Interval of convergence

Find the interval of convergence of each power series:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2\, 2^{2n+1}}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{42^n} x^n$

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2. $$\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2\, 2^{2n+1}}$$
3. $$\sum_{n=1}^{\infty} \frac{n^n}{42^n} x^n$$
4. (Hard!) $$\sum_{n=0}^{\infty} \frac{(3n)!}{n!(2n)!}\, x^n$$

What can you conclude?

Think of the power series $\sum_{n}^{\infty} a_n x^n$. Do not assume $a_n \geq 0$.

In each case, may the given series be absolutely convergent (AC)? conditionally convergent (CC)? divergent (D)? all of them?

IF	$\sum_{n}^{\infty} a_n 3^n$ is ...	AC	CC	D
THEN	$\sum_{n}^{\infty} a_n 2^n$ may be ...	???	???	???
	$\sum_{n}^{\infty} a_n (-3)^n$ may be ...	???	???	???
	$\sum_{n}^{\infty} a_n 4^n$ may be ...	???	???	???

Writing functions as power series

You know that $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

Manipulate it to write the following functions as power series centered at 0:

1. $g(x) = \dfrac{1}{1+x}$
2. $A(x) = \dfrac{1}{2-x}$

   Hint: Factor $1/2$.
3. $h(x) = \dfrac{1}{1-x^2}$
4. $F(x) = \ln(1+x)$

   Hint: Compute F'

Challenge

Compute $$A = \sum_{n=1}^{\infty} \frac{n}{3^n}$$

Challenge

Compute $$A = \sum_{n=1}^{\infty} \frac{n}{3^n}$$

---

1. What is the value of the sum $\displaystyle\sum_{n=0}^{\infty} x^n$?
2. Use derivatives to relate $\displaystyle\sum_{n}^{\infty} x^n$ and $\displaystyle\sum_{n}^{\infty} nx^{n-1}$.
3. Compute $\displaystyle\sum_{n=1}^{\infty} nx^{n-1}$. Then compute $\displaystyle\sum_{n=1}^{\infty} nx^n$.
4. Compute the value of series A.

Challenge

Compute $\quad A = \sum_{n=1}^{\infty} \frac{n}{3^n} \quad$ and $\quad B = \sum_{n=1}^{\infty} \frac{n^2}{3^n}$

Challenge

Compute $\quad A = \sum_{n=1}^{\infty} \frac{n}{3^n} \quad$ and $\quad B = \sum_{n=1}^{\infty} \frac{n^2}{3^n}$

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4. Compute the value of series A.
5. Compute the value of series B.

Challenge - 2

We want to calculate the value of

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,3^n}$$

Challenge - 2

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---

1. Write $F(x) = \arctan x$ as a power series.

   Hint: Compute $F'(x)$. Using the geometric series, write $F'(x)$ as a series. Then integrate.

2. Now calculate the original sum.

Warm up

Write down the (various equivalent) definitions of Taylor polynomial you have learned so far.

Tangent line

Let f be a C^1 function at $a \in \mathbb{R}$.
Then the tangent line of f at a is given by

$$y = L(x)$$

1. Recall the explicit formula for L
2. Prove that L is the 1-st Taylor polynomial for f at a using the 1st definition.
3. Prove that L is the 1-st Taylor polynomial for f at a using the 2nd definition.

Taylor polynomial of a polynomial

Let $f(x) = x^3$.
Let $Q_{n,a}$ be the n-th Taylor polynomial for f at a.

1. Using the 2nd definition, find $Q_{2,0}$.
   Then verify it also satisfies the 1st definition.

Taylor polynomial of a polynomial

Let $f(x) = x^3$.
Let $Q_{n,a}$ be the n-th Taylor polynomial for f at a.

1. Using the 2nd definition, find $Q_{2,0}$.
   Then verify it also satisfies the 1st definition.
2. Repeat for $Q_{3,0}$
3. Repeat for $Q_{3,1}$
4. Repeat for $Q_{2,1}$.

