Matrix Multiplication

Suppose $\mathcal{T}$ is a linear transformation and $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors with eigenvalues λ_1 and λ_2. With this setup, for any $\vec{a} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$, we can compute $\mathcal{T}(\vec{a})$ with minimal effort.

Let's get specific. Define $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^2$ to be the linear transformation with matrix $M = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. Let $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and notice that $\vec{v}_1$ is an eigenvector for $\mathcal{T}$ with eigenvalue -1 and that $\vec{v}_2$ is an eigenvector for $\mathcal{T}$ with eigenvalue 4. Let $\vec{a} = \vec{v}_1 + \vec{v}_2$.

Now,

$$\mathcal{T}(\vec{a}) = \mathcal{T}(\vec{v}_1 + \vec{v}_2) = \mathcal{T}(\vec{v}_1) + \mathcal{T}(\vec{v}_2) = -\vec{v}_1 + 4\vec{v}_2.$$

We didn't need to refer to the entries of M to compute $\mathcal{T}(\vec{a})$.

Exploring further, let $\mathcal{V} = \{\vec{v}_1, \vec{v}_2\}$ and notice that $\mathcal{V}$ is a basis for $\mathbb{R}^2$. By definition $[\vec{a}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and so we just computed

$$\mathcal{T}\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{V}} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}_{\mathcal{V}}.$$

When represented in the $\mathcal{V}$ basis, computing $\mathcal{T}$ is easy. In general,

$$\mathcal{T}(\alpha\vec{v}_1 + \beta\vec{v}_2) = \alpha\mathcal{T}(\vec{v}_1) + \beta\mathcal{T}(\vec{v}_2) = -\alpha\vec{v}_1 + 4\beta\vec{v}_2,$$

and so

$$\mathcal{T}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{\mathcal{V}} = \begin{bmatrix} -\alpha \\ 4\beta \end{bmatrix}_{\mathcal{V}}.$$

In other words, $\mathcal{T}$, when acting on vectors written in the $\mathcal{V}$ basis, just multiplies each coordinate by an eigenvalue. This is enough information to determine the matrix for $\mathcal{T}$ in the $\mathcal{V}$ basis:

$$[\mathcal{T}]_{\mathcal{V}} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}.$$

The matrix representations $[\mathcal{T}]_{\mathcal{E}} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ and $[\mathcal{T}]_{\mathcal{V}} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ are equally valid, but writing $\mathcal{T}$ in the $\mathcal{V}$ basis gives a very simple matrix!

Diagonalization

Recall that two matrices are similar if they represent the same transformation but in possibly different bases. The process of *diagonalizing* a matrix A is that of finding a diagonal matrix that is similar to A, and you can bet that this process is closely related to eigenvectors/values.

Let $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and suppose that $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis so that

$$[\mathcal{T}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{bmatrix}$$

is a diagonal matrix. This means that $\vec{b}_1, \dots, \vec{b}_n$ are eigenvectors for $\mathcal{T}$! The proof goes as follows:

$$[\mathcal{T}]_{\mathcal{B}}[\vec{b}_1]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \alpha_1[\vec{b}_1]_{\mathcal{B}} = [\alpha_1\vec{b}_1]_{\mathcal{B}},$$

and in general

$$[\mathcal{T}]_{\mathcal{B}}[\vec{b}_i]_{\mathcal{B}} = \alpha_i[\vec{b}_i]_{\mathcal{B}} = [\alpha_i\vec{b}_i]_{\mathcal{B}}.$$

Therefore, for $i = 1, \dots, n$, we have

$$\mathcal{T}\vec{b}_i = \alpha_i\vec{b}_i.$$

Since $\mathcal{B}$ is a basis, $\vec{b}_i \neq \vec{0}$ for any i, and so each $\vec{b}_i$ is an eigenvector for $\mathcal{T}$ with corresponding eigenvalue α_i.

We've just shown that if a linear transformation $\mathcal{T}:\mathbb{R}^n \to \mathbb{R}^n$ can be represented by a diagonal matrix, then there must be a basis for $\mathbb{R}^n$ consisting of eigenvectors for $\mathcal{T}$. The converse is also true.

