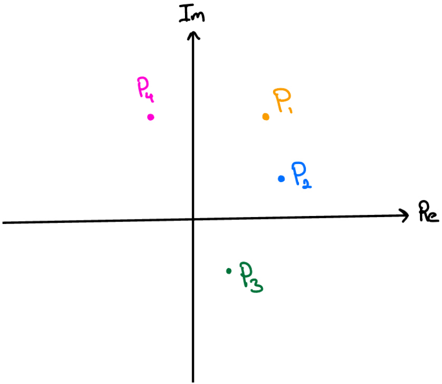
Calculus II

MAT187 Student Slides

Geoff McGregor
Arman Pannu
Jason Siefken

Exercise 1

Consider the plot of the complex numbers p_1, p_2, p_3, p_4 in the complex plane.



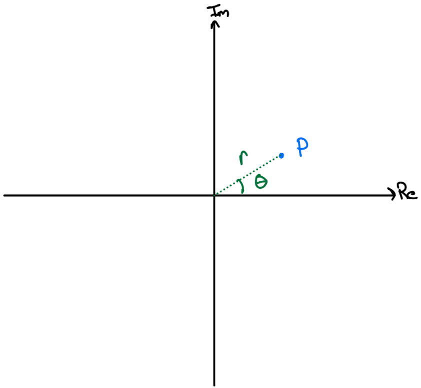
1.1 For which complex numbers is the real part grater than the imaginary part?

1.2 Which complex number has the smallest *modulus/absolute value*?

1.3 Which complex number has the largest *argument*? Is your answer at all ambiguous?

Exercise 2

Consider the plot of the complex number p in the complex plane.



2.1 Sketch the complex number $2p$.

2.2 Sketch the complex number p^2.

2.3 Sketch the complex numbers p^n for $n = 3, 4, \ldots$. Will your answer depend on r?

2.4 Use the geometry of the complex plane to find $\sqrt{i}$. Express your answer in both polar and rectangular form.

Exercise 3

Consider the equation

$$z^3 = -1 \tag{1}$$

3.1 Find a solution to Equation (1).

3.2 If $z = re^{i\theta}$ is a solution to Equation (1), what conditions must r and θ satisfy? Justify your conclusions.

3.3 Find all solutions to Equation (1).

Exercise 4

For each situation, decide whether *least squares* curve fitting or *polynomial interpolation* would be more appropriate.

4.1 You are modelling the arch used in the construction of a particular Roman aqueduct. You have collected several hundred data points of height of the arch vs. distance from the base of the aqueduct.

4.2 You are creating a function to govern the brightness of a light which will be used for signalling a computer. There are three different brightnesses that must be achieved exactly and the transition between those brightnesses must be smooth.

4.3 You are given exact data points from a lab and told that the data was created with a 4th degree polynomial. You are asked to find the coefficients of the polynomial.

Exercise 5

A baseball is thrown on the moon. You are trying to find the function

- $h(t)$, the height (in meters)of the baseball above the moon's surface at time t (in seconds).

You collected the following data

t	$h(t)$
1	4
2	3.8
3	2

5.1 What degree polynomial would best model h?

5.2 Use polynomial interpolation to find h.

5.3 Find the maximum height of the baseball above the moon's surface.

5.4 What would change (if anything) if you were given 4 data points?

Exercise 6

While developing a robotics control system, you find the need for a function f which satisfies the following properties:

(i) $f(0) = -1$ and $f(1) = 2$

(ii) $f'(0) = -1$ and $f'(1) = 2$

Your friend suggests that you could use the following polynomial to come up with f:

$$L_1(x) = -(x-1) \qquad L_2(x) = x$$

$$S_1(x) = (x-1)^2 x \qquad S_2(x) = (x-1)x^2$$

6.1 Can Lagrange interpolation be used to directly find f? Explain.

6.2 Complete the following table

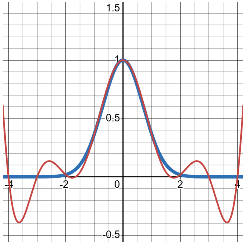
g	$g(0)$	$g(1)$	$g'(0)$	$g'(1)$
L_1				
L_2				
S_1				
S_2				

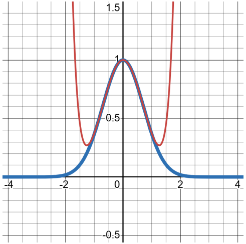
6.3 Use L_1, L_2, S_1, and S_2 to find a polynomial satisfying the properties of f.

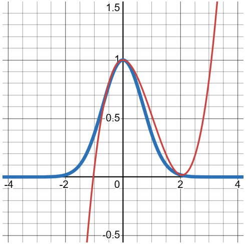
6.4 Explain how Lagrange interpolation can be generalized to allow finding a polynomial that passes through particular points and takes on particular derivatives at those points.

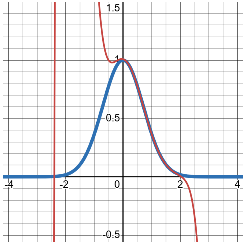
Exercise 7

7.1 For each polynomial approximation of the bell curve, is the approximation best at 0, best on the interval $[-2, 2]$, or best on the interval $[0, 2]$.

(A) 

(B) 

(C) 

(D) 

7.2 Based on the pictures, which polynomial(s) do you think come from a Taylor approximation?

Exercise 8

The function f satisfies

$$f(0) = 1 \qquad f'(0) = 0 \qquad f''(0) = -2$$

$$f'''(0) = 0 \qquad f''''(0) = 12$$

8.1 Write down T_4, the 4th degree Taylor approximation to f centered at 0.

8.2 Use Desmos to compare the graph of T_4 with the graphs of g_1, g_2, g_3, and g_4. Which of the g's do you think is most likely equal to f?

(a) $g_1(x) = e^{-|x|}$

(b) $g_2(x) = e^{-x^2}$

(c) $g_3(x) = \dfrac{1}{1+x^2}$

(d) $g_4(x) = \dfrac{1}{1+(2x)^4}$

Exercise 9

A bee is flying back and forth along a window sill trying to escape from your living room.

The bee's position at time t along the window sill is given by $r(t)$.

You know that a first-order Taylor approximation to $r(t)$ at time $t = 2$ is

$$A_1(t) = 3(t-2)+1$$

9.1 Estimate the position of the bee on the window sill at time 2.1. Is your answer exact or approximate?

9.2 Estimate the velocity of the bee at time 2.1. Is your answer exact or approximate?

9.3 Are there any times you can compute the *exact* position of the bee?

9.4 Are there any times you can compute the *exact* velocity?

9.5 What is your best estimate for the acceleration of the bee at time 2.1?

Exercise 10

A bee is flying back and forth along a window sill trying to escape from your living room.

The bee's position at time t along the window sill is given by $r(t)$.

You know that a second-order Taylor approximation to $r(t)$ at time $t = 2$ is

$$A_2(t) = 2(t-2)^2 + 3(t-2) + 1$$

10.1 Estimate the position of the bee on the window sill at time 2.1. Is your answer exact or approximate?

10.2 Estimate the velocity of the bee at time 2.1. Is your answer exact or approximate?

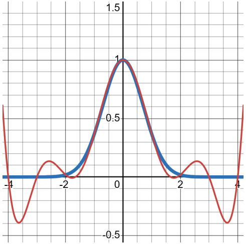
10.3 Are there any times you can compute the *exact* position of the bee?

10.4 Are there any times you can compute the *exact* velocity?

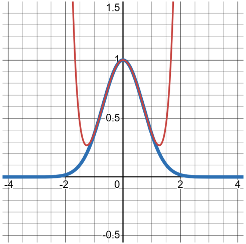
10.5 What is your best estimate for the acceleration of the bee at time 2.1?

Exercise 11

Based on the pictures, which polynomial approximations of the bell curve do you think are *Taylor* polynomials?