True or False – Taylor polynomials

Let f be a function defined at and near $a \in \mathbb{R}$. Let $n \in \mathbb{N}$.
Let P_n be the n-th Taylor polynomial for f at a.
Which ones of these are true?

1. P_n is an approximation for f of order n near a.
2. f is an approximation for P_n of order n near a.
3. P_3 is an approximation for f of order 4 near a.
4. P_4 is an approximation for f of order 3 near a.
5. $\lim_{x \to a} [f(x) - P_n(x)] = 0$
6. $\lim_{x \to a} \dfrac{f(x) - P_n(x)}{(x-a)^n} = 0$
7. If x is close to a, then $f(x) = P_n(x)$.

True or False – smooth functions

Let f be a function. Let $a \in \mathbb{R}$. Let $m \in \mathbb{N}$.

1. IF f is continuous, THEN f is C^0.
2. IF f is C^0, THEN f is continuous.
3. IF f is differentiable, THEN f is C^1.
4. IF f is C^1, THEN f is differentiable.
5. IF f is C^∞, THEN $\forall n \in \mathbb{N}$, f is C^n.
6. IF $\forall n \in \mathbb{N}$, f is C^n, THEN f is C^∞.
7. IF f is C^m at a,
   THEN f is C^m on some interval centered at a.
8. IF f is C^m at a,
   THEN f is C^{m-1} on some interval centered at a.

True or False – Operations with smooth functions

Let f and g be two functions with domain $\mathbb{R}$. Let $n \in \mathbb{N}$.

1. IF f and g are C^n, THEN $f + g$ is C^n.
2. IF f and g are C^n, THEN $f \cdot g$ is C^n.
3. IF f and g are C^n, THEN $f \circ g$ is C^n.
4. IF f and g are C^∞, THEN $f + g$ is C^∞.
5. IF f and g are C^∞, THEN $f \cdot g$ is C^∞.
6. IF f and g are C^∞, THEN $f \circ g$ is C^∞.

Approximating functions

Which one of the following functions is a better approximation for $F(x) = \sin x + \cos x$ near 0?

1. $f(x) = 1 + x - \frac{x^2}{2}$
2. $g(x) = e^x - x^2$
3. $h(x) = 1 + \ln(1 + x)$

♣

A polynomial given its derivatives

1. Consider the polynomial $P(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$. Find values of the coefficients that satisfy

$$P(0) = 1, \quad P'(0) = 5, \quad P''(0) = 3, \quad P'''(0) = -7$$

2. Find *all* polynomials P (of any degree) that satisfy

$$P(0) = 1, \quad P'(0) = 5, \quad P''(0) = 3, \quad P'''(0) = -7$$

3. Find a polynomial P of smallest possible degree that satisfies

$$P(0) = A, \quad P'(0) = B, \quad P''(0) = C, \quad P'''(0) = D$$

Competition!

- Do you prefer cats or dogs? You MUST choose one.

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Competition!

- Do you prefer cats or dogs? You MUST choose one. Now you are in the C-team or the D-team.
- Copy only one polynomial (C or D):

$$C(x) = -\frac{293}{8} + 29x + \frac{13}{4}x^2 - 3x^3 + \frac{3}{8}x^4$$

$$D(x) = 29 + 8(x-3) - \frac{7}{2}(x-3)^2 + \frac{9}{6}(x-3)^3 + \frac{9}{24}(x-3)^4$$

- I will ask you questions.
  Answer only about your polynomial (C or D).
  No calculators!

Competition!

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C-team compute...

- $C(3)$
- $C'(3)$
- $C''(3)$
- $C'''(3)$
- $C^{(4)}(3)$

D-team compute...

