Warm-up

What are the following sets?

1. $[2,4] \cup (2,5)$
2. $[2,4] \cap (2,5)$
3. $[\pi, e]$
4. $[0,0]$
5. $(0,0)$

Similar sets

What are the following sets?

1. $A = \{x \in \mathbb{Z} : x^2 < 6\}$
2. $B = \{x \in \mathbb{N} : x^2 < 6\}$
3. $C = \{x \in \mathbb{R} : x^2 < 6\}$

Describing a new set

An irrational number is a number that is real but not rational.
B is the set of positive, rational numbers and negative, irrational numbers.
Write a definition for B using only mathematical notation.
(You may use the words "and", "or", and "such that".)

Sets and quantifiers

What are the following sets?

1. $A = \{x \in \mathbb{R} : \forall y \in [0,1],\ x < y\}$
2. $B = \{x \in \mathbb{R} : \exists y \in [0,1] \text{ s.t. } x < y\}$
3. $C = \{x \in [0,1] : \forall y \in [0,1], x < y\}$
4. $D = \{x \in [0,1] : \exists y \in [0,1] \text{ s.t. } x < y\}$
5. $E = \{x \in [0,1] : \exists y \in \mathbb{R} \text{ s.t. } x < y\}$
6. $F = \{x \in [0,1] : y \in \mathbb{R},\ x < y\}$

Functions and quantifiers

Let f be a function with domain $\mathbb{R}$. Rewrite the following statements using $\forall$ or $\exists$:

1. The graph of f intercepts the x-axis.
2. f is the zero function.
3. f is not the zero function.
4. f never vanishes.
5. The equation $f(x) = 0$ has a solution.
6. The equation $f(x) = 0$ has no solutions.
7. f takes both positive and negative values.
8. f is never negative.

Negation 1

Write the negation of these statements as simply as possible:

1. My favourite integer number is greater than 7.
2. I know at least five students at U of T who have a cellphone.
3. There is a country in the European Union with fewer than 1000 inhabitants.
4. All of my friends like apples.
5. I like apples and oranges.

$$\text{Negation of } \boxed{\cdots} \quad = \quad \boxed{\cdots} \text{ is false.}$$

Negation 2

Write the negation of this statement without using any negative words ("no", "not", "none", etc.):

> *"Every page in this book contains at least one word whose first and last letters both come alphabetically before M."*

Negation 3

Negate the following statement without using any negative words ("no", "not", "none", etc.):

> *"I have a friend all of whose former boyfriends had at least two siblings with exactly three different vowels in their name."*

Symmetric difference

Given two sets A and B, we define

- $A \setminus B = \{x \in A \ : \ x \notin B\}$
- $A \triangle B = (A \setminus B) \cup (B \setminus A)$

Let

- $C_1 = \{$ students under 18 $\}$
- $C_2 = \{$ students born in Ontario $\}$

What is the set $C_1 \triangle C_2$?

Symmetric difference - 2

Given two sets A and B, we define

- $A \setminus B = \{x \in A \ : \ x \notin B\}$
- $A \triangle B = (A \setminus B) \cup (B \setminus A)$

Is the following equality

$$(A \triangle B) \triangle C \ = \ A \triangle (B \triangle C)$$

true for all sets A, B, and C?

Even numbers

Write a description of the set E of even integers using set-building notation.

Even numbers

Which of these is a correct description of the set E of even integers?

1. $E = \{n \in \mathbb{Z} \ : \ \forall a \in \mathbb{Z},\ n = 2a\}$
2. $E = \{n \in \mathbb{Z} \ : \ \exists a \in \mathbb{Z} \text{ s.t. } n = 2a\}$

Which of these statements is true?

3. $\forall a \in \mathbb{Z}$, the number $n = 2a$ is even.
4. $\exists a \in \mathbb{Z}$ s.t. the number $n = 2a$ is even.

Mother

Let

$$H = \{ \text{ humans } \}$$

True or False?

1. $\forall x \in H, \ \exists y \in H$ such that y gave birth to x
2. $\exists y \in H$ such that $\forall x \in H$, y gave birth to x

Elephants

True or False?

1. There is a pink elephant in this room.
2. All elephants in this room are pink.

Indecisive function

Construct a function f that satisfies all of the following properties at once:

- The domain of f is $\mathbb{R}$.
- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that

$$x < y \text{ and } f(x) < f(y)$$

- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that

$$x < y \text{ and } f(x) > f(y)$$

Conditionals - True or False?

Let $x \in \mathbb{R}$.

1. $x > 0 \implies x \geq 0$
2. $x \geq 0 \implies x > 0$

3. IF $2 > 3$ THEN Alfonso is in love.

Negation of conditionals

Write the negation of these statements:

1. If Justin Trudeau has a brother, then he also has a sister.
2. If a student in this class has a brother, then they also have a sister.

Cards

Take a look at the following cards.

E	3	P	8

Each card has a letter on one side and a number on the other, and I tell you:

> "***If*** *a card has a vowel on one side,*
> ***then*** *it has an odd number on the other side.*"

Which cards do you need to turn over in order to verify whether I am telling the truth or not?

Cards - 2

Four cards lie on the table in front of you. You know that each card has a letter on one side and a number on the other.

Negate the following statement:

> *"**If** a card has a vowel on one side,*
> ***then** it has an odd number on the other side."*

Hockey

Which of the following statements are equivalent to the statement *"Every Canadian man likes hockey"*?

1. If a man is Canadian, then he likes hockey.
2. If a man likes hockey, then he is Canadian.
3. If a man does not like hockey, then he is not Canadian.
4. If a man is not Canadian, then he likes hockey.
5. Non-Canadian men do not like hockey.
6. If a Canadian does not like hockey, then she is not a man.

Draw the graph of a function f with domain $\mathbb{R}$ that satisfies:

$$\text{If } 2 < x < 4 \text{ then } 1 < f(x) < 2.$$

Draw the graph of a function g with domain $\mathbb{R}$ that satisfies:

$$2 < x < 4 \text{ if and only if } 1 < g(x) < 2.$$

One-to-one functions

Let f be a function with domain D.

f is *one-to-one* means that ...

- ... different inputs (x) ...
- ... must produce different outputs ($f(x)$).

Write a formal definition of "one-to-one".

One-to-one functions

Definition: Let f be a function with domain D.
f is one-to-one means ...

1. $f(x_1) \neq f(x_2)$
2. $\exists x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$
3. $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$
4. $\forall x_1, x_2 \in D,\ x_1 \neq x_2,\ f(x_1) \neq f(x_2)$
5. $\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
6. $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2) \implies x_1 \neq x_2$
7. $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

One-to-one functions

Let f be a function with domain D.

What does each of the following mean?

1. $f(x_1) \neq f(x_2)$
2. $\exists x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$
3. $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$
4. $\forall x_1, x_2 \in D,\ x_1 \neq x_2,\ f(x_1) \neq f(x_2)$
5. $\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
6. $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2) \implies x_1 \neq x_2$
7. $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

Proving a function is one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $$\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$
- OR, equivalently, $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

Suppose I give you a specific function f and I ask you to prove it is one-to-one.

- Write the structure of your proof (how do you begin? what do you assume? what do you conclude?) if you use the first definition.
- Write the structure of your proof if you use the second definition.

Exercise

Prove that $f(x) = 3x + 2$, with domain $\mathbb{R}$, is one-to-one.

Proving a function is NOT one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $$\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$
- OR, equivalently, $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

Suppose I give you a specific function f and I ask you to prove it is not one-to-one. You need to prove f satisfies the *negation* of the definition.

- Write the negation of the first definition.
- Write the negation of the second definition.
- Write the structure of your proof.

Exercise

Prove that $f(x) = x^2$, with domain $\mathbb{R}$, is not one-to-one.

Proving a theorem

Theorem

Let f be a function with domain D.

- IF f is increasing on D
- THEN f is one-to-one on D

1. Remind yourself of the precise definition of "increasing" and "one-to-one".
2. To prove the theorem, what will you assume? what do you want to show?
3. Look at the part you want to show. Based on the definition, what is the structure of the proof?
4. Complete the proof.

DISproving a theorem

FALSE Theorem

Let f be a function with domain D.

- IF f is one-to-one on D
- THEN f is increasing on D

1. This theorem is false. What do you need to do to prove it is false?
2. Prove the theorem is false.

What is wrong with this proof? (1)

Theorem

The sum of two odd numbers is even.

Proof.

3 is odd.

5 is odd.

$3 + 5 = 8$ is even. $\square$

What is wrong with this proof? (2)

Theorem

The sum of two odd numbers is even.

Proof.

The sum of two odd numbers is always even.

even $+$ even $=$ even

even $+$ odd $=$ odd

odd $+$ even $=$ odd

odd $+$ odd $=$ even. $\square$

Definition of odd and even

Write a definition of "odd integer" and "even integer".

Definition

Let $x \in \mathbb{Z}$. We say that x is odd when ...

1. $x = 2a + 1$?
2. $\forall a \in \mathbb{Z}, x = 2a + 1$?
3. $\exists a \in \mathbb{Z}$ s.t. $x = 2a + 1$?

What is wrong with this proof? (3)

Theorem

The sum of two odd numbers is always even.

Proof.

$x = 2a + 1$ odd

$y = 2b + 1$ odd

$x + y = 2n$ even

$2a + 1 + 2b + 1 = 2n$

$2a + 2b + 2 = 2n$

$a + b + 1 = n$ $\square$

Write a correct proof!

Theorem

The sum of two odd numbers is always even.

Variations on induction

Let S_n be a statement depending on a positive integer n.

In each of the following cases, which statements are guaranteed to be true?

1. We have proven:
   - S_3
   - $\forall n \geq 1,\ S_n \implies S_{n+1}$
2. We have proven:
   - S_1
   - $\forall n \geq 3,\ S_n \implies S_{n+1}$
3. We have proven:
   - S_1
   - $\forall n \geq 1,\ S_n \implies S_{n+3}$
4. We have proven:
   - S_1
   - $\forall n \geq 1,\ S_{n+1} \implies S_n$

Variations on induction 2

We want to prove

$$\forall n \geq 1,\ S_n$$

So far we have proven

- S_1
- $\forall n \geq 1,\ S_n \implies S_{n+3}$.

What else do we need to do?

Variations on induction 3

We want to prove

$$\forall n \in \mathbb{Z},\ S_n$$

So far we have proven

- S_1

What else do we need to do?

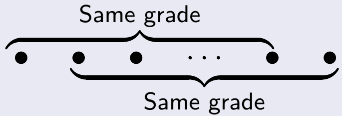
What is wrong with this proof by induction?

Theorem

$\forall N \geq 1$, *every set of N students in MAT137 will get the same grade.*

Proof.

- **Base case.** It is clearly true for $N = 1$.
- **Induction step.**
  Assume it is true for N. I'll show it is true for $N + 1$.
  Take a set of $N + 1$ students. By induction hypothesis:
  - The first N students get the same grade.
  - The last N students get the same grade.



Hence the $N + 1$ students all get the same grade.

What is wrong with this proof by induction?

For every $N \geq 1$, let

$$S_N = \text{"every set of } N \text{ students in MAT137} \\ \text{will get the same grade"}$$

What did we actually prove in the previous page?

- S_1 ?
- $\forall N \geq 1, \quad S_N \implies S_{N+1}$?

Properties of absolute value

Let $a, b \in \mathbb{R}$. What can we conclude?

1. $|ab| = |a||b|$
2. $|a + b| = |a| + |b|$

If any of the conclusions is wrong, fix it.

Properties of inequalities

Let $a, b, c \in \mathbb{R}$.
Assume $a < b$. What can we conclude?

1. $a + c < b + c$
2. $a - c < b - c$
3. $ac < bc$
4. $a^2 < b^2$
5. $\dfrac{1}{a} < \dfrac{1}{b}$

If any of the conclusions is wrong, fix it.

Sets described by distance

Let $a \in \mathbb{R}$. Let $\delta > 0$.
What are the following sets? Describe them using intervals

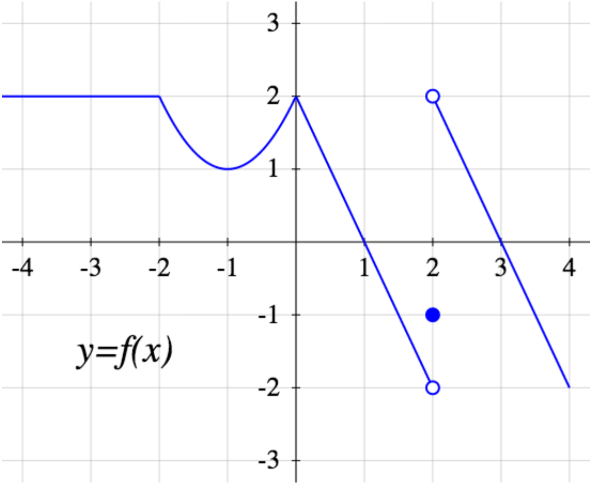
1. $A = \{x \in \mathbb{R} \ : \ |x| < \delta\}$
2. $B = \{x \in \mathbb{R} \ : \ |x| > \delta\}$
3. $C = \{x \in \mathbb{R} \ : \ |x - a| < \delta\}$
4. $D = \{x \in \mathbb{R} \ : \ 0 < |x - a| < \delta\}$

Implications

Find *all* positive values of A, B, and C which make the following implications true.

1. $|x - 3| < 1 \implies |2x - 6| < A$
2. $|x - 3| < B \implies |2x - 6| < 1$
3. $|x - 3| < 1 \implies |x + 5| < C$

Limits from a graph



Find the value of

1. $\lim_{x \to 2} f(x)$
2. $\lim_{x \to 0} f(f(x))$
3. $\lim_{x \to 2} [f(x)]^2$
4. $\lim_{x \to 0} f(2\cos x)$

Floor

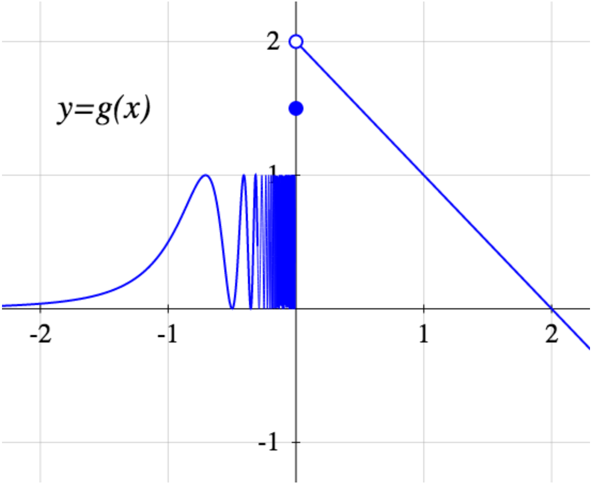
Given a real number x, we defined the *floor of* x, denoted by $\lfloor x \rfloor$, as the largest integer smaller than or equal to x. For example:

$$\lfloor \pi \rfloor = 3, \qquad \lfloor 7 \rfloor = 7, \qquad \lfloor -0.5 \rfloor = -1.$$

Sketch the graph of $y = \lfloor x \rfloor$. Then compute:

1. $\lim_{x \to 0^+} \lfloor x \rfloor$
2. $\lim_{x \to 0^-} \lfloor x \rfloor$
3. $\lim_{x \to 0} \lfloor x \rfloor$
4. $\lim_{x \to 0} \lfloor x^2 \rfloor$

More limits from a graph



Find the value of

1. $\lim_{x \to 0^+} g(x)$
2. $\lim_{x \to 0^+} \lfloor g(x) \rfloor$
3. $\lim_{x \to 0^+} g(\lfloor x \rfloor)$
4. $\lim_{x \to 0^-} g(x)$
5. $\lim_{x \to 0^-} \lfloor g(x) \rfloor$
6. $\lim_{x \to 0^-} \left\lfloor \frac{g(x)}{2} \right\rfloor$
7. $\lim_{x \to 0^-} g(\lfloor x \rfloor)$

Limit at a point

If a function f is not defined at $x = a$, then

1. $\lim_{x \to a} f(x)$ cannot exist
2. $\lim_{x \to a} f(x)$ could be 0
3. $\lim_{x \to a} f(x)$ must approach ∞
4. none of the above.

Evaluating Limits

- You're trying to guess $\lim_{x \to 0} f(x)$.
- You plug in $x = 0.1, 0.01, 0.001, \ldots$ and get $f(x) = 0$ for all these values.
- In fact, you're told that for all $n = 1, 2, \ldots,$ $f\left(\dfrac{1}{10^n}\right) = 0.$
- Can you conclude that $\lim_{x \to 0} f(x) = 0$?

Exponential limits

Compute:

$$\lim_{t\to 0^+} e^{1/t}, \qquad \lim_{t\to 0^-} e^{1/t}.$$

Suggestion: Sketch the graph of $y = e^x$ first.

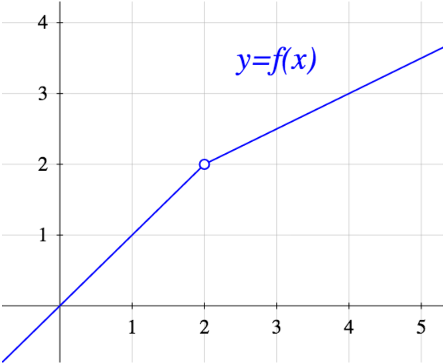
Rational limits

Consider the function

$$h(x) = \frac{(x-1)(2+x)}{x^2(x-1)(2-x)}.$$

- Find all real values a for which $h(a)$ is undefined.
- For each such value of a, compute $\lim_{x \to a^+} h(x)$ and $\lim_{x \to a^-} h(x)$.
- Based on your answer, and nothing else, try to sketch the graph of h.

δ from a graph



1. Find one value of $\delta > 0$ s.t. $0 < |x-2| < \delta \implies |f(x)-2| < 0.5$
2. Find *all* values of $\delta > 0$ s.t. $0 < |x-2| < \delta \implies |f(x)-2| < 0.5$

Warm-up

Write down the formal definition of

$$\lim_{x \to a} f(x) = L.$$

Side limits

Recall

Let $L, a \in \mathbb{R}$.
Let f be a function defined at least on an interval around a, except possibly at a.

$$\lim_{x \to a} f(x) = L$$

means

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \quad 0 < |x-a| < \delta \implies |f(x)-L| < \varepsilon.$$

Write, instead, the formal definition of

$$\lim_{x \to a^+} f(x) = L, \quad \text{and} \quad \lim_{x \to a^-} f(x) = L.$$

Infinite limits

Definition

Let $a \in \mathbb{R}$.
Let f be a function defined at least on an interval around a, except possibly at a.
Write a formal definition for

$$\lim_{x \to a} f(x) = \infty.$$

Infinite limits - 2

Which one(s) is the definition of $\lim_{x \to a} f(x) = \infty$?

1. $\forall M \in \mathbb{R}, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$

2. $\forall M \in \mathbb{Z}, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$

3. $\forall M > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$

4. $\forall M > 5, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$

Related implications

Let $a \in \mathbb{R}$. Let f be a function. Assume we know

$$0 < |x - a| < 0.1 \quad \Longrightarrow \quad f(x) > 100$$

1. Which values of $M \in \mathbb{R}$ satisfy ... ?

$$0 < |x - a| < 0.1 \quad \Longrightarrow \quad f(x) > M$$

2. Which values of $\delta > 0$ satisfy ... ?

$$0 < |x - a| < \delta \quad \Longrightarrow \quad f(x) > 100$$

Strict or non-strict inequality?

Let f be a function with domain $\mathbb{R}$. One of these statements implies the other. Which one?

1. $\forall M \in \mathbb{R}, \exists N \in \mathbb{R}$ s.t. $x > N \implies f(x) > M$
2. $\forall M \in \mathbb{R}, \exists N \in \mathbb{R}$ s.t. $x > N \implies f(x) \geq M$

Negation of conditionals

Write the negation of these statements:

1. If Justin Trudeau has a brother, then he also has a sister.
2. If a student in this class has a brother, then they also have a sister.

More negation

Let f be a function with domain $\mathbb{R}$. Write the negation of the statement:

$$\text{IF } \ 2 < x < 4, \qquad \text{THEN } \ 1 < f(x) < 3.$$

Existence

Write down the formal definition of the following statements:

1. $\lim_{x \to a} f(x) = L$
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x)$ does not exist

Preparation: choosing deltas

1. Find one value of $\delta > 0$ such that
$$|x - 3| < \delta \implies |5x - 15| < 1.$$
2. Find *all* values of $\delta > 0$ such that
$$|x - 3| < \delta \implies |5x - 15| < 1.$$
3. Find *all* values of $\delta > 0$ such that
$$|x - 3| < \delta \implies |5x - 15| < 0.1.$$
4. Let us fix $\varepsilon > 0$. Find *all* values of $\delta > 0$ such that
$$|x - 3| < \delta \implies |5x - 15| < \varepsilon.$$

What is wrong with this “proof”?