Suppose again that $\mathcal{T}:\mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and that $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis of eigenvectors for $\mathcal{T}$ with corresponding eigenvalues $\alpha_1, \dots, \alpha_n$. By definition,

$$\mathcal{T}(\vec{b}_i) = \alpha_i \vec{b}_i,$$

and so

$$\mathcal{T}\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \alpha_1 k_1 \\ \alpha_2 k_2 \\ \vdots \\ \alpha_n k_n \end{bmatrix}_{\mathcal{B}} \qquad \text{which is equivalent to} \qquad [\mathcal{T}]_{\mathcal{B}}\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} \alpha_1 k_1 \\ \alpha_2 k_2 \\ \vdots \\ \alpha_n k_n \end{bmatrix}.$$

The only matrix that does this is

$$[\mathcal{T}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{bmatrix},$$

which is a diagonal matrix.

What we've shown is summarized by the following theorem.

Theorem. A linear transformation $\mathcal{T}:\mathbb{R}^n \to \mathbb{R}^n$ can be represented by a diagonal matrix if and only if there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors for $\mathcal{T}$. If $\mathcal{B}$ is such a basis, then $[\mathcal{T}]_{\mathcal{B}}$ is a diagonal matrix.

Now that we have a handle on representing a linear transformation by a diagonal matrix, let's tackle the problem of diagonalizing a matrix itself.

Diagonalizable. A matrix is ***diagonalizable*** if it is similar to a diagonal matrix.

Suppose A is an $n \times n$ matrix. A induces some transformation $\mathcal{T}_A:\mathbb{R}^n \to \mathbb{R}^n$. By definition, this means $A = [\mathcal{T}_A]_{\mathcal{E}}$. The matrix B is similar to A if there is some basis $\mathcal{V}$ so that $B = [\mathcal{T}_A]_{\mathcal{V}}$. Using change-of-basis matrices, we see

$$A = [\mathcal{E} \leftarrow \mathcal{V}][\mathcal{T}_A]_{\mathcal{V}}[\mathcal{V} \leftarrow \mathcal{E}] = [\mathcal{E} \leftarrow \mathcal{V}]B[\mathcal{V} \leftarrow \mathcal{E}].$$

In other words, A and B are similar if there is some invertible change-of-basis matrix P so

$$A = PBP^{-1}.$$

Based on our earlier discussion, B will be a diagonal matrix if and only if P is the change-of-basis matrix for a basis of eigenvectors. In this case, we know B will be the diagonal matrix with eigenvalues along the diagonal (in the proper order).

Example. Let $A = \begin{bmatrix} 1 & 2 & 5 \\ -11 & 14 & 5 \\ -3 & 2 & 9 \end{bmatrix}$ be a matrix and notice that $\vec{v}_1 = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ are eigenvectors for A. Diagonalize A.

First, we find the eigenvalues that correspond to the eigenvectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$. Computing,

$$A\vec{v}_1 = \begin{bmatrix} 20 \\ 20 \\ 4 \end{bmatrix} = 4\vec{v}_1, \qquad A\vec{v}_2 = \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = 8\vec{v}_2, \qquad \text{and} \qquad A\vec{v}_3 = \begin{bmatrix} 12 \\ 36 \\ 12 \end{bmatrix} = 12\vec{v}_3,$$

and so the eigenvalue corresponding to $\vec{v}_1$ is 4, to $\vec{v}_2$ is 8, and to $\vec{v}_3$ is 12.
The change-of-basis matrix which converts from the $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ to the standard basis is

$$P = \begin{bmatrix} 5 & 1 & 1 \\ 5 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix},$$

and

$$P^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & \frac{5}{4} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

Define D to be the 3×3 matrix with the eigenvalues of A along the diagonal (in the order, $4, 8, 12$). That is, the matrix A written in the basis of eigenvectors is

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 12 \end{bmatrix}.$$

We now know

$$A = PDP^{-1} = \begin{bmatrix} 5 & 1 & 1 \\ 5 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & \frac{5}{4} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix},$$

and that D is the diagonalized form of A.

