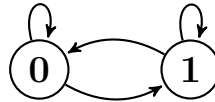
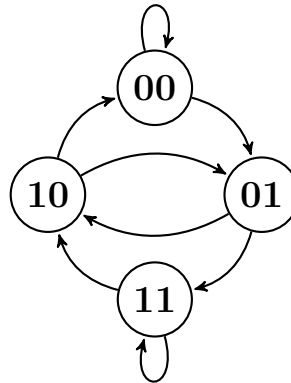


1. For a dynamical system (T, X) , a point $x \in X$ is called *eventually periodic* if there exists $m > m'$ so that $T^m x = T^{m'} x$.
 - (a) Let $(T, [0, 1))$ be the doubling map and let (σ, Ω) be the full shift on 2 symbols. For each system, give an example of (i) a point that is eventually periodic, and (ii) a point that is eventually periodic but *not* periodic.
 - (b) Let \mathbb{C} be the coding function for the doubling map with the usual partition $(\mathcal{P}_0 = [0, 1/2)$ and $\mathcal{P}_1 = [1/2, 1)$.
Prove $\mathbb{C}(x)$ is eventually periodic in (σ, Ω) if and only if x is eventually periodic in $(T, [0, 1))$.
 - (c) Prove that $\mathbb{C}(x)$ is eventually periodic if and only if x is a rational number.
 - (d) Prove that a binary expansion of a number in $[0, 1)$, viewed as an element of (σ, Ω) , is eventually periodic if and only if that number is rational. (*Hint: $\mathbb{C}(x)$ is always a binary expansion, but not all binary expansions come from codings*).
 - (e) Prove that the base- n expansion of a number in $[0, 1)$ is eventually periodic if and only if that number is rational.
2. Let (σ, Ω) be the full shift on 2 symbols. Let $G \subseteq \Omega$ be the set of sequences without two 1's in a row. Let $X \subseteq \Omega$ be the set of sequences without *three* 1's in a row.

- (a) Show that (σ, X) is a subshift.
- (b) A *two-step* Markov chain is a Markov chain where the transition probabilities between states depend on the current state *and* the previous state.
A *two-step* Markov chain on a state space S can be thought of as a one-step (regular) Markov chain on the state space $S \times S$. For example, let \mathcal{M} be the Markov chain on states 0 and 1 with equal transition probabilities. Normally, \mathcal{M} has a graph like this:



However, we can also model \mathcal{M} as a Markov chain \mathcal{M}' with state space 00, 01, 10, and 11 and transition graph like this:



Explain how \mathcal{M}' models \mathcal{M} . In particular, explain why the graph for \mathcal{M}' has no edge between 00 and 11.

- (c) Let \mathcal{M}_G be a Markov chain whose set of realizations is G (defined above).
 - i. Draw a graph for \mathcal{M}_G and find its associated transition matrix.
 - ii. Use the transition matrix for \mathcal{M}_G to compute the entropy of (σ, G) .

