MAT137 Lecture 65 — Power series

Before next class:

- **Watch videos 14.3, 14.4**

Interval of convergence

Find the interval of convergence of each power series:

(A) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

(C) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{42^n} x^n$

(B) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2\, 2^{2n+1}}$

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(C) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{42^n} x^n$

(D) (Hard!) $\displaystyle\sum_{n=0}^{\infty} \frac{(3n)!}{n!(2n)!}\, x^n$

Writing functions as power series

You know that $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

Manipulate it to write the following functions as power series centered at 0:

(A) $g(x) = \dfrac{1}{1+x}$

(B) $A(x) = \dfrac{1}{2-x}$

Hint: Factor $1/2$.

(C) $h(x) = \dfrac{1}{1-x^2}$

(D) $F(x) = \ln(1+x)$

Hint: Compute F'

Before next class:

- **Watch videos 14.5, 14.6**

Warm up

Write down the (various equivalent) definitions of Taylor polynomial you have learned so far.

Taylor polynomial of a polynomial

Let $f(x) = x^3$.
Let P_n be the n-th Taylor polynomial for f at 0.

(A) Find P_3.
Verify it satisfies the 1st and 2nd definition.

(B) Find P_2

More Taylor polynomial of a polynomial

Let $f(x) = x^3$.
Let Q_n be the n-th Taylor polynomial **for** f **at** 1.

(A) Find Q_3.
(B) Find Q_2.

A polynomial given its derivatives

(A) Consider the polynomial $P(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$. Find values of the coefficients that satisfy

$$P(0) = 1, \quad P'(0) = 5, \quad P''(0) = 3, \quad P'''(0) = -7$$

(B) Find *all* polynomials P (of any degree) that satisfy

$$P(0) = 1, \quad P'(0) = 5, \quad P''(0) = 3, \quad P'''(0) = -7$$

(C) Find a polynomial P of smallest possible degree that satisfies

$$P(0) = A, \quad P'(0) = B, \quad P''(0) = C, \quad P'''(0) = D$$

A polynomial given its derivatives 2

(A) Let f be a C^4 function at 0. Construct a polynomial P that satisfies

$$\begin{aligned} P(0) &= f(0) \\ P'(0) &= f'(0) \\ P''(0) &= f''(0) \\ P^{(3)}(0) &= f^{(3)}(0) \\ P^{(4)}(0) &= f^{(4)}(0) \end{aligned}$$

Before next class:

- **Watch videos 14.7, 14.8**

Competition!

- Do you prefer cats or dogs? You MUST choose one.

Competition!

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Competition!

- Do you prefer cats or dogs? You MUST choose one. Now you are in the C-team or the D-team.
- Copy only one polynomial (C or D):

$$C(x) = -\frac{293}{8} + 29x + \frac{13}{4}x^2 - 3x^3 + \frac{3}{8}x^4$$

$$D(x) = 29 + 8(x-3) - \frac{7}{2}(x-3)^2 + \frac{9}{6}(x-3)^3 + \frac{9}{24}(x-3)^4$$

- I will ask you questions.
  Answer only about your polynomial (C or D).
  No calculators!

Competition!

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C-team compute...

- $C(3)$
- $C'(3)$
- $C''(3)$
- $C'''(3)$
- $C^{(4)}(3)$

D-team compute...

- $D(3)$
- $D'(3)$
- $D''(3)$
- $D'''(3)$
- $D^{(4)}(3)$

Simplify your answers (write them as rational numbers)

I spy a polynomial with my little eye

I'm thinking of a cubic polynomial P. It satisfies

$$P(1) = 8, \quad P'(1) = -\pi, \quad P''(1) = 4, \quad P'''(1) = \sqrt{7}$$

What is $P(x)$?

arcsin

Let $f(x) = \dfrac{1}{\sqrt{1+x}}$.

(A) Find a formula for its derivatives $f^{(n)}(x)$.

(B) Write its Maclaurin series at 0. Call it $S(x)$.

(C) What is the radius of convergence of series $S(x)$?
Note: Use without proof that $f(x) = S(x)$ inside the interval of convergence.

(D) Use this result to write $h(x) = \text{arcsin}$ as a power series centered at 0.
Hint: Compute $h'(x)$.

(E) What is $h^{(7)}(0)$?

Before next class:

- **Watch videos 14.9, 14.10**

Warm up

(A) Write down the Maclaurin series for $f(x) = \sin x$. (Just recall it.)

(B) Compute the interval of convergence of this power series.

(C) Write down the statement of Lagrange's Remainder Theorem. (Just recall it. Look it up if needed.)

sin is analytic

Let $f(x) = \sin x$. You know its Maclaurin series is

$$S(x) \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

As you know, to prove that $\sin x = S(x)$ we need to show that

$$\forall x \in \mathbb{R}, \qquad \lim_{n\to\infty} R_n(x) = 0$$

Use Lagrange's Remainder Theorem to prove it!

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As you know, to prove that $\sin x = S(x)$ we need to show that

$$\forall x \in \mathbb{R}, \qquad \lim_{n\to\infty} R_n(x) = 0$$

Use Lagrange's Remainder Theorem to prove it!

Reminder: Lagrange's Remainder Theorem says that given f, a, x, and n with certain conditions,

$$\exists \xi \text{ between } a \text{ and } x \text{ s.t.} \qquad R_n(x) \;=\; \frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x-a)^{n+1}$$

Generalize your proof

Theorem

Let I be an open interval. Let $a \in I$. Let f be a C^∞ function on I.
Let $S(x)$ be the Taylor series for f centered at a.