Non-diagonalizable Matrices

Is every matrix diagonalizable? Unfortunately the world is not that sweet. But, we have a tool to tell if a matrix is diagonalizable—checking to see if there is a basis of eigenvectors.

Example. Is the matrix $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ diagonalizable?

Computing, $\operatorname{char}(R) = \lambda^2 + 1$ has no real roots. Therefore, R has no real eigenvalues. Consequently, R has no real eigenvectors, and so R is not diagonalizable.[a]

[a] If we allow complex eigenvalues, then R *is* diagonalizable and is similar to the matrix $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. So, to be more precise, we might say R is not *real* diagonalizable.

Example. Is the matrix $D = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ diagonalizable?

For every vector $\vec{v} \in \mathbb{R}^2$, we have $D\vec{v} = 5\vec{v}$, and so every non-zero vector in $\mathbb{R}^2$ is an eigenvector for D. Thus, $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ is a basis of eigenvectors for $\mathbb{R}^2$, and so D is diagonalizable.[a]

[a] Of course, every square matrix is similar to itself and D is already diagonal, so of course it's diagonalizable.

Example. Is the matrix $J = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ diagonalizable?

Computing, $\operatorname{char}(J) = (5 - \lambda)^2$ which has a double root at 5. Therefore, 5 is the only eigenvalue of J. The eigenvectors of J all lie in

$$\operatorname{null}(J - 5I) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}.$$

Since this is a one dimensional space, there is no basis for $\mathbb{R}^2$ consisting of eigenvectors for J. Therefore, J is not diagonalizable.

Example. Is the matrix $K = \begin{bmatrix} 5 & 1 \\ 0 & 2 \end{bmatrix}$ diagonalizable?

Computing, $\operatorname{char}(K) = (5 - \lambda)(2 - \lambda)$ which has roots at 5 and 2. Therefore, 5 and 2 are the eigenvalues of K. The eigenvectors of K lie in one of

$$\operatorname{null}(K - 5I) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \qquad \text{or} \qquad \operatorname{null}(K - 2I) = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}.$$

Picking one eigenvector from each null space, we have that $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$ consisting of eigenvectors of K. Thus, K is diagonalizable.

Takeaway. Not all matrices are diagonalizable, but you can check if an $n \times n$ matrix is diagonalizable by determining whether there is a basis of eigenvectors for $\mathbb{R}^n$.

Geometric and Algebraic Multiplicities

When analyzing linear transformations or matrices, we're often interested in studying the subspaces where vectors are stretched by only one eigenvalue. These are called the *eigenspaces*.

Eigenspace. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_m$. The ***eigenspace*** of A corresponding to the eigenvalue λ_i is the null space of $A - \lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.
The ***geometric multiplicity*** of an eigenvalue λ_i is the dimension of the corresponding eigenspace. The ***algebraic multiplicity*** of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times $x - \lambda_i$ occurs as a factor).

Now is the time when linear algebra and regular algebra (the solving of non-linear equations) combine. We know, every root of the characteristic polynomial of a matrix gives an eigenvalue for that matrix. Since the degree of the characteristic polynomial of an $n \times n$ matrix is always n, the fundamental theorem of algebra tells us exactly how many roots to expect.

Recall that the *multiplicity* of a root of a polynomial is the power of that root in the factored polynomial. So, for example $p(x) = (4-x)^3(5-x)$ has a root of 4 with multiplicity 3 and a root of 5 with multiplicity 1.

Example. Let $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and find the geometric and algebraic multiplicity of each eigenvalue of R.

Computing, $\mathrm{char}(R) = \lambda^2 + 1$ which has no real roots. Therefore, R has no real eigenvalues.[a]

[a] If we allow complex eigenvalues, then the eigenvalues i and $-i$ both have geometric and algebraic multiplicity of 1.

Example. Let $D = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ and find the geometric and algebraic multiplicity of each eigenvalue of D.