- $D(3)$
- $D'(3)$
- $D''(3)$
- $D'''(3)$
- $D^{(4)}(3)$

Simplify your answers (write them as rational numbers)

I spy a polynomial with my little eye

I'm thinking of a cubic polynomial P. It satisfies

$$P(1) = 8, \quad P'(1) = -\pi, \quad P''(1) = 4, \quad P'''(1) = \sqrt{7}$$

What is $P(x)$?

cosine

Obtain the Maclaurin series for $h(x) = \cos x$.
There are at least two ways to do this:

1. Use the general formula for Maclaurin series.
2. Use the Maclaurin series for sin to compute $\cos x = \dfrac{d}{dx} \sin x$.

Interval of convergence of Maclaurin series

1. (Recall) Write down the Maclaurin series for the following functions

$$f(x) = e^x, \qquad g(x) = \sin x, \qquad h(x) = \cos x$$

2. Compute the interval of convergence for each one of them.

Warm up

1. Write down the Maclaurin series for $f(x) = \sin x$. (Just recall it.)

2. Compute the interval of convergence of this power series.

3. Write down the statement of Lagrange's Remainder Theorem. (Just recall it. Look it up if needed.)

sin is analytic

Let $f(x) = \sin x$. You know its Maclaurin series is

$$S(x) \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

As you know, to prove that $\sin x = S(x)$ we need to show that

$$\forall x \in \mathbb{R}, \qquad \lim_{n\to\infty} R_n(x) = 0$$

Use Lagrange's Remainder Theorem to prove it!

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Use Lagrange's Remainder Theorem to prove it!

Reminder: Lagrange's Remainder Theorem says that given f, a, x, and n with certain conditions,

$$\exists \xi \text{ between } a \text{ and } x \text{ s.t.} \qquad R_n(x) \ = \ \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}$$

Generalize your proof

Theorem

Let I be an open interval. Let $a \in I$. Let f be a C^∞ function on I.
Let $S(x)$ be the Taylor series for f centered at a.

- IF $\boxed{???}$
- THEN $\forall x \in I,\ f(x) = S(x)$

Which condition can you write instead of "$\boxed{???}$" to make the theorem true?

Generalize your proof

Theorem

Let I be an open interval. Let $a \in I$. Let f be a C^∞ function on I.
Let $S(x)$ be the Taylor series for f centered at a.

- IF $\boxed{???}$
- THEN $\forall x \in I,\ f(x) = S(x)$

Which condition can you write instead of "$\boxed{???}$" to make the theorem true?

If you are thinking "the derivatives must be bounded", then you are on the right track, but you need to be much more precise. Which derivatives? On which domain? There are a lot of variables here; can the bounds depend on any variable?

Generalize your proof (continued)

Which one or ones of the following conditions can be written instead of "$\boxed{???}$" to make the theorem true?

1. $\forall n \in \mathbb{N}, \quad f^{(n)}$ is bounded on I
2. $\forall n \in \mathbb{N}, \quad \forall x \in I, \quad f^{(n)}$ is bounded on $J_{x,a}$
3. $\forall n \in \mathbb{N}, \quad \forall x \in I, \quad \exists A, B \in \mathbb{R}, \quad \forall \xi \in J_{x,a}, \quad A \leq f^{(n)}(\xi) \leq B$
4. $\forall x \in I, \quad \exists A, B \in \mathbb{R}, \quad \forall n \in \mathbb{N}, \quad \forall \xi \in J_{x,a}, \quad A \leq f^{(n)}(\xi) \leq B$
5. $\forall x \in I, \quad \exists M \geq 0, \quad \forall n \in \mathbb{N}, \quad \forall \xi \in J_{x,a}, \quad \left| f^{(n)}(\xi) \right| \leq M$
6. $\exists A, B \in \mathbb{R}, \quad \forall x \in I, \quad \forall n \in \mathbb{N}, \quad \forall \xi \in J_{x,a}, \quad A \leq f^{(n)}(\xi) \leq B$
7. $\exists A, B \in \mathbb{R}, \quad \forall x \in I, \quad \forall n \in \mathbb{N}, \quad A \leq f^{(n)}(x) \leq B$