- IF $\boxed{???}$
- THEN $\forall x \in I,\ f(x) = S(x)$

Which condition can you write instead of "$\boxed{???}$" to make the theorem true?

Generalize your proof

Theorem

Let I be an open interval. Let $a \in I$. Let f be a C^∞ function on I.
Let $S(x)$ be the Taylor series for f centered at a.

- IF $\boxed{???}$
- THEN $\forall x \in I,\ f(x) = S(x)$

Which condition can you write instead of "$\boxed{???}$" to make the theorem true?

If you are thinking "the derivatives must be bounded", then you are on the right track, but you need to be much more precise. Which derivatives? On which domain? There are a lot of variables here; can the bounds depend on any variable?

Generalize your proof (continued)

Which one or ones of the following conditions can be written instead of "$\boxed{???}$" to make the theorem true?

(A) $\forall n \in \mathbb{N}, \quad f^{(n)}$ is bounded on I

(B) $\forall n \in \mathbb{N}, \quad \forall x \in I, \quad f^{(n)}$ is bounded on $J_{x,a}$

(C) $\forall n \in \mathbb{N}, \quad \forall x \in I, \quad \exists A, B \in \mathbb{R}, \quad \forall \xi \in J_{x,a}, \quad A \leq f^{(n)}(\xi) \leq B$

(D) $\forall x \in I, \quad \exists A, B \in \mathbb{R}, \quad \forall n \in \mathbb{N}, \quad \forall \xi \in J_{x,a}, \quad A \leq f^{(n)}(\xi) \leq B$

(E) $\forall x \in I, \quad \exists M \geq 0, \quad \forall n \in \mathbb{N}, \quad \forall \xi \in J_{x,a}, \quad \left| f^{(n)}(\xi) \right| \leq M$

(F) $\exists A, B \in \mathbb{R}, \quad \forall x \in I, \quad \forall n \in \mathbb{N}, \quad \forall \xi \in J_{x,a}, \quad A \leq f^{(n)}(\xi) \leq B$

(G) $\exists A, B \in \mathbb{R}, \quad \forall x \in I, \quad \forall n \in \mathbb{N}, \quad A \leq f^{(n)}(x) \leq B$

Notation: $J_{x,a}$ is the interval between x and a

Before next class:

- **Watch videos 14.12, 14.14**

Taylor series gymnastics

Write the following functions as power series centered at 0. Write them first with sigma notation, and then write out the first few terms. Indicate the domain where each expansion is valid.

(A) $f(x) = e^{-x}$

(B) $f(x) = x^2 \cos x$

(C) $f(x) = \dfrac{1}{1+x}$

(D) $f(x) = \dfrac{1}{1-x^2}$

(E) $f(x) = \dfrac{x}{3+2x}$

(F) $f(x) = \sin\left(2x^3\right)$

(G) $f(x) = \dfrac{e^x + e^{-x}}{2}$

(H) $f(x) = \ln \dfrac{1+x}{1-x}$

Note: You do not need to take any derivatives. You can reduce them all to other Maclaurin series you know.

(A) Write the Maclaurin series for $G(x) = \arctan x$

Hint: Compute the first derivative. Then use the geometric series. Then integrate.

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Hint: Compute the first derivative. Then use the geometric series. Then integrate.

(B) What is $G^{(137)}(0)$?

(C) Use this previous results to compute

$$A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,3^n}$$

arcsin

Let $f(x) = \dfrac{1}{\sqrt{1+x}}$.

(A) Find a formula for its derivatives $f^{(n)}(x)$.

(B) Write its Maclaurin series at 0. Call it $S(x)$.

(C) What is the radius of convergence of series $S(x)$?
Note: Use without proof that $f(x) = S(x)$ inside the interval of convergence.

(D) Use this result to write $h(x) = \arcsin$ as a power series centered at 0.
Hint: Compute $h'(x)$.

(E) What is $h^{(7)}(0)$?

Before next class:

- **Watch videos 14.11, 14.13**

I want to calculate

$$A = \int_0^1 t^{10} \sin t \, dt.$$

There are two ways to do it. Choose your favourite one:

(A) Use integration by parts 10 times.

(B) Use power series.

Integrals

I want to calculate

$$A = \int_0^1 t^{10} \sin t \, dt.$$

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(B) Use power series.

Estimate A with an error smaller than 0.001.

Add these series

(A) $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^n}{(2n+1)!}$ *Hint:* Think of sin

(B) $\displaystyle\sum_{n=0}^{\infty} (4n+1)x^{4n+2}$ *Hint:* $\dfrac{d}{dx}\left[x^{4n+1}\right] = ???$

(C) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n)!}$ *Hint:* Write first few terms. Combine e^1 and e^{-1}.

(D) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!(n+1)}$ *Hint:* Integrate

Before next class:

- **Watch videos**

Limits

Use Maclaurin series to compute these limits:

(A) $$\lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$$

(B) $$\lim_{x \to 0} \frac{\cos(2x) - e^{-2x^2}}{x^4}$$

(C) $$\lim_{x \to 0} \frac{[\sin x - x]^3 x}{[\cos x - 1]^4 [e^x - 1]^2}$$

Estimations

I want to estimate these two numbers

$$A = \sin 1, \qquad B = \ln 0.9.$$

(A) Use Taylor series to write A and B as infinite sums.

(B) If you want to estimate A or B with a small error using a partial sum, the fastest way is to use different theorems for A and B. What are they?

(C) Estimate B with an error smaller than 0.001.

MAT137 Lecture 72 — Outroduction

Wecome to the end!