2$
- Therefore,  $2 \arctan \sqrt{x} - \arcsin \frac{x-1}{x+1} = \frac{\pi}{2}$

## Car race - 1

A driver competes in a race.

Use MVT to prove that at some point during the race the instantaneous velocity of the driver is exactly equal to the average velocity of the driver during the race.



## Car race - 2

Two drivers start a race at the same moment and finish in a tie.

Can you conclude that there was a time in the race (not counting the starting time) when the two drivers had exactly the same speed?

# Car race - Is this proof correct?

## Claim

IF two drivers start a race at the same moment and finish in a tie, THEN at some point in the race (not counting the starting time) they had exactly the same speed.

## Proof?

- Let  $f(t)$  and  $g(t)$  be the positions of the two cars at time  $t$ .
- Assume the race happens in the interval  $[t_1, t_2]$ . By hypothesis:

$$f(t_1) = g(t_1), \quad f(t_2) = g(t_2).$$

- Using MVT, there exists  $c \in (t_1, t_2)$  such that

$$f'(c) = \frac{f(t_2) - f(t_1)}{t_2 - t_1}, \quad g'(c) = \frac{g(t_2) - g(t_1)}{t_2 - t_1}.$$

- Then  $f'(c) = g'(c)$ .



## Car race - resolution

Two drivers start a race at the same moment and finish in a tie. Prove that at some point during the race (not counting the starting time) the two drivers had exactly the same speed.

**Before next class:**

- **Watch videos 6.1, 6.2**

## Definition of increasing

Let  $f$  be defined by  $f(x) = x^3$ .

Which statements are TRUE?

- (A)  $f$  is increasing on  $(0, \infty)$ .
- (B)  $f$  is increasing on  $[0, \infty)$ .
- (C)  $f$  is increasing on  $(-\infty, 0)$ .
- (D)  $f$  is increasing on  $(-\infty, 0]$ .
- (E)  $f$  is increasing on  $(-\infty, 0)$  and on  $(0, \infty)$ .
- (F)  $f$  is increasing on  $(-\infty, 0]$  and on  $[0, \infty)$ .
- (G)  $f$  is increasing on  $\mathbb{R}$ .
- (H)  $f$  is increasing on  $[1, 2]$ .

## True or False—Again, Again!

Let  $I$  be an **open** interval.

Let  $f$  be a function defined on  $I$ .

Let  $c \in I$ . Which implications are true?

- (A) IF  $f$  is increasing on  $I$ , THEN  $\forall x \in I, f'(x) > 0$ .
- (B) IF  $\forall x \in I, f'(x) > 0$ , THEN  $f$  is increasing on  $I$ .
- (C) IF  $f$  has a local extreme at  $c$ , THEN  $f'(c) = 0$ .
- (D) IF  $f'(c) = 0$ , THEN  $f$  has a local extreme at  $c$ .

## Preparation

- (A) Let  $f$  be a function defined on an interval  $I$ .  
Write the definition of “ $f$  is increasing on  $I$ ”.
  
  
  
  
  
  
  
  
  
  
- (B) Write the statement of the Mean Value Theorem

## Positive derivative implies increasing

Use the MVT to prove

### Theorem

Let  $a < b$ . Let  $f$  be a differentiable function on  $(a, b)$ .

- IF  $\forall x \in (a, b), f'(x) > 0$ ,
- THEN  $f$  is increasing on  $(a, b)$ .



## Positive derivative implies increasing

Use the MVT to prove

### Theorem

Let  $a < b$ . Let  $f$  be a differentiable function on  $(a, b)$ .

- IF  $\forall x \in (a, b), f'(x) > 0$ ,
- THEN  $f$  is increasing on  $(a, b)$ .

- (A) Recall the definition of what you are trying to prove.
- (B) **From that definition, figure out the structure of the proof.**
- (C) If you have used a theorem, did you verify the hypotheses?
- (D) Are there words in your proof, or just equations?

# What is wrong with this proof?

## Theorem

Let  $a < b$ . Let  $f$  be a differentiable function on  $(a, b)$ .

- IF  $\forall x \in (a, b), f'(x) > 0$ ,
- THEN  $f$  is increasing on  $(a, b)$ .

## Proof.

- From the MVT,  $f'(c) = \frac{f(b) - f(a)}{b - a}$
- We know  $b - a > 0$  and  $f'(c) > 0$
- Therefore  $f(b) - f(a) > 0$ . Thus  $f(b) > f(a)$ .
- $f$  is increasing.



Prove that, for every  $x \in \mathbb{R}$

$$e^x \geq 1 + x$$

*Hint:* Where is the function  $f(x) = e^x - 1 - x$  increasing or decreasing? What is its minimum?