Computing, $\mathrm{char}(D) = (5-\lambda)^2$, so 5 is an eigenvalue of D with algebraic multiplicity 2. The eigenspace of D corresponding to 5 is $\mathbb{R}^2$. Thus, the geometric multiplicity of 5 is 2.

Example. Let $J = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ and find the geometric and algebraic multiplicity of each eigenvalue of J.

Computing, $\mathrm{char}(J) = (5-\lambda)^2$, so 5 is an eigenvalue of J with algebraic multiplicity 2. The eigenspace of J corresponding to 5 is $\mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. Thus, the geometric multiplicity of 5 is 1.

Example. Let $K = \begin{bmatrix} 5 & 1 \\ 0 & 2 \end{bmatrix}$ and find the geometric and algebraic multiplicity of each eigenvalue of K.

Computing, $\mathrm{char}(K) = (5-\lambda)(2-\lambda)$, so 5 and 2 are eigenvalues of K, both with algebraic multiplicity 1. The eigenspace of K corresponding to 5 is $\mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ and the eigenspace corresponding to 2 is $\mathrm{span}\left\{\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}$. Thus, both 5 and 2 have a geometric multiplicity of 1.

Consider the following two theorems.

Theorem (Fundamental Theorem of Algebra). Let p be a polynomial of degree n. Then, if complex roots are allowed, the sum of the multiplicities of the roots of p is n.

Theorem. Let λ be an eigenvalue of the matrix A. Then

$$\text{geometric mult}(\lambda) \le \text{algebraic mult}(\lambda).$$

We can now deduce the following.

Theorem. An $n \times n$ matrix A is diagonalizable if and only if the sum of its geometric multiplicities is equal to n. Further, provided complex eigenvalues are permitted, A is diagonalizable if and only if its geometric multiplicities are equal to its corresponding algebraic multiplicities.

Proof. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_k$. Let $E_1, \ldots, E_k$ be bases for the eigenspaces corresponding to $\lambda_1, \ldots, \lambda_k$. We will start by showing $E = E_1 \cup \cdots \cup E_k$ is a linearly independent set using the following two lemmas.

No New Eigenvalue Lemma. Suppose that $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent eigenvectors of a matrix A with eigenvalues $\lambda_1, \ldots, \lambda_k$. Then, any eigenvector for A contained in $\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ must have one of $\lambda_1, \ldots, \lambda_k$ as its eigenvalue.

The proof goes as follows. Suppose $\vec{v} = \sum_{i \leq k} \alpha_i \vec{v}_i$ is an eigenvector for A with eigenvalue λ. We now compute $A\vec{v}$ in two different ways: once by using the fact that $\vec{v}$ is an eigenvector, and again by using the fact that $\vec{v}$ is a linear combination of other eigenvectors. Observe

$$A\vec{v} = \lambda\vec{v} = \lambda\left(\sum_{i \leq k} \alpha_i \vec{v}_i\right) = \sum_{i \leq k} \alpha_i \lambda \vec{v}_i$$

and

$$A\vec{v} = A\left(\sum_{i \leq k} \alpha_i \vec{v}_i\right) = \sum_{i \leq k} \alpha_i A\vec{v}_i = \sum_{i \leq k} \alpha_i \lambda_i \vec{v}_i.$$

We now have

$$\vec{0} = A\vec{v} - A\vec{v} = \sum_{i \leq k} \alpha_i \lambda \vec{v}_i - \sum_{i \leq k} \alpha_i \lambda_i \vec{v}_i = \sum_{i \leq k} \alpha_i (\lambda - \lambda_i) \vec{v}_i.$$

Because $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent, we know $\alpha_i(\lambda - \lambda_i) = 0$ for all $i \leq k$. Further, because $\vec{v}$ is non-zero (it's an eigenvector), we know at least one α_i is non-zero. Therefore $\lambda - \lambda_i = 0$ for at least one i. In other words, $\lambda = \lambda_i$ for at least one i, which is what we set out to show.[1]

Basis Extension Lemma. Let $P = \{\vec{p}_1, \ldots, \vec{p}_a\}$ and $Q = \{\vec{q}_1, \ldots, \vec{q}_b\}$ be linearly independent sets, and suppose $P \cup \{\vec{q}\}$ is linearly independent for all non-zero $\vec{q} \in \operatorname{span} Q$. Then $P \cup Q$ is linearly independent.

