

## Set

A **set** is a (possibly infinite) collection of items and is notated with curly braces (for example,  $\{1, 2, 3\}$  is the set containing the numbers 1, 2, and 3). We call the items in a set **elements**.

If  $X$  is a set and  $a$  is an element of  $X$ , we may write  $a \in X$ , which is read “ $a$  is an element of  $X$ .”

If  $X$  is a set, a **subset**  $Y$  of  $X$  (written  $Y \subseteq X$ ) is a set such that every element of  $Y$  is an element of  $X$ . Two sets are called **equal** if they are subsets of each other (i.e.,  $X = Y$  if  $X \subseteq Y$  and  $Y \subseteq X$ ).

We can define a subset using **set-builder notation**. That is, if  $X$  is a set, we can define the subset

$$Y = \{a \in X : \text{some rule involving } a\},$$

which is read “ $Y$  is the set of  $a$  in  $X$  **such that** some rule involving  $a$  is true.” If  $X$  is intuitive, we may omit it and simply write  $Y = \{a : \text{some rule involving } a\}$ . You may equivalently use “|” instead of “:”, writing  $Y = \{a | \text{some rule involving } a\}$ .

Some common sets are

$\mathbb{N} = \{\text{natural numbers}\} = \{\text{non-negative whole numbers}\}.$

$\mathbb{Z} = \{\text{integers}\} = \{\text{whole numbers, including negatives}\}.$

$\mathbb{R} = \{\text{real numbers}\}.$

$\mathbb{R}^n = \{\text{vectors in } n\text{-dimensional Euclidean space}\}.$

1 Which of the following statements are true?

- 1.1 (a)  $3 \in \{1, 2, 3\}$ .
- (b)  $1.5 \in \{1, 2, 3\}$ .
- (c)  $4 \in \{1, 2, 3\}$ .
- (d) “b” $\in \{x : x \text{ is an English letter}\}$ .
- (e) “ò” $\in \{x : x \text{ is an English letter}\}$ .
- (f)  $\{1, 2\} \subseteq \{1, 2, 3\}$ .
- (g) For some  $a \in \{1, 2, 3\}$ ,  $a \geq 3$ .
- (h) For any  $a \in \{1, 2, 3\}$ ,  $a \geq 3$ .
- (i)  $1 \subseteq \{1, 2, 3\}$ .
- (j)  $\{1, 2, 3\} = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$ .
- (k)  $\{1, 2, 3\} = \{x \in \mathbb{Z} : 1 \leq x \leq 3\}$ .

2

Write the following in set-builder notation

2.1 The subset  $A \subseteq \mathbb{R}$  of real numbers larger than  $\sqrt{2}$ .

2.2 The subset  $B \subseteq \mathbb{R}^2$  of vectors whose first coordinate is twice the second.

# Unions & Intersections

Two common set operations are *unions* and *intersections*. Let  $X$  and  $Y$  be sets.

$$\text{(union)} \quad X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)} \quad X \cap Y = \{a : a \in X \text{ and } a \in Y\}.$$

3 Let  $X = \{1, 2, 3\}$  and  $Y = \{2, 3, 4, 5\}$  and  $Z = \{4, 5, 6\}$ . Compute

3.1  $X \cup Y$

3.2  $X \cap Y$

3.3  $X \cup Y \cup Z$

3.4  $X \cap Y \cap Z$

4 Draw the following subsets of  $\mathbb{R}^2$ .

4.1  $V = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$ .

4.2  $H = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$ .

4.3  $D = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$ .

4.4  $N = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for all } t \in \mathbb{R} \right\}$ .

4.5  $V \cup H$ .

4.6  $V \cap H$ .

4.7 Does  $V \cup H = \mathbb{R}^2$ ?

# Linear Combination

A *linear combination* of the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  is a vector

$$\vec{w} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n.$$

The scalars  $\alpha_1, \alpha_2, \dots, \alpha_n$  are called the *coefficients* of the linear combination.

5

Let  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , and  $\vec{w} = 2\vec{v}_1 + \vec{v}_2$ .

- 5.1 Write  $\vec{w}$  as a column vector. When  $\vec{w}$  is written as a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ , what are the coefficients of  $\vec{v}_1$  and  $\vec{v}_2$ ?
- 5.2 Is  $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$  a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ ?
- 5.3 Is  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$  a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ ?
- 5.4 Is  $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$  a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ ?
- 5.5 Can you find a vector in  $\mathbb{R}^2$  that isn't a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ ?
- 5.6 Can you find a vector in  $\mathbb{R}^2$  that isn't a linear combination of  $\vec{v}_1$ ?

6

Recall the *Magic Carpet Ride* task where the hover board could travel in the direction  $\vec{h} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$  and the magic carpet could move in the

direction  $\vec{m} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .

- 6.1 Rephrase the sentence “*Gauss can be reached using just the magic carpet and the hover board*” using formal mathematical language.
- 6.2 Rephrase the sentence “*There is nowhere Gauss can hide where he is inaccessible by magic carpet and hover board*” using formal mathematical language.
- 6.3 Rephrase the sentence “ $\mathbb{R}^2$  is the set of all linear combinations of  $\vec{h}$  and  $\vec{m}$ ” using formal mathematical language.

# Non-negative & Convex Linear Combinations

The linear combination  $\vec{w} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n$  is called a **non-negative** linear combination of  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  if  $\alpha_1, \alpha_2, \dots, \alpha_n \geq 0$ .

If  $\alpha_1, \alpha_2, \dots, \alpha_n \geq 0$  and  $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$ , then  $\vec{w}$  is called a **convex** linear combination of  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ .

Let

$$\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \vec{d} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \vec{e} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

7.1 Out of  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$ ,  $\vec{d}$ , and  $\vec{e}$ , which vectors are

- (a) linear combinations of  $\vec{a}$  and  $\vec{b}$ ?
- (b) non-negative linear combinations of  $\vec{a}$  and  $\vec{b}$ ?
- (c) convex linear combinations of  $\vec{a}$  and  $\vec{b}$ ?

7.2 If possible, find two vectors  $\vec{u}$  and  $\vec{v}$  so that

- (a)  $\vec{a}$  and  $\vec{c}$  are non-negative linear combinations of  $\vec{u}$  and  $\vec{v}$  but  $\vec{b}$  is not.
- (b)  $\vec{a}$  and  $\vec{e}$  are non-negative linear combinations of  $\vec{u}$  and  $\vec{v}$ .
- (c)  $\vec{a}$  and  $\vec{b}$  are non-negative linear combinations of  $\vec{u}$  and  $\vec{v}$  but  $\vec{d}$  is not.
- (d)  $\vec{a}$ ,  $\vec{c}$ , and  $\vec{d}$  are convex linear combinations of  $\vec{u}$  and  $\vec{v}$ .

Otherwise, explain why it's not possible.

- 8 Let  $A$  be the set of points  $(x, y) \in \mathbb{R}^2$  such that  $y = 2x + 1$ .
- 8.1 Describe  $A$  using set-builder notation.
- 8.2 Draw  $A$  as a subset of  $\mathbb{R}^2$ .
- 8.3 Add the vectors  $\vec{a} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ ,  $\vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$  and  $\vec{d} = \vec{b} - \vec{a}$  to your drawing.
- 8.4 For which  $t \in \mathbb{R}$  is it true that  $\vec{a} + t\vec{d} \in A$ ? Explain using your picture.

## Vector Form of a Line

### DEFINITION

A line  $\ell$  is written in *vector form* if it is expressed as

$$\vec{x} = t\vec{d} + \vec{p}$$

for some vector  $\vec{d}$  and point  $\vec{p}$ . That is,  $\ell = \{\vec{x} : \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \in \mathbb{R}\}$ . The vector  $\vec{d}$  is called a *direction vector* for  $\ell$ .

9

Let  $\ell \subseteq \mathbb{R}^2$  be the line with equation  $2x + y = 3$ , and let  $L \subseteq \mathbb{R}^3$  be the line with equations  $2x + y = 3$  and  $z = y$ .

9.1

Write  $\ell$  in vector form. Is vector form of  $\ell$  unique?

9.2

Write  $L$  in vector form.

9.3

Find another vector form for  $L$  where both “ $\vec{d}$ ” and “ $\vec{p}$ ” are different from before.

10

Let  $A$ ,  $B$ , and  $C$  be given in vector form by

$$\overrightarrow{x} = t \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_A + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\overrightarrow{x} = t \underbrace{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}_B + \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\overrightarrow{x} = t \underbrace{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}_C + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

- 10.1 Do the lines  $A$  and  $B$  intersect? Justify your conclusion.
- 10.2 Do the lines  $A$  and  $C$  intersect? Justify your conclusion.
- 10.3 Let  $\vec{p} \neq \vec{q}$  and suppose  $X$  has vector form  $\overrightarrow{x} = t\vec{d} + \vec{p}$  and  $Y$  has vector form  $\overrightarrow{x} = t\vec{d} + \vec{q}$ . Is it possible that  $X$  and  $Y$  intersect?

## Vector Form of a Plane

DEFINITION

A plane  $\mathcal{P}$  is written in *vector form* if it is expressed as

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p}$$

for some vectors  $\vec{d}_1$  and  $\vec{d}_2$  and point  $\vec{p}$ . That is,  $\mathcal{P} = \{\vec{x} : \vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p} \text{ for some } t, s \in \mathbb{R}\}$ . The vectors  $\vec{d}_1$  and  $\vec{d}_2$  are called *direction vectors* for  $\mathcal{P}$ .