- iii. Model \mathcal{M}_G as a two-step Markov chain \mathcal{M}'_G , which can be viewed as a Markov chain with state space $\{00, 01, 10, 11\}$.
 Draw the graph associated with \mathcal{M}'_G and find its transition matrix.
- iv. Using the transition matrix for \mathcal{M}'_G , compute the entropy of (σ, G) .
- (d) Can X be modeled by a one-step Markov chain? Why or why not?
- (e) Find the entropy of (σ, X) .
3. Let (σ, Ω) be the full shift on 2 symbols.
- (a) Prove that (σ, Ω) is *expansive*.
- (b) Prove that (σ, Ω) is *transitive*.
- (c) Show that (σ, Ω) is *chaotic*.
- (d) Let (T, X) be a dynamical system and let $x, y \in X$. We say x and y are δ -correlated for n steps if the distance between $T^i x$ and $T^i y$ is at most δ for $i = 0, \dots, n$.
 Let $x, y \in \Omega$ be points that are at distance 2^{-k} of each other. For how many steps will x and y be δ -correlated when $\delta = 1/4$?
- (e) If (T, X) is a chaotic dynamical system, can two points be δ -correlated indefinitely? What implications does this have for measurement error?
4. Two dynamical systems (T, X) and (S, Y) are called *conjugate* if there exists a continuous, invertible function $\Phi: Y \rightarrow X$ so that $T = \Phi \circ S \circ \Phi^{-1}$. In this case, Φ is called a *conjugacy*.
 The systems are called *semi-conjugate* if there exists a continuous, onto function $\Phi: Y \rightarrow X$ so that $T \circ \Phi = \Phi \circ S$. In this case, Φ is called a *semi-conjugacy*.
- (a) Let (T, X) and (S, Y) be conjugate dynamical systems.
- i. Show that if there exists a point of period k in Y , there exists a point of period k in X .
- ii. Show that if (S, Y) is transitive, then (T, X) is transitive.
- iii. Is it true that if (S, Y) is expansive, then (T, X) is necessarily expansive?
- (b) Prove that if (T, X) and (S, Y) are conjugate dynamical systems, that they are also semi-conjugate dynamical systems.
- (c) Let (T, X) and (S, Y) be semi-conjugate dynamical systems.
- i. Show that if there exists a point of period k in Y , there exists a point of period $\leq k$ in X .
- ii. Show that if (S, Y) is transitive, then (T, X) is transitive.
- iii. Is it true that if (S, Y) is expansive, then (T, X) is necessarily expansive?
- (d) Let $(T, [0, 1])$ be the doubling map and let (σ, Ω) be the full shift on 2 symbols.
- i. Show that $(T, [0, 1])$ and (σ, Ω) are semi-conjugate.
- ii. Show that $(T, [0, 1])$ is chaotic.
- (e) Let $(T, [0, 1])$ be the doubling map and let $(L, [0, 1])$ be the *logistic map* defined by $L(x) = rx(1 - x)$ for $r = 4$.
- i. Define $f: [0, 1] \rightarrow [0, 1]$ by $f(x) = \sin^2(2\pi x)$. Show that f is a semi-conjugacy between $(L, [0, 1])$ and $(T, [0, 1])$.
- ii. Show that $(L, [0, 1])$ contains points of every period.
- iii. Show that $(L, [0, 1])$ is chaotic.

Programming Problems

For the programming problems, please use the Jupyter notebook available at

<https://utoronto.syzygy.ca/jupyter/user-redirect/git-pull?repo=https://github.com/siefkenj/2023-MAT-335-webpage&subPath=homework/homework4-exercises.ipynb>

Make sure to comment your code and use “Markdown” style cells to explain your answers.

1. The *logistic map with parameter r* is the function $f: [0, 1] \rightarrow [0, 1]$ defined by $f(x) = rx(1 - x)$ (for $r \in [0, 4]$). It was invented as a simple model for population in biology. The idea is that if x is your original population, there will be some birth rate r governing population growth; however, when the population outstretches its resources, its growth will be constrained. The $1 - x$ parameter models this constraint on growth.

Despite being so simple, the logistic map can have amazingly complex behaviour!

- (a) A two-parameter logistic function `f` has been predefined. Create a function `orbit_f` which inputs a starting value, a parameter r , and an orbit length n , and returns a list with the first n points along the orbit of `f`.
 - (b) Given a dynamical system (T, X) , the n -orbit of x is $\mathcal{O}^n(x) = \{x, Tx, \dots, T^{n-1}x\}$. Using $r = 2$, plot the 20-orbits of x for at least 10 different x 's (plot time vs. value). What do you notice?
 - (c) Does the logistic map with $r = 2$ have a *basin of attraction*¹? Justify your conclusion.
 - (d) Repeat part 1b using $r = 3$. What do you notice? Is there a basin of attraction consisting of a single point?
 - (e) In general, a *basin of attraction* for a dynamical system is a set which all points limit to. Does the logistic map have a basin of attraction when $r = 3$? If so, describe it.
 - (f) Repeat part 1b with $r = 4$. Is there a basin of attraction? Why or why not?
2. We are going to plot the basins of attraction for the logistic map as a function of r . We can approximate a basin of attraction by taking the 1000-orbit of several points, and then taking the set consisting of the last 100 points in each orbit.
 - (a) Create a function `approximate_basin` which takes in a parameter r and returns a list of points that approximate the basin of attraction of the logistic map with parameter r .
Hint: It may be useful to round your results to 3 or 4 decimal places and then use `np.unique` to get a list of manageable length.
 - (b) Plot the basins of attraction of the logistic map vs. r for at 1000 different values of r between 0 and 4.
 - (c) What do you notice about the basins of attraction? Did you expect this?
 - (d) In the *proofs* part of this homework set, you proved that the logistic map is chaotic when $r = 4$. What does that imply about the basin of attraction?
 - (e) If a population is well-modeled by the logistic map with a parameter of $r = 3.82$, what can you say about the population after 1000 days? What if it is modeled by a logistic map with parameter $r = 3.83$? Is the population more or less predictable?

¹Look back in the notes if you've forgotten this term.