To show this, suppose $\vec{0} = \alpha_1 \vec{p}_1 + \cdots + \alpha_a \vec{p}_a + \beta_1 \vec{q}_1 + \cdots + \beta_b \vec{q}_b$ is a linear combination of vectors in $P \cup Q$. Let $\vec{q} = \beta_1 \vec{q}_1 + \cdots + \beta_b \vec{q}_b$. First, note that $\vec{q}$ must be the zero vector. If not, $\vec{0} = \alpha_1 \vec{p}_1 + \cdots + \alpha_a \vec{p}_a + \vec{q}$ is a non-trivial linear combination of vectors in $P \cup \{\vec{q}\}$, which contradicts the assumption that $P \cup \{\vec{q}\}$ is linearly independent. Since we've established $\vec{0} = \vec{q} = \beta_1 \vec{q}_1 + \cdots + \beta_b \vec{q}_b$, we conclude $\beta_1 = \cdots = \beta_b = 0$ because Q is linearly independent. It follows that since $\vec{0} = \alpha_1 \vec{p}_1 + \cdots + \alpha_a \vec{p}_a + \vec{q} = \alpha_1 \vec{p}_1 + \cdots + \alpha_a \vec{p}_a + \vec{0}$, we must have that $\alpha_1 = \cdots = \alpha_a = 0$ because P is linearly independent. This shows that the only way to express $\vec{0}$ as a linear combination of vectors in $P \cup Q$ is as the trivial linear combination, and so $P \cup Q$ is linearly independent.

Now we can put our lemmas to good use. We will use induction to show that $E = E_1 \cup \cdots \cup E_k$ is linearly independent. By assumption E_1 is linearly independent. Now, suppose $U = E_1 \cup \cdots \cup E_j$ is linearly independent. By construction, every non-zero vector $\vec{v} \in \operatorname{span} E_{j+1}$ is an eigenvector for A with eigenvalue λ_{j+1}. Therefore, since $\lambda_{j+1} \neq \lambda_i$ for $1 \leq i \leq j$, we may apply the *No New Eigenvalue Lemma* to see that $\vec{v} \notin \operatorname{span} U$. It follows that $U \cup \{\vec{v}\}$ is linearly independent. Since E_{j+1} is itself linearly independent, we may now apply the *Basis Extension Lemma* to deduce that $U \cup E_{j+1}$ is linearly independent. This shows that $E = E_1 \cup \cdots \cup E_k$ is linearly independent.

To conclude notice that by construction, geometric $\operatorname{mult}(\lambda_i) = |E_i|$. Since $E = E_1 \cup \cdots \cup E_k$ is linearly independent, the E_i's must be disjoint and so $\sum$ geometric $\operatorname{mult}(\lambda_i) = \sum |E_i| = |E|$. If $\sum$ geometric $\operatorname{mult}(\lambda_i) = n$, then $E \subseteq \mathbb{R}^n$ is a linearly independent set of n vectors and so is a basis for $\mathbb{R}^n$. Finally, because we have a basis for $\mathbb{R}^n$ consisting of eigenvectors for A, we know A is diagonalizable.

Conversely, if there is a basis E for $\mathbb{R}^n$ consisting of eigenvectors, we must have a linearly independent set of n eigenvectors. Grouping these eigenvectors by eigenvalue, an application of the *No New Eigenvalue Lemma* shows that each group must actually be a basis for its eigenspace. Thus, the sum of the geometric multiplicities must be n.

Finally, if complex eigenvalues are allowed, the algebraic multiplicities sum to n. Since the algebraic multiplicities bound the geometric multiplicities, the only way for the geometric multiplicities to sum to n is if corresponding geometric and algebraic multiplicities are equal.