11 Recall the intersecting lines  $A$  and  $B$  given in vector form by

$$\overrightarrow{x} = t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\overrightarrow{x} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

Let  $\mathcal{P}$  the plane that contains the lines  $A$  and  $B$ .

- 11.1 Find two direction vectors in  $\mathcal{P}$ .
- 11.2 Write  $\mathcal{P}$  in vector form.
- 11.3 Describe how vector form of a plane relates to linear combinations.
- 11.4 Write  $\mathcal{P}$  in vector form using different direction vectors and a different point.

12 Let  $Q \subseteq \mathbb{R}^3$  be a plane with equation  $x + y + z = 1$ .

12.1 Find three points in  $Q$ .

12.2 Find two direction vectors for  $Q$ .

12.3 Write  $Q$  in vector form.

# Span

The *span* of a set of vectors  $V$  is the set of all linear combinations of vectors in  $V$ . In set notation:  $\text{span } V$  is defined to be the set of vectors:

$$\{\vec{v} : \vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n \text{ for some } \vec{v}_1, \dots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \dots, \alpha_n\}$$

13 Let  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , and  $\vec{v}_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ .

13.1 Draw  $\text{span}\{\vec{v}_1\}$ .

13.2 Draw  $\text{span}\{\vec{v}_2\}$ .

13.3 Describe  $\text{span}\{\vec{v}_1, \vec{v}_2\}$ .

13.4 Describe  $\text{span}\{\vec{v}_1, \vec{v}_3\}$ .

13.5 Describe  $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ .

- 14 Let  $\ell_1 \subseteq \mathbb{R}^2$  be the line with equation  $x - y = 0$  and  $\ell_2 \subseteq \mathbb{R}^2$  the line with equation  $x - y = 4$ .
- 14.1 If possible, describe  $\ell_1$  as a span. Otherwise explain why it's not possible.
- 14.2 If possible, describe  $\ell_2$  as a span. Otherwise explain why it's not possible.
- 14.3 Does the expression  $\text{span}(\ell_1)$  make sense? If so, what is it? How about  $\text{span}(\ell_2)$ ?

## DEFINITION

# Set Addition

If  $A$  and  $B$  are sets of vectors, then the *set sum* of  $A$  and  $B$ , denoted  $A + B$ , is

$$A + B = \{\vec{x} : \vec{x} = \vec{a} + \vec{b} \text{ for some } \vec{a} \in A \text{ and } \vec{b} \in B\}.$$

- 15 Let  $A = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ ,  $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ , and  $\ell = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ .
- 15.1 Draw  $A$ ,  $B$ , and  $A + B$  in the same picture.
- 15.2 Is  $A + B$  the same as  $B + A$ ?
- 15.3 Draw  $\ell + A$ .
- 15.4 Consider the line  $\ell_2$  given in vector form by  $\vec{x} = t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . Can  $\ell_2$  be described using only a span? What about using a span and set addition?

## Linearly Dependent & Independent (Geometric)

### DEFINITION

We say the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  are *linearly dependent* if for at least one  $i$ ,

$$\vec{v}_i \in \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}.$$

Otherwise, they are called *linearly independent*.

16

Let  $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ , and  $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ .

- 16.1 Describe  $\text{span}\{\vec{u}, \vec{v}, \vec{w}\}$ .
- 16.2 Is  $\{\vec{u}, \vec{v}, \vec{w}\}$  linearly independent? Why or why not?  
Let  $X = \{\vec{u}, \vec{v}, \vec{w}\}$ .
- 16.3 Give a subset  $Y \subseteq X$  so that  $\text{span } Y = \text{span } X$  and  $Y$  is linearly independent.
- 16.4 Give a subset  $Z \subseteq X$  so that  $\text{span } Z = \text{span } X$  and  $Z$  is linearly independent and  $Z \neq Y$ .

# Trivial Linear Combination

DEF

We say a linear combination  $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$  is *trivial* if  
 $a_1 = a_2 = \cdots = a_n = 0.$

17

Recall  $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ , and  $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ .

- 17.1 Consider the linearly dependent set  $\{\vec{u}, \vec{v}, \vec{w}\}$  (where  $\vec{u}, \vec{v}, \vec{w}$  are defined as above). Can you write  $\vec{0}$  as a non-trivial linear combination of vectors in this set?
- 17.2 Consider the linearly independent set  $\{\vec{u}, \vec{v}\}$ . Can you write  $\vec{0}$  as a non-trivial linear combination of vectors in this set?

We now have an equivalent definition of linear dependence.

# Linearly Dependent & Independent (Algebraic)

DEF

The vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  are *linearly dependent* if there is a non-trivial linear combination of  $\vec{v}_1, \dots, \vec{v}_n$  that equals the zero vector.

- 18.1 Explain how this new definition implies the old one.
- 18.2 Explain how the old definition implies this new one.

Since we have old def  $\implies$  new def, and new def  $\implies$  old def ( $\implies$  should be read aloud as ‘implies’), the two definitions are *equivalent* (which we write as new def  $\iff$  old def).

19 Suppose for some unknown  $\vec{u}, \vec{v}, \vec{w}$ , and  $\vec{a}$ ,

$$\vec{a} = 3\vec{u} + 2\vec{v} + \vec{w} \quad \text{and} \quad \vec{a} = 2\vec{u} + \vec{v} - \vec{w}.$$

19.1 Could the set  $\{\vec{u}, \vec{v}, \vec{w}\}$  be linearly independent?

Suppose that

$$\vec{a} = \vec{u} + 6\vec{r} - \vec{s}$$

is the *only* way to write  $\vec{a}$  using  $\vec{u}, \vec{r}, \vec{s}$ .

19.2 Is  $\{\vec{u}, \vec{r}, \vec{s}\}$  linearly independent?

19.3 Is  $\{\vec{u}, \vec{r}\}$  linearly independent?

19.4 Is  $\{\vec{u}, \vec{v}, \vec{w}, \vec{r}\}$  linearly independent?

# Norm

The **norm** of a vector  $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$  is the length/magnitude of  $\vec{v}$ .

It is written  $\|\vec{v}\|$  and can be computed from the Pythagorean formula

$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

## Dot Product

### DEFINITION

If  $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$  and  $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$  are two vectors in  $n$ -dimensional space, then the *dot product* of  $\vec{a}$  and  $\vec{b}$  is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Equivalently, the dot product is defined by the geometric formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

where  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ .

20

Let  $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ , and  $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ .

- 20.1 (a) Draw a picture of  $\vec{a}$  and  $\vec{b}$ .  
 (b) Compute  $\vec{a} \cdot \vec{b}$ .  
 (c) Find  $\|\vec{a}\|$  and  $\|\vec{b}\|$  and use your knowledge of the multiple ways to compute the dot product to find  $\theta$ , the angle between  $\vec{a}$  and  $\vec{b}$ . Label  $\theta$  on your picture.
- 20.2 Draw the graph of  $\cos$  and identify which angles make  $\cos$  negative, zero, or positive.
- 20.3 Draw a new picture of  $\vec{a}$  and  $\vec{b}$  and on that picture draw
- (a) a vector  $\vec{c}$  where  $\vec{c} \cdot \vec{a}$  is negative.
  - (b) a vector  $\vec{d}$  where  $\vec{d} \cdot \vec{a} = 0$  and  $\vec{d} \cdot \vec{b} < 0$ .
  - (c) a vector  $\vec{e}$  where  $\vec{e} \cdot \vec{a} = 0$  and  $\vec{e} \cdot \vec{b} > 0$ .
  - (d) Could you find a vector  $\vec{f}$  where  $\vec{f} \cdot \vec{a} = 0$  and  $\vec{f} \cdot \vec{b} = 0$ ? Explain why or why not.
- 20.4 Recall the vector  $\vec{u}$  whose coordinates are given at the beginning of this problem.
- (a) Write down a vector  $\vec{v}$  so that the angle between  $\vec{u}$  and  $\vec{v}$  is  $\pi/2$ . (Hint, how does this relate to the dot product?)
  - (b) Write down another vector  $\vec{w}$  (in a different direction from  $\vec{v}$ ) so that the angle between  $\vec{w}$  and  $\vec{u}$  is  $\pi/2$ .
  - (c) Can you write down other vectors different than both  $\vec{v}$  and  $\vec{w}$  that still form an angle of  $\pi/2$  with  $\vec{u}$ ? How many such vectors are there?

## THEOREM

For a vector  $\vec{v} \in \mathbb{R}^n$ , the formula

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

always holds.



# Distance

The *distance* between two vectors  $\vec{u}$  and  $\vec{v}$  is  $||\vec{u} - \vec{v}||$ .

# Unit Vector

A vector  $\vec{v}$  is called a *unit vector* if  $\|\vec{v}\| = 1$ .

21

Let  $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ .

- 21.1 Find the distance between  $\vec{u}$  and  $\vec{v}$ .
- 21.2 Find a unit vector in the direction of  $\vec{u}$ .
- 21.3 Does there exist a *unit vector*  $\vec{x}$  that is distance 1 from  $\vec{u}$ ?
- 21.4 Suppose  $\vec{y}$  is a unit vector and the distance between  $\vec{y}$  and  $\vec{u}$  is 2.  
What is the angle between  $\vec{y}$  and  $\vec{u}$ ?

# Orthogonal

DEF Two vectors  $\vec{u}$  and  $\vec{v}$  are *orthogonal* if  $\vec{u} \cdot \vec{v} = 0$ . The word orthogonal is synonymous with the word perpendicular.