(A)



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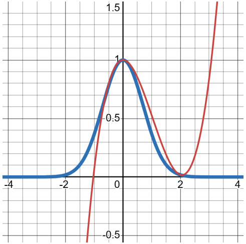

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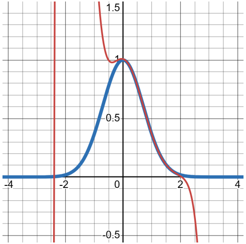


(D)

Exercise 12

Let $f(x) = e^x$ and let $P_n(x)$ be the nth Taylor approximation to f centered at 0. In particular

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Let $R_n(x)$ be the (signed) error in $P_n(x)$.

12.1 Find $R_3(1.5)$ (you may use a calculator).

12.2 What is the largest value of $R_3(x)$ when $0 \le x \le 2$?

12.3 Is there a value of x for which $R_3(x) = 0$? What does this say about P_3?

12.4 Given that $|f^{(5)}(x)| \le 8$ when $x \in [0,2]$, find an upper bound for $R_4(x)$ that

(a) works for a fixed $x \in [0,2]$

(b) works simultaneously for all $x \in [0,2]$

12.5 Given what you know from the previous part(s), can you bound $R_n(x)$?

Exercise 13

Let f be an infinitely differentiable function, and let P_n be a a Taylor polynomial for f of degree n centered at a.

We approximate $f(x) \approx P_n(x)$. Which of the following affect the size of the error in $P_n(x)$ (i.e., the magnitude of $R_n(x)$)?

(A) The degree of P_n, i.e., n.

(B) The magnitude of $f(a)$, i.e., $|f(a)|$.

(C) The magnitudes of the derivatives of f at a, i.e., the size of $|f'(a)|$, $|f''(a)|$, etc..

(D) The distance from a that you are approximating at, i.e., the size of $|x-a|$.

Exercise 14

Use Desmos to conjecture about the following questions.

`https://www.desmos.com/calculator/nrru5n0gqq`

14.1 True/False? When approximating $\sin(x)$ using Taylor polynomials centered at $x=0$, higher degree polynomials will approximate $\sin(2)$ better.

14.2 True/False? When approximating $\tan(x)$ using Taylor polynomials centered at $x=0$, higher degree polynomials will approximate $\tan(2)$ better.

14.3 True/False? When approximating $f(x)=\frac{1}{1+x^2}$ using Taylor polynomials centered at $x=0$, higher degree polynomials will approximate $f(2)$ better.

14.4 Make a conjecture about the relationship between the degree of your Taylor approximation and the accuracy of its values. Does this contradict what you know from Taylor's remainder theorem?

Exercise 15

Consider the function $f(x) = \frac{1}{2}x^2 + 1$ and the value $I = \int_0^3 f(x)\,\mathrm{d}x$.

15.1 Make three sketches: one where the left-endpoint rule is used to approximate I, one where the right-endpoint rule is used, and one where the trapezoid rule is used. (Use at least three intervals.)

15.2 For the left-endpoint, right-endpoint, and trapezoid rules, which will give over estimates of I and which will give underestimates? Will any give an exact value?

15.3 Consider the following estimates of I:

- $E_1 = 8.6875$
- $E_2 = 6.4375$
- $E_3 = 7.5625$

Each estimate comes from using the same partition.

Which estimates come from a left-endpoint approximation, a right-endpoint approximation, and a trapezoid approximation?

Hint: you know calculus!

15.4 (Homework) Will the midpoint rule produce an over or under estimate of I?

Exercise 16

In a classic problem, you are trying to find the volume of a wine barrel. Let $r(\ell)$ represent the radius of the barrel ℓ cm from the base. The total length of the barrel is 80cm.

You know the volume of the barrel can be computed exactly by

$$\int_0^{80} \pi[r(\ell)]^2\,\mathrm{d}\ell.$$

You have measured the barrel in several places and gotten the following data

r(0)	r(20)	r(40)	r(60)	r(80)
12.8	21.2	22.7	21.4	13.4

16.1 Make a sketch of the barrel's profile. Make a second sketch of $\pi[r(\ell)]^2$.

16.2 Based on your sketch, do you think using a trapezoid approximation will produce an over or under estimate for the volume?

16.3 Use a trapezoid approximation to estimate the volume of the barrel.

16.4 Use a Simpson's approximation to estimate the volume of the barrel.

Reminder: if p is a quadratic polynomial,

$$\int_a^b p(x)\,\mathrm{d}x = \frac{b-a}{6}\left(p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right)$$

16.5 The exact (rounded) volume of the barrel is 104384cm^3. What approximation method was most accurate? Why?

Exercise 17

For this question, the domain of integration will be 0 to 5 and you will be using a uniform partition with 5 pieces.

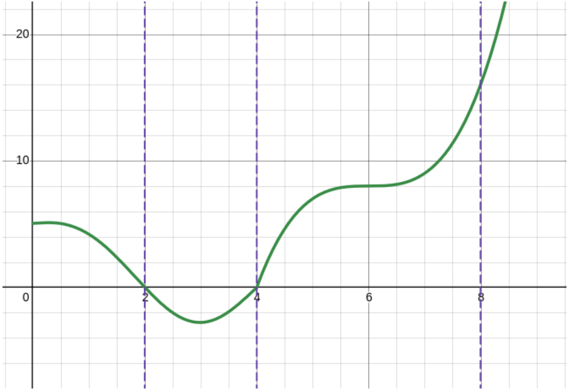
17.1 Draw a function where the left endpoint approximation is an *under estimate*.

17.2 Draw a function where the right endpoint approximation is an *under estimate*.

17.3 Draw a function where the trapezoid approximation is an *under estimate*.

17.4 Draw a function where the midpoint approximation is an *under estimate*.

Exercise 18



The graph above is of the function f. Marked on the graph are the intervals $A = [0,2]$, $B = [2,4]$, and $C = [4,8]$. We are interested in the quantity $I = \int_0^8 f(x)\,\mathrm{d}x$.

18.1 On each interval, identify whether left/right/midpoint/trapezoid approximations will produce an

(a) Underestimate

(b) Overestimate

(c) Cannot be determined

18.2 Is there any interval where you're confident that Simpson's rule would produce an over/under estimate?

18.3 Come up with a strategy (i.e., a choice of integration method for each interval) that gives the best possible upper and lower bounds for I.

Exercise 19

Given an interval $[a,b]$ the midpoint-rule (with one interval) says to use $(b-a)f(0.5a+0.5b)$ as an estimate for $I=\int_a^b f(x)\,\mathrm{d}x$.

A *biased* midpoint rule with bias $\alpha\in[0,1]$ uses $(b-a)f(\alpha a+(1-\alpha)b)$ as an estimate for I.

19.1 Is there a bias α so that with a single partition, $\int_0^1 x^2\,\mathrm{d}x$ is *perfectly* approximated? If so, what is the bias?

19.2 Is there a bias α so that with *two* partitions, $\int_0^1 x^2\,\mathrm{d}x$ is *perfectly* approximated? If so, what is the bias?

19.3 How do your biases compare with the standard midpoint rule?

Exercise 20

20.1 Explain to your table: What is the difference between a sequence and a series?

20.2 How can you produce a sequence from a series?

20.3 How can you produce a series from a sequence?

20.4 Give an example of a bounded sequence that when summed produces an unbounded series.

Exercise 21

Define the sequence a_n by $a_n = \sin(\pi n)$ and the function f by $f(x) = \sin(\pi x)$.

21.1 Find $\lim\limits_{n\to\infty} a_n$, if it exists.

21.2 Find $\lim\limits_{x\to\infty} f(x)$, if it exists.

21.3 What is the difference between a sequence and a function.

Exercise 22

Define

$$a_n = \frac{4+n}{2+n} \qquad b_n = \frac{(-1)^n}{n^2}$$

for $n \geq 1$.