Prove that

$$\lim_{x \to 3} (5x + 1) = 16$$

“Proof:”

Let $\varepsilon > 0$.
WTS $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$0 < |x - 3| < \delta \implies |(5x + 1) - (16)| < \varepsilon$$

$$\begin{aligned} |(5x + 1) - (16)| < \varepsilon &\iff |5x + 15| < \varepsilon \\ &\iff 5|x + 3| < \varepsilon \implies \delta = \frac{\varepsilon}{3} \end{aligned}$$

$\square$

Your first $\varepsilon - \delta$ proof

Goal

We want to prove that

$$\lim_{x \to 3} (5x + 1) = 16 \tag{1}$$

directly from the definition.

1. Write down the formal definition of the statement (**??**).
2. Write down what the structure of the formal proof should be, without filling the details.
3. Write down a complete formal proof.

A harder proof

Goal

We want to prove that

$$\lim_{x \to 0} \left(x^3 + x^2\right) = 0 \tag{2}$$

directly from the definition.

1. Write down the formal definition of the statement (**??**).
2. Write down what the structure of the formal proof should be, without filling the details.
3. Rough work: What is δ?
4. Write down a complete formal proof.

Is this proof correct?

Claim:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \quad 0 < |x| < \delta \implies |x^3 + x^2| < \varepsilon.$$

Proof:

- Let $\varepsilon > 0$.
- Take $\delta = \sqrt{\dfrac{\varepsilon}{|x+1|}}$.
- Let $x \in \mathbb{R}$. Assume $0 < |x| < \delta$. Then

$$|x^3+x^2| = x^2|x+1| < \delta^2|x+1| = \frac{\varepsilon}{|x+1|}|x+1| = \varepsilon.$$

- I have proven that $|x^3 + x^2| < \varepsilon$. $\square$

Choosing deltas again

Let us fix numbers $A, \varepsilon > 0$. Find:

1. a value of $\delta > 0$ s.t. $|x| < \delta \implies |Ax^2| < \varepsilon$
2. *all* values of $\delta > 0$ s.t. $|x| < \delta \implies |Ax^2| < \varepsilon$
3. a value of $\delta > 0$ s.t. $|x| < \delta \implies |x+1| < 10$
4. *all* values of $\delta > 0$ s.t. $|x| < \delta \implies |x+1| < 10$
5. a value of $\delta > 0$ s.t. $|x| < \delta \implies \begin{cases} |Ax^2| < \varepsilon \\ |x+1| < 10 \end{cases}$
6. a value of $\delta > 0$ s.t. $|x| < \delta \implies |(x+1)x^2| < \varepsilon$

Indeterminate form

Let $a \in \mathbb{R}$. Let f and g be positive functions defined near a, except maybe at a.

Assume $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$.

What can we conclude about $\lim_{x \to a} \frac{f(x)}{g(x)}$?

1. The limit is 1.
2. The limit is 0.
3. The limit is ∞.
4. The limit does not exist.
5. We do not have enough information to decide.

A theorem about limits

Let f be a function with domain $\mathbb{R}$ such that

$$\lim_{x \to 0} f(x) = 3$$

Prove that

$$\lim_{x \to 0} [5f(2x)] = 15$$

directly from the definition of limit. Do not use any of the limit laws.

1. Write down the formal definition of the statement you want to prove.
2. Write down what the structure of the formal proof should be, without filling the details.
3. Rough work.
4. Write down a complete proof.

Proof feedback

1. Is the structure of the proof correct?
   (First fix ε, then choose δ, then ...)
2. Did you say exactly what δ is?
3. Is the proof self-contained?
   (I do not need to read the rough work)
4. Are all variables defined? In the right order?
5. Do all steps follow logically from what comes before?
   Do you start from what you know and prove what you have to prove?
6. Are you proving your conclusion or assuming it?

A new squeeze

This is the Squeeze Theorem, as you know it:

The (classical) Squeeze Theorem

Let $a, L \in \mathbb{R}$.
Let f, g, and h be functions defined near a, except possibly at a.

IF
- For x close to a but not a, $\quad h(x) \leq g(x) \leq f(x)$
- $\lim\limits_{x \to a} f(x) = L \quad$ and $\quad \lim\limits_{x \to a} h(x) = L$

THEN
- $\lim\limits_{x \to a} g(x) = L$

Come up with a new version of the theorem about limits being infinity. (The conclusion should be $\lim\limits_{x \to a} g(x) = \infty$.)

Hint: Draw a picture for the classical Squeeze Theorem. Then draw a picture for the new theorem.

A new squeeze

The (new) Squeeze Theorem

Let $a \in \mathbb{R}$.
Let g and h be functions defined near a, except possibly at a.

IF
- For x close to a but not a, $h(x) \leq g(x)$
- $\lim_{x \to a} h(x) = \infty$

THEN
- $\lim_{x \to a} g(x) = \infty$

1. Replace the first hypothesis with a more precise mathematical statement.
2. Write down the definition of what you want to prove.
3. Write down the structure of the formal proof.
4. Rough work
5. Write down a complete, formal proof.

True or False?

Is this theorem true?

Claim

Let $a \in \mathbb{R}$.

Let f and g be functions defined near a.

- IF $\lim_{x \to a} f(x) = 0$,
- THEN $\lim_{x \to a} [f(x)g(x)] = 0$.

A new theorem about products

Theorem

Let $a \in \mathbb{R}$. Let f and g be functions with domain $\mathbb{R}$, except possibly a. Assume

- $\lim_{x \to a} f(x) = 0$, and
- g is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN $\lim_{x \to a} [f(x)g(x)] = 0$

1. Write down the formal definition of what you want to prove.
2. Write down what the structure of the formal proof.
3. Rough work.
4. Write down a complete formal proof.

Critique this "proof" – #1

- WTS $\lim_{x \to a} [f(x)g(x)] = 0$:
  $\forall \varepsilon > 0, \exists \delta > 0 \quad \text{s.t.} \quad 0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon.$
- We know $\lim_{x \to a} f(x) = 0$
  $\forall \varepsilon_1 > 0, \exists \delta_1 > 0 \quad \text{s.t.} \quad 0 < |x - a| < \delta_1 \implies |f(x)| < \varepsilon_1.$
- We know $\exists M > 0 \quad \text{s.t.} \quad \forall x \neq 0, \ |g(x)| \leq M.$
- $|f(x)g(x)| = |f(x)||g(x)| < \varepsilon_1 M$
- $\varepsilon = \varepsilon_1 M \implies \varepsilon_1 = \dfrac{\varepsilon}{M}$
- Take $\delta = \delta_1$

Critique this "proof" – #2

- WTS $\lim_{x \to a} [f(x)g(x)] = 0$. By definition, WTS:
$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x-a| < \delta \implies |f(x)g(x)| < \varepsilon$$
- Let $\varepsilon > 0$.
- Use the value $\frac{\varepsilon}{M}$ as "epsilon" in the definition of $\lim_{x \to a} f(x) = 0$
$$\exists \delta_1 \in \mathbb{R} \text{ s.t. } 0 < |x-a| < \delta_1 \implies |f(x)| < \frac{\varepsilon}{M}.$$
- Take $\delta = \delta_1$.
- Let $x \in \mathbb{R}$. Assume $0 < |x-a| < \delta$
- Since $\exists M > 0$ s.t. $\forall x \neq 0, |g(x)| \leq M$
$$|f(x)g(x)| < \frac{\varepsilon}{M} \cdot M = \varepsilon.$$

Critique this "proof" – #3

- Since g is bounded, $\exists M > 0 \quad \text{s.t.} \quad \forall x \neq 0, \ |g(x)| \leq M$

- Since $\lim_{x \to a} f(x) = 0$, there exists $\delta_1 > 0$ s.t.
  if $0 < |x - a| < \delta_1$, then $|f(x) - 0| = |f(x)| < \varepsilon_1 = \dfrac{\varepsilon}{M}$.

- $|f(x)g(x)| = |f(x)| \cdot |g(x)| \leq |f(x)| \cdot M < \varepsilon_1 \cdot M = \dfrac{\varepsilon}{M} \cdot M = \varepsilon$

- In summary, by setting $\delta = \min\{\delta_1\}$, we find that
  if $0 < |x - a| < \delta$, then $|f(x) \cdot g(x)| < \varepsilon$.

Limits involving $\sin(1/x)$ Part I

The reason that $\lim_{x \to 0} \sin(1/x)$ does not exist is:

1. because the function values oscillate around 0
2. because $1/0$ is undefined
3. because no matter how close x gets to 0, there are x's near 0 for which $\sin(1/x) = 1$, and some for which $\sin(1/x) = -1$
4. all of the above

Limits involving $\sin(1/x)$ Part II

The limit $\lim_{x \to 0} x^2 \sin(1/x)$

1. does not exist because the function values oscillate around 0
2. does not exist because $1/0$ is undefined
3. does not exist because no matter how close x gets to 0, there are x's near 0 for which $\sin(1/x) = 1$, and some for which $\sin(1/x) = -1$
4. equals 0
5. equals 1

Absolute value and the Squeeze Theorem

Use the Squeeze Theorem to prove:

Theorem

IF $\lim_{x \to a} |f(x)| = 0$, *THEN* $\lim_{x \to a} f(x) = 0$.

Hint: Recall that $-|c| \leq c \leq |c|$ for every $c \in \mathbb{R}$.

Undefined function

Let $a \in \mathbb{R}$ and let f be a function. Assume $f(a)$ is undefined.

What can we conclude?

1. $\lim_{x \to a} f(x)$ exist
2. $\lim_{x \to a} f(x)$ doesn't exist.
3. No conclusion. $\lim_{x \to a} f(x)$ may or may not exist.

What else can we conclude?

4. f is continuous at a.
5. f is not continuous at a.
6. No conclusion. f may or may not be continuous at a.

A new function

- Let $x, y \in \mathbb{R}$. What does the following expression calculate? Prove it.

$$f(x,y) = \frac{x + y + |x - y|}{2}$$

  Suggestion: If you don't know how to start, try some sample values of x and y.

- Write a similar expression to compute $\min\{x, y\}$.

More continuous functions

We want to prove the following theorem

Theorem

IF f and g are continuous functions
THEN $h(x) = \max\{f(x), g(x)\}$ is also a continuous function.

You are allowed to use all results that we already know.
What is the fastest way to prove this?

Hint: There is a way to prove this quickly without writing any epsilons.

True or False? – Discontinuities

1. IF f and g have removable discontinuities at a
   THEN $f + g$ has a removable discontinuity at a
2. IF f and g have non-removable discontinuities at a
   THEN $f + g$ has a non-removable discontinuity at a

Which one is the correct claim?

Claim 1?

(Assuming these limits exist)

$$\lim_{x \to a} g(f(x)) \;=\; g\left(\lim_{x \to a} f(x)\right)$$

Claim 2?

IF (A) $\lim_{x \to a} f(x) = L$, and (B) $\lim_{t \to L} g(t) = M$

THEN (C) $\lim_{x \to a} g(f(x)) = M$

A difficult example

Construct a pair of functions f and g such that

$$\begin{aligned} \lim_{x \to 0} f(x) &= 1 \\ \lim_{t \to 1} g(t) &= 2 \\ \lim_{x \to 0} g(f(x)) &= 42 \end{aligned}$$

Transforming limits

The only thing we know about the function g is that

$$\lim_{x \to 0} \frac{g(x)}{x^2} = 2.$$

Use it to compute the following limits:

1. $\displaystyle\lim_{x \to 0} \frac{g(x)}{x}$
2. $\displaystyle\lim_{x \to 0} \frac{g(x)}{x^4}$
3. $\displaystyle\lim_{x \to 0} \frac{g(3x)}{x^2}$

Limits at infinity

Compute:

1. $\lim_{x \to \infty} \left(x^7 - 2x^5 + 11\right)$
2. $\lim_{x \to \infty} \left(x^2 - \sqrt{x^5 + 1}\right)$
3. $\lim_{x \to \infty} \dfrac{x^2 + 11}{x + 1}$
4. $\lim_{x \to \infty} \dfrac{x^2 + 2x + 3}{3x^2 + 4x + 5}$
5. $\lim_{x \to \infty} \dfrac{x^3 + \sqrt{2x^6 + 1}}{2x^3 + \sqrt{x^5 + 1}}$

Trig computations

Using that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, compute the following limits:

1. $\lim_{x \to 2} \frac{\sin x}{x}$
2. $\lim_{x \to 0} \frac{\sin(5x)}{x}$
3. $\lim_{x \to 0} \frac{\tan^2(2x^2)}{x^4}$
4. $\lim_{x \to 0} \frac{\sin e^x}{e^x}$
5. $\lim_{x \to 0} \frac{1 - \cos x}{x}$
6. $\lim_{x \to 0} \frac{\tan^{10}(2x^{20})}{\sin^{200}(3x)}$
7. $\lim_{x \to 0} \left[(\sin x)\,(\cos(2x))\,(\tan(3x))\,(\sec(4x))\,(\csc(5x))\,(\cot(6x))\right]$

Plus or minus infinity?

Compute:

1. $\displaystyle\lim_{x\to -3^+} \frac{x^2-9}{3-2x-x^2}$
2. $\displaystyle\lim_{x\to 1^+} \frac{x^2-9}{3-2x-x^2}$

A harder limit

Calculate

$$\lim_{x \to 2} \frac{\left[\sqrt{2+x} - 2\right]\left[\sqrt{3+x} - 3\right]}{\sqrt{x-1} - 1}$$

Which solution is right?

Compute $L = \lim_{x \to -\infty} \left[x - \sqrt{x^2 + x} \right]$.

- **Solution 1**

$$L = \lim_{x \to -\infty} \frac{\left[x - \sqrt{x^2 + x} \right] \; \left[x + \sqrt{x^2 + x} \right]}{\left[x + \sqrt{x^2 + x} \right]} = \lim_{x \to -\infty} \frac{x^2 - (x^2 + x)}{\left[x + \sqrt{x^2 + x} \right]}$$

$$= \lim_{x \to -\infty} \frac{-x}{x \left[1 + \sqrt{1 + \frac{1}{x}} \right]} = \lim_{x \to -\infty} \frac{-1}{\left[1 + \sqrt{1 + \frac{1}{x}} \right]} = \frac{-1}{2}$$

- **Solution 2**

$$L = \lim_{x \to -\infty} \left[{\color{red} x} - {\color{blue} \sqrt{x^2 + x}} \right] = {\color{red} (-\infty)} - {\color{blue} \infty} = -\infty$$

Can we conclude this?

- Consider the function $(x) = \dfrac{4}{x}$.
- We have $f(-1) = -4 < 0$ and $f(1) = 4 > 0$.
- Use IVT.
  Can we conclude $f(c) = 0$ for some $c \in (-1, 1)$?

Existence of solutions

Prove that the equation

$$x^4 - 2x = 100$$

has at least two solutions.

Can this be proven? (Use IVT)

1. Prove that there exists a time of the day when the hour hand and the minute hand of a clock form an angle of exactly 23 degrees.

2. During a Raptors basketball game, at half time the Raptors have 52 points. Prove that at some point they had exactly 26 points.

3. Prove that at some point during Alfonso's life, his height in centimetres was exactly equal to 10 times his weight in kilograms. Some data:
   - His height at birth: 47 cm
   - His weight at birth: 3.2 kg
   - His height today: 172 cm

Temperature

On June 09, 2016, the outside temperature in Toronto at 6 AM was 10°. At 4 PM, the temperature was 20°.

1. Must there have been a time between 6 AM and 4 PM when the temperature was 15°? Explain how you know. Which assumption about temperature allows you to reach your conclusion?
2. Must there have been a time between 6 AM and 4 PM when the temperature was 22°? Explain how you know.
3. Could there have been a time between 6 AM and 4 PM when the temperature was 22°? Explain how you know.

Extrema

In each of the following cases, does the function f have a maximum and a minimum on the interval I?

1. $f(x) = x^2, \quad I = (-1, 1).$
2. $f(x) = \dfrac{(e^x + 2)\sin x}{x} - \cos x + 3, \quad I = [2, 6]$
3. $f(x) = \dfrac{(e^x + 2)\sin x}{x} - \cos x + 3, \quad I = (0, 5]$

Definition of maximum

Let f be a function with domain I.
Which one (or ones) of the following is (or are) a definition of "f has a maximum on I"?

1. $\forall x \in I, \exists C \in \mathbb{R}$ s.t. $f(x) \leq C$
2. $\exists C \in I$ s.t. $\forall x \in I, f(x) \leq C$
3. $\exists C \in \mathbb{R}$ s.t. $\forall x \in I, f(x) \leq C$
4. $\exists C \in \mathbb{R}$ s.t. $\forall x \in I, f(x) < C$

More on the definition of maximum

Let f be a function with domain I.
What does each of the following mean?

1. $\exists C \in \mathbb{R}$ s.t. $\forall x \in I$, $f(x) \leq C$
2. $\exists C \in \mathbb{R}$ s.t. $\forall x \in I$, $f(x) < C$
3. $\exists a \in I$ s.t. $\forall x \in I$, $f(x) \leq f(a)$
4. $\exists a \in I$ s.t. $\forall x \in I$, $f(x) < f(a)$

Tangent line to a line?

What is the equation of the line tangent to the graph of $y = x$ at the point with x–coordinate 7?

1. $y = x + 7$
2. $y = x$
3. $y = 7$
4. $x = 7$
5. There is no tangent line at that point.
6. There is more than one tangent line at that point.

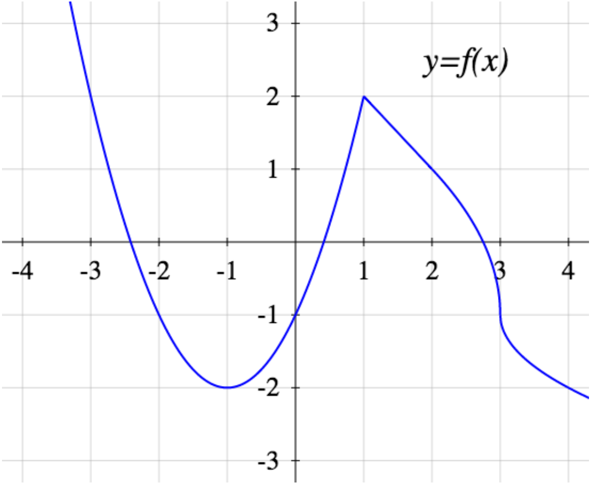
Prove these statements are false with counterexamples

Let C be a curve. Let P be a point in C.

1. The line tangent to C at P
   intersects C at only one point: P.
2. If a line intersects C only at P,
   then that line must be the tangent line to C at P.
3. The tangent line to C at P intersects C at P
   and "does not cross" C at P.
   (This means that, near P, it stays on one side of C.)
4. If a line intersects C at P
   and "does not cross" C at P,
   then it is the tangent line to C at P.

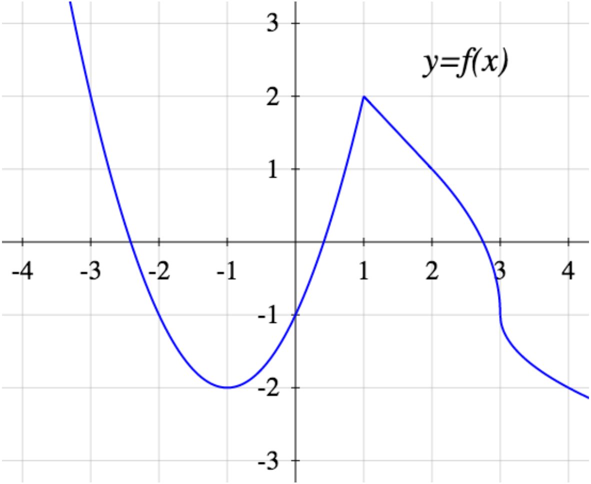
Tangent line from a graph

This is the graph of the function f. Write the equation of the line tangent to it at the point with x–coordinate -2.



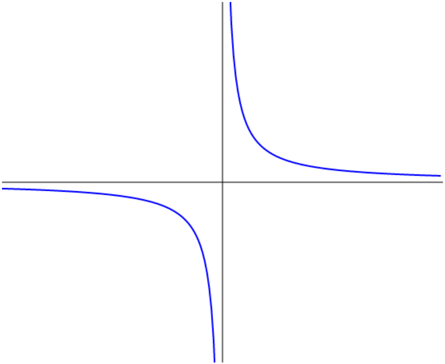
Derivative from a graph

This is the graph of the function f.
Sketch the graph of its derivative f'.