- 22.1 Find two vectors orthogonal to  $\vec{a} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ . Can you find two such vectors that are not parallel?
- 22.2 Find two vectors orthogonal to  $\vec{b} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$ . Can you find two such vectors that are not parallel?
- 22.3 Suppose  $\vec{x}$  and  $\vec{y}$  are orthogonal to each other and  $\|\vec{x}\| = 5$  and  $\|\vec{y}\| = 3$ . What is the distance between  $\vec{x}$  and  $\vec{y}$ ?

23

- 23.1 Draw  $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  and *all* vectors orthogonal to it. Call this set  $A$ .
- 23.2 If  $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$  and  $\vec{x}$  is orthogonal to  $\vec{u}$ , what is  $\vec{x} \cdot \vec{u}$ ?
- 23.3 Expand the dot product  $\vec{u} \cdot \vec{x}$  to get an equation for  $A$ .
- 23.4 If possible, express  $A$  as a span.

# Normal Vector

DEFINITION

A *normal vector* to a line (or plane or hyperplane) is a non-zero vector that is orthogonal to all direction vectors for the line (or plane or hyperplane).

24

Let  $\vec{d} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\vec{p} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , and define the lines

$$\ell_1 = \text{span}\{\vec{d}\} \quad \text{and} \quad \ell_2 = \text{span}\{\vec{d}\} + \{\vec{p}\}.$$

- 24.1 Find a vector  $\vec{n}$  that is a normal vector for both  $\ell_1$  and  $\ell_2$ .
- 24.2 Let  $\vec{v} \in \ell_1$  and  $\vec{u} \in \ell_2$ . What is  $\vec{n} \cdot \vec{v}$ ? What about  $\vec{n} \cdot \vec{u}$ ?
- 24.3 A line is expressed in *normal form* if it is represented by an equation of the form  $\vec{n} \cdot (\vec{x} - \vec{q}) = 0$  for some  $\vec{n}$  and  $\vec{q}$ . Express  $\ell_1$  and  $\ell_2$  in normal form.

25

Let  $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ .

- 25.1 Use set-builder notation to write down the set,  $X$ , of all vectors orthogonal to  $\vec{n}$ . Describe this set geometrically.
- 25.2 Describe  $X$  using an equation.
- 25.3 Describe  $X$  as a span.

# Projection

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Let  $X$  be a set. The *projection* of the vector  $\vec{v}$  onto  $X$ , written  $\text{proj}_X \vec{v}$ , is the closest point in  $X$  to  $\vec{v}$ .

26

Let  $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\vec{b} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$  and  $\ell = \text{span}\{\vec{a}\}$ .

- 26.1 Draw  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{v}$  in the same picture.
- 26.2 Find  $\text{proj}_{\{\vec{b}\}} \vec{v}$ ,  $\text{proj}_{\{\vec{a}, \vec{b}\}} \vec{v}$ .
- 26.3 Find  $\text{proj}_{\ell} \vec{v}$ . (Recall that a quadratic  $at^2 + bt + c$  has a minimum at  $t = -\frac{b}{2a}$ ).
- 26.4 Is  $\vec{v} - \text{proj}_{\ell} \vec{v}$  a normal vector for  $\ell$ ? Why or why not?

27

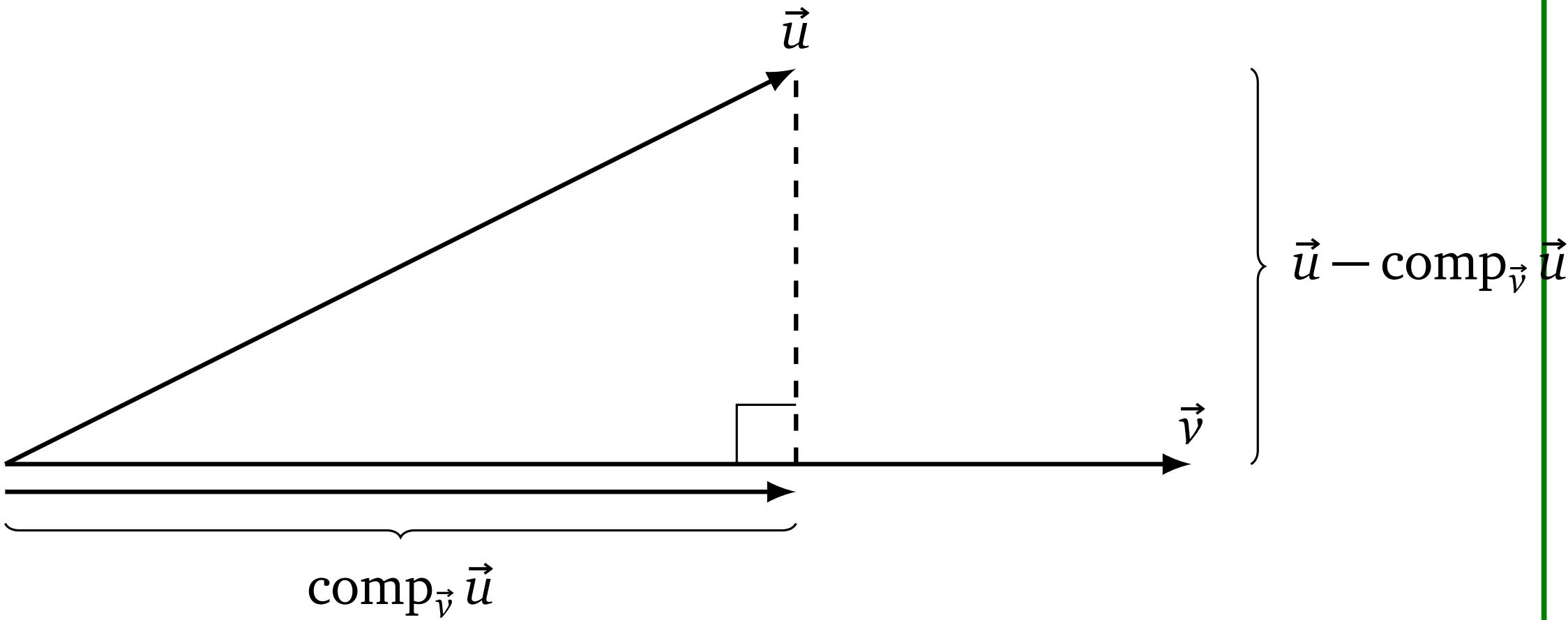
Let  $K$  be the line given in vector form by  $\vec{x} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and let  $\vec{c} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ .

- 27.1 Make a sketch with  $\vec{c}$ ,  $K$ , and  $\text{proj}_K \vec{c}$  (you don't need to compute  $\text{proj}_K \vec{c}$  exactly).
- 27.2 What should  $(\vec{c} - \text{proj}_K \vec{c}) \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  be? Explain.
- 27.3 Use your formula from the previous part to find  $\text{proj}_K \vec{c}$  without computing any distances.

## Component

Let  $\vec{u}$  and  $\vec{v} \neq \vec{0}$  be vectors. The *component of  $\vec{u}$  in the  $\vec{v}$  direction*, written  $\text{comp}_{\vec{v}} \vec{u}$ , is the vector in the direction of  $\vec{v}$  so that  $\vec{u} - \text{comp}_{\vec{v}} \vec{u}$  is orthogonal to  $\vec{v}$ .

DEFINITION



28 Let  $\vec{a}, \vec{b} \in \mathbb{R}^3$  be unknown vectors.

28.1 List two conditions that  $\text{comp}_{\vec{b}} \vec{a}$  must satisfy.

28.2 Find a formula for  $\text{comp}_{\vec{b}} \vec{a}$ .

29

Let  $\vec{d} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$  and  $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .

- 29.1 Draw  $\vec{d}$ ,  $\vec{u}$ ,  $\text{span}\{\vec{d}\}$ , and  $\text{proj}_{\text{span}\{\vec{d}\}} \vec{u}$  in the same picture.
- 29.2 How do  $\text{proj}_{\text{span}\{\vec{d}\}} \vec{u}$  and  $\text{comp}_{\vec{d}} \vec{u}$  relate?
- 29.3 Compute  $\text{proj}_{\text{span}\{\vec{d}\}} \vec{u}$  and  $\text{comp}_{\vec{d}} \vec{u}$ .
- 29.4 Compute  $\text{comp}_{-\vec{d}} \vec{u}$ . Is this the same as or different from  $\text{comp}_{\vec{d}} \vec{u}$ ? Explain.

# Subspace

A **subspace**  $V \subseteq \mathbb{R}^n$  is a non-empty subset such that

(i)  $\vec{u}, \vec{v} \in V$  implies  $\vec{u} + \vec{v} \in V$ .

(ii)  $\vec{u} \in V$  implies  $k\vec{u} \in V$  for all scalars  $k$ .

Subspaces give a mathematically precise definition of a “flat space through the origin.”

- 30 For each set, draw it and explain whether or not it is a subspace of  $\mathbb{R}^2$ .
- 30.1  $A = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} a \\ 0 \end{bmatrix} \text{ for some } a \in \mathbb{Z} \right\}$ .
- 30.2  $B = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ .
- 30.3  $C = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$ .
- 30.4  $D = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$ .
- 30.5  $E = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ or } \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$ .
- 30.6  $F = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$ .
- 30.7  $G = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ .
- 30.8  $H = \text{span}\{\vec{u}, \vec{v}\}$  for some unknown vectors  $\vec{u}, \vec{v} \in \mathbb{R}^2$ .