22.1 If a_n and b_n define sequences, what values can n take on? (E.g., any number in $\mathbb{R}$, any number in $\mathbb{Z}$, etc.)

22.2 Make a plot of a_n vs. n and b_n vs. n.

22.3 Which sequences (out of a_n and b_n) are (i) bounded above, (ii) bounded below, (iii) strictly increasing, (iv) strictly decreasing, (v) alternating.

22.4 Define $c_n = a_{n-1} + b_{2n}$ for $n \geq 2$. Find a formula for c_n.

22.5 Based on your answer to Part 3, will c_n be bounded above or below? Neither?

22.6 Find $\lim_{n\to\infty} c_n$.

Exercise 23

Let a_n (for $n \geq 1$) be a sequence and define

$$S_n = \sum_{i=1}^{n} a_n.$$

Let $S_\infty = \lim_{n\to\infty} S_n$.

23.1 Which of the following statements must be true?

(a) If $|a_n| \geq 1$ for all n, then S_n converges.

(b) If $|a_n| \leq 1$ for all n, then S_n converges.

(c) If $|S_n| \geq 1$ for all n, then a_n diverges.

(d) If $|S_n| \leq 1$ for all n, then a_n diverges.

(e) If $a_n \to 0$ then S_n converges.

23.2 If you switch *converges* $\longleftrightarrow$ *diverges*, which statements change their truth value? (I.e., switch from being true to false or false to true.)

Exercise 24

Consider the function $f(x) = 1/x$, the sequence $a_n = 1/n$ and the sequence of partial sums $S_n = \sum_{i=1}^{n} a_i$.

In this question we want to get bounds on the *series*

$$\sum_{i=1}^{\infty} a_i$$

24.1 Use Σ-notation to write down a formula for the left-endpoint approximation of $\int_1^n \frac{1}{x}\,\mathrm{d}x$ using a partition whose intervals are width 1.

24.2 Use Σ-notation to write down a formula for the right-endpoint approximation of $\int_1^n \frac{1}{x}\,\mathrm{d}x$ using a partition whose intervals are width 1.

24.3 Use the actual value of $\int_1^n \frac{1}{x}\,\mathrm{d}x$ to give upper and lower bounds for S_n.

24.4 Does S_n converge or diverge? Explain.

Exercise 25

Consider the function $f(x) = 1/x^2$, the sequence $a_n = 1/n^2$ and the sequence of partial sums $S_n = \sum_{i=1}^{n} a_i$.

In this question we want to get bounds on the *series*

$$\sum_{i=1}^{\infty} a_i$$

25.1 Use Σ-notation to write down a formula for the left-endpoint approximation of $\int_1^n \frac{1}{x^2}\,\mathrm{d}x$ using a partition whose intervals are width 1.

25.2 Use Σ-notation to write down a formula for the right-endpoint approximation of $\int_1^n \frac{1}{x^2}\,\mathrm{d}x$ using a partition whose intervals are width 1.

25.3 Use the actual value of $\int_1^n \frac{1}{x^2}\,\mathrm{d}x$ to give upper and lower bounds for S_n.

25.4 Does S_n converge or diverge? Explain.

25.5 Conjecture about the convergence of $\sum_{i=1}^{\infty} i^{\alpha}$ for $\alpha > 0$. Can you justify your answer by comparing with known integrals?

Exercise 26

Let

$$a_n = \frac{1}{\sqrt{n}} \qquad b_n = \frac{1}{n^3} \qquad c_n = e^{-n} \qquad d_n = e^{-n^2}$$

and consider the corresponding sequences of partial sums A_n, B_n, C_n, and D_n. (I.e., $A_n = \sum_{i=1}^{n} a_n$, etc.)

26.1 Use a comparison with known integrals to decide the convergence of A_n, B_n, C_n.

26.2 Can you decide the convergence of D_n using a comparison to a known integral? Explain.

Exercise 27

Consider the function $f(x) = \sin(x)$.

27.1 Write down $T_k(x)$, the kth Taylor approximation to f centered at 0. You may use "$\cdots$" notation or Σ-notation.

27.2 Write down, using Σ-notation, $T(x)$, the Taylor series for f centered at 0.

27.3 In general a Taylor series may be written as $\displaystyle\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$, where a_n is a sequence. Find a_n in this case.

27.4 Let $R_k(x) = f(x) - T_k(x)$. Find an expression for $R_k(x)$ using Taylor's Remainder Theorem. Use your expression to find an upper bound for $|R_k(x)|$ (Hint: your bound may depend on x).

27.5 Using the fact that for any $\alpha \in \mathbb{R}$, $\displaystyle\lim_{n\to\infty} \frac{\alpha^n}{n!} = 0$, find $\displaystyle\lim_{k\to\infty} R_k(x)$.

27.6 For which x is $f(x) = T(x)$? Justify your answer.

Exercise 28

Consider the function $g(x) = \frac{1}{1-x}$. The kth Taylor approximation of g centered at 0 is

$$T_k(x) = \sum_{i=0}^{k} x^k$$

and the remainder $R_k(x) = g(x) - T_k(x)$ satisfies

$$|R_k(x)| \leq \frac{1}{1-x}\left(\frac{x}{1-x}\right)^{k+1}$$

when $x \geq 0$ and

$$|R_k(x)| \leq x^{k+1}$$

when $x < 0$.

28.1 For which x is $\lim_{k\to\infty} R_k(x) = 0$?

28.2 Let $T(x)$ be the Taylor series for g centered at 0. For which x can you guarantee that $g(x) = T(x)$?

28.3 Use the following Desmos link to numerically answer the question: for which x does $g(x) = T(x)$?

`https://www.desmos.com/calculator/yi4qczkxqn`

28.4 Does your answer to the previous part contradict Taylor's remainder theorem?

Exercise 29

Let $f(x) = \sin(x)$ and let

$$T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

be the Taylor series for f centered at 0. We know that $f(x) = T(x)$ for all $x \in \mathbb{R}$.

29.1 Find a series representation for $g_1(x) = f(2x)$ (without computing any derivatives).

29.2 Find a series representation for $g_2(x) = f(x^2)$.

29.3 Use WolframAlpha to integrate g_2. Does WolframAlpha's solution make sense?

29.4 Compute $g_3(x) = \displaystyle\int g_2(x)\,\mathrm{d}x$ by integrating your series for $g_2(x)$ term by term. What should you do with the constants of integration?

29.5 For which x do you expect $g_3(x)$ to be valid? Explain.

29.6 When would it be advantageous to integrate a Taylor series term by term instead of integrating the original function? Explain.

Exercise 30

Let $f(x) = \frac{1}{1-x}$ and let

$$T(x) = \sum_{n=0}^{\infty} x^n$$

be the Taylor series for f centered at 0. We know that $f(x) = T(x)$ for all $x \in (-1, 1)$.

30.1 Find a series representation for $g_1(x) = f(2x)$ (without computing any derivatives).

30.2 For which x do you expect your series for $g_1(x)$ to be valid (i.e. to equal $f(2x)$? Explain.

30.3 Find a series representation for $g_2(x) = f(x^2)$.

30.4 Compute $g_3(x) = \displaystyle\int g_2(x)\,\mathrm{d}x$ by integrating your series for $g_2(x)$ term by term.

30.5 For which x do you expect $g_3(x)$ to be valid? Explain.

Exercise 31

The function f has a Taylor series centered at 0 of the form

$$T(x) = -\frac{1}{2} + \frac{x}{3} - \frac{x^2}{4} + \frac{x^3}{5} - \frac{x^4}{6} + \cdots.$$

31.1 Express T using Σ-notation.

31.2 Find a series representation for $f'(x)$ and $\displaystyle\int f(x)\,\mathrm{d}x$.

31.3 Modify the following Desmos link and make a conjecture: for which values of x is $f(x) = T(x)$?

`https://www.desmos.com/calculator/try63qzvo5`

31.4 Based on your conjecture, for which values of x should your series for $f'(x)$ and $\displaystyle\int f(x)\,\mathrm{d}x$ be valid?