From the derivative to the function

1. Sketch the graph of a continuous function with domain $\mathbb{R}$, whose derivative has the graph below.
2. Sketch the graph of a non-continuous function whose derivative has the graph below.



Estimations - 1

Let f be a continuous function with domain $\mathbb{R}$.

1. We know $f(4) = 3$ and $f(4.2) = 2.2$.
   Based only on this, give your best estimate for $f(4.1)$.

2. We know $f(4) = 3$ and $f'(4) = 0.6$.
   Based only on this, give your best estimate for $f(4.1)$.

3. We know $f(4) = 3$ and $f(4.1) = 4$.
   Based only on this, give your best estimate for $f'(4)$.

Estimations – 2

Without using a calculator, estimate $\sqrt[20]{1.01}$ as well as you can.
Hint: You know the value of $f(x) = \sqrt[20]{x}$ and its derivative at one point very close to 1.01. Use the tangent line at that point as an approximation.

Estimations – 3

1. We know
   $f(0) = 2, \quad f'(0) = 3, \quad g(0) = 7, \quad g'(0) = 5.$

   Compute $\displaystyle\lim_{x\to 0} \frac{f(x)}{g(x)}$.

2. We know
   $f(0) = 0, \quad f'(0) = 3, \quad g(0) = 0, \quad g'(0) = 5.$

   - When x is close to 0, give estimates for $f(x)$ and $g(x)$ using the tangent lines at 0.
   - Use those estimates to compute $\displaystyle\lim_{x\to 0} \frac{f(x)}{g(x)}$.

Derivatives from the definition

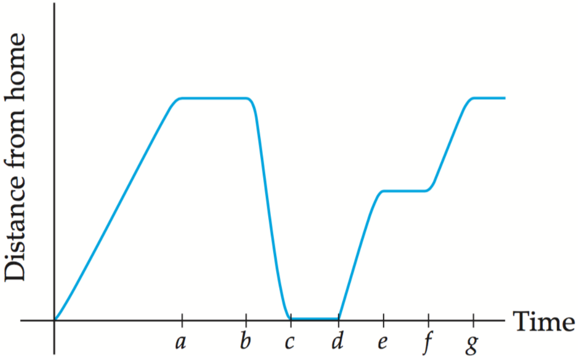
Let

$$g(x) = \frac{2}{\sqrt{x}}$$

Calculate $g'(4)$ directly from the definition of derivative as a limit.

Bella

The graph below describes Bella's distance from home one morning as she drives drive between her home and school.
Describe a possible scenario for her travels that morning.
Then sketch the corresponding graph of his velocity.



Edward and Jacob

Jacob walked at 5 km/h for 20 minutes and then sprinted at 15 km/h for 8 minutes.

1. How fast would Edward have to walk or run to go the same distance as Jacob did in the same time while moving at a constant speed?
2. Sketch a graph of Jacob's and Edward's positions over time on the same set of axes.

Computations: Basic differentiation rules

Compute the derivative of the following functions:

1. $f(x) = x^{100} - 3x^9 - 2$
2. $f(x) = \sqrt[3]{x} + 6$
3. $f(x) = \dfrac{4}{x^4}$
4. $f(x) = \sqrt{x}\,(1 + 2x)$
5. $f(x) = \dfrac{x^6 + 1}{x^3}$
6. $f(x) = \dfrac{x^2 - 2}{x^2 + 2}$

Compute the derivative of

1. $f(x) = \left(2x^2 + x + 1\right)^8$
2. $f(x) = \dfrac{1}{\left(x + \sqrt{x^2 + x}\right)^{137}}$

A long chain

The function below has 137 square roots:

$$f(x) = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots + \sqrt{x + \sqrt{x + 1}}}}}}$$

Find the equation of the line tangent to the graph of f at the point with x-coordinate 0.

Computations: Trig derivatives

Compute the derivatives of the following functions:

1. $f(x) = \tan(3x^2 + 1)$

2. $f(x) = (\cos x)(\sin 2x)(\tan 3x)$

3. $f(x) = \cos(\sin(\tan x))$

4. $f(x) = \cos\left(3x + \sqrt{1 + \sin^2 x^2}\right)$

Differentiable functions

Let $a \in \mathbb{R}$.
Let f be a function with domain $\mathbb{R}$.
Assume f is differentiable everywhere.
What can we conclude?

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. f is continuous at a.
4. $f'(a)$ exists.
5. $\lim_{x \to a} f'(x)$ exists.
6. f' is continuous at a.

True or False - Differentiability vs Continuity

Let f be a function with domain $\mathbb{R}$. Let $c \in \mathbb{R}$.
Which of these implications are true?

1. IF f is continuous at c, THEN f is differentiable at c
2. IF f is differentiable at c, THEN f is continuous at c
3. IF f is differentiable at c, THEN f' is continuous at c
4. IF f' is continuous at c, THEN f is continuous at c
5. IF f is differentiable at c, THEN f is continuous at and near c.
6. IF f is continuous at and near c, THEN f is differentiable at c.

True or False - Differentiability and Operations

Let f be a function with domain $\mathbb{R}$. Let $c \in \mathbb{R}$.
Let $g(x) = f(x)^2$. Which of these implications are true?

1. IF f is differentiable at c, THEN $f + f'$ is continuous at c
2. IF f is differentiable at c, THEN $3f$ is differentiable at c.
3. IF f is differentiable at c, THEN g is differentiable at c.
4. IF g is differentiable at c, THEN f is differentiable at c.
5. IF f is differentiable at c, THEN $1/f$ is differentiable at c.

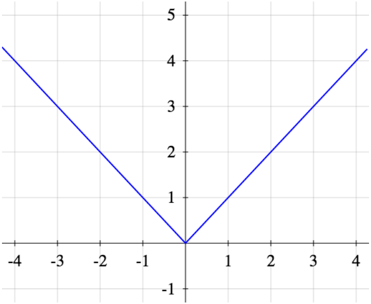
Vertical things

- Construct a function f that has a vertical asymptote at $x = 2$.
- Construct a function g that has a vertical tangent line at $x = 2$.

Absolute value and tangent lines

At (0,0) the graph of $y = |x|$...

1. ... has one tangent line: $y = 0$
2. ... has one tangent line: $x = 0$
3. ... has two tangent lines $y = x$ and $y = -x$
4. ... has no tangent line



Absolute value and derivatives - 1

Let $h(x) = x|x|$. What is $h'(0)$?

1. It is 0.
2. It doesn't exist because $|x|$ is not differentiable at 0.
3. It doesn't exist because the right- and left-limits, when computing the derivative, are different.
4. It doesn't exist because it has a corner.
5. It doesn't exist for a different reason.

Absolute value and derivatives - 2

True or False?

For all $n \in \mathbb{Z}$ and all x, $\frac{d}{dx}|x|^n = nx|x|^{n-2}$.

Write a proof for the quotient rule for derivatives

Theorem

- Let $a \in \mathbb{R}$.
- Let f and g be functions defined at and near a. Assume $g(x) \neq 0$ for x close to a.
- We define the function h by $h(x) = \dfrac{f(x)}{g(x)}$.

IF f and g are differentiable at a,
THEN h is differentiable at a, and

$$h'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Write a proof directly from the definition of derivative.
Hint: Imitate the proof of the product rule in Video 3.6.

Check your proof

1. Did you use the *definition* of derivative?
2. Are there words or only equations?
3. Does every step follow logically?
4. Did you only assume things you could assume?
5. Did you assume at some point that a function was differentiable? If so, did you justify it?
6. Did you assume at some point that a function was continuous? If so, did you justify it?

If you answered "no" to Q**??**, you probably missed something important.

Critique this proof

$$\begin{aligned}
h'(a) &= \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} \frac{\dfrac{f(x)}{g(x)} - \dfrac{f(a)}{g(a)}}{x - a} \\
&= \lim_{x \to a} \frac{f(x)g(a) - f(a)g(x)}{g(x)g(a)\,(x-a)} \\
&= \lim_{x \to a} \frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(x)}{g(x)g(a)\,(x-a)} \\
&= \lim_{x \to a} \left\{ \left[\frac{f(x) - f(a)}{x - a} g(a) - f(a) \frac{g(x) - g(a)}{x - a} \right] \frac{1}{g(x)g(a)} \right\} \\
&= \left[f'(a)g(a) - f(a)g'(a) \right] \frac{1}{g(a)g(a)}
\end{aligned}$$

Higher order derivatives

Let $g(x) = \dfrac{1}{x^3}$.

- Calculate the first few derivatives.
- Make a conjecture for a formula for the n-th derivative $g^{(n)}(x)$.
- Prove it by induction.

Nixon

Richard Nixon, during the 1972 US Presidential campaign, (paraphrased):

> *Inflation is increasing, but the rate of increase of inflation is decreasing.*

Let

- C = cost of life
- t = time

What did Nixon say in terms of derivatives?

Quick composition

Let f and g be differentiable functions and let $h = f \circ g$. What is $h'(2)$?

1. $f'(2) \circ g'(2)$
2. $f'(2)g'(2)$
3. $f'(g(2))g'(2)$
4. $f'(g(x))g'(2)$

True or False - Differentiability and Composition

Let f and g be functions with domain $\mathbb{R}$. Let $c \in \mathbb{R}$.
Assume f and g are differentiable at c.
What can we conclude?

1. $f \circ g$ is differentiable at c.

2. $f \circ f$ is differentiable at c.

3. $f \circ \sin$ is differentiable at c.

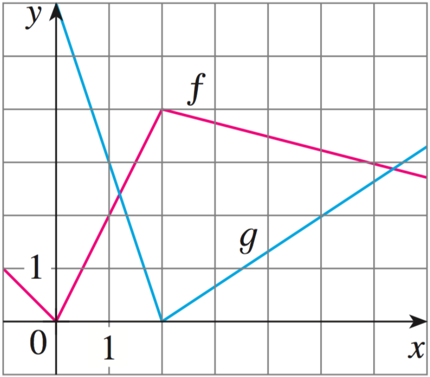
4. $\sin \circ f$ is differentiable at c.

Chain rule from a graph

If f and g are the functions whose graphs are shown.
Let $u(x) = f(g(x))$ and $v(x) = g(f(x))$.

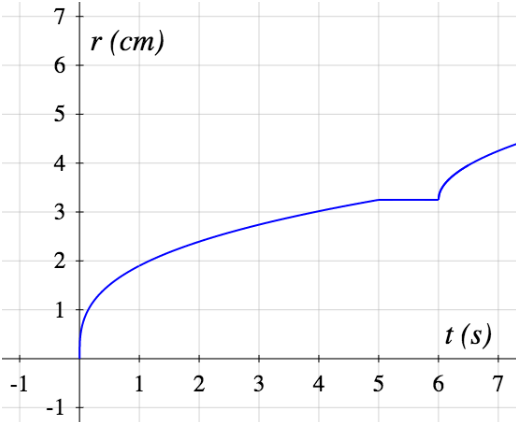
Find each derivative, if it exists.
If it does not exist, explain why.

1. $u'(1)$
2. $v'(1)$



Balloon

I am inflating a spherical balloon. Below is the graph of the radius r (in cm) as a function of time t (in s). At what rate is the volume of the balloon increasing at time $4s$?



Balloon

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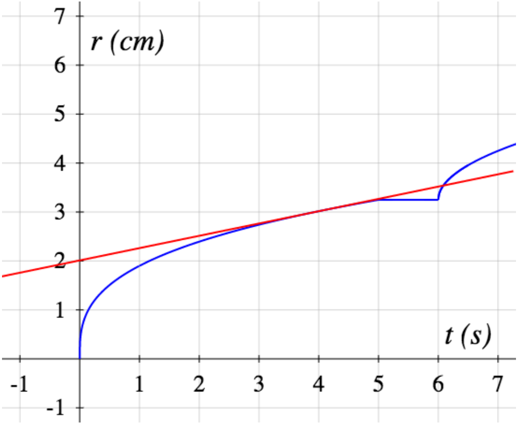


An alternative proof of the quotient rule

Assume we have already proven the product rule, the power rule, and the chain rule.

Obtain a formula for the derivative of $h(x) = \dfrac{f(x)}{g(x)}$.

Hint: $\dfrac{f(x)}{g(x)} = f(x) \cdot g(x)^{-1}$

Derivatives of $(f \circ g)$

Assume f and g are functions that have all their derivatives. Find formulas for

1. $(f \circ g)'(x)$
2. $(f \circ g)''(x)$
3. $(f \circ g)'''(x)$

in terms of the values of f, g and their derivatives.
Hint: The first one is simply the chain rule.

Challenge: Find a formula for $(f \circ g)^{(n)}(x)$
(This is beyond the scope of this course).

Derivative of cos

Let $g(x) = \cos x$.
Obtain and prove a formula for its derivative directly from the definition of derivative as a limit.

Hint: Imitate the derivation in Video 3.12.
If you need a trig identity that you do not know, google it or ask another student.

Derivatives of the other trig functions

Use the basic differentiation rules, as well as

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x,$$

to quickly obtain and prove formulas for the derivatives of tan, cot, sec, and csc.

Product of trig functions

Let $f(x) = \sin x \cos x$. What is its derivative $f'(x)$?

1. $1 - 2\sin^2(x)$
2. $2\cos^2(x) - 1$
3. $\cos 2x$
4. all of the above
5. none of the above

A pesky function

Let $h(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

1. Calculate $h'(x)$ for any $x \neq 0$.
2. Using the definition of derivative, calculate $h'(0)$.
3. Calculate $\lim_{x \to 0} h'(x)$

   Hint: Questions **??** and **??** have different answers.
4. Is h continuous at 0?
5. Is h differentiable at 0?
6. Is h' continuous at 0?

Implicit differentiation

The equation

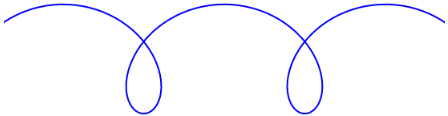
$$\sin(x + y) + xy^2 = 0$$

defines a function $y = h(x)$ near $(0, 0)$. graph
Using implicit differentiation, compute

1. $h(0)$
2. $h'(0)$
3. $h''(0)$
4. $h'''(0)$

Worm up

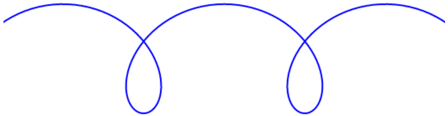
A worm is crawling accross the table. The path of the worm looks something like this:



True or False?

The position of the worm is a function.

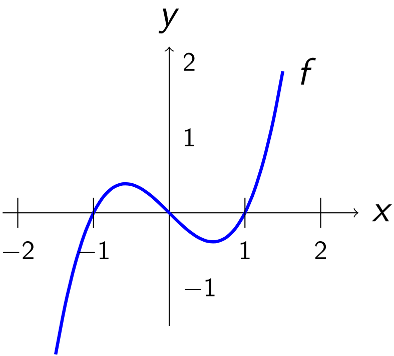
Worm function



A worm is crawling accross the table.
For any time t, let $f(t)$ be the position of the worm.
This defines a function f.

1. What is the domain of f?
2. What is the codomain of f?
3. What is the range of f?

Finding a Restricted Domain on which a Function is Invertible



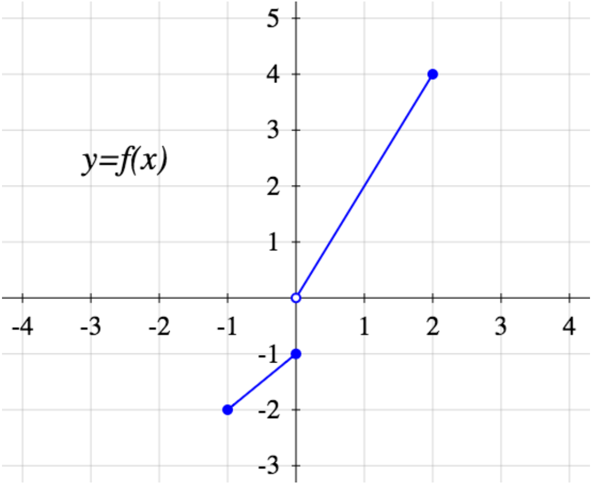
1. Find the largest interval containing 0 on which f is invertible.
2. Find the largest interval containing 1 on which f is invertible.

Fill in the Blank

Given that f is an invertible function, fill in the blanks.

1. If $f(-1) = 0$, then $f^{-1}(0) =$ ——.
2. If $f^{-1}(2) = 1$, then $f(1) =$ ——.
3. If $(2, 3)$ is on the graph of f, then —— is on the graph of f^{-1}.
4. If —— is on the graph of f, then $(-2, 4)$ is on the graph of f^{-1}.

Inverse function from a graph



Calculate:

1. $f(2)$
2. $f(0)$
3. $f^{-1}(2)$
4. $f^{-1}(0)$
5. $f^{-1}(-1)$

Absolute value and inverses

Let

$$h(x) = x|x| + 1$$

1. Calculate $h^{-1}(-8)$.
2. Sketch the graph of h.
3. Find an equation for h^{-1}.
4. Sketch the graph of h^{-1}.
5. Verify that
   - for every $t \in \boxed{???}$, $\quad h(h^{-1}(t)) = t$.
   - for every $t \in \boxed{???}$, $\quad h^{-1}(h(t)) = t$.

Functions, inverses, and graphs

Sketch the graph of a function g satisfying all the following properties:

1. The domain of g is $\mathbb{R}$.
2. g is continuous everywhere except at -2.
3. g is differentiable everywhere except at -2 and 1.
4. g has an inverse function.
5. $g(0) = 2$
6. $g'(0) = 2$
7. $\left(g^{-1}\right)'(-3) = -2$.

Functions, inverses, and graphs - 2

Draw the graph of a function f satisfying all of the following:

1. The domain of f is $\mathbb{R}$.
2. f is differentiable everywhere.
3. The restriction of f to $[0, \infty)$ is one-to-one, and its INVERSE has a vertical tangent line at 2.
4. The restriction of f to $(-\infty, 0]$ is one-to-one, and its INVERSE has derivative 2 at 2.

Composition and inverses

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$.

Let f and g be functions. Assume they each have an inverse.

Is $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$?

- If YES, prove it.
- If NO, fix the statement.

If you do not know how to start, experiment with the functions

$$f(x) = x + 1, \qquad g(x) = 2x.$$

Composition of one-to-one functions

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$. Prove the following theorem.

Theorem A

Let f and g be functions.
IF f and g are one-to-one,
THEN $f \circ g$ is one-to-one.

Suggestion:

1. Write the definition of what you want to prove.
2. Figure out the formal structure of the proof.
3. Complete the proof (use the hypotheses!)

Composition of one-to-one functions – 2

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$. Prove the following theorem.

Theorem B

Let f and g be functions.
IF $f \circ g$ is one-to-one, THEN g is one-to-one.

Suggestion:

1. Transform the "$P \implies Q$" theorem into an equivalent "(not Q) $\implies$ (not P)" theorem. You will prove that one instead.
2. Write the definition of the hypotheses and of the conclusion.
3. Write the proof.

Composition of one-to-one functions – 3

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$.

Prove the following claim is FALSE with a counterexample.

Claim

Let f and g be functions.
IF $f \circ g$ is one-to-one,
THEN f is one-to-one.

Increasing and one-to-one

Definition

Let f be a function with domain D. We say that f is *increasing* on D when

$$\forall x_1, x_2 \in D, \quad x_1 < x_2 \implies f(x_1) < f(x_2).$$

1. Prove that if a function is increasing, then it is one-to-one.
2. Use this to show that $g(x) = x^5 + 4x^3 + 2x + 1$ has an inverse.
3. Find $(g^{-1})'(1)$.

Where is the error?

- We know that $\boxed{(f^{-1})' = \dfrac{1}{f'}}$
- Let $f(x) = x^2$, restricted to the domain $x \in (0, \infty)$

$$f'(x) = 2x \quad \text{and} \quad \boxed{f'(4) = 8}$$

- Then $f^{-1}(x) = \sqrt{x}$

$$(f^{-1})'(x) = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \boxed{(f^{-1})'(4) = \frac{1}{4}}$$

- So $\boxed{(f^{-1})'(4) \neq \dfrac{1}{f'(4)}}$

Derivatives of the inverse function

Let f be a one-to-one function.
Let $a, b \in \mathbb{R}$ such that $b = f(a)$.

1. Obtain a formula for $\left(f^{-1}\right)'(b)$ in terms of $f'(a)$.

   Hint: This was done in Video 4.4
   Take $\dfrac{d}{dy}$ of both sides of $\quad f(f^{-1}((y)) = y$.