# Basis

---

DEF

A *basis* for a subspace  $V$  is a linearly independent set of vectors,  $\mathcal{B}$ , so that  $\text{span } \mathcal{B} = V$ .

# Dimension

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The **dimension** of a subspace  $V$  is the number of elements in a basis for  $V$ .

31

Let  $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ,  $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ , and  $V = \text{span}\{\vec{u}, \vec{v}, \vec{w}\}$ .

- 31.1 Describe  $V$ .
- 31.2 Is  $\{\vec{u}, \vec{v}, \vec{w}\}$  a basis for  $V$ ? Why or why not?
- 31.3 Give a basis for  $V$ .
- 31.4 Give another basis for  $V$ .
- 31.5 Is  $\text{span}\{\vec{u}, \vec{v}\}$  a basis for  $V$ ? Why or why not?
- 31.6 What is the dimension of  $V$ ?

32

Let  $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ,  $\vec{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ ,  $\vec{c} = \begin{bmatrix} 7 \\ 8 \\ 8 \end{bmatrix}$  (notice these vectors are linearly independent) and let  $P = \text{span}\{\vec{a}, \vec{b}\}$  and  $Q = \text{span}\{\vec{b}, \vec{c}\}$ .

- 32.1 Give a basis for and the dimension of  $P$ .
- 32.2 Give a basis for and the dimension of  $Q$ .
- 32.3 Is  $P \cap Q$  a subspace? If so, give a basis for it and its dimension.
- 32.4 Is  $P \cup Q$  a subspace? If so, give a basis for it and its dimension.

33

Let  $A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$ ,  $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ , and  $\vec{b} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ .

- 33.1 Compute the product  $A\vec{x}$ .
- 33.2 Write down a system of equations that corresponds to the matrix equation  $A\vec{x} = \vec{b}$ .
- 33.3 Let  $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$  be a solution to  $A\vec{x} = \vec{b}$ . Explain what  $x_0$  and  $y_0$  mean in terms of *linear combinations* (hint: think about the columns of  $A$ ).
- 33.4 Let  $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$  be a solution to  $A\vec{x} = \vec{b}$ . Explain what  $x_0$  and  $y_0$  mean in terms of *intersecting lines* (hint: think about systems of equations).

34

Let  $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ ,  $\vec{w} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ .

- 34.1 How could you determine if  $\{\vec{u}, \vec{v}, \vec{w}\}$  was a linearly independent set?
- 34.2 Can your method be rephrased in terms of a matrix equation? Explain.

35 Consider the system represented by

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{b}.$$

35.1 If  $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ , is the set of solutions to this system a point, line, plane, or other?

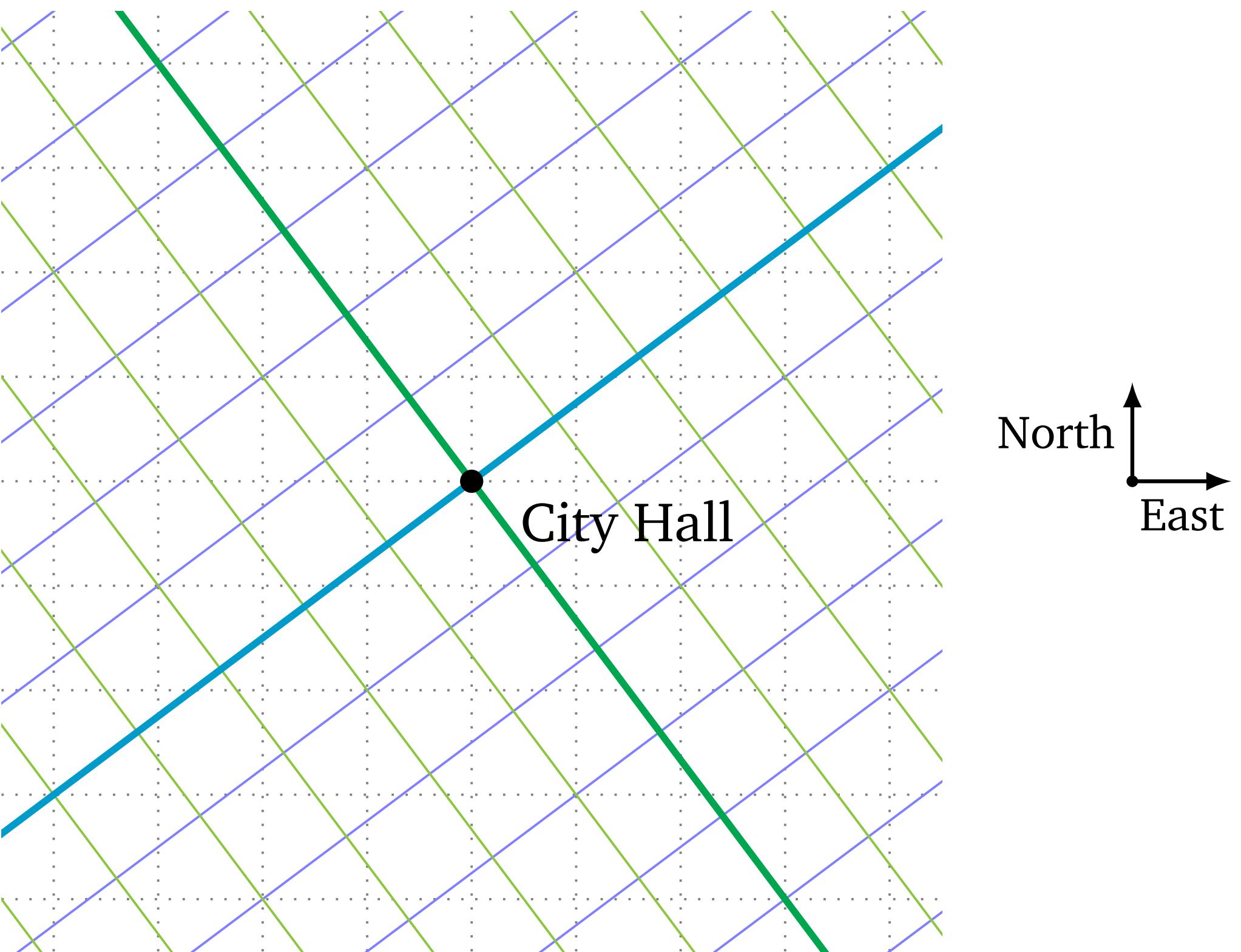
35.2 If  $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ , is the set of solutions to this system a point, line, plane, or other?

36

Let  $\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$  and  $\vec{d}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ . Let  $\mathcal{P}$  be the plane given in vector form by  $\vec{x} = t\vec{d}_1 + s\vec{d}_2$ . Further, suppose  $M$  is a matrix so that  $M\vec{r} \in \mathcal{P}$  for any  $\vec{r} \in \mathbb{R}^2$ .

- 36.1 How many rows does  $M$  have?
- 36.2 Find such an  $M$ .
- 36.3 Find necessary and sufficient conditions (phrased as equations) for  $\vec{n}$  to be a normal vector for  $\mathcal{P}$ .
- 36.4 Find a matrix  $K$  so that non-zero solutions to  $K\vec{x} = \vec{0}$  are normal vectors for  $\mathcal{P}$ . How do  $K$  and  $M$  relate?

- 37 The fictional town of Oronto is not aligned with the usual compass directions. The streets are laid out as follows:



Instead, every street is parallel to the vector  $\vec{d}_1 = \frac{1}{5} \begin{bmatrix} 4 \text{ east} \\ 3 \text{ north} \end{bmatrix}$  or  $\vec{d}_2 = \frac{1}{5} \begin{bmatrix} -3 \text{ east} \\ 4 \text{ north} \end{bmatrix}$ . The center of town is City Hall at  $\vec{0} = \begin{bmatrix} 0 \text{ east} \\ 0 \text{ north} \end{bmatrix}$ .

Locations in Oronto are typically specified in *street coordinates*. That is, as a pair  $(a, b)$  where  $a$  is how far you walk along streets in the  $\vec{d}_1$  direction and  $b$  is how far you walk in the  $\vec{d}_2$  direction, provided you start at city hall.

- 37.1 The points  $A = (2, 1)$  and  $B = (3, -1)$  are given in street coordinates. Find their east-north coordinates.
- 37.2 The points  $X = (4, 3)$  and  $Y = (1, 7)$  are given in east-north coordinates. Find their street coordinates.
- 37.3 Define  $\vec{e}_1 = \begin{bmatrix} 1 \text{ east} \\ 0 \text{ north} \end{bmatrix}$  and  $\vec{e}_2 = \begin{bmatrix} 0 \text{ east} \\ 1 \text{ north} \end{bmatrix}$ . Does  $\text{span}\{\vec{e}_1, \vec{e}_2\} = \text{span}\{\vec{d}_1, \vec{d}_2\}$ ?
- 37.4 Notice that  $Y = 5\vec{d}_1 + 5\vec{d}_2 = \vec{e}_1 + 7\vec{e}_2$ . Is the point  $Y$  better represented by the pair  $(5, 5)$  or by the pair  $(1, 7)$ ? Explain.

## Representation in a Basis

Let  $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$  be a basis for a subspace  $V$  and let  $\vec{v} \in V$ . The *representation of  $\vec{v}$  in the  $\mathcal{B}$  basis*, notated  $[\vec{v}]_{\mathcal{B}}$ , is the column matrix

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

where  $\alpha_1, \dots, \alpha_n$  uniquely satisfy  $\vec{v} = \alpha_1 \vec{b}_1 + \dots + \alpha_n \vec{b}_n$ .