Exercise 32

Recall

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

for all $x \in \mathbb{R}$.

Let $f(x) = \cos(\sqrt{x})$.

32.1 Write down a Taylor series, T, for f.

Hint: you don't need to take any derivatives.

32.2 Find $f^{(6)}(0)$.

32.3 For what x is $T(x) = f(x)$? Explain.

32.4 Using Desmos, make a conjecture: for which values of x does your series converge?

`https://www.desmos.com/calculator/try63qzvo5`

32.5 Let T be a Taylor series for an unknown function g. If T converges at a value x_0, must it be true that $T(x_0) = g(x_0)$? Explain.

Exercise 33

The sequence a_n is defined by $a_0 = 10$ and

$$\frac{a_{n+1}}{a_n} = \frac{1}{4}$$

Define $S_n = \sum_{i=0}^{n} a_i$ and $S = \lim_{n\to\infty} S_n$.

33.1 Find an expression for a_n.

33.2 Is S_n bounded? Explain.

33.3 Compute S.

Recall: $\sum_{i=0}^{n} \alpha^i = \dfrac{1-\alpha^{n+1}}{1-\alpha}$

Exercise 34

Recall the sequence a_n from Exercise 33 defined by $a_0 = 10$ and $\dfrac{a_{n+1}}{a_n} = \dfrac{1}{4}$.

Consider the unknown, positive, sequence b_n. You know that $b_0 = 5$ and $\dfrac{b_{n+1}}{b_n} < \dfrac{1}{5}$.

34.1 Do you have enough information to write down an expression for b_n?

34.2 Which (if any) of the following relationships must hold for all n?

$$a_n < b_n \qquad b_n < a_n \qquad a_n = b_n$$

Justify your answer.

34.3 Consider the series $\displaystyle\sum_{n=0}^{\infty} b_n$. Does the series converge? Justify your answer by comparing with a known series.

34.4 If you were told that, actually, $b_0 = 100$, would that change your answer to the previous part?

Exercise 35

The ratio test states for a sequence c_n if

$$\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} < 1$$

then $\displaystyle\sum_{n=0}^{\infty} c_n$ converges.

Recall the sequence a_n from Exercise 33 defined by $a_0 = 10$ and $\dfrac{a_{n+1}}{a_n} = \dfrac{1}{4}$.

You know the following about the positive sequence d_n: $\displaystyle\lim_{n\to\infty} \frac{d_{n+1}}{d_n} = \rho < \frac{1}{5}$.

35.1 Which (if any) of the following relationships must hold for all n?

$$a_n < d_n \qquad d_n < a_n \qquad a_n = d_n$$

35.2 Which (if any) of the following relationships *eventually* hold (i.e. hold for all sufficiently large n)?

$$a_n < d_n \qquad d_n < a_n \qquad a_n = d_n$$

Justify your answer.

35.3 Justify, without the ratio test, whether $\displaystyle\sum_{n=0}^{\infty} d_n$ converges.

35.4 Prove the ratio test.

Exercise 36

Theorem (Ratio Test). If c_n is a sequence and

$$\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \rho$$

then $\sum_{n=0}^{\infty} c_n$

- converges if $\rho < 1$
- diverges if $\rho > 1$
- could converge or diverge if $\rho = 1$

36.1 The Ratio Test talks about the convergence of $\sum_{n=0}^{\infty} c_n$. Does it also apply to sums that don't start at $n = 0$? Explain.

36.2 Apply the ratio test to determine the convergence of the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{8^n}{(-2)^{n+1} n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

Exercise 37

The Taylor series for $f(x) = \frac{1}{1-2x}$ is

$$T(x) = \sum_{n=0}^{\infty} 2^n x^n$$

37.1 Apply the ratio test to $T(x)$. Does $T(x)$ converge? Does your answer depend on x?

37.2 Let $G(x)$ be the Taylor series for $g(x) = e^x$. Apply the ratio test to $G(x)$. Does your answer depend on x?

37.3 Write down the largest (open) interval of convergence and the radius of convergence for T and G.

Exercise 38

The Taylor series for $h(x) = \frac{1}{1-x}$ centered at $a > 1$ is

$$H(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{(1-a)^{n+1}}.$$

38.1 Find the largest (open) interval of converge and radius of convergence for H.

38.2 Graph h. Just looking at the graph, can you determine whether a Taylor series for h should have an infinite or finite radius of converge?

Theorem (Integration by Parts). If $f(x)$ and $g(x)$ are differentiable functions, then

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$$

39.1 The integration by parts formula comes from reversing one of the differentiation rules (e.g., chain rule/product rule/quotient rule). Which rule does the integration by parts formula come from?

39.2 Let $h_1(x) = x\sin x$.

(a) For h_1, write down all the ways to divide it into a product of "parts" f and g' so that $h_1 = f \cdot g'$.

(b) Pick the decomposition into parts that you think will be most useful and integrate h_1.

39.3 Let $h_2(x) = x^3 e^{x^2}$.

(a) For h_2, write down all the ways to divide it into a product of "parts" f and g' so that $h_2 = f \cdot g'$.

(b) Pick the decomposition into parts that you think will be most useful and integrate h_2.

Exercise 40

Theorem (Integration by Parts). If $f(x)$ and $g(x)$ are differentiable functions, then

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$$

40.1 Use integration by parts to find $\displaystyle\int e^x \sin x\,\mathrm{d}x$.

Hint: *if at first you don't succeed, try, try again.*

40.2 Use integration by parts to find $\displaystyle\int_1^2 \ln x\,\mathrm{d}x$.

Hint: *sometimes g is hiding in plain sight.*

Exercise 41

We would like to compute

$$F(\theta) = \int \sin^2(\theta)\,\mathrm{d}\theta$$

41.1 (Review) Use integration by parts to find $F(\theta)$.

Hint: *The identity* $1 = \cos^2\theta + \sin^2\theta$ *may reduce your workload.*

41.2 Use the trig identity $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$ to find $F(\theta)$.

41.3 Find $\displaystyle\int \cos^2(\theta)\,\mathrm{d}\theta$ using any method you like.

Exercise 42

Let $f(x) = \sqrt{1-x^2}$ and consider

$$I = \int_0^1 f(x)\,\mathrm{d}x$$

42.1 If we define a change of variables $x = \sin\theta$, what would $\mathrm{d}x$ equal?

42.2 Apply the substitution $x = \sin\theta$ to get a new function $g(\theta)$ so that

$$\int_0^1 f(x)\,\mathrm{d}x = \int_?^? g(\theta)\,\mathrm{d}\theta$$

Find the function g and the bounds for the new integral (i.e. fill in the ?'s).

42.3 Find I.

42.4 Graph f. What shape does the graph make? Use your knowledge of geometry to find I.

Exercise 43

In Exercise 42 we computed

$$I = \int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \int_0^{\pi/2} \cos^2\theta\,\mathrm{d}\theta$$

43.1 Explain why the bounds changed from $[0,1]$ to $[0,\pi/2]$.

43.2 Would it be okay to change the bounds from $[0,1]$ to $[2\pi, 5\pi/2]$?

43.3 Would it be okay to change the bounds from $[0,1]$ to $[0, 5\pi/2]$?

43.4 Compute I using the substitution $x = \cos\theta$. Pay close attention to make sure you use the correct bounds.

Exercise 44

Let $f(x) = \dfrac{\sqrt{9-x^2}}{x^2}$ and consider

$$F(x) = \int f(x)\,\mathrm{d}x$$

Using a substitution of $x = 3\sin\theta$, we arrive at

$$F(x) = -\frac{\cos\theta}{\sin\theta} - \theta + C.$$

44.1 Find an expression for $F(x)$ that involves only x.

44.2 Are there restrictions on domain of x for which your answer makes sense?

44.3 The domain of arcsin is $[-1,1]$. Does this change your restrictions on the domain of x?

Exercise 45

Let $f(x) = \sqrt{\cos x}$ and consider

$$I = \int_0^{\sqrt{2}} f(x)\,\mathrm{d}x$$

You would like to find I.