2. Obtain a formula for $\left(f^{-1}\right)''(b)$ in terms of $f'(a)$ and $f''(a)$.

3. *Challenge:* Obtain a formula for $\left(f^{-1}\right)'''(b)$ in terms of $f'(a)$, $f''(a)$, and $f'''(a)$.

Computations - Exponentials and logarithms

Compute the derivative of the following functions:

1. $f(x) = e^{\sin x + \cos x} \ln x$

2. $f(x) = \pi^{\tan x}$

3. $f(x) = \ln\left[e^{x} + \ln \ln \ln x\right]$

4. $f(x) = \log_{10}(2x + 3)$

Logarithm and Absolute Value

The function F is defined by the equation

$$F(x) = \ln |x|.$$

What is its derivative?

1. $F'(x) = \dfrac{1}{x}$

2. $F'(x) = \dfrac{1}{|x|}$

3. F is not differentiable

A different type of logarithm

Calculate the derivative of

$$f(x) = \log_{x+1}(x^2 + 1)$$

Note: This is a new function. We have not given you a formula for it yet, That is on purpose.

Hint: If you do not know where to start, remember the definition of logarithm:

$$\log_a b = c \iff a^c = b.$$

Logarithmic differentiation

Calculate the derivative of

$$g(x) = x^{\tan x}.$$

More logarithmic differentiation

Calculate the derivative of

$$f(x) = (\sin x)^{\cos x} + (\cos x)^{\sin x}.$$

What is wrong with this answer?

$$\ln f(x) = (\cos x)\ln(\sin x) + (\sin x)(\ln \cos x)$$

$$\frac{d}{dx}[\ln f(x)] = \frac{d}{dx}[(\cos x)\ln(\sin x)] + \frac{d}{dx}[(\sin x)(\ln \cos x)]$$

$$\frac{f'(x)}{f(x)} = -(\sin x)\ln(\sin x) + (\cos x)\frac{\cos x}{\sin x}$$

$$+ (\cos x)\ln(\cos x) + (\sin x)\frac{-\sin x}{\cos x}$$

$$f'(x) = f(x)\left[-(\sin x)\ln(\sin x) + (\cos x)\ln(\cos x) + \frac{\cos^2 x}{\sin x} - \frac{\sin^2 x}{\cos x}\right]$$

Hard derivatives made easier

Calculate the derivative of

$$h(x) = \sqrt[3]{\frac{\left(\sin^6 x\right) \sqrt{x^7 + 6x + 2}}{3^x \left(x^{10} + 2x\right)^{10}}}$$

An Implicit Function

Find y' if $x^y = y^x$.

Definition of arctan

1. Sketch the graph of tan.
2. Prove that tan is not one-to-one.
3. Select the largest interval containing 0 such that the restriction of tan to it is one-to-one. We define arctan as the inverse of this restriction. Let $x, y \in \mathbb{R}$

$$\arctan y = x \quad \iff \quad ???$$

4. What is the domain of arctan? What is the range of arctan?
   Sketch the graph of arctan.
5. Compute

   5.1 $\arctan(\tan(1))$
   5.2 $\arctan(\tan(3))$
   5.3 $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$
   5.4 $\arctan(\tan(-6))$
   5.5 $\tan(\arctan(0))$
   5.6 $\tan(\arctan(10))$

Derivative of arctan

Obtain (and prove) a formula for the derivative of arctan.

Hint: Call $f(t) = \arctan t$ and differentiate

$$\forall t \in \ldots \qquad \tan(f(t)) = t$$

Computations - Inverse trig functions

Compute the derivatives of these functions, and simplify them as much as possible:

1. $f(x) = \arcsin\left(x^{3/2}\right)$

2. $f(x) = 2x^2 \arctan(x^2) - \ln(x^4 + 1)$

Trig-inverse-trig

Find simple expressions for these quantities and state the domain on which they are valid:

1. $\sin(\arccos x)$
2. $\sec(\arccos x)$
3. $\sec(\arctan x)$
4. $\tan(\operatorname{arcsec} x)$

Hint: There are two standard ways to attack these problems:

- Use a trig identity
  e.g.: a trig identity relating sin and cos for (1)
- Or draw a right triangle with side lengths 1 and x
  e.g.: with an angle θ such that $\cos\theta = x$ for (1)

If you need to take a square root, you must justify which branch ($+$ or $-$) you are choosing.

1. Complete: "We define arcsec as the inverse of the restriction of sec to ..."
   Hint: Sketch the graph of sec.

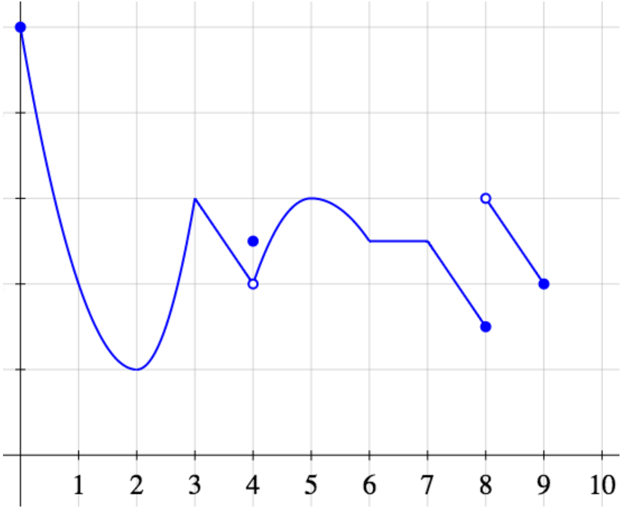
2. What are the domain and range of arcsec?
   Sketch its graph.

3. Obtain (and prove) a formula for the derivative of arcsec in the same way you did for arctan.

4. Now obtain the same formula in a different way:
   use $\sec x = \dfrac{1}{\cos x}$ to write arcsec in terms of arccos.

Definition of local extremum

Find local and global extrema of the function with this graph:



Where is the maximum?

We know the following about the function h:

- The domain of h is $(-4, 4)$.
- h is continuous on its domain.
- h is differentiable on its domain, except at 0.
- $h'(x) = 0 \quad \iff \quad x = -1$ or 1.

What can you conclude about the maximum of h?

1. h has a maximum at $x = -1$, or 1.
2. h has a maximum at $x = -1, 0$, or 1.
3. h has a maximum at $x = -4, -1, 0, 1$, or 4.
4. None of the above.

What can you conclude?

We know the following about the function f.

- f has domain $\mathbb{R}$.
- f is continuous
- $f(0) = 0$
- For every $x \in \mathbb{R}$, $f(x) \geq x$.

What can you conclude about $f'(0)$? Prove it.

Hint: Sketch the graph of f. Looking at the graph, make a conjecture.
To prove it, imitate the proof of the Local EVT from Video 5.3.

Fractional exponents

Let $g(x) = x^{2/3}(x-1)^3$.
Find local and global extrema of g on $[-1, 2]$.

Trig extrema

Let $f(x) = \dfrac{\sin x}{3 + \cos x}$.

Find the maximum and minimum of f.

How many zeroes?

Let

$$f(x) = e^x - \sin x + x^2 + 10x$$

How many zeroes does f have?

Zeroes of the derivative

Sketch the graph of a function f that is differentiable on $\mathbb{R}$ and such that

1. f has exactly 3 zeroes and f' has exactly 2 zeroes.
2. f has exactly 3 zeroes and f' has exactly 3 zeroes.
3. f has exactly 3 zeroes and f' has exactly 1 zero.
4. f has exactly 3 zeroes and f' has infinitely many zeroes.

Zeroes of a polynomial

You probably learned in high school that a polynomial of degree n has at most n real zeroes. Now you can prove it!

Hint: Use induction. If you are having trouble, try the case $n = 3$ first.

The second Theorem of Rolle

Complete statement for this theorem and prove it.

Rolle's Theorem 2

Let $a < b$. Let f be a function defined on $[a, b]$.
IF

- (Some conditions on continuity and derivatives)
- $f(a) = f'(a) = 0$
- $f(b) = 0$

THEN $\exists c \in (a, b)$ such that $f''(c) = 0$.

Hint: Apply the 1st Rolle's Theorem to f, then do something else.

The N-th Theorem of Rolle

Complete the statement for this theorem and prove it.

Rolle's Theorem N

Let N be a positive integer.
Let $a < b$. Let f be a function defined on $[a, b]$.
IF

- (Some conditions on continuity and derivatives)
- (Some conditions at a)
- $f(b) = 0$

THEN $\exists c \in (a, b)$ such that $f^{(N)}(c) = 0$.

A new theorem

We want to prove this theorem:

Theorem 1

Let f be a differentiable function on an open interval I.
IF $\forall x \in I, f'(x) \neq 0$
THEN f is one-to-one on I.

1. Transform $[P \implies Q]$ into $[(\text{not } Q) \implies (\text{not } P)]$.
   You get an equivalent Theorem (call it "Theorem 2").
   We are going to prove Theorem 2 instead.
2. Write the definition of "f is not one-to-one on I". You will need it.
3. Recall the statement of Rolle's Theorem. You will need it.
4. Do some rough work if needed.
5. Write a complete proof for Theorem 2.

A variant

Complete this variation on Theorem 2.
Use the weakest conditions you can to make it true.

Theorem 3

Let $a < b$. Let f be a function defined on $[a, b]$.
IF

- (Some conditions on continuity and differentiability)
- f is not one-to-one on $[a, b]$

THEN $\exists c \in (a, b)$ such that $f'(c) = 0$.

Why the three hypotheses are necessary

You have proven

Theorem 3

Let $a < b$. Let f be a function defined on $[a, b]$.
IF

1. f is continuous on $[a, b]$
2. f is differentiable on (a, b)
3. f is not one-to-one on $[a, b]$

THEN $\exists c \in (a, b)$ such that $f'(c) = 0$.

Give three examples to justify that each of the three hypotheses are necessary for the theorem to be true. (Graphs of the examples are enough).

MVT – True or False?

True or False

Consider $f(x) = |x|$ on the interval $[-\frac{1}{2}, 2]$.
There exists c in $(-\frac{1}{2}, 2)$ such that

$$f'(c) = \frac{f(2) - f(-\frac{1}{2})}{2 - (-\frac{1}{2})}$$

Car race - 1

A driver competes in a race.
Use MVT to prove that at some point during the race the instantaneous velocity of the driver is exactly equal to the average velocity of the driver during the race.

Car race - 2

Two drivers start a race at the same moment and finish in a tie. Can you conclude that there was a time in the race (not counting the starting time) when the two drivers had exactly the same speed?

Car race - Is this proof correct?

Claim

IF two drivers start a race at the same moment and finish in a tie, THEN at some point in the race (not counting the starting time) they had exactly the same speed.

Proof?

- Let $f(t)$ and $g(t)$ be the positions of the two cars at time t.
- Assume the race happens in the interval $[t_1, t_2]$. By hypothesis:
$$f(t_1) = g(t_1), \qquad f(t_2) = g(t_2).$$
- Using MVT, there exists $c \in (t_1, t_2)$ such that
$$f'(c) = \frac{f(t_2) - f(t_1)}{t_2 - t_1}, \qquad g'(c) = \frac{g(t_2) - g(t_1)}{t_2 - t_1}.$$
- Then $f'(c) = g'(c)$. $\square$

Car race - resolution

Two drivers start a race at the same moment and finish in a tie. Prove that at some point during the race (not counting the starting time) the two drivers had exactly the same speed.

Speeding ticket!

On a toll road Barney takes a time stamped toll-card from the starting booth and drives directly to the end of the toll section.

After paying the required toll, Barney is surprised to receive a speeding ticket along with the toll receipt.

Which of the following are true?

1. The booth attendant does not have enough information to prove that Barney was speeding.
2. The booth attendant can prove that Barney was speeding during his trip.
3. Barney's ticket is for a lower speed than his actual maximum speed.

Proving difficult identities

Prove that, for every $x \geq 0$,

$$2 \arctan \sqrt{x} - \arcsin \frac{x-1}{x+1} = \frac{\pi}{2}$$

Hint: Derivatives.

Critique this "proof"

- $$\left[2\arctan\sqrt{x} - \arcsin\frac{x-1}{x+1}\right] = \left[\frac{\pi}{2}\right]$$
- $$\frac{d}{dx}\left[2\arctan\sqrt{x} - \arcsin\frac{x-1}{x+1}\right] = \frac{d}{dx}\left[\frac{\pi}{2}\right]$$
- $$\frac{2}{1+\left(\sqrt{x}\right)^2}\cdot\frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}\cdot\frac{(x+1)-(x-1)}{(x+1)^2} = 0$$
- $$\frac{1}{(1+x)\sqrt{x}} - \frac{1}{\sqrt{\frac{4x}{(x+1)^2}}}\cdot\frac{2}{(x+1)^2} = 0$$
- $0 = 0$
- So $2\arctan\sqrt{x} - \arcsin\dfrac{x-1}{x+1}$ is constant.
- Evaluate at $x = 0$ to find the value of the constant.
- $2\arctan 0 - \arcsin(-1) = 0 - (-\pi/2) = \pi/2$
- Therefore, $2\arctan\sqrt{x} - \arcsin\dfrac{x-1}{x+1} = \dfrac{\pi}{2}$

Warm up

1. Let f be a function defined on an interval I.
   Write the definition of "f is increasing on I".
2. Write the statement of the Mean Value Theorem.

Positive derivative implies increasing

Use the MVT to prove

Theorem

Let $a < b$. Let f be a differentiable function on (a, b).

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN f is increasing on (a, b).

1. Recall the definition of what you are trying to prove.
2. **From that definition, figure out the structure of the proof.**
3. If you have used a theorem, did you verify the hypotheses?
4. Are there words in your proof, or just equations?

What is wrong with this proof?

Theorem

Let $a < b$. Let f be a differentiable function on (a, b).

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN f is increasing on (a, b).

Proof.

- From the MVT, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$
- We know $b - a > 0$ and $f'(c) > 0$
- Therefore $f(b) - f(a) > 0$. Thus $f(b) > f(a)$.
- f is increasing.

$\square$

Your first integration

Find all functions f such that, for all $x \in \mathbb{R}$:
$f''(x) = x + \sin x$.

Intervals of monotonicity

Let $g(x) = x^3(x^2 - 4)^{1/3}$.
Find out on which intervals this function is increasing or decreasing.
Using that information, sketch its graph.
To save time, here is the first derivative:

$$g'(x) = \frac{x^2(11x^2 - 36)}{3(x^2 - 4)^{2/3}}$$

True or False – Monotonicity and local extrema

Let I be an interval. Let f be a function defined on I. Let $c \in I$. Which implications are true?

1. IF f is increasing on I, THEN $\forall x \in I,\ f'(x) > 0$.
2. IF $\forall x \in I,\ f'(x) > 0$, THEN f is increasing on I.
3. IF f has a local extremum at c, THEN $f'(c) = 0$.
4. IF $f'(c) = 0$, THEN f has a local extremum at c.
5. IF f has local extremum at c, THEN f has an extremum at c
6. IF f has an extremum at c, THEN f has local extremum at c

Inequalities

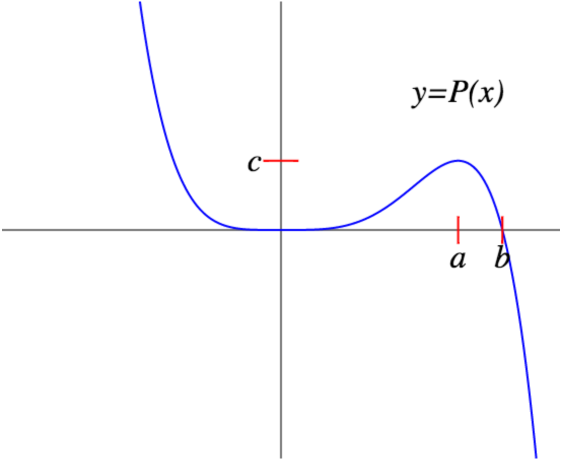
Prove that, for every $x \in \mathbb{R}$

$$e^x \geq 1 + x$$

Hint: Where is the function $f(x) = e^x - 1 - x$ increasing or decreasing? What is its minimum?

Backwards graphing

Below is the graph of a polynomial P. Notice that it is not at scale. The coordinates in the graph are $a = 24$, $b = 25$, and $c = 1$. Find the equation of P.



A sneaky function

Construct a function f satisfying all the following properties:

- Domain $f = \mathbb{R}$
- f is continuous
- $f'(0) = 0$
- f does not have a local extremum at 0.
- There isn't an interval centered at 0 on which f is increasing.
- There isn't an interval centered at 0 on which f is decreasing.

Lake ripple

We drop a pebble into a lake. It produces a circular ripple. When the radius is 2 meters and is increasing at a rate of $10cm/s$, at what rate is the area increasing?

Sliding ladder

A ten-meter long ladder is leaning against a vertical wall and sliding. The top end of the ladder is 8 meters high and sliding down at a rate of 1 meter per second. At which rate is the bottom end sliding?

Math party

The MAT137 TAs wanted to rent a disco ball for their upcoming party. However, since they are poor, they could only afford a flashlight. At the party, one TA is designated the "human disco ball". The TA stands in the center of the room pointing the flashlight horizontally and spins at 3 revolutions per second. (Yes, they are that fast. Ask your TA to demonstratel if you don't believe me!) The room is square with side length 8 meters. At which speed is the light from the flashlight moving across the wall when it is 3 meters away from a corner?

Sleepy ants

Two ants are taking a nap. The first one is resting at the tip of the minute hand of a cuckoo clock, which is 25 cm long. The second one is resting at the tip of the hour hand, which is half the length. At what rate is the distance between the two ants changing at 3:30?

The kite

Mary Poppins is flying a kite. The kite is 21 meters above the ground and it is being blown horizontally by the wind at 2 m/s. Mary's hands are 1 meter above the ground. Right now 30 meters of string are out. At what rate is the string being released from Mary's hands?

Coffee

A coffee filter is shaped like an inverted cone. It has a radius at the top of $4cm$ and it is $6cm$ in height. Coffee flows out of at the bottom at a rate of $2cm^3/s$. If the filter begins completely filled, how fast is the coffee level decreasing after 30 seconds?

The classic farmer problem

A farmer has $300m$ of fencing and wants to fence off a rectangular field and add an extra fence that divides the rectangular area in two equal parts down the middle. What is the largest area that the field can have?

Distance

Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

A matter of perspective

A painting in an art gallery has height h and is hung so that its lower edge is a distance a above your eye. How far from the wall should you stand to get the best view?

Airplane

The cost of fuel per hour for a certain airplane is proportional to the square of its speed and is \$1200 per hour for a speed of $600 km/h$. After every 5,000 hours flown, the aircraft must undergo an \$8 million dollar safety inspection. What speed should the airplane fly at in order to achieve the lowest cost per kilometre?

Fire

You hear a scream. You turn around and you see Alfonso is on fire. Literally! Luckily, you are next to a straight river. Alfonso is 10 meters away from the river and you are 5 meters away from the point P on the river closest to Alfonso. You are carrying an empty bucket. You can run twice as fast with an empty bucket as you can run with a full bucket. How far from the point P should you fill your bucket in order to get to Alfonso with a bucket full of water as fast as possible?

Dominion

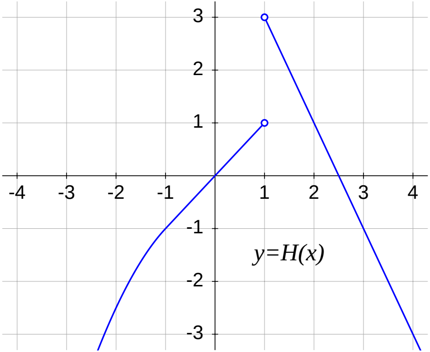
Dominion is a board game where, among other things, players buy cards worth victory points. The player with the most victory points wins.
It is your last turn and you can only buy "duchies" and "dukes". A duchy is worth 3 victory points. A duke is worth as many victory points as duchies you have. Each duchy costs 3 coins, and each duke costs 3 coins. You have not bough any duke or duchy yet.
If you have N coins, how many dukes and how many duchies should you buy?

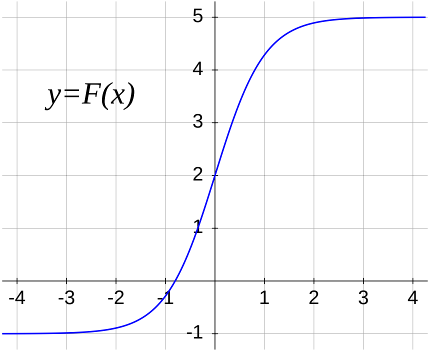
Limits from graphs

Compute:

1. $\displaystyle\lim_{x\to 0} \frac{H(x)}{H(2+3x)-1}$

2. $\displaystyle\lim_{x\to 2} \frac{F^{-1}(x)}{x-2}$





Polynomial vs Exponential

1. Use L'Hôpital Rule to compute

$$\lim_{x \to \infty} \frac{x^7 + 5x^3 + 2}{e^x}$$

2. Make a conjecture for the value of

$$\lim_{x \to \infty} \frac{x^N}{e^x}$$

where N is a positive integer. Prove it by induction.