Conversely,

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}_{\mathcal{B}} = \alpha_1 \vec{b}_1 + \dots + \alpha_n \vec{b}_n$$

is notation for the linear combination of  $\vec{b}_1, \dots, \vec{b}_n$  with coefficients  $\alpha_1, \dots, \alpha_n$ .

38 Let  $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$  be the standard basis for  $\mathbb{R}^2$  and let  $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$  where  $\vec{c}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{E}}$  and  $\vec{c}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{E}}$  be another basis for  $\mathbb{R}^2$ .

- 38.1 Express  $\vec{c}_1$  and  $\vec{c}_2$  as a linear combination of  $\vec{e}_1$  and  $\vec{e}_2$ .
- 38.2 Express  $\vec{e}_1$  and  $\vec{e}_2$  as a linear combination of  $\vec{c}_1$  and  $\vec{c}_2$ .
- 38.3 Let  $\vec{v} = 2\vec{e}_1 + 2\vec{e}_2$ . Find  $[\vec{v}]_{\mathcal{E}}$  and  $[\vec{v}]_{\mathcal{C}}$ .
- 38.4 Can you find a matrix  $X$  so that  $X[\vec{w}]_{\mathcal{C}} = [\vec{w}]_{\mathcal{E}}$  for any  $\vec{w}$ ?
- 38.5 Can you find a matrix  $Y$  so that  $Y[\vec{w}]_{\mathcal{E}} = [\vec{w}]_{\mathcal{C}}$  for any  $\vec{w}$ ?
- 38.6 What is  $YX$ ?

# Orientation of a Basis

The ordered basis  $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$  is *right-handed* or *positively oriented* if it can be continuously transformed to the standard basis (with  $\vec{b}_i \mapsto \vec{e}_i$ ) while remaining linearly independent throughout the transformation. Otherwise,  $\mathcal{B}$  is called *left-handed* or *negatively oriented*.

- 39 Let  $\{\vec{e}_1, \vec{e}_2\}$  be the standard basis for  $\mathbb{R}^2$  and let  $\vec{u}_\theta$  be a unit vector. Let  $\theta$  be the angle between  $\vec{u}_\theta$  and  $\vec{e}_1$  measured counter-clockwise starting at  $\vec{e}_1$ .
- 39.1 For which  $\theta$  is  $\{\vec{e}_1, \vec{u}_\theta\}$  a linearly independent set?
- 39.2 For which  $\theta$  can  $\{\vec{e}_1, \vec{u}_\theta\}$  be continuously transformed into  $\{\vec{e}_1, \vec{e}_2\}$  and remain linearly independent the whole time?
- 39.3 For which  $\theta$  is  $\{\vec{e}_1, \vec{u}_\theta\}$  right-handed? Left-handed?
- 39.4 For which  $\theta$  is  $\{\vec{u}_\theta, \vec{e}_1\}$  (in that order) right-handed? Left-handed?
- 39.5 Is  $\{2\vec{e}_1, 3\vec{e}_2\}$  right-handed or left-handed? What about  $\{2\vec{e}_1, -3\vec{e}_2\}$ ?

40  $\mathcal{R} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is the transformation that rotates vectors counter-clockwise by  $90^\circ$ .

40.1 Compute  $\mathcal{R} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\mathcal{R} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

40.2 Compute  $\mathcal{R} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . How does this relate to  $\mathcal{R} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\mathcal{R} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ?

40.3 What is  $\mathcal{R} \left( a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ ?

40.4 Write down a matrix  $R$  so that  $R\vec{v}$  is  $\vec{v}$  rotated counter-clockwise by  $90^\circ$ .

# Linear Transformation

Let  $V$  and  $W$  be subspaces. A function  $T : V \rightarrow W$  is called a *linear transformation* if

$$T(\vec{u} + \vec{v}) = T\vec{u} + T\vec{v} \quad \text{and} \quad T(\alpha\vec{v}) = \alpha T\vec{v}$$

for all vectors  $\vec{u}, \vec{v} \in V$  and all scalars  $\alpha$ .

41 Classify the following as linear transformations or not.

41.1 (a)  $\mathcal{R}$  from before (rotation counter-clockwise by  $90^\circ$ ).

(b)  $W : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  where  $W \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y \end{bmatrix}$ .

(c)  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  where  $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2 \\ y \end{bmatrix}$ .

(d)  $\mathcal{P} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  where  $\mathcal{P} \begin{bmatrix} x \\ y \end{bmatrix} = \text{comp}_{\vec{u}} \begin{bmatrix} x \\ y \end{bmatrix}$  and  $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ .

# Image of a Set

DEFINITION

Let  $L : V \rightarrow W$  be a transformation and let  $X \subseteq V$  be a set. The *image of the set  $X$  under  $L$* , denoted  $L(X)$ , is the set

$$L(X) = \{\vec{x} \in W : \vec{x} = L(\vec{y}) \text{ for some } \vec{y} \in X\}.$$

42

Let  $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 0 \leq x, y \leq 1 \right\} \subseteq \mathbb{R}^2$  be the filled-in unit square and let  $C = \{\vec{0}, \vec{e}_1, \vec{e}_2, \vec{e}_1 + \vec{e}_2\} \subseteq \mathbb{R}^2$  be the corners of the unit square.

- 42.1 Find  $\mathcal{R}(C)$ ,  $W(C)$ , and  $T(C)$  (where  $\mathcal{R}$ ,  $W$ , and  $T$  are from the previous question).
- 42.2 Draw  $\mathcal{R}(S)$ ,  $T(S)$ , and  $\mathcal{P}(S)$  (where  $\mathcal{R}$ ,  $T$ , and  $\mathcal{P}$  are from the previous question).
- 42.3 Let  $\ell = \{\text{all convex combinations of } \vec{a} \text{ and } \vec{b}\}$  be a line segment with endpoints  $\vec{a}$  and  $\vec{b}$  and let  $A$  be a linear transformation. Must  $A(\ell)$  be a line segment? What are its endpoints?
- 42.4 Explain how images of sets relate to the *Italicizing N* task.

43

Define  $\mathcal{P}$  to be projection onto  $\text{span}\{\vec{u}\}$  where  $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ , and let  $\mathcal{R}$  be rotation counter-clockwise by  $90^\circ$ .

- 43.1 Find a matrix  $P$  so that  $P\vec{x} = \mathcal{P}(\vec{x})$  for all  $\vec{x} \in \mathbb{R}^2$ .
- 43.2 Find a matrix  $R$  so that  $R\vec{x} = \mathcal{R}(\vec{x})$  for all  $\vec{x} \in \mathbb{R}^2$ .
- 43.3 Write down matrices  $A$  and  $B$  for  $\mathcal{P} \circ \mathcal{R}$  and  $\mathcal{R} \circ \mathcal{P}$ .
- 43.4 How do the matrices  $A$  and  $B$  relate to the matrices  $P$  and  $R$ ?

# Range

The *range* (or *image*) of a linear transformation  $T : V \rightarrow W$  is the set of vectors that  $T$  can output. That is,

$$\text{range}(T) = \{\vec{y} \in W : \vec{y} = T\vec{x} \text{ for some } \vec{x} \in V\}.$$

# Null Space

DEFINITION

The *null space* (or *kernel*) of a linear transformation  $T : V \rightarrow W$  is the set of vectors that get mapped to zero under  $T$ . That is,

$$\text{null}(T) = \{\vec{x} \in V : T\vec{x} = \vec{0}\}.$$

44 Let  $\mathcal{P} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be projection onto  $\text{span}\{\vec{u}\}$  where  $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  (like before).

44.1 What is the range of  $\mathcal{P}$ ?

44.2 What is the null space of  $\mathcal{P}$ ?

45 Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be an arbitrary linear transformation.

45.1 Show that the null space of  $T$  is a subspace.

45.2 Show that the range of  $T$  is a subspace.

## Induced Transformation

Let  $M$  be an  $n \times m$  matrix. We say  $M$  *induces* a linear transformation  $T_M : \mathbb{R}^m \rightarrow \mathbb{R}^n$  defined by

$$[T_M \vec{v}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}},$$

where  $\mathcal{E}$  is the standard basis for  $\mathbb{R}^m$  and  $\mathcal{E}'$  is the standard basis for  $\mathbb{R}^n$ .

46 Let  $M$  be a  $2 \times 2$  matrix and let  $\vec{v} \in \mathbb{R}^2$ . Further, let  $T_M$  be the transformation induced by  $M$ .

46.1 What is the difference between “ $M\vec{v}$ ” and “ $M[\vec{v}]_{\mathcal{E}}$ ”?

46.2 What is  $[T_M \vec{e}_1]_{\mathcal{E}}$ ?

46.3 Can you relate the columns of  $M$  to the range of  $T_M$ ?

# Fundamental Subspaces

Associated with any matrix  $M$  are three fundamental subspaces: the *row space* of  $M$ , denoted  $\text{row}(M)$ , is the span of the rows of  $M$ ; the *column space* of  $M$ , denoted  $\text{col}(M)$ , is the span of the columns of  $M$ ; and the *null space* of  $M$ , denoted  $\text{null}(M)$ , is the set of solutions to  $M\vec{x} = \vec{0}$ .

47

Consider  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ .

- 47.1 Describe the row space of  $A$ .
- 47.2 Describe the column space of  $A$ .
- 47.3 Is the row space of  $A$  the same as the column space of  $A$ ?
- 47.4 Describe the set of all vectors perpendicular to the rows of  $A$ .
- 47.5 Describe the null space of  $A$ .
- 47.6 Describe the range and null space of  $T_A$ , the transformation induced by  $A$ .