45.1 Use WolframAlpha to find an anti-derivative of f. Does WolframAlpha give you a useful answer?

45.2 Using a 2^{nd} degree Taylor approximation for cos, write down an integral that will approximate I.

45.3 Find an approximation for I.

45.4 Use a 2^{nd} degree Taylor approximation for f to approximate I.

45.5 Desmos claims $I \approx 1.15686930348$. Which of your estimates is more accurate?

Exercise 46

Let

$$f(x) = \frac{1}{x^2-1} \qquad \text{and} \qquad g(x) = \frac{A}{x-1} + \frac{B}{x+1}.$$

46.1 You would like to know if there are constants A and B such that $f(x) = g(x)$ for all x.

Set up a system of linear equations (or a matrix equation) which has a solution if and only if there are constants A and B that make $f(x) = g(x)$ for all x (in the domain of f).

46.2 Find A and B, if possible.

46.3 Compute $\displaystyle\int f(x)\,\mathrm{d}x$ using any method of your choice.

Exercise 47

Let

$$f(x)=\frac{1}{(x-1)x^2} \qquad \text{and} \qquad g(x)=\frac{A}{x-1}+\frac{B}{x}.$$

47.1 You would like to know if there are constants A and B such that $f(x)=g(x)$ for all x.

Set up a system of linear equations (or a matrix equation) which has a solution if and only if there are constants A and B that make $f(x)=g(x)$ for all x (in the domain of f).

47.2 Find A and B, if possible.

47.3 Let $h(x)=\frac{A}{x-1}+\frac{B}{x}+\frac{C}{x^2}$. Can you find constants A, B, and C so that $f(x)=h(x)$ for all x (in the domain of f)? If so, do it.

47.4 Compute $\displaystyle\int f(x)\,\mathrm{d}x$ using any method of your choice.

Exercise 48

We know $\int \frac{1}{x^2+1}\,\mathrm{d}x = \arctan x + C$. However, we can also use partial fraction decomposition over the complex numbers to integrate.

Let $f(x) = \frac{1}{1+x^2}$ and $g(x) = \frac{A}{1+ix} + \frac{B}{1-ix}$.

48.1 Find A and B so that $f(x) = g(x)$ for all x (in the domain of f).

48.2 Compute $\displaystyle\int f(x)\,\mathrm{d}x$ using your result from the previous part.

However: use $\ln x$ as the antiderivative of $\frac{1}{x}$ rather than $\ln|x|$.

48.3 Use the fact that $\ln(re^{i\theta}) = \ln r + i\theta$ to simplify your answer.

Exercise 49

An *improper integral* formalizes the concept of the area under an “infinite” curve.

Suppose f is a bounded function and let $I = \int_0^\infty f(x)\mathrm{d}x$.

49.1 Write down a formal definition of I.

49.2 Compute, using the definition, $\int_0^\infty \frac{1}{(x+1)^2}\mathrm{d}x$

49.3 Compute, using the definition, $\int_0^\infty \frac{1}{(x+1)}\mathrm{d}x$

Exercise 50

Let $f(x) = \dfrac{x}{x^2+1}$

In this question, we will try to compute

$$Q = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

50.1 Graph f. Make a guess on what you think the "total area under the curve" (i.e. Q) should be.

50.2 Find $\displaystyle\int f(x)\,\mathrm{d}x$

50.3 Compute $\displaystyle\lim_{N\to\infty}\int_{-N}^{N} f(x)\,\mathrm{d}x$. Should your result be equal to Q?

50.4 Should $\displaystyle\lim_{N\to\infty}\int_{-N}^{2N} f(x)\,\mathrm{d}x$ correspond to Q? Compute it and compare with the previous part.

50.5 Compute $A = \displaystyle\int_{0}^{\infty} f(x)\,\mathrm{d}x$ and $B = \displaystyle\int_{-\infty}^{0} f(x)\,\mathrm{d}x$.

50.6 By the properties of integrals, we must have

$$Q = A + B.$$

Do the properties of integrals hold for this improper integral? What does this say about Q?

Exercise 51

The moral of improper integral is:

> *Wherever an infinity might appear, take a separate limit.*

51.1 Rewrite $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,\mathrm{d}x$ using limits of definite integrals.

51.2 Let $g(x) = \frac{1}{x^{1/3}}$ and consider $I = \displaystyle\int_{-8}^{27} g(x)\,\mathrm{d}x$.

(a) Identify all regions where $\displaystyle\int g(x)\,\mathrm{d}x$ *could* produce infinities.

(b) Rewrite I using limit(s) of definite integrals.

(c) Find I.

Exercise 52

Let $f(x) = \dfrac{\ln|x|}{x^4+1}$

$$J = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

52.1 Rewrite J using limit(s) of definite integrals.

52.2 Consider the functions

$$b_1(x) = \ln x \qquad b_2(x) = \frac{\ln x}{2}$$

$$b_3(x) = \frac{x}{x^4+1} \qquad b_4(x) = 0$$

(a) Let $A = \displaystyle\int_0^1 f(x)\,\mathrm{d}x$. Find upper and lower bounds for A by comparing with the appropriate $b_?$ functions.

(b) Let $B = \displaystyle\int_1^{\infty} f(x)\,\mathrm{d}x$. Find upper and lower bounds for B.

52.3 Use your results from the previous part to find upper and lower bounds for J.

Exercise 53

Consider the functions

$$A(t)=t \qquad B(t)=t^2 \qquad C(t)=t^{1/2}$$

$$D(t)=2t \qquad E(t)=2t^2 \qquad F(t)=2t^{1/2}$$

53.1 Consider the parametric equations $x(t)=A(t)$ and $y(t)=D(t)$.

Graph, by hand, (x,y) for $t\in[0,4]$.

53.2 Consider the parametric equations $x(t)=B(t)$ and $y(t)=A(t)$.

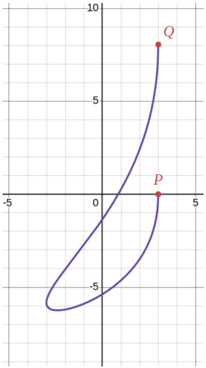
Graph, by hand, (x,y) for $t\in[0,4]$.

53.3 Identify all possible assignments of $x(t)=??$ and $y(t)=??$ (where $??$ come from the functions above) so that the graph of (x,y) for $t\in[0,4]$ is a line segment.

53.4 Out of your examples above, which example produces the *longest* line segment?

Exercise 54

Shown is the graph of $\begin{cases} x(t) = 3\cos t \\ y(t) = t^2 - 5t \end{cases}$ for $t \in [0, 2\pi]$.



The parametric equations describe the position of a particle at time t.

54.1 Is the particle moving from P to Q or Q to P? Explain.

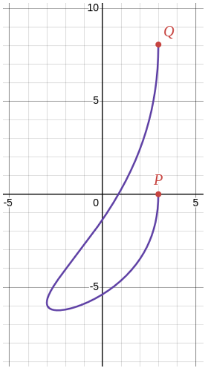
54.2 At what time(s) is the particle moving up *and* to the right?

54.3 At what time(s) is the particle moving parallel to the x-axis?

54.4 Find the tangent line to the particles path at time $t = \pi/2$.

Exercise 55

Shown is the graph of $\begin{cases} x(t) = 3\cos t \\ y(t) = t^2 - 5t \end{cases}$ for $t \in [0, 2\pi]$.