Computations

Calculate:

1. $\lim_{x \to 2} \dfrac{x^2 + 2x - 6}{x^2 + 3x - 10}$
2. $\lim_{x \to 0} \dfrac{e^{2x^2} - \cos x}{x \sin x}$
3. $\lim_{x \to \infty} x^3 e^{-x}$
4. $\lim_{x \to \infty} \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
5. $\lim_{x \to 0} x \sin \dfrac{2}{x}$
6. $\lim_{x \to \infty} x \sin \dfrac{2}{x}$
7. $\lim_{x \to \infty} x \cos \dfrac{2}{x}$
8. $\lim_{x \to 1} \left[(\ln x) \tan \dfrac{\pi x}{2} \right]$

Infinity minus infinity

Calculate:

1. $\lim_{x \to 0} \left[\dfrac{\csc x}{x} - \dfrac{\cot x}{x} \right]$

2. $\lim_{x \to \infty} \left[\ln(x+2) - \ln(3x+4) \right]$

3. $\lim_{x \to 1} \left[\dfrac{2}{x^2 - 1} - \dfrac{1}{x - 1} \right]$

4. $\lim_{x \to -\infty} \left[\sqrt{x^2 + 3x} - \sqrt{x^2 - 3x} \right]$

Exponential indeterminate forms

Calculate:

1. $\lim_{x \to 0} [1 + 2\sin(3x)]^{4\cot(5x)}$
2. $\lim_{x \to \infty} \left(\frac{x+2}{x-2} \right)^{3x}$
3. $\lim_{x \to 0^+} x^x$
4. $\lim_{x \to \frac{\pi}{2}^-} (\tan x)^{\cos x}$
5. $\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{1/x^2}$

Backwards L'Hôpital

Construct a polynomial P such that

$$\lim_{x \to 1} \frac{P(x)}{e^x - e \cdot x} = \frac{1}{e}$$

Come to the dark side

Help us write a difficult question for Test 3! We will ask you to compute a limit like this

$$\lim_{x \to 0} \frac{e^x + e^{-x} - 2\cos x + bx^N}{x^6}$$

where b is a real number and N is a natural number that we have not chosen yet.
We do not want the answer to be 0 or ∞ or $-\infty$ or "DNE", because you could guess that randomly.
What values of b and N should we choose? What will the value of the limit be?

Indeterminate?

Which of the following are indeterminate forms for limits?
If any of them isn't, then what is the value of such limit?

1. $\frac{0}{0}$
2. $\frac{0}{\infty}$
3. $\frac{0}{1}$
4. $\frac{\infty}{0}$
5. $\frac{\infty}{\infty}$
6. $\frac{1}{\infty}$
7. $0 \cdot \infty$
8. $\infty \cdot \infty$
9. $\sqrt{\infty}$
10. $\infty - \infty$
11. 1^{∞}
12. $1^{-\infty}$
13. 0^{0}
14. 0^{∞}
15. $0^{-\infty}$
16. ∞^{0}
17. ∞^{∞}
18. $\infty^{-\infty}$

Proving something is an indeterminate form

1. Prove that $\forall c \in \mathbb{R}$, $\exists a \in \mathbb{R}$ and functions f and g such that

$$\lim_{x\to a} f(x) = 0, \quad \lim_{x\to a} g(x) = 0, \quad \lim_{x\to a} \frac{f(x)}{g(x)} = c$$

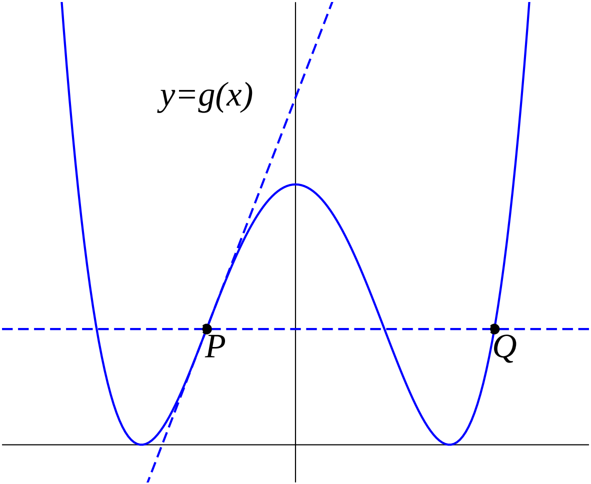
   This is how you show that $\frac{0}{0}$ is an indeterminate form.

2. Prove the same way that $\frac{\infty}{\infty}$, $0 \cdot \infty$, and $\infty - \infty$ are also indeterminate forms.

3. Prove that 1^∞, 0^0, and ∞^0 are indeterminate forms. (You will only get $c \geq 0$ this time)

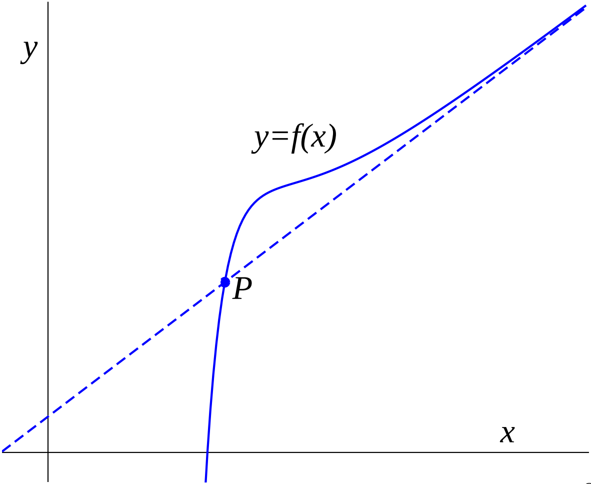
Find the coordinates of P and Q

$$g(x) = x^4 - 6x^2 + 9$$



Find the coordinates of P

$$f(x) = 3x + 4 + \frac{2x - 10}{x^2}$$



True or False – Concavity and inflection points

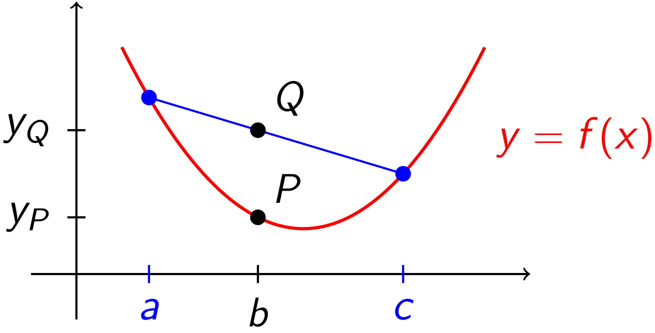
Let f be a differentiable function with domain $\mathbb{R}$.
Let $c \in \mathbb{R}$. Let I be an interval. Which implications are true?

1. IF f is concave up on I, THEN $\forall x \in I,\ f''(x) > 0$.
2. IF $\forall x \in I,\ f''(x) > 0$, THEN f is concave up on I.
3. IF f is concave up on I THEN f' is increasing on I.
4. IF f' is increasing on I, THEN f is concave up on I.
5. IF f has an I.P. at c, THEN $f''(c) = 0$.
6. IF $f''(c) = 0$, THEN f has an I.P. at c.
7. IF f has an I.P. at c, THEN f' has a local extremum at c
8. IF f' has a local extremum at c, THEN f has an I.P. at c.

I.P. = "inflection point"

"Secant segments are above the graph"

Let f be a function defined on an interval I.
In Video 6.13 you learned that an alternative to our definition of "f is concave up on I" is "the secant segments stay above the graph".



Rewrite this as a precise mathematical statement of the form

"$\forall a, b, c \in I, \quad a < b < c \implies$ $\boxed{\text{an inequality involving } f,\ a,\ b,\ c}$

A polynomial from 3 points

Construct a polynomial that satisfies the following three properties at once:

1. It has an inflection point at $x = 2$
2. It has a a local extremum at $x = 1$
3. It has y-intercept at $y = 1$.

Monotonicity and concavity

Let $f(x) = xe^{-x^2/2}$.

1. Find the intervals where f is increasing or decreasing, and its local extrema.
2. Find the intervals where f is concave up or concave down, and its inflection points.
3. Calculate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
4. Using this information, sketch the graph of f.

Fractional exponents

Let $h(x) = \dfrac{x^{2/3}}{(x-1)^{2/3}}$. Its first two derviatives are

$$h'(x) = \frac{-2}{3x^{1/3}(x-1)^{5/3}} \qquad h''(x) = \frac{2(6x-1)}{9x^{4/3}(x-1)^{8/3}}$$

1. Find all asymptotes of h
2. Study the monotonicity of h and local extrema
3. Study the concavity of h and inflection points
4. With this information, sketch the graph of h

Hyperbolic tangent

The function tanh, defined by

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

is called the "hyperbolic tangent".

1. Find its two asymptotes
2. Study its monotonicity
3. Study its concavity
4. With this information, sketch its graph.

A very hard function to graph

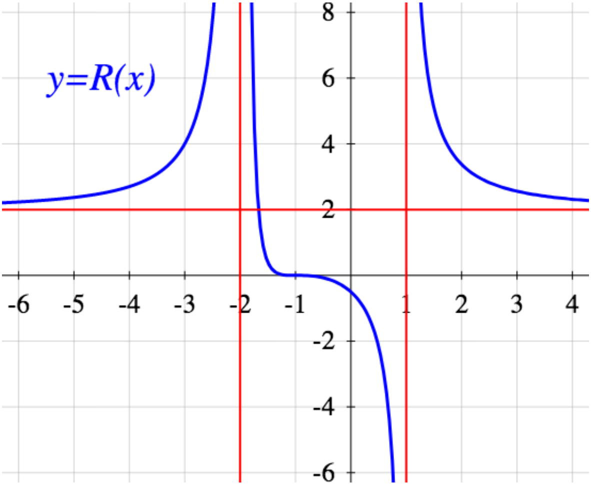
The function $G(x) = xe^{1/x}$ is deceiving. To help you out:

$$G'(x) = \frac{x-1}{x}e^{1/x}, \qquad G''(x) = \frac{e^{1/x}}{x^3}$$

1. Carefully study the behaviour as $x \to \pm\infty$.
   You should find an asymptote, but it is not easy.
2. Carefully study the behaviour as $x \to 0^+$ and $x \to 0^-$. The two are very different.
3. Use G' to study monotonocity.
4. Use G'' to study concavity.
5. Sketch the graph of G.

Backwards graphing

R is a rational function (a quotient of polynomials). Find its equation.



Unexpected asymptotes

Find the two asymptotes of the function

$$F(x) = x + \sqrt{x^2 + 2x + 2}$$

Hint: The behaviour as $x \to \infty$ is very different from $x \to -\infty$.

graph

Unusual examples

Construct three functions f, g, and h.

1. f has domain at least $(0, \infty)$, is continuous, is always concave up, and satisfies $\lim_{x \to \infty} f(x) = -\infty$
2. g has domain $\mathbb{R}$, is continuous, has a local minimum at $x = 0$, and has an inflection point also at $x = 0$.
3. h has domain $\mathbb{R}$, is differentiable, is strictly increasing. In addition, h' is periodic with period 2, and h' is not constant.

Warm-up: sums

Compute

1. $$\sum_{i=2}^{4}(2i+1)$$

2. $$\sum_{i=2}^{4}2i+1$$

3. $$\sum_{j=2}^{4}(2i+1)$$

Write these sums with Σ notation

1. $1^5 + 2^5 + 3^5 + 4^5 + \ldots + 100^5$

2. $\dfrac{2}{4^2} + \dfrac{2}{5^2} + \dfrac{2}{6^2} + \dfrac{2}{7^2} + \ldots + \dfrac{2}{N^2}$

3. $\cos 0 - \cos 1 + \cos 2 - \cos 3 + \ldots \pm \cos(N+1)$

4. $\dfrac{1}{0!} + \dfrac{1}{2!} + \dfrac{1}{4!} + \dfrac{1}{6!} + \ldots + \dfrac{1}{(2N)!}$

5. $\dfrac{1}{1!} - \dfrac{1}{3!} + \dfrac{1}{5!} - \dfrac{1}{7!} + \ldots + \dfrac{1}{81!}$

6. $\dfrac{2x^3}{4!} + \dfrac{3x^4}{5!} + \dfrac{4x^5}{6!} + \ldots + \dfrac{999x^{1000}}{1001!}$

Re-writing sums

1. $$\sum_{i=1}^{100} \tan i \;-\; \sum_{i=1}^{50} \tan i \;=\; \sum_{\boxed{}}^{\boxed{}} \boxed{}$$

2. $$\sum_{i=1}^{N} (2i-1)^5 \;=\; \sum_{i=0}^{N-1} \boxed{}$$

3. $$\left[\sum_{k=1}^{N} x^k\right] + \left[\sum_{k=0}^{N} k\, x^{k+1}\right] = \left[\sum_{k=\boxed{}}^{\boxed{}} \boxed{}\, x^k\right] + \boxed{}$$

Hint: Write out the sums on the left hand side first, simplify if possible, then write them back into sigma notation.

Telescopic sum

- Calculate the exact value of

$$\sum_{i=1}^{137} \left[\frac{1}{i} - \frac{1}{i+1} \right]$$

  Hint: Write down the first few terms.

- Calculate the exact value of

$$\sum_{i=1}^{10{,}000} \frac{1}{i(i+1)}$$

Double sums

Compute:

1. $$\sum_{i=1}^{N}\sum_{k=1}^{N} 1$$
2. $$\sum_{i=1}^{N}\sum_{k=1}^{i} 1$$
3. $$\sum_{i=1}^{N}\sum_{k=1}^{i} i$$
4. $$\sum_{i=1}^{N}\sum_{k=1}^{i} k$$
5. $$\sum_{i=1}^{N}\sum_{k=1}^{i} (ik)$$

Useful formulas:

$$\sum_{j=1}^{N} j = \frac{N(N+1)}{2}, \quad \sum_{j=1}^{N} j^2 = \frac{N(N+1)(2N+1)}{6}, \quad \sum_{j=1}^{N} j^3 = \frac{N^2(N+1)^2}{4}$$

Harmonic sums

We define the N-th Harmonic term as the sum

$$H_N = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N} = \sum_{i=1}^{N} \frac{1}{i}.$$

Write the following sums in terms of harmonic terms.

1. $\displaystyle\sum_{i=k}^{N} \frac{1}{i}$
2. $\displaystyle\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2N}$
3. $\displaystyle\sum_{i=1}^{N} \frac{1}{2i-1}$
4. $\displaystyle\sum_{i=1}^{2N} \frac{(-1)^{i+1}}{i}$

Fubini-Tonelli

- $A_{i,k}$ is a function of 2 variables. For example, $A_{i,k} = \dfrac{i}{k+i^2}$.
- Decide what to write instead of each "$\boxed{?}$" so that the following identity is true:

$$\sum_{i=1}^{N} \sum_{k=1}^{i} A_{i,k} = \sum_{k=\boxed{?}}^{\boxed{?}} \sum_{i=\boxed{?}}^{\boxed{?}} A_{i,k}$$

Warm up: suprema and infima

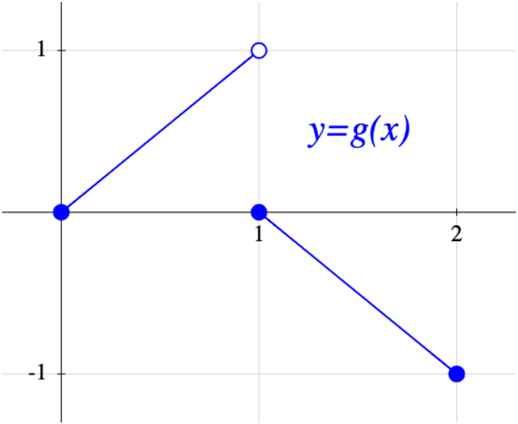
Find the supremum, infimum, maximum, and minimum of the following sets (if they exist):

1. $A = [-1, 5)$
2. $B = (-\infty, 6] \cup (8, 9)$
3. $C = \{2, 3, 4\}$
4. $D = \left\{ \frac{1}{n} \ : \ n \in \mathbb{Z}, \ n > 0 \right\}$
5. $E = \left\{ \frac{(-1)^n}{n} \ : \ n \in \mathbb{Z}, \ n > 0 \right\}$
6. $F = \{2^n \ : \ n \in \mathbb{Z}\}$

Suprema from a graph

Calculate, for the function g on the interval $[0.5, 1.5]$:

1. supremum 2. infimum 3. maximum 4. minimum



Trig suprema

Let $f(x) = \sin x$.
Find four open intervals I_1, I_2, I_3, I_4 such that

1. f has a supremum and an infimum on I_1.
2. f has a supremum and no infimum on I_2.
3. f has a maximum and a minimum on I_3.
4. f has a maximum and no minimum on I_4.

Empty set

1. Does $\emptyset$ have an upper bound ?
2. Does $\emptyset$ have a supremum?
3. Does $\emptyset$ have a maximum?
4. Is $\emptyset$ bounded above?

Equivalent definitions of supremum

Assume S is an upper bound of the set A.
Which of the following is equivalent to "S is the supremum of A"?

1. If R is an upper bound of A, then $S \leq R$.
2. $\forall R \geq S$, R is an upper bound of A.
3. $\forall R \leq S$, R is not an upper bound of A.
4. $\forall R < S$, R is not an upper bound of A.
5. $\forall R < S$, $\exists x \in A$ such that $R < x$.
6. $\forall R < S$, $\exists x \in A$ such that $R \leq x$.
7. $\forall R < S$, $\exists x \in A$ such that $R < x \leq S$.
8. $\forall R < S$, $\exists x \in A$ such that $R < x < S$.
9. $\forall \varepsilon > 0$, $\exists x \in A$ such that $S - \varepsilon < x$.
10. $\forall \varepsilon > 0$, $\exists x \in A$ such that $S - \varepsilon < x \leq S$.

Fix these FALSE statements

1. Let f and g be bounded functions on $[a, b]$. Then

$$\begin{matrix}\text{sup of } (f+g) \\ \text{on } [a,b]\end{matrix} \quad = \quad \begin{matrix}\text{sup of } f \\ \text{on } [a,b]\end{matrix} \quad + \quad \begin{matrix}\text{sup of } g \\ \text{on } [a,b]\end{matrix}$$

2. Let $a < b < c$. Let f be a bounded function on $[a, c]$. Then

$$\begin{matrix}\text{sup of } f \\ \text{on } [a,c]\end{matrix} \quad = \quad \begin{matrix}\text{sup of } f \\ \text{on } [a,b]\end{matrix} \quad + \quad \begin{matrix}\text{sup of } f \\ \text{on } [b,c]\end{matrix}$$

3. Let f be a bounded function on $[a, b]$. Let $c \in \mathbb{R}$. Then:

$$\begin{matrix}\text{sup of } (cf) \\ \text{on } [a,b]\end{matrix} \quad = \quad c \left(\begin{matrix}\text{sup of } f \\ \text{on } [a,b]\end{matrix} \right)$$

True or False - Suprema and infima

Let $A, B, C \subseteq \mathbb{R}$. Assume $C \subseteq A$. Which statements are true?
If possible, fix the false statements

1. IF A is bounded above, THEN C is bounded above.
2. IF C is bounded below, THEN A is bounded below.
3. IF A and C are bounded above, THEN $\sup C \leq \sup A$.
4. IF A and C are bounded below, THEN $\inf C \leq \inf A$.
5. IF A and B are bounded, $\sup B \leq \sup A$, and $\inf A \leq \inf B$, THEN $B \subseteq A$.
6. IF A and B are bounded above, THEN $\sup(A \cup B) = \max\{\sup A, \sup B\}$.
7. IF A and B are bounded above, THEN $\sup(A \cap B) = \min\{\sup A, \sup B\}$.

Warm up: partitions

Which ones are partitions of $[0, 2]$?

1. $[0, 2]$
2. $\{0.5,\ 1,\ 1.5\}$
3. $\{0,\ 2\}$
4. $\{1,\ 2\}$
5. $\{0,\ e,\ 2\}$
6. $\{0,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2\}$
7. $\left\{\dfrac{n}{n+1}\ :\ n \in \mathbb{N}\right\} \cup \{2\}$

Partitions of different intervals

Let $a < b < c$. Which of these statements are true?
If any are false, fix them.