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$C = \text{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

- 48.1 How does the row space of  $B$  relate to the row space of  $C$ ?
- 48.2 How does the null space of  $B$  relate to the null space of  $C$ ?
- 48.3 Compute the null space of  $B$ .

49

$$P = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

$$Q = \text{rref}(P) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

- 49.1 How does the column space of  $P$  relate to the column space of  $Q$ ?
- 49.2 Describe the column space of  $P$  and the column space of  $Q$ .

# Rank

For a linear transformation  $T : V \rightarrow W$ , the *rank* of  $T$ , denoted  $\text{rank}(T)$ , is the dimension of the range of  $T$ .

For an  $n \times m$  matrix  $M$ , the *rank* of  $M$ , denoted  $\text{rank}(M)$ , is the number of pivots in  $\text{rref}(M)$ .

50

Let  $\mathcal{P}$  be projection onto  $\text{span}\{\vec{u}\}$  where  $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ , and let  $\mathcal{R}$  be rotation counter-clockwise by  $90^\circ$ .

- 50.1 Describe  $\text{range}(\mathcal{P})$  and  $\text{range}(\mathcal{R})$ .
- 50.2 What is the rank of  $\mathcal{P}$  and the rank of  $\mathcal{R}$ ?
- 50.3 Let  $P$  and  $R$  be the matrices corresponding to  $\mathcal{P}$  and  $\mathcal{R}$ . What is the rank of  $P$  and the rank of  $R$ ?
- 50.4 Make a conjecture about how the rank of a transformation and the rank of its corresponding matrix relate. Can you justify your claim?

51 Determine the rank of

- 51.1 (a)  $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$  (b)  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  (c)  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  (d)  $\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$  (e)  $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

52 Consider the homogeneous system

$$\begin{array}{cccc} x & +2y & +z & = 0 \\ x & +2y & +3z & = 0 \\ -x & -2y & +z & = 0 \end{array} \quad (1)$$

and the non-augmented matrix of coefficients  $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ -1 & -2 & 1 \end{bmatrix}$ .

- 52.1 What is  $\text{rank}(A)$ ?
- 52.2 Give the general solution to system (1).
- 52.3 Are the column vectors of  $A$  linearly independent?
- 52.4 Give a non-homogeneous system with the same coefficients as (1) that has
  - (a) infinitely many solutions
  - (b) no solutions.

- 53.1 The rank of a  $3 \times 4$  matrix  $A$  is 3. Are the column vectors of  $A$  linearly independent?
- 53.2 The rank of a  $4 \times 3$  matrix  $B$  is 3. Are the column vectors of  $B$  linearly independent?

# Rank-nullity Theorem

THEOREM

The *nullity* of a matrix is the dimension of the null space.

The rank-nullity theorem for a matrix  $A$  states

$$\text{rank}(A) + \text{nullity}(A) = \# \text{ of columns in } A.$$

54.1 Is there a version of the rank-nullity theorem that applies to linear transformations instead of matrices? If so, state it.

55 The vectors  $\vec{u}, \vec{v} \in \mathbb{R}^9$  are linearly independent and  $\vec{w} = 2\vec{u} - \vec{v}$ .  
Define  $A = [\vec{u} | \vec{v} | \vec{w}]$ .

55.1 What is the rank and nullity of  $A^T$ ?

55.2 What is the rank and nullity of  $A$ ?

- 56.1 Apply the row operation  $R_3 \mapsto R_3 + 2R_1$  to the  $3 \times 3$  identity matrix and call the result  $E_1$ .
- 56.2 Apply the row operation  $R_3 \mapsto R_3 - 2R_1$  to the  $3 \times 3$  identity matrix and call the result  $E_2$ .

An *elementary matrix* is the identity matrix with a single row operation applied.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

- 56.3 Compute  $E_1A$  and  $E_2A$ . How do the resulting matrices relate to row operations?
- 56.4 Without computing, what should the result of applying the row operation  $R_3 \mapsto R_3 - 2R_1$  to  $E_1$  be? Compute and verify.
- 56.5 Without computing, what should  $E_2E_1$  be? What about  $E_1E_2$ ? Now compute and verify.

**DEF**

The *inverse* of a matrix  $A$  is a matrix  $B$  such that  $AB = I$  and  $BA = I$ . In this case,  $B$  is called the inverse of  $A$  and is notated by  $A^{-1}$ .

57

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -3 & -6 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

57.1 Which pairs of matrices above are inverses of each other?

$$B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

- 58.1 Use two row operations to reduce  $B$  to  $I_{2 \times 2}$  and write an elementary matrix  $E_1$  corresponding to the first operation and  $E_2$  corresponding to the second.
- 58.2 What is  $E_2 E_1 B$ ?
- 58.3 Find  $B^{-1}$ .
- 58.4 Can you outline a procedure for finding the inverse of a matrix using elementary matrices?

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$C = [A|\vec{b}] \quad A^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

- 59.1 What is  $A^{-1}A$ ?
- 59.2 What is  $\text{rref}(A)$ ?
- 59.3 What is  $\text{rref}(C)$ ? (Hint, there is no need to actually do row reduction!)
- 59.4 Solve the system  $A\vec{x} = \vec{b}$ .

60

60.1 For two square matrices  $X, Y$ , should  $(XY)^{-1} = X^{-1}Y^{-1}$ ?

60.2 If  $M$  is a matrix corresponding to a non-invertible linear transformation  $T$ , could  $M$  be invertible?

61 Let  $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$  where  $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\vec{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  and let  $X = [\vec{b}_1 | \vec{b}_2]$  be the matrix whose columns are  $\vec{b}_1$  and  $\vec{b}_2$ .

- 61.1 Compute  $[\vec{e}_1]_{\mathcal{B}}$  and  $[\vec{e}_2]_{\mathcal{B}}$ .
- 61.2 Compute  $X[\vec{e}_1]_{\mathcal{B}}$  and  $X[\vec{e}_2]_{\mathcal{B}}$ . What do you notice?
- 61.3 Find the matrix  $X^{-1}$ . How does  $X^{-1}$  relate to change of basis?

62 Let  $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$  be the standard basis for  $\mathbb{R}^n$ . Given a basis  $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$  for  $\mathbb{R}^n$ , the matrix  $X = [\vec{b}_1 | \vec{b}_2 | \cdots | \vec{b}_n]$  converts vectors from the  $\mathcal{B}$  basis into the standard basis. In other words,

$$X[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{E}}.$$

62.1 Should  $X^{-1}$  exist? Explain.

62.2 Consider the equation

$$X^{-1}[\vec{v}]_? = [\vec{v}]_?.$$

Can you fill in the “?” symbols so that the equation makes sense?

62.3 What is  $[\vec{b}_1]_{\mathcal{B}}$ ? How about  $[\vec{b}_2]_{\mathcal{B}}$ ? Can you generalize to  $[\vec{b}_i]_{\mathcal{B}}$ ?

63 Let  $\vec{c}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{E}}$ ,  $\vec{c}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{E}}$ ,  $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ , and  $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ . Note that  $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$  and that  $A$  changes vectors from the  $\mathcal{C}$  basis to the standard basis and  $A^{-1}$  changes vectors from the standard basis to the  $\mathcal{C}$  basis.

63.1 Compute  $[\vec{c}_1]_{\mathcal{C}}$  and  $[\vec{c}_2]_{\mathcal{C}}$ .

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation that stretches in the  $\vec{c}_1$  direction by a factor of 2 and doesn't stretch in the  $\vec{c}_2$  direction at all.

63.2 Compute  $T \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{E}}$  and  $T \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{E}}$ .

63.3 Compute  $[T\vec{c}_1]_{\mathcal{C}}$  and  $[T\vec{c}_2]_{\mathcal{C}}$ .

63.4 Compute the result of  $T \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{\mathcal{C}}$  and express the result in the  $\mathcal{C}$  basis (i.e., as a vector of the form  $\begin{bmatrix} ? \\ ? \end{bmatrix}_{\mathcal{C}}$ ).