The parametric equations describe the position of a particle at time t.

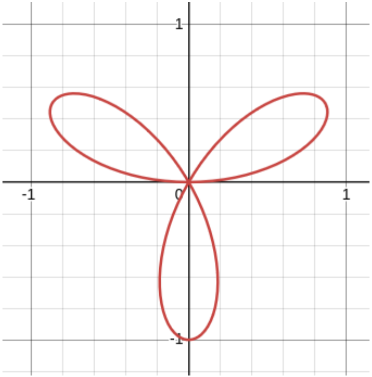
55.1 Find a parameterization for a particle that traces the same path, but starts at Q and ends at P.

55.2 Find a parameterization so that the particle finishes its journey in π seconds instead of 2π seconds.

55.3 Is there a parameterization so that the particle finishes its journey in π seconds but starts its journey at the same speed as the original particle? If such a parameterization exists, how would you come up with it?

Exercise 56

Shown is the graph of $r(\theta) = \sin(3\theta)$ in polar coordinates for $\theta \in [0, \pi]$.



The curve models the boundary of a propeller.

56.1 The blade in the first quadrant achieves a maximum length at an angle of $\theta = \pi/6$. Find the rectangular coordinates of the tip of the blade in the first quadrant.

56.2 Find parametric equations $(x(t), y(t))$ that trace out the propeller.

56.3 Find the tangent line to the propeller when $\theta = 0$ and when $\theta = \pi/3$.

56.4 We know that the propeller is contained in a circle with area π.

Come up with a better upper bound for the area of the propeller.

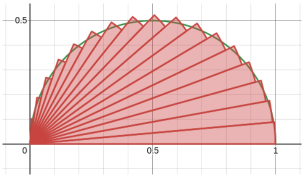
Exercise 57

The same semi-circle can be described in polar coordinates by

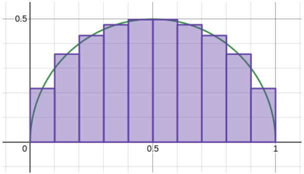
$$r(\theta) = \cos\theta \quad \text{with} \quad \theta \in [0, \pi/2]$$

or in rectangular coordinates by

$$y(x) = \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2} \quad \text{with} \quad x \in [0,1].$$



Below are two ways to divide up the semicircle to approximate its area: one with rectangles and one with circular *sectors*.



57.1 Write down a Riemann sum that approximates the area using *rectangles*. Use Δx as the width of a rectangle.

57.2 Write down a Riemann sum that approximates the area using *sectors*. Use $\Delta\theta$ as the sector angle.

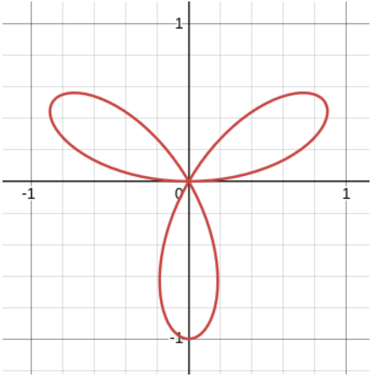
57.3 Take limits of your previous two Riemann sums to find integrals that represent the *exact* area of the semicircle.

Do not evaluate your integrals.

57.4 Which integral would you rather do?

Exercise 58

Shown is the graph of $r(\theta) = \sin(3\theta)$ in polar coordinates for $\theta \in [0, \pi]$.



The curve models the boundary of a propeller.

58.1 The first propeller blade is traced out for $\theta \in [0, \pi/3]$

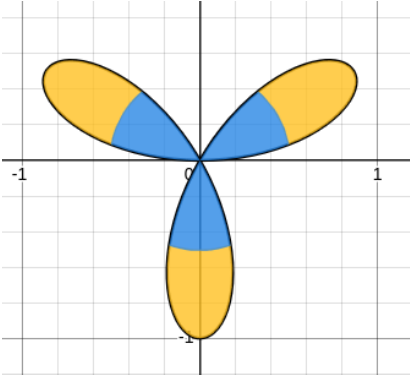
Set up a Riemann sum that approximates the area of the first propeller blade.

58.2 Set up an integral that will give the *exact* area of the first propeller blade. Then, find the area of the first propeller blade.

58.3 Find the total area of the propeller.

Exercise 59

Shown is the graph of $r(\theta) = \sin(3\theta)$ in polar coordinates for $\theta \in [0, \pi]$.



The curve models the boundary of a propeller. The parts of the propeller within distance $1/2$ of the origin are painted blue. The rest is painted yellow.

59.1 Consider the propeller blade in the first quadrant. At what angle does the yellow paint start to appear? At what angle does it disappear?

59.2 Set up an integral that will give the amount of yellow paint needed for the first blade.

59.3 Set up an expression with integral(s) that will give the amount of blue paint needed for the first blade.

59.4 Find the amounts of each paint needed to paint the whole propeller.

Exercise 60

60.1 Which of the following are solutions to the ODE

$$ty' - 6y = 18$$

(a) $y = 8 - 3t^6$

(b) $y = 3t^6 - 8$

(c) $y = 8t^6 - 3$

(d) $y = 3 - 8t^6$

(e) $y = t^6 - 3$

60.2 Consider the differential equation

$$y'' = \cos(t)y' + t^3$$

(a) What is the order of the ODE?

(b) Is the ODE linear?

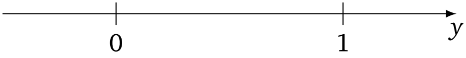
Exercise 61

Consider $y'(t) = 5y(1-y)$

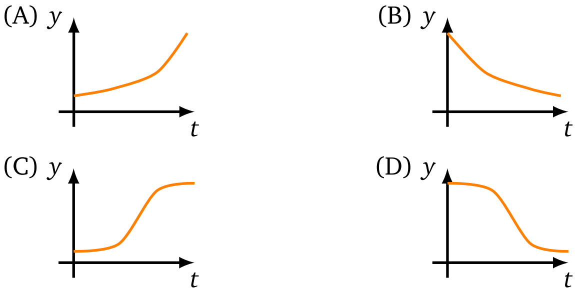
61.1 What do we call the special values $y = 0$ and $y = 1$?

(*Hint: think back to MAT186.*)

61.2 Draw a phase diagram for the ODE.



61.3 Which of the following plots show a possible solution with initial condition $y(0) = 1.5$? What about initial condition $y(0) = 0.2$?

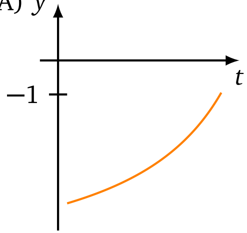


Exercise 62

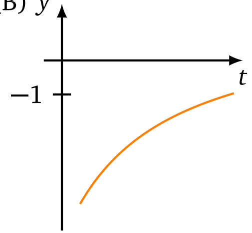
Consider the ODE

$$y' = y^2$$

(A)



(B)



62.1 Which of the following could be sketches of solutions to this ODE?

62.2 Sketch a *slope field* for this ODE.

62.3 Make sketches of all *qualitatively different* solutions to this ODE (i.e., all solutions whose shape is fundamentally different).

Level 1 Modelling

Let $y(t)$ denote the mass (measured in grams) at time t of a chemical substance in a full 150 litre tank. A solution of water and the chemical with a concentration of 10 grams/litre is pumped into the tank at a rate of 3 litres/minute. The contents is pumped out of the tank at a rate of 3 litre/minute.

Assume the tank is well mixed at all times.

63.1 What are the units of $\frac{dy}{dt}$?

63.2 At what rate is the chemical substance **entering** the tank? What should these units be?

63.3 At what rate is the chemical substance **exiting** the tank? What should these units be?

63.4 Assuming there is initially 100 grams of the chemical in the 150 litre tank, write down an initial value problem modeling the amount of chemical in the tank.