1. IF P and Q are partitions of $[a, b]$,
   THEN $P \cup Q$ is a partition of $[a, b]$.
2. IF P and Q are partitions of $[a, b]$,
   THEN $P \cap Q$ is a partition of $[a, b]$.
3. IF P is a partition of $[a, b]$ and Q is a partition of $[b, c]$
   THEN $P \cup Q$ is a partition of $[a, c]$
4. IF P is a partition of $[a, c]$,
   THEN $P \cap [a, b]$ is a partition of $[a, b]$

Warm up: lower and upper sums

Let $f(x) = \sin x$.
Consider the partition $P = \{0, 1, 3\}$ of the interval $[0, 3]$.
Calculate $L_P(f)$ and $U_P(f)$.

Equations for lower and upper sums

Let f be a **decreasing**, bounded function on $[a, b]$.
Let $P = \{x_0, x_1, \dots, x_N\}$ be a partition of $[a, b]$
Which ones are a valid equation for $L_P(f)$? For $U_P(f)$?

1. $\displaystyle\sum_{i=0}^{N} f(x_i)\,\Delta x_i$
2. $\displaystyle\sum_{i=1}^{N} f(x_i)\,\Delta x_i$
3. $\displaystyle\sum_{i=0}^{N-1} f(x_i)\,\Delta x_i$
4. $\displaystyle\sum_{i=1}^{N} f(x_{i+1})\,\Delta x_i$
5. $\displaystyle\sum_{i=1}^{N} f(x_{i-1})\,\Delta x_i$
6. $\displaystyle\sum_{i=0}^{N-1} f(x_i)\,\Delta x_{i+1}$

Recall: $\Delta x_i = x_i - x_{i-1}$.

Easier than it looks

Let f be a bounded function on $[a, b]$.
Assume f is not constant.
Prove that there exists a partition P of $[a, b]$ such that

$$L_P(f) \neq U_P(f).$$

Joining partitions

Assume

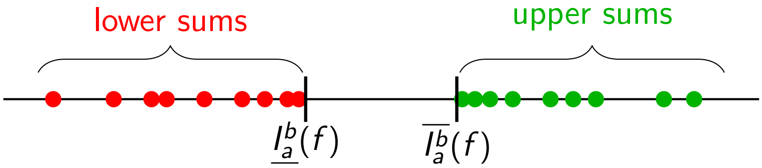
$$L_P(f) = 2, \quad U_P(f) = 6$$
$$L_Q(f) = 3, \quad U_Q(f) = 8$$

1. Is $P \subseteq Q$?
2. Is $Q \subseteq P$?
3. What can you say about $L_{P \cup Q}(f)$ and $U_{P \cup Q}(f)$?

A tricky question

Let f be a bounded function on $[a, b]$. Which statement is true?

1. There exists a partition P of $[a, b]$ such that
$$\underline{I_a^b}(f) = L_P(f) \qquad \text{and} \qquad \overline{I_a^b}(f) = U_P(f).$$
2. There exist partitions P and Q of $[a, b]$ such that
$$\underline{I_a^b}(f) = L_P(f) \qquad \text{and} \qquad \overline{I_a^b}(f) = U_Q(f).$$
3. There exists a partition P of $[a, b]$ such that
$$\underline{I_a^b}(f) = L_P(f).$$



An alternative definition

Let f be a bounded function on the interval $[a, b]$. Let $M \in \mathbb{R}$. Some of these four statements imply others. What implies what?

1. $\forall$ partition P of $[a, b]$, $\quad L_P(f) \leq M$,
2. $\forall \varepsilon > 0$, $\exists$ partition P of $[a, b]$ $\quad$ s.t. $\quad M - \varepsilon < L_P(f)$
3. $M \leq \underline{I_a^b}(f)$
4. $\underline{I_a^b}(f) \leq M$

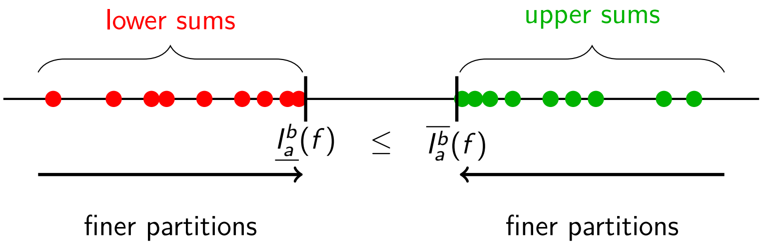
Based on this exercise, we could have defined $\underline{I_a^b}(f)$ as "the only number $M \in \mathbb{R}$ satisfying these two properties: ..."
Use the same idea to write an alternative definition of $\overline{I_a^b}(f)$.

The "ε–characterization" of integrability

True or False?

Let f be a bounded function on $[a, b]$.

1. IF "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$ s.t. $U_P(f) - L_P(f) < \varepsilon$",
   THEN f is integrable on $[a, b]$
2. IF f is integrable on $[a, b]$
   THEN "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$ s.t. $U_P(f) - L_P(f) < \varepsilon$".



The "ε–characterization" of integrability - Part 1

True or False?

Let f be a bounded function on $[a, b]$.

- IF "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$ s.t. $U_P(f) - L_P(f) < \varepsilon$",
- THEN f is integrable on $[a, b]$

Hints:

1. Recall the definition of "f is integrable on $[a, b]$".
2. Let P be a partition.
   Order the numbers $U_P(f)$, $L_P(f)$, $\overline{I_a^b}(f)$, $\underline{I_a^b}(f)$.
   (Draw a picture of these numbers in the real line.)

The "ε–characterization" of integrability - Part 2

True or False?

Let f be a bounded function on $[a, b]$.

- IF f is integrable on $[a, b]$
- THEN "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$ s.t. $U_P(f) - L_P(f) < \varepsilon$".

Hints: Assume f is integrable on $[a, b]$. Let I be the integral. Fix $\varepsilon > 0$.

1. Recall the definition of "f is integrable on $[a, b]$".
2. There exist a partition P_1 s.t. $U_{P_1}(f) < I + \dfrac{\varepsilon}{2}$. Why?
3. There exist a partition P_2 s.t. $L_{P_2}(f) > I - \dfrac{\varepsilon}{2}$. Why?
4. What can you say about $U_{P_1}(f) - L_{P_2}(f)$?
5. Construct a partition P s.t. $L_{P_2}(f) \leq L_P(f) \leq U_P(f) \leq U_{P_1}(f)$.

Example 1: a constant function

Consider the function $f(x) = 2$ on $[0, 4]$.

1. Given $P = \{0, 1, e, \pi, 4\}$, compute $L_P(f)$ and $U_P(f)$.
2. Explicitly compute *all* the upper sums and *all* the lower sums.
3. Compute $\overline{I_0^4}(f)$
4. Compute $\underline{I_0^4}(f)$
5. Is f integrable on $[0, 4]$?

Example 2: a non-continuous function

Consider the function $f(x) = \begin{cases} 0 & x = 0 \\ 5 & 0 < x \leq 1 \end{cases}$, defined on $[0, 1]$.

1. Let $P = \{0, 0.2, 0.5, 0.9, 1\}$.
   Calculate $L_P(f)$ and $U_P(f)$ for this partition.
2. Fix an arbitrary partition $P = \{x_0, x_1, \dots, x_N\}$ of $[0, 1]$.
   What is $U_P(f)$? What is $L_P(f)$? (Draw a picture!)
3. Find a partition P with exactly 3 points (2 subintervals) such that $L_P(f) = 4.99$.
4. What is the upper integral, $\overline{I_0^1}(f)$?
5. What is the lower integral, $\underline{I_0^1}(f)$?
6. Is f integrable on $[0, 1]$?

Example 3: a very non-continuous function

Consider the function f defined on $[0,1]$:

$$f(x) = \begin{cases} 1/2 & \text{if } 0 \leq x \leq 1/2 \\ 1 & \text{if } 1/2 < x \leq 1 \text{ and } x \in \mathbb{Q} \\ 0 & \text{if } 1/2 < x \leq 1 \text{ and } x \notin \mathbb{Q} \end{cases}$$

1. Draw a picture!
2. Let $P = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. Calculate $L_P(f)$ and $U_P(f)$.
3. Construct a partition P such that $L_P(f) = \dfrac{1}{4}$ and $U_P(f) = \dfrac{3}{4}$
4. What is the upper integral, $\overline{I_0^1}(f)$?
5. What is the lower integral, $\underline{I_0^1}(f)$?
6. Is f integrable on $[0,1]$?

Sum of non-integrable functions

Find bounded functions f and g on $[0, 1]$ such that

- f is non-integrable on $[0, 1]$,
- g is non-integrable on $[0, 1]$,
- $f + g$ is integrable on $[0, 1]$.

or prove this is impossible.

Properties of the integral

Assume we know the following

$$\int_0^2 f(x)dx = 3, \qquad \int_0^4 f(x)dx = 9, \qquad \int_0^4 g(x)dx = 2.$$

Compute:

1. $\displaystyle\int_0^2 f(t)dt$
2. $\displaystyle\int_0^2 f(x)dx$
3. $\displaystyle\int_0^2 f(t)dx$
4. $\displaystyle\int_2^0 f(x)dx$
5. $\displaystyle\int_2^4 f(x)dx$
6. $\displaystyle\int_{-2}^0 f(x)dx$
7. $\displaystyle\int_0^4 [f(x) - 2g(x)]\ dx$

The norm of a partition

1. Construct a partition P of $[0, 1]$ such that $||P|| = \dfrac{\pi}{10}$.
2. Construct a sequence of partitions of $[0, 1]$

   $$P_1, P_2, P_3, \ldots$$

   as simple as possible, such that $\lim_{n\to\infty} ||P_n|| = 0$.
3. Construct a *different* sequence of partitions of $[0, 1]$

   $$Q_1, Q_2, Q_3, \ldots$$

   such that $\lim_{n\to\infty} ||Q_n|| = 0$.

Compute $\int_1^2 x^2 dx$ using Riemann sums

Let $f(x) = x^2$ on $[1,2]$. Let P_n be the partition that breaks $[1,2]$ into n subintervals of equal length.

1. Write a explicit formula for P_n.
2. What is Δx_i?
3. Write the Riemann sum $S^*_{P_n}(f)$ with sigma notation (choose x_i^* as the right endpoint).
4. Add the sum
5. Compute $\lim_{n\to\infty} S^*_{P_n}(f)$.
6. Repeat the last 3 questions when we choose x_i^* as the left endpoint.

Helpful identities: $$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}, \qquad \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$$

Riemann sums backwards

Interpret the following limits as integrals:

1. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \sin \frac{i}{n}$
2. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{n+i}{n^2}$

Hint: Let f be a continuous function on $[0, 1]$. Write a formula for $\displaystyle\int_0^1 f(x)dx$ as a limit of Riemann sums, making the simplest choices you can.

The Mean Value Theorem for integrals

Prove the following theorem.

Theorem

Let $a < b$. Let f be a continuous function on $[a, b]$.
There exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t)dt$$

Hints:

1. Compute $L_P(f)$ and $U_P(f)$ for the partition $P = \{a, b\}$.
2. Use that $L_P(f) \leq \int_a^b f(t)dt \leq U_P(f)$ to prove that

$$??? \leq \frac{1}{b-a} \int_a^b f(t)dt \leq ???$$

3. Use EVT and IVT.

Initial Value Problem

Find a function f such that

- For every $x \in \mathbb{R}$, $f''(x) = \sin x + x^2$,
- $f'(0) = 5$,
- $f(0) = 7$.

The most misunderstood antiderivative

1. Find the *domain* and the derivative of $F_1(x) = \ln x$
2. Find the *domain* and the derivative of $F_2(x) = \ln(-x)$
3. Find the *domain* and the derivative of $F_3(x) = \ln|x|$
   Suggestion: Break the domain into two pieces.
4. Based on your answers, what is $\displaystyle\int \frac{1}{x}\,dx$?
5. Find the *domain* and the derivative of $F_4(x) = \ln|2x|$
   Why doesn't this contradict your answer to 4 ?

Compute these antiderivatives by guess 'n check

1. $\int x^5 dx$
2. $\int \left(3x^8 - 18x^5 + 1\right) dx$
3. $\int \sqrt[3]{x}\, dx$
4. $\int \frac{1}{x^9}\, dx$
5. $\int \sqrt{x}\left(x^2 + 5\right) dx$
6. $\int \frac{1}{e^{2x}}\, dx$
7. $\int \sin(3x)\, dx$
8. $\int \cos(3x + 2)\, dx$
9. $\int \sec^2 x\, dx$
10. $\int \sec x \tan x\, dx$
11. $\int \frac{1}{x}\, dx$
12. $\int \frac{1}{x+3}\, dx$

Integration by parts 1

1. $\dfrac{d}{dx}[x \sin x] =$
2. $\dfrac{d}{dx}[\cos x] =$

Use the previous answers to calculate

3. $\displaystyle\int x \cos x \, dx =$

Integration by parts 2

1. $\dfrac{d}{dx}\left[xe^x\right] =$

2. ???

3. $\displaystyle\int xe^x\,dx =$

Integration by parts 3

1. ???
2. ???
3. $\displaystyle\int xe^{-x}\,dx =$

Integration by parts 4

1. $\dfrac{d}{dx}\left[x^2 e^x\right] =$

2. $\dfrac{d}{dx}[xe^x] =$

3. ???

4. $\displaystyle\int x^2 e^x \, dx =$

Trig-exp antiderivatives

1. $\dfrac{d}{dx}\left[e^x \sin x\right] =$
2. $\dfrac{d}{dx}\left[e^x \cos x\right] =$

Use the previous answers to calculate:

3. $\displaystyle\int e^x \sin x \, dx =$
4. $\displaystyle\int e^x \cos x \, dx =$

A challenge for guess-and-check ninjas

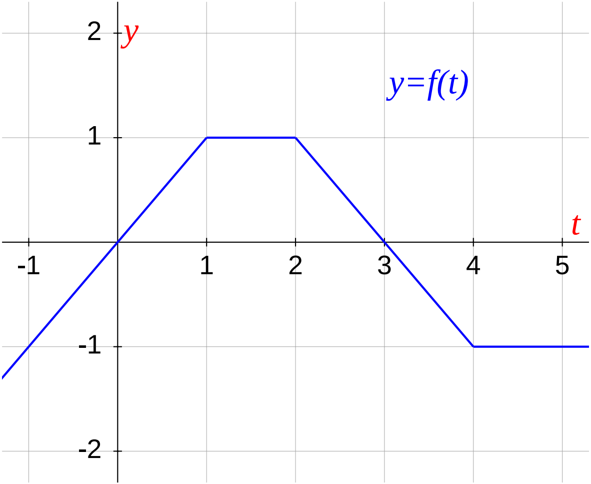
$$\int x\, e^x \cos x \, dx \;=\; ???$$

Functions defined by integrals

Which ones of these are valid ways to define functions?

1. $F(x) = \int_0^x \frac{t}{1+t^8}\,dt$
2. $F(x) = \int_0^x \frac{x}{1+x^8}\,dx$
3. $F(x) = \int_0^x \frac{x}{1+t^8}\,dt$
4. $F(x) = \int_0^{x^2} \frac{t}{1+t^8}\,dt$
5. $F(x) = \int_{\sin x}^{e^x} \frac{t}{1+t^8}\,dt$
6. $F(x) = \int_0^3 \frac{t}{1+x^2+t^8}\,dt$
7. $F(x) = x\int_{\sin x}^{e^x} \frac{t}{1+x^2+t^8}\,dt$
8. $F(x) = t\int_{\sin x}^{e^x} \frac{t}{1+x^2+t^8}\,dt$

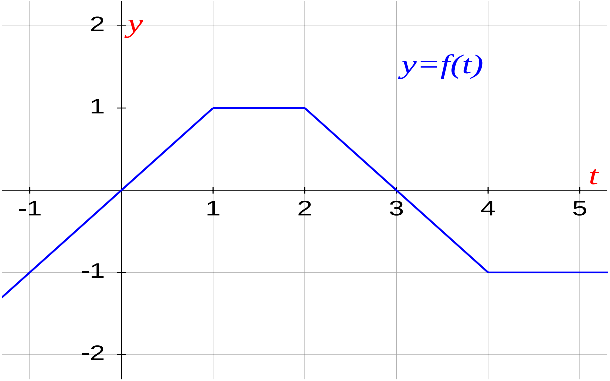
Towards FTC



Compute:

1. $\displaystyle\int_0^1 f(t)dt$
2. $\displaystyle\int_0^2 f(t)dt$
3. $\displaystyle\int_0^3 f(t)dt$
4. $\displaystyle\int_0^4 f(t)dt$
5. $\displaystyle\int_0^5 f(t)dt$

Towards FTC (continued)



Call $F(x) = \int_0^x f(t)dt$. This is a new function.

- Sketch the graph of $y = F(x)$.
- Using the graph you just sketched, sketch the graph of $y = F'(x)$.

Filling the tank

A tank is being filled with water. At time t water flows into the tank at a rate of

$$A\, e^{-bt} \arctan(ct)$$

litres per second, where A, b, and c are constants. The amount of water in the tank at time $t = 0s$ is V_0. Write an expression for the amount of water V in the tank at time t.

True or False?

1. If f is continuous on the interval $[a, b]$, then

$$\frac{d}{dx}\left(\int_a^b f(t)dt\right) = f(x).$$

2. If f is differentiable, then

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = \int_a^x f'(t)\,dt.$$

More True or False

Let f and g be differentiable functions with domain $\mathbb{R}$.
Assume that $f'(x) = g(x)$ for all x.
Which of the following statements must be true?

1. $f(x) = \int_0^x g(t)dt$.
2. If $f(0) = 0$, then $f(x) = \int_0^x g(t)dt$.
3. If $g(0) = 0$, then $f(x) = \int_0^x g(t)dt$.
4. There exists $C \in \mathbb{R}$ such that $f(x) = C + \int_0^x g(t)dt$.
5. There exists $C \in \mathbb{R}$ such that $f(x) = C + \int_1^x g(t)dt$.

True, False, or Shrug?

We want to find a function H with domain $\mathbb{R}$ such that $H(1) = -2$ and such that $H'(x) = e^{\sin x}$ for all x. Decide whether each of the following statements is true, false, or we do not have enough information to decide.

1. The function $H(x) = \int_0^x e^{\sin t} dt$ is a solution.
2. The function $H(x) = \int_2^x e^{\sin t} dt$ is a solution.
3. $\forall C \in \mathbb{R}$, the function $H(x) = \int_0^x e^{\sin t} dt + C$ is a solution.
4. $\exists C \in \mathbb{R}$ s.t. the function $H(x) = \int_0^x e^{\sin t} dt + C$ is a solution.
5. The function $H(x) = \int_1^x e^{\sin t} dt - 2$ is a solution.
6. There is more than one solution.

Examples of FTC-1

Compute the derivative of the following functions

1. $F_1(x) = \displaystyle\int_0^1 e^{-t^2} dt.$

2. $F_2(x) = \displaystyle\int_0^x e^{-\sin t} dt.$

3. $F_3(x) = \displaystyle\int_1^{x^2} \frac{\sin t}{t^2} dt.$

4. $F_4(x) = \displaystyle\int_x^7 \sin^3\left(\sqrt{t}\right) dt.$

5. $F_5(x) = \displaystyle\int_{2x}^{x^2} \frac{1}{1+t^3} dt.$

A generalized version of FTC-1

Let f, u, v be differentiable functions with domain $\mathbb{R}$.
Let us call

$$F(x) = \int_{u(x)}^{v(x)} f(t)dt$$

Find a formula for

$$F'(x)$$

in terms of f, u, v, f', u', v'.

An integral equation

Assume f is a continuous function that satisfies, for every $x \in \mathbb{R}$:

$$\int_0^x e^t f(t) = \frac{\sin x}{x^2+1}$$

Find an explicit expression for $f(x)$.

Compute these definite integrals

1. $\displaystyle\int_{1}^{2} x^3 dx$

2. $\displaystyle\int_{0}^{1} \left[e^x + e^{-x} - \cos(2x)\right] dx$

3. $\displaystyle\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}} dx$

4. $\displaystyle\int_{\pi/4}^{\pi/3} \sec^2 x \ dx$

5. $\displaystyle\int_{1}^{2} \left[\frac{d}{dx}\left(\frac{\sin^2 x}{1 + \arctan^2 x + e^{-x^2}}\right)\right] dx$

Find the error

$$\int_{-1}^{1} \frac{1}{x^4} dx \; = \; \left. \frac{-1}{3x^3} \right|_{-1}^{1} = \frac{-2}{3}$$

However, x^4 is always positive, so the integral should be positive.

Areas

Calculate the area of the bounded region...