63.5 Find  $[T]_{\mathcal{C}}$ , the matrix for  $T$  in the  $\mathcal{C}$  basis.

63.6 Find  $[T]_{\mathcal{E}}$ , the matrix for  $T$  in the standard basis.

# Similar Matrices

A matrices  $A$  and  $B$  are called *similar matrices*, denoted  $A \sim B$ , if  $A$  and  $B$  represent the same linear transformation but in possibly different bases. Equivalently,  $A \sim B$  if there is an invertible matrix  $X$  so that

$$A = XBX^{-1}.$$

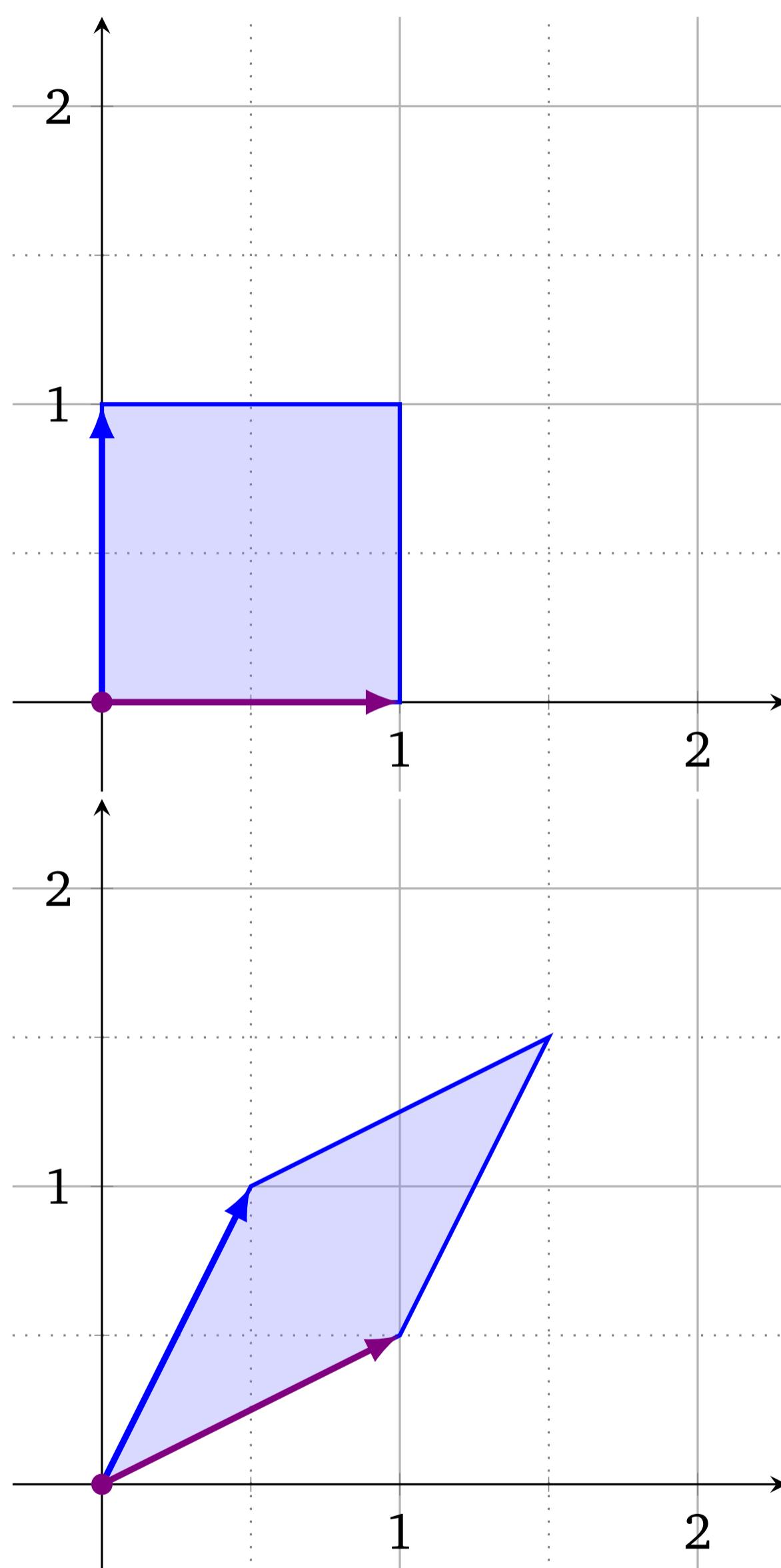
## Unit $n$ -cube

The **unit  $n$ -cube** is the  $n$ -dimensional cube with sides given by the standard basis vectors and lower-left corner located at the origin. That is

$$C_n = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^n \alpha_i \vec{e}_i \text{ for some } \alpha_1, \dots, \alpha_n \in [0, 1] \right\} = [0, 1]^n.$$

The sides of the unit  $n$ -cube are always length 1 and its volume is always 1.

64 The picture shows what the linear transformation  $T$  does to the unit square (i.e., the unit 2-cube).



64.1 What is  $T \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  $T \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ?

64.2 Write down a matrix for  $T$ .

64.3 What is the volume of the image of the unit square (i.e., the volume of  $T(C_2)$ )? You may use trigonometry.

# Determinant

The *determinant* of a linear transformation  $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the oriented volume of the image of the unit  $n$ -cube. The determinant of a square matrix is the determinant of its induced transformation.

65 We know the following about the transformation  $A$ :

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

- 65.1 Draw  $C_2$  and  $A(C_2)$ , the image of the unit square under  $A$ .
- 65.2 Compute the area of  $A(C_2)$ .
- 65.3 Compute  $\det(A)$ .

- 66 Suppose  $R$  is a rotation counter-clockwise by  $30^\circ$ .
- 66.1 Draw  $C_2$  and  $R(C_2)$ .
- 66.2 Compute the area of  $R(C_2)$ .
- 66.3 Compute  $\det(R)$ .

67 We know the following about the transformation  $F$ :

$$F \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and}$$

$$F \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

67.1 What is  $\det(F)$ ?

# Volume Theorem I

THM

For a square matrix  $M$ ,  $\det(M)$  is the oriented volume of the parallelepiped ( $n$ -dimensional parallelogram) given by the column vectors of  $M$ .

## Volume Theorem II

THM

For a square matrix  $M$ ,  $\det(M)$  is the oriented volume of the parallelepiped ( $n$ -dimensional parallelogram) given by the row vectors of  $M$ .

- 68.1 Explain Volume Theorem I using the definition of determinant.
- 68.2 Based on Volume Theorems I and II, how should  $\det(M)$  and  $\det(M^T)$  relate for a square matrix  $M$ ?

69 Let  $D = \{\vec{x} : \|\vec{x}\| \leq 1\}$  be the unit disk. You know the following about the linear transformations  $M$ ,  $T$ , and  $S$ .

$M$  is defined by  $\vec{x} \mapsto 2\vec{x}$ ;  $T$  has determinant 2; and  $S$  has determinant 3.

- 69.1 Find the oriented volumes of  $M(C_2)$ ,  $T(C_2)$ , and  $S(C_2)$ .
- 69.2 How does the volume of  $S(C_2 + \{\vec{e}_1\})$  compare to the volume of  $S(C_2)$ ?
- 69.3 What is the oriented volume of  $S(D)$ ?
- 69.4 What is the oriented volume of  $T \circ M(C_2)$ ? What is  $\det(T \circ M)$ ?

- $E_f$  is  $I_{3 \times 3}$  with the first two rows swapped.
- $E_m$  is  $I_{3 \times 3}$  with the third row multiplied by 6.
- $E_a$  is  $I_{3 \times 3}$  with  $R_1 \mapsto R_1 + 2R_2$  applied.

70.1 What is  $\det(E_f)$ ?

70.2 What is  $\det(E_m)$ ?

70.3 What is  $\det(E_a)$ ?

70.4 What is  $\det(E_f E_m)$ ?

70.5 What is  $\det(4I_{3 \times 3})$ ?

70.6 What is  $\det(W)$  where  $W = E_f E_a E_f E_m E_m$ ?

71

$$U = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

- 71.1 What is  $\det(U)$ ?
- 71.2  $V$  is a square matrix and  $\text{rref}(V)$  has a row of zeros. What is  $\det(V)$ ?
- 71.3  $P$  is projection onto  $\text{span}\left\{\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right\}$ . What is  $\det(P)$ ?

- 72 Suppose you know  $\det(X) = 4$ .
- 72.1 What is  $\det(X^{-1})$ ?
- 72.2 Derive a relationship between  $\det(Y)$  and  $\det(Y^{-1})$  for an arbitrary matrix  $Y$ .
- 72.3 Suppose  $Y$  is not invertible. What is  $\det(Y)$ ?