Notation: $J_{x,a}$ is the interval between x and a

A C^∞ but not analytic function

Consider the function $F(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$

1. Prove that, for every $n \in \mathbb{N}$, $\lim_{t \to \infty} t^n e^{-t} = 0$.
2. Prove that, for every $n \in \mathbb{N}$, $\lim_{x \to 0^+} \dfrac{e^{-1/x}}{x^n} = 0$.
3. Calculate $F'(x)$ for $x > 0$.
4. Calculate $F'(x)$ for $x < 0$.
5. Calculate $F'(0)$ from the definition.
6. Calculate $F''(0)$ from the definition.
7. Prove that for every $n \in \mathbb{N}$, $F^{(n)}(0) = 0$.
8. Write the Maclaurin series for F at 0.
9. Is F analytic? Is it C^∞?

Taylor series gymnastics

Write the following functions as power series centered at 0. Write them first with sigma notation, and then write out the first few terms. Indicate the domain where each expansion is valid.

1. $f(x) = e^{-x}$
2. $f(x) = x^2 \cos x$
3. $f(x) = \dfrac{1}{1+x}$
4. $f(x) = \dfrac{1}{1-x^2}$
5. $f(x) = \dfrac{x}{3+2x}$
6. $f(x) = \sin\left(2x^3\right)$
7. $f(x) = \dfrac{e^x + e^{-x}}{2}$
8. $f(x) = \ln \dfrac{1+x}{1-x}$

Note: You do not need to take any derivatives. You can reduce them all to other Maclaurin series you know.

Taylor series not at 0

Write the Taylor series...

1. for $f(x) = e^x$ at $a = -1$
2. for $g(x) = \sin x$ at $a = \pi/4$
3. for $H(x) = 1/x$ at $a = 3$

You can do these problems in two ways:

1. Compute first few derivatives, guess the pattern, use general formula
2. Use substitution $u = x - a$, use known Maclaurin series (without computing any derivative).

1. Write the Maclaurin series for $G(x) = \arctan x$

   Hint: Compute the first derivative. Then use the geometric series. Then integrate.

arctan

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2. What is $G^{(137)}(0)$?

1. Write the Maclaurin series for $G(x) = \arctan x$

   Hint: Compute the first derivative. Then use the geometric series. Then integrate.

2. What is $G^{(137)}(0)$?

3. Use this previous results to compute

$$A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\, 3^n}$$

arcsin

Let $f(x) = \dfrac{1}{\sqrt{1+x}}$.

1. Find a formula for its derivatives $f^{(n)}(x)$.
2. Write its Maclaurin series at 0. Call it $S(x)$.
3. What is the radius of convergence of series $S(x)$?
   Note: Use without proof that $f(x) = S(x)$ inside the interval of convergence.
4. Use this result to write $h(x) = \text{arcsin}$ as a power series centered at 0.
   Hint: Compute $h'(x)$.
5. What is $h^{(7)}(0)$?

Parity

1. Write down the definition of odd function and even function. (Assume the domain is $\mathbb{R}$.)
2. Let f be an odd, C^∞ function. What can you say about its Maclaurin series? What if f is even?

   Hint: Think of $\sin$ and $\cos$.
3. Prove it.

   Hints:

   - Use the general formula for the Maclaurin series.
   - If h is odd then what is $h(0)$?
   - The derivative of an even function is ...?
   - The derivative of an odd function is ...?

Product of Taylor series

Let $f(x) = e^x \ln(1+x)$

1. Write the 4-th Taylor polynomial for f at $a = 0$.

   Hint: Write the first few terms of the Maclaurin series for each factor and multiply them.

2. What is $f^{(4)}(0)$?

Product of Taylor series

Let $f(x) = e^x \ln(1+x)$

1. Write the 4-th Taylor polynomial for f at $a = 0$.

   Hint: Write the first few terms of the Maclaurin series for each factor and multiply them.