63.5 Without solving the IVP, do we expect the amount of chemical in the tank to initially increase or decrease?

63.6 You got a differential equation that looks like $y' = F(y)$. Can you solve this differential equation by integrating both sides? I.e., is $\displaystyle\int F(y)\,\mathrm{d}y$ a solution? Explain.

Exercise 64

Level 2 Modelling

In this problem we use the same setup as before but with one minor tweak: substance is being pumped out at 2 litres/minute (instead of 3). The same 3 litres/minute is being pumped into the tank at the same concentration stated previously. Assume the 150L tank initially has only 50 litres of fluid in it with 100 grams of chemical inside.

64.1 Write down an initial value problem modeling the amount of chemical in the tank.

64.2 For what interval of time is this an appropriate model?

Level 3 Modelling

In this problem we use the same setup as in Level 2 but now when the tank reaches 150 litres it begins to overflow, spilling the well mixed solution out of the top of the tank. (Assume an initial volume of 50 litres in the tank with 100 grams of chemical, the 3 litre/min inflow and 2L/min outflow; everything is still perfectly mixed).

65.1 Write down an initial value problem modeling the amount of chemical in the tank. What range of t values is your IVP valid for?

65.2 Write down an Initial value problem with a single ODE describing the amount of the chemical in the tank for $t \geq 100$. You may use the solution from Level 2:

$$y(t) = \frac{10(t^3 + 150t^2 + 7500t + 25000)}{(t+50)^2},$$

which satisfies $y(100) = 1455.555\ldots$.

65.3 Let $y(t)$ denote the IVP solution to part (a) of this question. Compute $\lim_{t\to\infty} y(t)$.

Apple Products

We are modelling the popularity, $P(t)$, of Apple's newest product over time. Market research shows that a reasonable model could be created with the following assumptions:

(A1) Ignoring all else, popularity grows in proportion to current popularity.

(A2) Ignoring all else, popularity decreases in proportion to elapsed time since release.

Popularity is non-negative and can be unbounded.

66.1 Are the marketing team's assumptions reasonable? Come up with explanations for each.

66.2 Write a differential equation based on assumptions (A1) and (A2).

66.3 Assume the constants of proportionality are 1 and make a slope field for your model using Desmos.

`https://www.desmos.com/calculator/yeusl7guu3`

66.4 Use Desmos to explore which initial conditions give rise to qualitatively different solutions.

66.5 Is $P = 0$ an equilibrium solution to your model? Explain.

Apple Products II

Recall the assumptions for modelling the popularity of a new Apple product.

(A1) Ignoring all else, popularity grows in proportion to current popularity.

(A2) Ignoring all else, popularity decreases in proportion to elapsed time since release.

67.1 From real-world data, we know that popularity of products eventually ends up near zero.

Modify your model so that popularity always ends up at zero.

67.2 For certain *timeless* products, the time limit of popularity approaches a positive constant.

Modify your model so that popularity will always tend towards a positive constant.

Exercise 68

Which of the following ODEs are separable? For each separable ODE, rewrite it in a form that can be "integrated" to find a solution.

68.1 $e^{t+y^2} \cdot y' - 1 = 0$

68.2 $y'(t) + \tan(t) \cdot y(t) = \tan(t)$

68.3 $y'(t) = y(t) \cdot t + 1$

68.4 $y'(t) - (t-1)(y(t))^2 = t - 1$

Exercise 69

General Solution. The general solution to an ODE is the set of *all* solutions to that ODE.

Consider the ODE $y' = 2y$.

69.1 Is $y(t) = e^C e^{2t}$ a solution to the ODE for all $C \in \mathbb{R}$?

69.2 Is $y(t) = \pm e^C e^{2t}$ a solution to the ODE for all $C \in \mathbb{R}$?

69.3 Are there any solutions missing from the two forms above?

69.4 Write the general solution to the ODE.

Exercise 70

Consider the IVP $y' = ty^2$ with $y(0) = 0$.

A student notices it is separable and does the following work:

$$y' = ty^2 \quad \Longrightarrow \quad \frac{y'}{y^2} = t$$

$$\Longrightarrow \quad \int \frac{y'}{y^2}\,\mathrm{d}t = \int t\,\mathrm{d}t \quad \Longrightarrow \quad \int \frac{1}{y^2}\,\mathrm{d}y = \int t\,\mathrm{d}t$$

$$\Longrightarrow \quad \frac{-1}{y} = \frac{t^2}{2} + C \quad \Longrightarrow \quad y = \frac{-2}{t^2 + C}$$

Plugging in $y(0) = 0$ we get $0 = \frac{-2}{0^2+C}$, which cannot be solved.

70.1 Was there a mistake made?

70.2 Find the solution to the IVP.

Exercise 71

Consider the ODE $y' = ty^2$.

Graph the solutions to the following IVPs. Pay attention to the domain of each solution.

71.1 $y(0) = 1$

71.2 $y(0) = 0$

71.3 $y(0) = -1$

71.4 The ODE models bacteria on a far-off planet. We measure that $y(0) = 1$.

Define $y(t) = \begin{cases} \frac{-2}{t^2-2} & \text{if } t \in (-\sqrt{2}, \sqrt{2}) \\ 0 & \text{if } t \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty) \end{cases}$

(a) Is y a solution to the ODE?

(b) Does y satisfy $y(0) = 1$?

(c) The scientists want to predict the number of bacteria at time 2. They compute $y(2) = 0$. Is their prediction valid?

Exercise 72

Consider the differential equation

$$\left[e^{-t^2}y(t)\right]' = te^{-t^2}$$

72.1 Solve the differential equation for y.

72.2 Expand the differential equation and rewrite it as $y'(t) = \cdots$. Simplify your solution.

72.3 In finding $y'(t)$, did you divide by any quantities that could possibly be 0? Do you need to correct for this? Explain.

72.4 Find the general solution to $y' = t + 2t \cdot y$.

Exercise 73

Let h and f be functions and consider the differential equation

$$\left[e^{h(t)}y(t)\right]' = f(t)e^{h(t)}$$

73.1 Write down the general solution to this differential equation. (Your answer may involve integrals/f/h.)

73.2 Rewrite the differential equation in the form $y'(t) = \cdots$.

73.3 In finding $y'(t)$, did you divide by any quantities that could possibly be 0? Do you need to correct for this? Explain.

73.4 Let $a(t)$ be a function. Write down a general solution to

$$y'(t) = a(t) \cdot y(t) + f(t).$$

Congratulations! You just derived integrating factors.

Exercise 74

Consider the ODE

$$(t^2-1)y'+2ty=t.$$

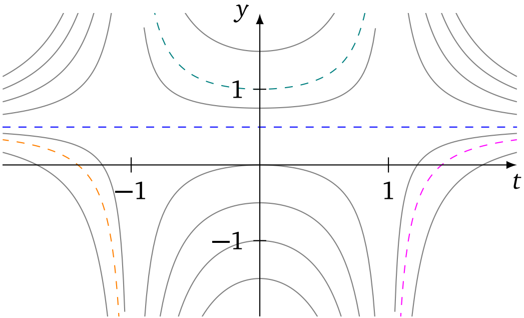
74.1 Rewrite the ODE so that you can apply the method of integrating factors to solve it.

74.2 Solve the ODE.

Consider the ODE

$$(t^2-1)y'+2ty=t$$

with several solutions graphed below.



75.1 For each dashed solution, find a pair (a,b) so that it is the solution to the IVP $y(a)=b$.

75.2 For each IVP you created, find the domain of the solution.

Exercise 76

Consider the line at the Myhal Second Cup. $P(t)$ describes the number of people in line at time t (in minutes since 10:00 am).