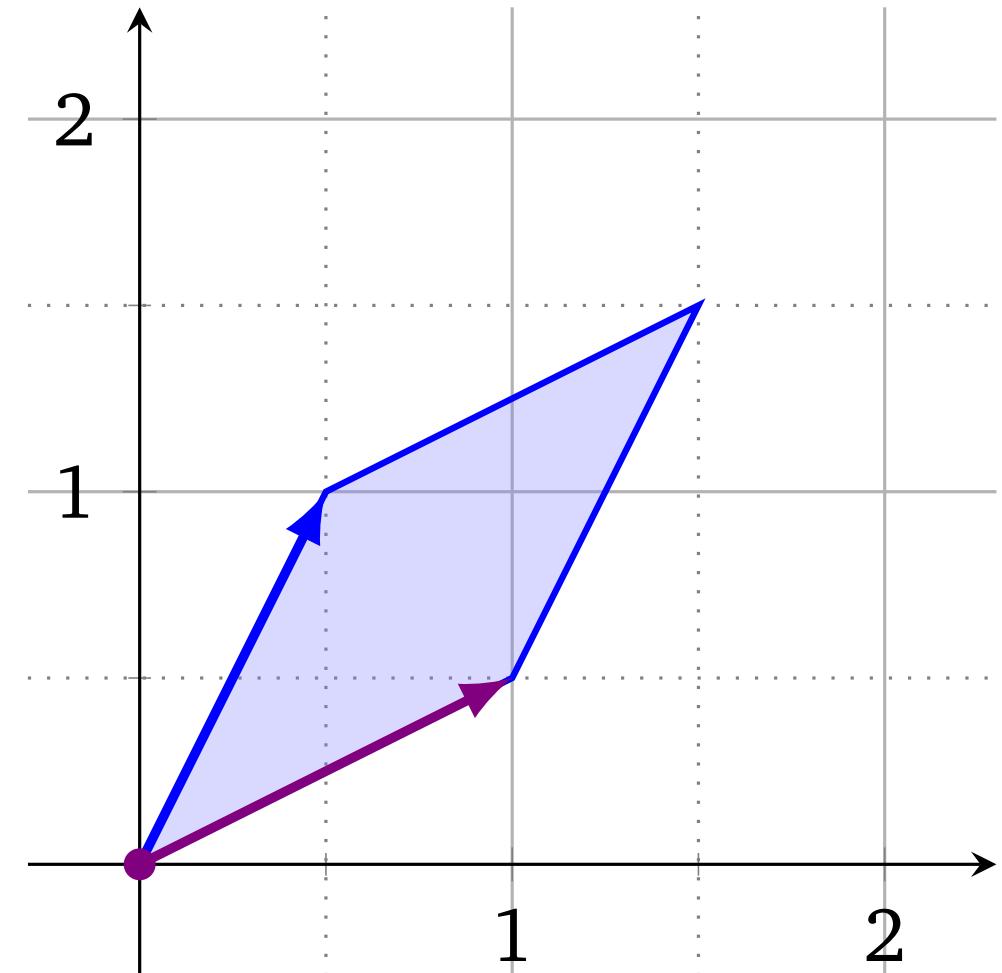
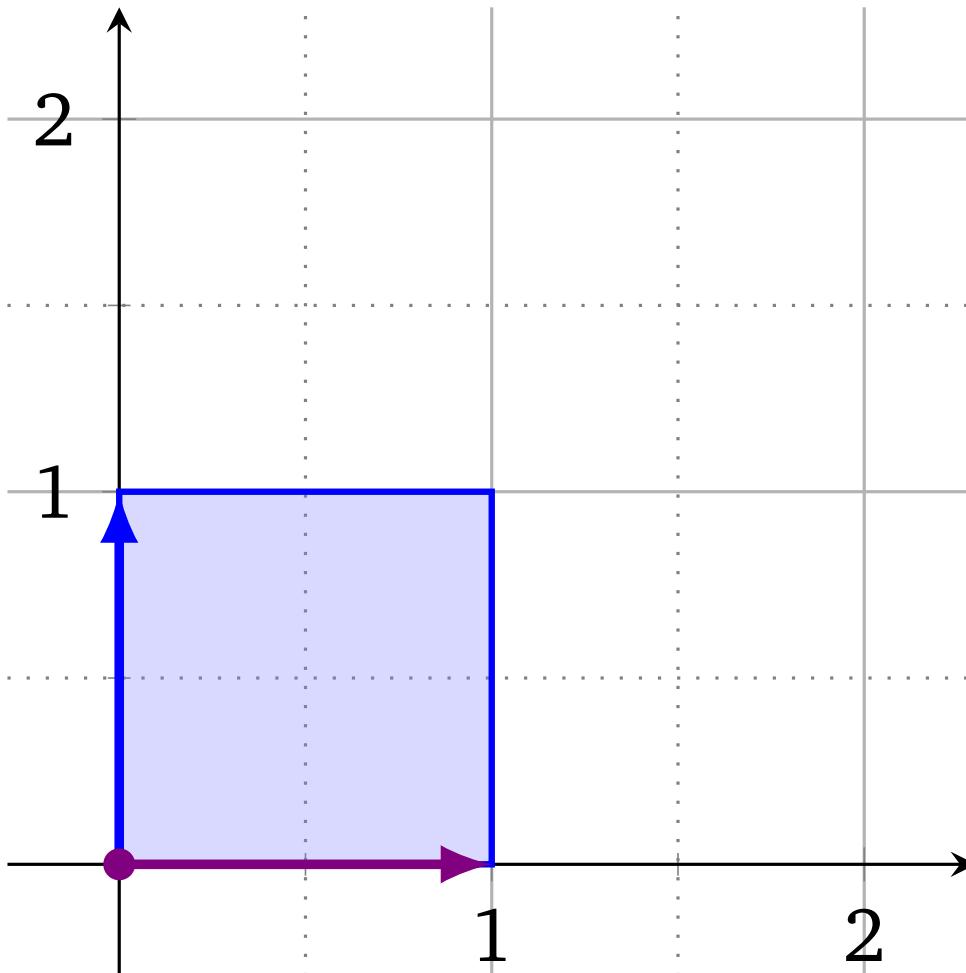
# Eigenvector

Let  $X$  be a linear transformation. An *eigenvector* for  $X$  is a non-zero vector that doesn't change directions when  $X$  is applied. That is,  $\vec{v} \neq \vec{0}$  is an eigenvector for  $X$  if

$$X\vec{v} = \lambda\vec{v}$$

for some scalar  $\lambda$ . We call  $\lambda$  the *eigenvalue* of  $X$  corresponding to the eigenvector  $\vec{v}$ .

73 The picture shows what the linear transformation  $T$  does to the unit square (i.e., the unit 2-cube).



73.1 Give an eigenvector for  $T$ . What is the eigenvalue?

73.2 Can you give another?

74 For some matrix  $A$ ,

$$A \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2/3 \end{bmatrix} \quad \text{and} \quad B = A - \frac{2}{3}I.$$

74.1 Give an eigenvector and a corresponding eigenvalue for  $A$ .

74.2 What is  $B \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$ ?

74.3 What is the dimension of  $\text{null}(B)$ ?

74.4 What is  $\det(B)$ ?

75 Let  $C = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$  and  $E_\lambda = C - \lambda I$ .

- 75.1 For what values of  $\lambda$  does  $E_\lambda$  have a non-trivial null space?
- 75.2 What are the eigenvalues of  $C$ ?
- 75.3 Find the eigenvectors of  $C$ .

# Characteristic Polynomial

DEF

For a matrix  $A$ , the *characteristic polynomial* of  $A$  is

$$\text{char}(A) = \det(A - \lambda I).$$

76

Let  $D = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ .

76.1

Compute  $\text{char}(D)$ .

76.2

Find the eigenvalues of  $D$ .

77 Suppose  $\text{char}(E) = -\lambda(\lambda-2)(\lambda+3)$  for some unknown  $3 \times 3$  matrix  $E$ .

77.1 What are the eigenvalues of  $E$ ?

77.2 Is  $E$  invertible?

77.3 What can you say about  $\text{nullity}(E)$ ,  $\text{nullity}(E - 3I)$ ,  $\text{nullity}(E + 3I)$ ?

## 78 Consider

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and notice that  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are eigenvectors for  $A$ . Let  $T_A$  be the transformation induced by  $A$ .

- 78.1 Find the eigenvalues of  $A$ .
- 78.2 Find the characteristic polynomial of  $A$ .
- 78.3 Compute  $A\vec{w}$  where  $w = 2\vec{v}_1 - \vec{v}_2$ .
- 78.4 Compute  $T_A\vec{u}$  where  $\vec{u} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$  for unknown scalar coefficients  $a, b, c$ .

Notice that  $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is a basis for  $\mathbb{R}^3$ .

- 78.5 If  $[\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$  is  $\vec{x}$  written in the  $\mathcal{V}$  basis, compute  $T_A\vec{x}$  in the  $\mathcal{V}$  basis.

79 The transformation  $P^{-1}$  takes vectors in the standard basis and outputs vectors in their  $\mathcal{V}$ -basis representation. Here,  $A$ ,  $T_A$ , and  $\mathcal{V}$  come from Problem .

79.1 Describe in words what  $P$  does.

79.2 Describe how you can use  $P$  and  $P^{-1}$  to easily compute  $T_A \vec{y}$  for any  $\vec{y} \in \mathbb{R}^3$ .

79.3 Can you find a matrix  $D$  so that

$$PDP^{-1} = A?$$

79.4  $[\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ . Compute  $A^{100} \vec{x}$ .

# Diagonalizable



A matrix is *diagonalizable* if it is similar to a diagonal matrix.

80

For an  $n \times n$  matrix  $T$ , suppose its eigenvectors  $\{\vec{v}_1, \dots, \vec{v}_n\}$  form a basis for  $\mathbb{R}^n$ . Let  $\lambda_1, \dots, \lambda_n$  be the corresponding eigenvalues.

80.1

Is  $T$  diagonalizable (i.e., similar to a diagonal matrix)? If so, explain how to obtain its diagonalized form.

80.2

What if one of the eigenvalues of  $T$  is zero? Is  $T$  diagonalizable?

80.3

What if the eigenvectors of  $T$  did not form a basis for  $\mathbb{R}^n$ . Would  $T$  be diagonalizable?

## Eigenspace

Let  $A$  be a matrix with eigenvalues  $\{\lambda_1, \dots, \lambda_m\}$ . The *eigenspace* of  $A$  corresponding to the eigenvalue  $\lambda_i$  is the null space of  $A - \lambda_i I$ . That is, it is the space spanned by all eigenvectors that have the eigenvalue  $\lambda_i$ .

The *geometric multiplicity* of an eigenvalue  $\lambda_i$  is the dimension of the eigenspace corresponding to  $\lambda_i$ . The *algebraic multiplicity* of  $\lambda_i$  is the number of times  $\lambda_i$  occurs as a root of the characteristic polynomial of  $A$  (i.e., the number of times  $x - \lambda_i$  occurs as a factor).

81 Let  $F = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$  and  $G = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ .

81.1 Is  $F$  diagonalizable? Why or why not?

81.2 Is  $G$  diagonalizable? Why or why not?

81.3 What is the geometric and algebraic multiplicity of each eigenvalue of  $F$ ? What about the multiplicities for each eigenvalue of  $G$ ?

81.4 Suppose  $A$  is a matrix where the geometric multiplicity of one of its eigenvalues is smaller than the algebraic multiplicity of the same eigenvalue. Is  $A$  diagonalizable? What if all the geometric and algebraic multiplicities match?