1. ... between the x-axis and $y = 4x - x^2$.
2. ... between $y = \cos x$, the x-axis, from $x = 0$ to $x = \pi$.
3. ... between $y = x^2 + 3$ and $y = 3x + 1$.
4. ... between $y = 1$, the y-axis, and $y = \ln(x + 1)$.

Minimizing area

For each $a > 0$ consider the function

$$f_a(x) = 1 + a - ax^2$$

Find the value of a that minimizes the area of the region bounded by the graph of f_a and the x-axis.

desmos

Symmetry

Calculate the value of these integrals *without computing any antiderivative*.

1. $\int_{-2}^{2} \sin x^3 dx$ 2. $\int_{0}^{\pi} \cos^2 x \, dx$ 3. $\int_{-1}^{1} \arccos x \, dx$

Hint: Sketch the graphs (use desmos) and use symmetry to compute the integral.
Once you guess the symmetry of the graph, try to write it algebraically.

▸ 1 ▸ 2 ▸ 3

Average Velocity

You are traveling.
Your position at time t is $s(t)$.
Your velocity at time t is $v(t)$.
The function v is continuous on an interval $[a, b]$.
Which of the following represent your average velocity on $[a, b]$?

1. $\dfrac{s(b) - s(a)}{b - a}$
2. $\dfrac{1}{b - a} \displaystyle\int_a^b v(t)dt$
3. $v(c)$ for at least one c between a and b

The Mean Value Theorem for integrals is back

Prove the following theorem.

Theorem

Let $a < b$. Let f be a continuous function on $[a, b]$.
There exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t)dt$$

Hint: Use MVT for the function $F(x) = \displaystyle\int_a^x f(t)dt$.

Warm up

Calculate

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$$

Hint: Use the substitution $u = \sqrt{x}$.

Computation practice: integration by substitution

Use substitutions to compute:

1. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$
2. $\displaystyle\int e^x \cos\left(e^x\right) dx$
3. $\displaystyle\int \cot x \, dx$
4. $\displaystyle\int x^2\sqrt{x+1}\,dx$
5. $\displaystyle\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$
6. $\displaystyle\int \frac{(\ln\ln x)^2}{x\ln x}\,dx$
7. $\displaystyle\int x e^{-x^2}\,dx$
8. $\displaystyle\int e^{-x^2}\,dx$

Definite integral via substitution

This final answer is right, but the write-up is WRONG. Why?

Calculate $I = \int_0^2 \sqrt{x^3+1}\, x^2 dx$

Wrong answer

Substitution: $u = x^3 + 1,\ du = 3x^2 dx.$

$$I = \frac{1}{3}\int_0^2 \sqrt{x^3+1}\,(3x^2dx) \qquad = \frac{1}{3}\int_0^2 u^{1/2}\, du$$

$$= \frac{1}{3}\,\frac{2}{3}\ u^{3/2}\Big|_0^2 \qquad = \frac{1}{9}\left(x^3+1\right)^{2/3}\Big|_0^2$$

$$= \frac{2}{9}\left(2^3+1\right)^{3/2} - \frac{2}{9}\left(0+1\right)^{3/2} \qquad = \frac{52}{9}$$

Integral of products of sin and cos

We want to compute

$$I = \int \sin^3 x \cos^2 x \, dx$$

1. Attempt the substitution $u = \sin x$
2. Attempt the substitution $u = \cos x$
3. One worked better than the other. Which one? Why? Finish the problem.
4. Assume we want to compute

$$\int \sin^n x \cos^m x \, dx$$

When will the substitution $u = \sin x$ be helpful?
When will the substitution $u = \cos x$ be helpful?

Odd functions

Theorem

Let f be a continuous function. Let $a > 0$. IF f is odd, THEN

$$\int_{-a}^{a} f(x)dx = 0$$

1. Write down the definition of "odd function".
2. Draw a picture to interpret the theorem geometrically.
3. Prove the theorem!
   Hint: Write the integral as sum of two pieces. Use a substitution to show that one of the two pieces equals minus the other.

Computation practice: Integration by parts

Use integration by parts (possibly in combination with other methods) to compute:

1. $\displaystyle\int xe^{-2x}dx$
2. $\displaystyle\int x^2 \sin x \, dx$
3. $\displaystyle\int \ln x \, dx$
4. $\displaystyle\int \sin \sqrt{x} \, dx$
5. $\displaystyle\int x \arctan x \, dx$
6. $\displaystyle\int x^2 \arcsin x \, dx$
7. $\displaystyle\int e^{\cos x} \sin^3 x \, dx$
8. $\displaystyle\int e^{ax} \sin(bx) dx$

Persistence

Compute

- $\displaystyle\int_1^e (\ln x)^3\,dx$
- $\displaystyle\int_1^e (\ln x)^{10}\,dx$

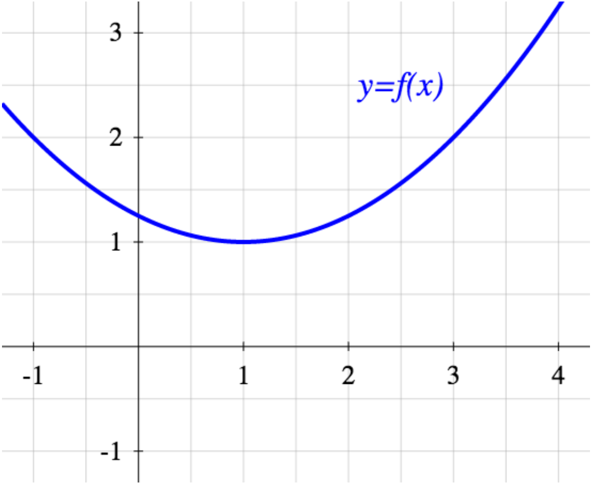
There is a more efficient approach. Call

$$I_n = \int_1^e (\ln x)^n\,dx$$

Use integration by parts on I_n. You will get an equation with I_n and I_{n-1}. Now solve the previous questions.

OEIS

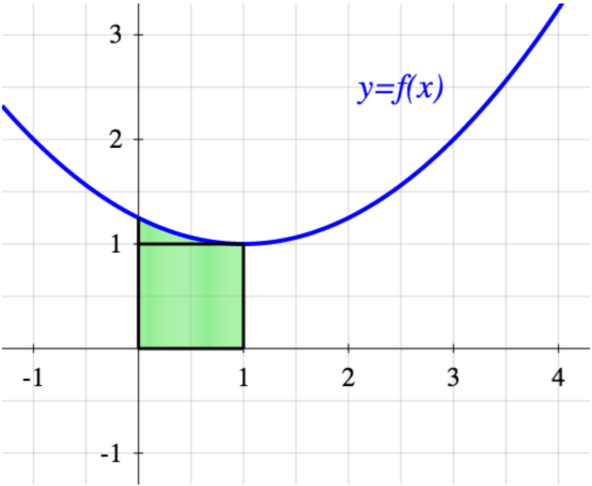
Integrals from a graph



Estimate:

1. $\displaystyle\int_0^1 f(x)dx$
2. $\displaystyle\int_0^1 f'(x)dx$
3. $\displaystyle\int_0^3 x\, f'(x)dx$
4. $\displaystyle\int_0^1 f(3x)dx$

Integrals from a graph



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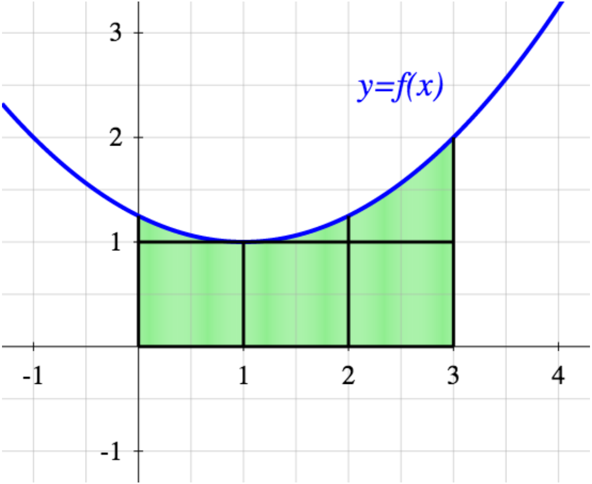
Integrals from a graph



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4. $\displaystyle\int_0^1 f(3x)dx$

The error function

The following function is tabulated.

$$E(x) = \int_0^x e^{-t^2} dt.$$

Write the following quantities in terms of E:

1. $\displaystyle\int_1^2 e^{-t^2} dt$
2. $\displaystyle\int_0^x t^2 e^{-t^2} dt$
3. $\displaystyle\int_0^x e^{-2t^2} dt$
4. $\displaystyle\int_0^1 e^{-t^2+6t} dt$
5. $\displaystyle\int_{x_1}^{x_2} e^{-\frac{(t-\mu)^2}{\sigma^2}} dt$
6. $\displaystyle\int_1^2 \frac{e^{-t}}{\sqrt{t}} dt$

Exp-trig antiderivative

We want to compute

$$I = \int e^{ax} \sin(bx)\, dx$$

- Try once integration by parts choosing $u = e^{ax}$. Stop.
- Go back to I. Now try integration by parts once choosing $u = \sin(bx)$ instead. Stop.
- Look at what you did. Think.

Practice: Integrals with trigonometric functions

Compute the following antiderivatives. (Once you get them to a form from where you see a path to finish them, even if long, you may stop.)

1. $\displaystyle\int \sin^{10} x \cos x \; dx$
2. $\displaystyle\int \sin^{10} x \cos^7 x \, dx$
3. $\displaystyle\int e^{\cos x} \cos x \sin^3 x \, dx$
4. $\displaystyle\int \cos^2 x \, dx$
5. $\displaystyle\int \cos^4 x \, dx$
6. $\displaystyle\int \csc x \, dx$

Useful trig identities

$$\sin^2 x + \cos^2 x = 1 \qquad \sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\tan^2 x + 1 = \sec^2 x \qquad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

Integral of products of secant and tangent

To integrate

$$\int \sec^n x \tan^m x \, dx$$

- If $\boxed{}$, then use the substitution $u = \tan x$.
- If $\boxed{}$, then use the substitution $u = \sec x$.

Hint: You will need

- $\frac{d}{dx}[\tan x] = \ldots$
- $\frac{d}{dx}[\sec x] = \ldots$

- The trig identity involving $\sec$ and $\tan$

A reduction formula

Let $I_n = \displaystyle\int_0^{2\pi} \sin^n x \, dx$.

1. Compute I_0 and I_1.
2. Write an equation for I_n in terms of I_{n-2}. This is called a reduction formula.

   Hint: Starting with I_n, use integration by parts once. Then use $\sin^2 x + \cos^2 x = 1$ to rewrite the new integral in terms of I_n and I_{n-2}.
3. Write a a formula for I_n for all natural numbers n.

A different kind of substitution

Calculate

$$\int_0^1 \sqrt{1-x^2}\, dx$$

using the substitution

$$\begin{cases} x = \sin\theta \\ dx = ?? \end{cases}$$

Rational integrals

1. Calculate $\displaystyle\int \frac{1}{x+a}\,dx$
2. Reduce to common denominator $\displaystyle\frac{2}{x} - \frac{3}{x+3}$
3. Calculate $\displaystyle\int \frac{-x+6}{x^2+3x}\,dx$
4. Calculate $\displaystyle\int \frac{1}{x^2+3x}\,dx$
5. Calculate $\displaystyle\int \frac{1}{x^3-x}\,dx$

Repeated factors

1. Calculate $\displaystyle\int \frac{1}{(x+1)^n}\,dx \quad \text{for } n > 1$
2. Calculate $\displaystyle\int \frac{(x+1)-1}{(x+1)^2}\,dx$
3. Calculate $\displaystyle\int \frac{2x+6}{(x+1)^2}\,dx$
4. Calculate $\displaystyle\int \frac{x^2}{(x+1)^3}\,dx$

Irreducible quadratics

1. Calculate $\displaystyle\int \frac{1}{x^2+1}\,dx$ and $\displaystyle\int \frac{x}{x^2+1}\,dx$.

   Hint: These two are very short.

2. Calculate $\displaystyle\int \frac{2x+3}{x^2+1}\,dx$

3. Calculate $\displaystyle\int \frac{x^2}{x^2+1}\,dx$

4. Calculate $\displaystyle\int \frac{x}{x^2+x+1}\,dx$

   Hint: Complete the square in the denominator and use a substitution to transform into one of the previous ones.

Repeated quadratics

1. Calculate

$$\frac{d}{dx}\left[\arctan x\right], \qquad \frac{d}{dx}\left[\frac{x}{1+x^2}\right].$$

2. Use the previous answer to calculate

$$\int \frac{1}{\left(1+x^2\right)^2}\,dx$$

The integral of secant

Compute

$$\int \sec x \, dx$$

using the substitution $u = \sin x$.

Messier rational functions

1. How could we compute an integral of the form

$$\int \frac{\text{polynomial}}{(x+1)^3(x+2)}\, dx \ ?$$

2. How could we compute an integral of the form

$$\int \frac{\text{polynomial}}{x^4(x+1)^3(x+2)(x^2+1)(x^2+4)}\, dx \ ?$$

An equation for volumes by the carrot method

Let $a < b$.
Let f be a continuous, positive function defined on $[a, b]$.
Let R be the region in the first quadrant bounded between the graph of f and the x-axis.
Find a formula for the volume of the solid of revolution obtained by rotation the region R around the x-axis.

Sphere

You know the formula for the volume of a sphere with radius R. Now you are able to prove it!

1. Write an equation for the circle with radius R centered at $(0, 0)$.
2. If you rotate this circle around the x-axis, it will produce a sphere. Compute its volume as an integral by slicing it like a carrot.

Pyramid

Compute the volume of a pyramid with height H and square base with side length L.
Hint: Slice the pyramid like a carrot with cuts parallel to the base.

Many axis of rotation

Let R be the region in the first quadrant bounded between the curves with equations $y = x^3$ and $y = \sqrt{32x}$.
Compute the volume of the solid of revolution obtained by rotating R around...

1. ... the x-axis
2. ... the y-axis
3. ... the line $y = -1$

An equation for volumes by "cylindrical shells"

Let $a < b$.
Let f be a continuous, positive function defined on $[a, b]$.
Let R be the region in the first quadrant bounded between the graph of f and the x-axis.
Find a formula for the volume of the solid of revolution obtained by rotation the region R around the y-axis.

A hat

Let R be the region in the first quadrant bounded between the graphs of $y = x^5 + x - 2$, $x = 2$, and the x-axis.
Compute the volume of the solid of revolution obtained by rotating R around the y-axis.

Doughnut

Let R be the region inside the curve with equation

$$(x-1)^2 + y^2 = 1.$$

Rotate R around the line with equation $x = 4$. The resulting solid is called a *torus*.

1. Draw a picture and convince yourself that a torus looks like a doughnut.
2. Set up the volume of the torus as an integral using x as the variable ("cylindrical shell method"). You do not need to compute the integral.
3. Set up the volume of the torus as an integral using y as the variable ("carrot method"). You do not need to compute the integral.

Challenge

Two cylinders have the same radius R and their axes are perpendicular. Find the volume of their intersection.
Hint: You can slice the resulting solid by parallel cuts in three different directions. One of the three makes the problem much, much simpler than the other two.

Warm up

Write a formula for the general term of these sequences

1. $\{a_n\}_{n=0}^{\infty} = \{\, 1,\ 4,\ 9,\ 16,\ 25,\ \ldots \,\}$

2. $\{b_n\}_{n=1}^{\infty} = \{\, 1,\ -2,\ 4,\ -8,\ 16,\ -32,\ \ldots \,\}$

3. $\{c_n\}_{n=1}^{\infty} = \left\{\, \frac{2}{1!},\ \frac{3}{2!},\ \frac{4}{3!},\ \frac{5}{4!},\ \ldots \,\right\}$

4. $\{d_n\}_{n=1}^{\infty} = \{\, 1,\ 4,\ 7,\ 10,\ 13,\ \ldots \,\}$

Sequences vs functions – convergence

For any function f with domain $[0, \infty)$,
we define a sequence as $a_n = f(n)$.
Let $L \in \mathbb{R}$. Which of these implications is true?

1. IF $\lim_{x\to\infty} f(x) = L$, THEN $\lim_{n\to\infty} a_n = L$.

2. IF $\lim_{n\to\infty} a_n = L$, THEN $\lim_{x\to\infty} f(x) = L$.

3. IF $\lim_{n\to\infty} a_n = L$, THEN $\lim_{n\to\infty} a_{n+1} = L$.

Definition of limit of a sequence

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$.
Which statements are equivalent to "$\{a_n\}_{n=0}^{\infty} \longrightarrow L$"?

1. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
2. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n > n_0 \implies |L - a_n| < \varepsilon.$
3. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{R},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
4. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{R}, \quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
5. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| \leq \varepsilon.$
6. $\forall \varepsilon \in (0,1),\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
7. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < \dfrac{1}{\varepsilon}.$
8. $\forall k \in \mathbb{Z}^+,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < k.$
9. $\forall k \in \mathbb{Z}^+,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies |L - a_n| < \dfrac{1}{k}.$

Definition of limit of a sequence (continued)

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$.
Which statements are equivalent to "$\{a_n\}_{n=0}^{\infty} \longrightarrow L$"?

10. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains all the elements of the sequence, except the first few.
11. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains infinitely many of the elements of the sequence.
12. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains *almost all* the elements of the sequence.
13. $\forall \varepsilon > 0$, the interval $[L - \varepsilon, L + \varepsilon]$ contains *almost all* the elements of the sequence.
14. Every interval that contains L must contain *almost all* all the elements of the sequence.
15. Every open interval that contains L must contain *almost all* all the elements of the sequence.

Notation: "*almost all*" = "all, except finitely many"

Convergence and divergence

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence.
Write the formal definition of the following concepts:

1. $\{a_n\}_{n=0}^{\infty}$ is convergent.

2. $\{a_n\}_{n=0}^{\infty}$ is divergent.

3. $\{a_n\}_{n=0}^{\infty}$ is divergent to ∞.

Proof from the definition of limit

Prove, directly from the definition of limit, that

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1.$$

Suggestion:

1. Write down the definition of what you want to show.
2. Use itto decide the structure of the proof.
3. Do some rough work if necessary.
4. Write down the formal proof.

Sequences vs functions – monotonicity and boundness

For any function f with domain $[0, \infty)$,
we define a sequence as $a_n = f(n)$.
Which of these implications is true?

1. IF f is increasing, THEN $\{a_n\}_{n=0}^{\infty}$ is increasing.
2. IF $\{a_n\}_{n=0}^{\infty}$ is increasing, THEN f is increasing.
3. IF f is bounded, THEN $\{a_n\}_{n=0}^{\infty}$ is bounded.
4. IF $\{a_n\}_{n=0}^{\infty}$ is bounded, THEN f is bounded.

Examples

Construct 8 examples of sequences.
If any of them is impossible, cite a theorem to justify it.

		convergent	divergent
monotonic	bounded		
	unbounded		
not monotonic	bounded		
	unbounded		

A sequence defined by recurrence

Consider the sequence $\{R_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} & R_0 = 1 \\ \forall n \in \mathbb{N}, & R_{n+1} = \dfrac{R_n + 2}{R_n + 3} \end{cases}$$

Compute R_1, R_2, R_3.

Is this proof correct?

Let $\{R_n\}_{n=0}^{\infty}$ be the sequence in the previous slide.

Claim:

$\{R_n\}_{n=0}^{\infty} \longrightarrow -1 + \sqrt{3}$.

Proof.

- Let $L = \lim_{n\to\infty} R_n$.
- $R_{n+1} = \dfrac{R_n + 2}{R_n + 3}$
- $\lim_{n\to\infty} R_{n+1} = \lim_{n\to\infty} \dfrac{R_n + 2}{R_n + 3}$
- $L = \dfrac{L+2}{L+3}$
- $L(L+3) = L+2$
- $L^2 + 2L - 2 = 0$
- $L = -1 \pm \sqrt{3}$
- L must be positive, so $L = -1 + \sqrt{3}$

$\square$

Another sequence defined by recurrence

Consider the sequence $\{a_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} & a_0 = 1 \\ \forall n \in \mathbb{N}, & a_{n+1} = 1 - a_n \end{cases}$$

- Use the same method as in the previous slide to compute its limit.
- **After** you have computed the limit, calculate a_1, a_2, a_3, and a_4.
- What happened?

The original sequence defined by recurrence – done right

Consider the sequence $\{R_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} & R_0 = 1 \\ \forall n \in \mathbb{N}, & R_{n+1} = \dfrac{R_n + 2}{R_n + 3} \end{cases}$$

1. Prove $\{R_n\}_{n=0}^{\infty}$ is bounded below by 0.
2. Prove $\{R_n\}_{n=0}^{\infty}$ is decreasing (use induction)
3. Prove $\{R_n\}_{n=0}^{\infty}$ is convergent (use a theorem)
4. Now the calculation in the earlier slide is correct, and we can get the value of the limit.