There are three effects:

(E1) Every minute, 3 students join the line

(E2) Every minute, 1 student leaves the line (as they are being served)

(E3) Every minute, 20% of the students in line get impatient and leave the line

Initially, no one is in line.

76.1 Write down an ODE that models P.

76.2 Solve your ODE.

76.3 On what domain does your solution makes sense?

Exercise 77

Consider the ODE

$$y'' - y' - 6y = 0$$

77.1 Which of the following functions are solutions to this ODE?

(a) $f_1(t) = e^t$

(b) $f_3(t) = e^{3t}$

(c) $f_{-3}(t) = e^{-3t}$

(d) $f_2(t) = e^{2t}$

77.2 Find all solutions to the ODE of the form $f(t) = e^{rt}$.

77.3 If f is a solution to the ODE, is $f + 1$ also a solution? Justify your answer.

77.4 If f is a solution to the ODE, is $2f$ also a solution? Justify your answer.

Exercise 78

Consider the ODE

$$y'' - y' - 6y = 0$$

Recall from MAT188 that $\operatorname{span}\{a, b\}$ is the set of all linear combinations of a and b. This is true even if a and b are both functions.

78.1 Suppose f and g are solutions to the ODE. Let $h \in \operatorname{span}\{f, g\}$. Write h in terms of f and g.

78.2 Suppose f and g are solutions to the ODE. Show that any function $h \in \operatorname{span}\{f, g\}$ is also a solution.

78.3 Write down the general solution to the ODE. Justify your answer.

You may use the following facts:

- The solution space to the ODE is two dimensional.
- The functions $a(t) = e^{k_1 t}$ and $b(t) = e^{k_2 t}$ are linearly independent if and only if $k_1 \neq k_2$.

Exercise 79

Consider the ODE

$$y'' - y' - 6y = 0$$

79.1 Write down the general solution to this ODE. (Hint: you'v already done this.)

79.2 Find a solution to the IVP

$$\begin{cases} y'' - y' - 6y = 0 \\ y(0) = 7 \\ y'(0) = 1 \end{cases}$$

79.3 Does $y'(0)$ need to be specified to find a unique solution to the IVP? Explain.

79.4 If you changed the initial condition to $y(0) = 7$ and $y'(0) = a$ for $a \in \mathbb{R}$, will there always be a unique solution (regardless of a)?

Exercise 80

Hooke's Law states that the force a spring exerts is proportional to its displacement. When combined with Newton's Laws, we arrive at a differential equation for a particular spring of

$$p'' + 4p = 0$$

where $p(t)$ is the spring's displacement from equilibrium in metres at time t.

80.1 Based on your knowledge of how springs work, sketch a guess of what solutions to this ODE will look like.

80.2 Based on your physical intuition, could an exponential curve be a potential solution to this ODE?

80.3 Find the characteristic equation for this ODE.

80.4 Write down the general (complex) solution to the ODE.

80.5 Should solutions to the spring equation be real or complex? Explain.

80.6 Find the general real solution to the ODE. Do your solutions make sense?

Exercise 81

Consider the differential equation

$$y'' + 4y' + 5y = 0.$$

This equation has complex solutions $f_1(t) = e^{(-2+i)t}$ and $f_2(t) = e^{(-2-i)t}$.

81.1 Apply Euler's Formula to f_1 and f_2 to get complex solutions in terms of sines and cosines.

81.2 Find real solutions to the ODE.

81.3 Suppose g is a solution to the ODE. What is $\lim_{t\to\infty} g(t)$?

81.4 This ODE is a model for a "dampened spring". Do your real solutions make sense in this context? Explain.

Exercise 82

82.1 Come up with a second order homogeneous ODE where all non-zero solutions grow exponentially.

82.2 Come up with a second order homogeneous ODE where all non-zero solutions are periodic.

82.3 Come up with a second order homogeneous ODE where all non-zero solutions oscillate and decay.

82.4 Come up with a **third** order homogeneous ODE where some solutions are periodic and others grow exponentially.

Exercise 83

Consider the ODE

$$y'' - y' = t + 4$$

83.1 Is there a solution to this ODE of the form $f(t) = e^{rt}$? Explain.

83.2 Find a *polynomial* solution to this ODE.

Hint: try guessing a general quadratic polynomial and figuring out what the coefficients should be.

83.3 If f and g are solutions to the ODE, must $f + g$ be a solution?

83.4 Suppose f is a solution to the ODE and h is a solution to the related ODE $y'' - y' = 0$.

Must $f + h$ be a solution to the original ODE? Justify your answer.

83.5 Find a non-polynomial solution to the ODE.

83.6 Find the general solution to the ODE.

Exercise 84

Your MAT186 instructor, Sean, and your MAT188 instructor, Camelia, are arguing.

Sean says: *the general solution to a non-homogeneous linear ODE is the set consisting of a particular solution plus the homogeneous solutions.*

Camelia says: *the general solution to a non-homogeneous linear ODE is a particular solution plus vectors in the kernel of a particular linear transformation.*

84.1 Which one is right? Are they both right? Explain.

84.2 For the equation $y'' - y' = t + 4$ what linear transformation could Camelia be talking about?

Exercise 85

Let $y(t)$ represent the displacement of an object at time t. A general model comes from Newton's laws:

$$y''(t) = V(y'(t)) + P(y(t)) + F_{\text{external}}(t)$$

where V and P are forces that depend on the velocity and the position of the object.

85.1 For a frictionless spring there are no external forces and no forces proportional to velocity. The only force is in the opposite direction of displacement. Write down a general equation for a frictionless spring.

85.2 In a dampened spring there is friction that depends on velocity. Write down an equation for a dampened spring.

85.3 A *forced* spring is one where there an external force applied. Write a differential equation for a spring that is forced with an exponentially decaying force.

85.4 Come up with strategies to solve each differential equation you wrote down.

Exercise 86

Consider the equation for a dampened spring

$$y'' + 4y' + 5y = 0$$

86.1 Find a general solution to the ODE.

86.2 If the spring *weren't* dampened, what would you expect its period to be.

86.3 Rewrite the differential equation as if there were an external force applied proportional to a function $F(t)$.

86.4 Can you find an external force so that the spring's motion would be periodic (i.e., the dampened spring would act like an undampened spring)? If so, add such a force to your equation and solve it. Otherwise explain why no such force exists.

Exercise 87

Each of the following ODEs has an IVP solution which corresponds to one of the graphs below. Match the ODEs with their graphs.

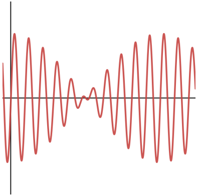
(1) $y'' + 4y = 10e^{-\frac{t}{10}}$

(2) $y'' + 4y = \sin(2t)$

(3) $y'' + 2y' + 4y = 0$

(4) $y'' + 4y = -1.2\sin(2.2t)$

(1)



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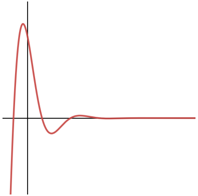

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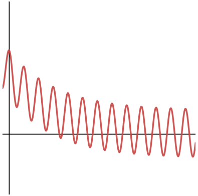

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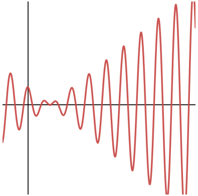

Exercise 88

Consider the ODE

$$y'' + 4y = \sin(2t)$$

88.1 Find the general solution to the corresponding homogeneous ODE.

88.2 What happens if you "guess" $A \sin 2t + B \cos 2t$ as a particular solution to the ODE? Can you determine A and B?

88.3 Guess $At \sin 2t + Bt \cos 2t$ as a particular solution to the ODE. Can you determine A and B?

88.4 Write down the general solution to the ODE.

88.5 When there is an external force matching an object's natural frequency, a phenomenon called *resonance* occurs. When resonance occurs, what happens to solutions? How might this manifest in the real world?