■

[1]You may notice that we've proved something stronger than we needed: if an eigenvector is a linear combination of linearly independent eigenvectors, the only non-zero coefficients of that linear combination must belong to eigenvectors with the same eigenvalue.

Computing 2×2 and 3×3 Determinants

Practice Problems

1 For each of the matrices below, find the geometric and algebraic multiplicity of each eigenvalue.

(a) $A = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}$

(b) $B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $C = \begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix}$

(d) $D = \begin{bmatrix} 0 & 3/2 & 4 \\ 0 & 1 & 0 \\ -1 & 1 & 4 \end{bmatrix}$

(e) $E = \begin{bmatrix} 2 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 2 \end{bmatrix}$

2 For each matrix from question 1, diagonalize the matrix if possible. Otherwise explain why the matrix cannot be diagonalized.

3 Give an example of a 4×4 matrix with 2 and 7 as its only eigenvalues.

4 Can the geometric multiplicity of an eigenvalue ever be 0? Explain.

5 (a) Show that if $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors for a matrix M corresponding to different eigenvalues, then $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

(b) If possible, give an example of a non-diagonalizable 3×3 matrix where 1 and -1 are the only eigenvalues.

(c) If possible, give an example of a non-diagonalizable 2×2 matrix where 1 and -1 are the only eigenvalues.

Practice Problem Solutions

Solutions for Module 2

1 (a) i. $\lambda = 1$, Algebraic: 1, Geometric: 1
   ii. $\lambda = 2$, Algebraic: 1, Geometric: 1
 (b) i. $\lambda = 3$, Algebraic: 2, Geometric: 2
 (c) i. $\lambda = 0$, Algebraic: 1, Geometric: 1
   ii. $\lambda = 3$, Algebraic: 1, Geometric: 1
 (d) i. $\lambda = 1$, Algebraic: 1, Geometric: 1
   ii. $\lambda = 2$, Algebraic: 2, Geometric: 1
 (e) i. $\lambda = 2$, Algebraic: 2, Geometric: 2
   ii. $\lambda = 1$, Algebraic: 1, Geometric: 1

2 (a) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$
 (b) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$
 (c) $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$
 (d) Not diagonalizable.
 (e) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3 $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

4 No. If λ is an eigenvalue of a matrix A, then $det(A-\lambda I) = 0$ and therefore $A - \lambda I$ is not invertible. Specifically, nullity$(A-\lambda I) \geq 1$ and hence there exists at least one eigenvector for the eigenvalue λ. Therefore the geometric multiplicity of λ is at least one.

5 (a) Let $\vec{v}_1$ and $\vec{v}_2$ be eigenvectors for a matrix M corresponding to distinct eigenvalues λ_1 and λ_2 respectively. Let $a, b \in \mathbb{R}$ be such that $a\vec{v}_1 + b\vec{v}_2 = \vec{0}$. Multiplying both sides by $M - \lambda_1 I$, we get

$$b(\lambda_2 - \lambda_1)\vec{v}_2 = \vec{0}.$$

Since $\vec{v}_2$ is an eigenvector, it is nonzero. Hence, either $b = 0$ or $\lambda_2 - \lambda_1 = 0$. Since $\lambda_1 \neq \lambda_2$ we know $\lambda_2 - \lambda_1 \neq 0$ and so $b = 0$.

We have deduced that $a\vec{v}_1 = \vec{0}$. However, since $\vec{v}_1$ is nonzero (because it is an eigenvector), we must have that $a = 0$. This means that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

 (b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

 (c) This is impossible. Suppose that for some matrix M, $\vec{v}_1$ is an eigenvector corresponding to 1 and $\vec{v}_2$ is an eigenvector corresponding to -1. By 5a, $\vec{v}_1$ and $\vec{v}_2$ are linearly independent and thus form a basis for $\mathbb{R}^2$. Since, $\mathbb{R}^2$ has a basis consisting of eignvectors of M, we know M is diagonalizable.