2. What is $f^{(4)}(0)$?

3. Use it to calculate the limit

$$\lim_{x \to 0} \frac{e^x \ln(1+x) + \ln(1-x)}{x^4}$$

Composition of Taylor series

Let $g(x) = e^{\sin x}$.

1. Write the 4-th Taylor polynomial for g at $a = 0$.

   Hint: First use the Maclaurin series for the exponential. Then use the Maclaurin series for sin and treat it like a polynomial. You only need to keep the first few terms.

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2. What is $g^{(4)}(0)$?

3. Find a value of $a \in \mathbb{R}$ such that the limit

   $$\lim_{x \to 0} \frac{e^{\sin x} - e^x + ax^3}{x^4}$$

   exists and is not 0. Then compute the limit.

Tangent

There is no nice, compact formula for the Maclaurin series of tan, but we can obtain the first few terms. Set

$$\tan x = c_1 x + c_3 x^3 + c_5 x^5 + \dots$$

By definition of tan, we have:

$$\sin x = (\cos x)(\tan x)$$

Thus

$$\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right] = \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right] \cdot \left[c_1 x + c_3 x^3 + c_5 x^5 + \dots \right]$$

Multiply the two series on the right. Obtain equations for the coefficients c_n and solve for the first few ones.

Secant

I want to obtain the first few terms of the Maclaurin series of $f(x) = \sec x$. Notice that

$$\sec x \;=\; \frac{1}{\cos x} \;=\; \frac{1}{1-[1-\cos x]} \;=\; \frac{1}{1-u} \tag{1}$$

where I have called $u = 1 - \cos x$. Notice that as $x \to 0$, $u \to 0$.

Use the geometric series in (1). Then write u as a power series centered at 0. Then expand and regroups terms.

1. Use the above to obtain the 6-th Maclaurin polynomial for f.

Secant

I want to obtain the first few terms of the Maclaurin series of $f(x) = \sec x$. Notice that

$$\sec x = \frac{1}{\cos x} = \frac{1}{1 - [1 - \cos x]} = \frac{1}{1-u} \qquad (1)$$

where I have called $u = 1 - \cos x$. Notice that as $x \to 0$, $u \to 0$.

Use the geometric series in (1). Then write u as a power series centered at 0. Then expand and regroups terms.

1. Use the above to obtain the 6-th Maclaurin polynomial for f.
2. Without taking any derivative, what is $f^{(6)}(0)$?

Integrals

I want to calculate

$$A = \int_0^1 t^{10} \sin t \; dt.$$

There are two ways to do it. Choose your favourite one:

1. Use integration by parts 10 times.
2. Use power series.

Integrals

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Estimate A with an error smaller than 0.001.

Add these series

1. $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^n}{(2n+1)!}$ *Hint:* Think of sin

2. $\displaystyle\sum_{n=0}^{\infty} (4n+1)x^{4n+2}$ *Hint:* $\dfrac{d}{dx}\left[x^{4n+1}\right] = ???$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n)!}$ *Hint:* Write first few terms. Combine e^1 and e^{-1}.

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!(n+1)}$ *Hint:* Integrate

Add more series

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+1)(n+2)}$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n+2)n!}$

9. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)}{(2n)!} 2^n$

10. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$

Hint: Take derivatives or antiderivatives of series whose values you know.

Limits

Use Maclaurin series to compute these limits:

1. $$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$$

2. $$\lim_{x\to 0} \frac{\cos(2x) - e^{-2x^2}}{x^4}$$

3. $$\lim_{x\to 0} \frac{[\sin x - x]^3\, x}{[\cos x - 1]^4\, [e^x - 1]^2}$$

Estimations

I want to estimate these two numbers

$$A = \sin 1, \qquad B = \ln 0.9.$$

1. Use Taylor series to write A and B as infinite sums.
2. If you want to estimate A or B with a small error using a partial sum, the fastest way is to use different theorems for A and B. What are they?
3. Estimate B with an error smaller than 0.001.