True or False - convergence, monotonicity, and boundedness

1. If a sequence is convergent, then it is bounded above.
2. If a sequence is bounded, then it is convergent
3. If a sequence is convergent, then it is eventually monotonic.
4. If a sequence is positive and converges to 0, then it is eventually monotonic.
5. If a sequence diverges to ∞, then it is eventually monotonic.
6. If a sequence diverges, then it is unbounded.
7. If a sequence diverges and is unbounded above, then it diverges to ∞.
8. If a sequence is eventually monotonic, then it is either convergent, divergent to ∞, or divergent to $-\infty$.

True or False - Rapid fire

1. (convergent) $\implies$ (bounded)
2. (convergent) $\implies$ (monotonic)
3. (convergent) $\implies$ (eventually monotonic)
4. (bounded) $\implies$ (convergent)
5. (monotonic) $\implies$ (convergent)
6. (bounded + monotonic) $\implies$ (convergent)
7. (divergent to ∞) $\implies$ (eventually monotonic)
8. (divergent to ∞) $\implies$ (unbounded above)
9. (unbounded above) $\implies$ (divergent to ∞)

Fill in the blanks

Let $\{a_n\}$ be a decreasing, bounded sequence.
Assume $a_1 = 1$ and a_n is never 0.
Let m be the greatest lower bound of $\{a_n\}$.

For each of the statements below, find **all** the values of m that make the statement true.

1. IF $\boxed{}$ THEN $\{1/a_n\}$ is bounded
2. IF $\boxed{}$ THEN $\{1/a_n\}$ is increasing
3. IF $\boxed{}$ THEN $\{\sin a_n\}$ is bounded
4. IF $\boxed{}$ THEN $\{\sin a_n\}$ is decreasing

Proof of Theorem 3

Write a proof for the following Theorem

Theorem 3

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence.

- IF $\{a_n\}_{n=0}^{\infty}$ is increasing AND unbounded above,
- THEN $\{a_n\}_{n=0}^{\infty}$ is divergent to ∞

1. Write the definitions of "increasing", "unbounded above", and "divergent to ∞"
2. Using the definition of what you want to prove, write down the structure of the formal proof.
3. Do some rough work if necessary.
4. Write a formal proof.

Proof feedback

1. Does your proof have the correct structure?
2. Are all your variables fixed (not quantified)? In the right order? Do you know what depends on what?
3. Is the proof self-contained? Or do I need to read the rough work to understand it?
4. Does each statement follow logically from previous statements?
5. Did you explain what you were doing? Would your reader be able to follow your thought process without reading your mind?

Critique this proof - #1

- $\forall M \in \mathbb{R},\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies x_n > M$

- M is not an upper bound: $\exists n_0 \in \mathbb{N}$ s.t. $x_{n_0} > M$

- $n \geq n_0 \implies x_n \geq x_{n_0} > M$

Critique this proof - #2

- WTS $a_n \to \infty$. This means:
  $\forall M \in \mathbb{R},\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \implies x_n > M$

- bounded above: $\exists M \in \mathbb{R},\ \forall n \in \mathbb{N},\ x_n \leq M$

- negation: $\forall M \in \mathbb{R},\ \exists n \in \mathbb{N},\ x_n > M$

- $\forall n \in \mathbb{N}$, take $n = n_0$.

Composition law

Write a proof for the following Theorem

Theorem

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$. Let f be a function.

- IF $\begin{cases} \{a_n\}_{n=0}^{\infty} \longrightarrow L \\ f \text{ is continuous at } L \end{cases}$
- THEN $\{f(a_n)\}_{n=0}^{\infty} \longrightarrow f(L)$.

1. Write the definition of your hypotheses and your conclusion.
2. Using the definition of your conclusion, figure out the structure of the proof.
3. Do some rough work if necessary.
4. Write a formal proof.

Calculations

1. $$\lim_{n\to\infty} \frac{n! + 2e^n}{3n! + 4e^n}$$

2. $$\lim_{n\to\infty} \frac{2^n + (2n)^2}{2^{n+1} + n^2}$$

3. $$\lim_{n\to\infty} \frac{5n^5 + 5^n + 5n!}{n^n}$$

True or False – The Big Theorem

Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be positive sequences.

1. IF $a_n \ll b_n$, THEN $\forall m \in \mathbb{N},\ a_m < b_m$.

2. IF $a_n \ll b_n$, THEN $\exists m \in \mathbb{N}$ s.t. $a_m < b_m$.

3. IF $a_n \ll b_n$, THEN $\exists n_0 \in \mathbb{N}$ s.t.
   $\forall m \in \mathbb{N},\ m \geq n_0 \implies a_m < b_m$.

4. IF $\forall m \in \mathbb{N},\ a_m < b_m$, THEN $a_n \ll b_n$.

5. IF $\exists m \in \mathbb{N}$ s.t. $a_m < b_m$, THEN $a_n \ll b_n$.

6. IF $\exists n_0 \in \mathbb{N}$ s.t. $\forall m \in \mathbb{N},\ m \geq n_0 \implies a_m < b_m$,
   THEN $a_n \ll b_n$.

Refining the Big Theorem - 1

1. Construct a sequence $\{u_n\}_n$ such that

$$\begin{cases} \forall a < 0, & n^a \ll u_n \\ \forall a \geq 0, & u_n \ll n^a \end{cases}$$

2. Construct a sequence $\{v_n\}_n$ such that

$$\begin{cases} \forall a \leq 0, & n^a \ll v_n \\ \forall a > 0, & v_n \ll n^a \end{cases}$$

Refining the Big Theorem - 2

1. Construct a sequence $\{u_n\}_n$ such that

$$\begin{cases} \forall a < 2, & n^a \ll u_n \\ \forall a \geq 2, & u_n \ll n^a \end{cases}$$

2. Construct a sequence $\{v_n\}_n$ such that

$$\begin{cases} \forall a \leq 2, & n^a \ll v_n \\ \forall a > 2, & v_n \ll n^a \end{cases}$$

True or False - Review

1. If $\{a_n\}_{n=0}^{\infty}$ diverges and is increasing, then $\exists n \in \mathbb{N}$ s.t. $a_n > 100$.
2. If $\lim\limits_{n\to\infty} a_n = L$, then $\forall n \in \mathbb{N}$, $a_n < L + 1$.
3. If $\lim\limits_{n\to\infty} a_n = L$, then $\exists n \in \mathbb{N}$ s.t. $a_n < L + 1$.
4. If $\lim\limits_{n\to\infty} a_n = L$, then $\exists \varepsilon > 0$ s.t. $\forall n \in \mathbb{N}$, $a_n < L + \varepsilon$.
5. If $\{a_n\}_{n=0}^{\infty}$ is convergent and $b_n = a_n$ for *almost all* $n \in \mathbb{N}$, then $\{b_n\}_{n=0}^{\infty}$ is convergent.
6. If $a_n \ll b_n$, then $\exists n \in \mathbb{N}$ s.t. $a_n < b_n$.
7. If $a_n \ll b_n$, then $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ s.t. $a_n < \varepsilon b_n$.
8. If $a_n \ll b_n$, then $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $n \geq n_0 \implies a_n < \varepsilon b_n$,

Recall the definitions

1. **Type-1 improper integrals.** Let f be a bounded, continuous function on $[c, \infty)$. How do we define the improper integral

$$\int_c^\infty f(x)dx\,?$$

2. **Type-2 improper integrals.** Let f be a continuous function on $(a, b]$, possibly with $x = a$ as a vertical asymptote. How do we define the improper integral

$$\int_a^b f(x)dx\,?$$

Computation

Calculate, using the definition of improper integral

$$\int_1^\infty \frac{1}{x^2+x}dx$$

Hint: $\dfrac{1}{x^2+x} = \dfrac{(x+1)-(x)}{x(x+1)}$

The most important improper integrals

Use the definition of improper integral to determine for which values of $p \in \mathbb{R}$ each of the following improper integrals converges.

1. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$

2. $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$

3. $\displaystyle\int_0^\infty \frac{1}{x^p}\,dx$

Quick review

For which values of $p \in \mathbb{R}$ is each of the following improper integrals convergent?

1. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$

2. $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$

3. $\displaystyle\int_0^\infty \frac{1}{x^p}\,dx$

Examples

1. Let f be continuous on $[a, \infty)$. Let $A = \displaystyle\int_a^\infty f(x)dx$

   $$\text{Then } A \text{ may be } \begin{cases} \text{convergent (a number)} \\ \text{divergent} \begin{cases} \text{to } \infty \\ \text{to } -\infty \\ \text{“oscillating”} \end{cases} \end{cases}$$

   Give one example of each of the four results.

2. Now do the same thing for “type 2” improper integrals.

Positive functions

- Let f be continuous on $[a, \infty)$. Let $A = \displaystyle\int_a^\infty f(x)dx$

$$\text{Then } A \text{ may be } \begin{cases} \text{convergent (a number)} \\ \text{divergent} \begin{cases} \text{to } \infty \\ \text{to } -\infty \\ \text{“oscillating”} \end{cases} \end{cases}$$

- Assume $\forall x \geq a, f(x) \geq 0$.

$\boxed{\text{Which of the four options are still possible?}}$

- Assume $\exists M \geq a$, s.t. $\forall x \geq M, f(x) \geq 0$.

$\boxed{\text{Which of the four options are still possible?}}$

A “simple” integral

What is $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx$?

1. $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx \;=\; (\ln|x|)\Big|_{-1}^{1} \;=\; \ln|1| - \ln|-1| = 0$

2. $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx = 0$ because $f(x) = \dfrac{1}{x}$ is an odd function.

3. $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx$ is divergent.

What is wrong with this computation?

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{x}\,dx &= \lim_{\varepsilon\to 0^+} \left[\int_{-1}^{-\varepsilon} \frac{1}{x}\,dx + \int_{\varepsilon}^{1} \frac{1}{x}\,dx \right] \\
&= \lim_{\varepsilon\to 0^+} \left[\ln|x| \Big|_{-1}^{-\varepsilon} + \ln|x| \Big|_{\varepsilon}^{1} \right] \\
&= \lim_{\varepsilon\to 0^+} \left[\ln|-\varepsilon| - \ln|\varepsilon| \right] \\
&= \lim_{\varepsilon\to 0^+} [\,0\,] = 0
\end{aligned}$$

Probability

A nonnegative function f defined on $(-\infty, \infty)$ is called a **probability density function** if

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

The *mean* of a probability density function is defined as

$$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx.$$

Let $f(x) = \begin{cases} Ce^{-kx} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

1. For $k > 0$, find a constant C such that the function f is a probability density function.
2. Calculate the mean μ.

Collection of antiderivatives

Let f be a positive, continuous function with domain $\mathbb{R}$.
We know two ways to describe a collection of antiderivatives:

1. $G(x) + C$ for $C \in \mathbb{R}$, where G is any one antiderivative.
2. The collection of functions F_a for $a \in \mathbb{R}$, where

$$F_a(x) = \int_a^x f(t)dt$$

These two collections are not always the same. Why not? Are they the same for some functions f? When are they the same?

Hint: https://tinyurl.com/137antiderivatives

A simple BCT application

We want to determine whether $\displaystyle\int_1^\infty \frac{1}{x+e^x}\,dx$ is convergent or divergent.

We can try at least two comparisons:

1. Compare $\dfrac{1}{x}$ and $\dfrac{1}{x+e^x}$.
2. Compare $\dfrac{1}{e^x}$ and $\dfrac{1}{x+e^x}$.

Try both. What can you conclude from each one of them?

True or False - Comparisons

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, \infty)$.
Assume that $\boxed{\forall x \geq a, \quad 0 \leq f(x) \leq g(x)}$.
What can we conclude?

1. IF $\int_a^{\infty} f(x)dx$ is convergent, THEN $\int_a^{\infty} g(x)dx$ is convergent.
2. IF $\int_a^{\infty} f(x)dx = \infty$, THEN $\int_a^{\infty} g(x)dx = \infty$.
3. IF $\int_a^{\infty} g(x)dx$ is convergent, THEN $\int_a^{\infty} f(x)dx$ is convergent.
4. IF $\int_a^{\infty} g(x)dx = \infty$, THEN $\int_a^{\infty} f(x)dx = \infty$.

True or False - Comparisons II

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, \infty)$.

Assume that $\boxed{\forall x \geq a, \quad f(x) \leq g(x)}$.

What can we conclude?

1. IF $\displaystyle\int_a^{\infty} f(x)dx$ is convergent, THEN $\displaystyle\int_a^{\infty} g(x)dx$ is convergent.
2. IF $\displaystyle\int_a^{\infty} f(x)dx = \infty$, THEN $\displaystyle\int_a^{\infty} g(x)dx = \infty$.
3. IF $\displaystyle\int_a^{\infty} g(x)dx$ is convergent, THEN $\displaystyle\int_a^{\infty} f(x)dx$ is convergent.
4. IF $\displaystyle\int_a^{\infty} g(x)dx = \infty$, THEN $\displaystyle\int_a^{\infty} f(x)dx = \infty$.

True or False - Comparisons III

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, \infty)$.
Assume that $\boxed{\exists M \geq a \text{ s.t. } \forall x \geq M, \quad 0 \leq f(x) \leq g(x)}$.
What can we conclude?

1. IF $\int_a^\infty f(x)dx$ is convergent, THEN $\int_a^\infty g(x)dx$ is convergent.
2. IF $\int_a^\infty f(x)dx = \infty$, THEN $\int_a^\infty g(x)dx = \infty$.
3. IF $\int_a^\infty g(x)dx$ is convergent, THEN $\int_a^\infty f(x)dx$ is convergent.
4. IF $\int_a^\infty g(x)dx = \infty$, THEN $\int_a^\infty f(x)dx = \infty$.

What can you conclude?

Let $a \in \mathbb{R}$. Let f be a continuous, **positive** function on $[a, \infty)$. In each of the following cases, what can you conclude about $\displaystyle\int_a^\infty f(x)dx$? Is it convergent, divergent, or we do not know?

1. $\forall b \geq a,\ \exists M \in \mathbb{R}$ s.t. $\displaystyle\int_a^b f(x)\,dx \leq M.$
2. $\exists M \in \mathbb{R}$ s.t. $\forall b \geq a,$ $\displaystyle\int_a^b f(x)\,dx \leq M.$
3. $\exists M > 0$ s.t. $\forall x \geq a,\ f(x) \leq M.$
4. $\exists M > 0$ s.t. $\forall x \geq a,\ f(x) \geq M.$

BCT calculations

Use BCT to determine whether each of the following is convergent or divergent

1. $\displaystyle\int_1^\infty \frac{1+\cos^2 x}{x^{2/3}}\,dx$
2. $\displaystyle\int_1^\infty \frac{1+\cos^2 x}{x^{4/3}}\,dx$
3. $\displaystyle\int_0^\infty \frac{\arctan x^2}{1+e^x}\,dx$
4. $\displaystyle\int_0^\infty e^{-x^2}\,dx$
5. $\displaystyle\int_2^\infty \frac{(\ln x)^{10}}{x^2}\,dx$

Rapid questions: convergent or divergent?

1. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

2. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

3. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

4. $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$

5. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

6. $\displaystyle\int_0^1 \frac{1}{x}\,dx$

7. $\displaystyle\int_1^\infty \frac{3}{x^2}\,dx$

8. $\displaystyle\int_1^\infty \frac{1}{x^2+3}\,dx$

9. $\displaystyle\int_1^\infty \left(\frac{1}{x^2}+3\right)dx$

Slow questions: convergent or divergent?

1. $\displaystyle\int_1^\infty \frac{x^3 + 2x + 7}{x^5 + 11x^4 + 1}\, dx$
2. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2 + x + 1}}\, dx$
3. $\displaystyle\int_0^1 \frac{3\cos x}{x + \sqrt{x}}\, dx$
4. $\displaystyle\int_0^1 \sqrt{\cot x}\, dx$
5. $\displaystyle\int_0^1 \frac{\sin x}{x^{3/2}}\, dx$
6. $\displaystyle\int_0^\infty e^{-x^2}\, dx$
7. $\displaystyle\int_2^\infty \frac{(\ln x)^{10}}{x^2}\, dx$

A harder calculation

For which values of $a > 0$ is the integral

$$\int_0^\infty \frac{\arctan x}{x^a}\, dx$$

convergent?

A variation on LCT

This is the theorem you have learned:

Theorem (Limit-Comparison Test)

Let $a \in \mathbb{R}$. Let f and g be positive, continuous functions on $[a, \infty)$.

- IF the limit $L = \lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)}$ exists and $L > 0$
- THEN $\displaystyle\int_a^\infty f(x)dx$ and $\displaystyle\int_a^\infty g(x)dx$ are both convergent or both divergent.

What if we change the hypotheses to $L = 0$?

1. Write down the new theorem (different conclusion).
2. Prove it.

Hint: If $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$, what is larger $f(x)$ or $g(x)$?

A variation on LCT - 2

This is the theorem you have learned:

Theorem (Limit-Comparison Test)

Let $a \in \mathbb{R}$. Let f and g be positive, continuous functions on $[a, \infty)$.

- IF the limit $L = \lim_{x \to \infty} \dfrac{f(x)}{g(x)}$ exists and $L > 0$
- THEN $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ are both convergent or both divergent.

What if we change the hypotheses to $L = \infty$?

1. Write down the new theorem (different conclusion).
2. Prove it.

Absolute Convergence

Definition

The integral $\int_a^\infty f(x)\,dx$ is called **absolutely convergent** when $\int_a^\infty |f(x)|\,dx$ converges.

Prove that

- IF an improper integral is absolutely convergent
- THEN it is convergent

Hint: Consider the functions

$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) \leq 0 \end{cases} \qquad f_-(x) = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ |f(x)| & \text{if } f(x) \leq 0 \end{cases}$$

Write $f(x)$ and $|f(x)|$ in terms of $f_+(x)$ and $f_-(x)$. Use BCT.

Dirichlet integral

Let $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

1. Is $\displaystyle\int_0^1 f(x)\,dx$ an inproper integral?
2. Show that $\displaystyle\int_1^\infty \frac{\cos x}{x^2}\,dx$ is absolutely convergent.
   Hint: Use BCT.
3. The same argument is inconclusive for $\displaystyle\int_1^\infty f(x)\,dx$. Why?
4. Show that $\displaystyle\int_1^\infty f(x)dx$ is convergent
   Hint: Use the definition of improper integral, not comparison tests. Use integration by parts with $u = \frac{1}{x}$ and $dv = \sin x\,dx$.

Note: It is possible to prove that $\displaystyle\int_1^\infty \frac{\sin x}{x}\,dx$ is not absolutely convergent.

A telescopic series

I want to calculate the value of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$.

1. Find a formula for the k-th partial sum $S_k = \displaystyle\sum_{n=1}^{k} \frac{1}{n^2+2n}$.

   Hint: $\displaystyle\frac{1}{n^2+2n} = \frac{A}{n} + \frac{B}{n+2}$

2. Using the definition of series, compute the value of

$$\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$$

What is wrong with this calculation? Fix it

Claim: $$\sum_{n=2}^{\infty} \ln \frac{n}{n+1} = \ln 2$$

"Proof"

$$\begin{aligned}
\sum_{n=2}^{\infty} \ln \frac{n}{n+1} &= \sum_{n=2}^{\infty} [\ln n - \ln(n+1)] \\
&= \sum_{n=2}^{\infty} \ln(n) - \sum_{n=2}^{\infty} \ln(n+1) \\
&= (\ln 2 + \ln 3 + \ln 4 + \dots) - (\ln 3 + \ln 4 + \dots) \\
&= \ln 2
\end{aligned}$$

Trig series: convergent or divergent?

1. $\displaystyle\sum_{n=0}^{\infty} \sin(n\pi)$

2. $\displaystyle\sum_{n=0}^{\infty} \cos(n\pi)$

Help me write the next assignment

In the next assignment I want to give you a series and ask you to calculate its value from the definition. I want the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ to be

$$\forall n \geq 1, \; S_n = n^2$$

What series should I ask you to calculate?

What can you conclude?

Assume $\forall n \in \mathbb{N},\ a_n > 0$. Consider the series $\displaystyle\sum_{n=0}^{\infty} a_n$.

Let $\{S_n\}_{n=0}^{\infty}$ be its sequence of partial sums.

In each of the following cases, what can you conclude about the *series*? Is it convergent, divergent, or we do not know?

1. $\forall n \in \mathbb{N}, \quad \exists M \in \mathbb{R}$ s.t. $\quad S_n \leq M$.
2. $\exists M \in \mathbb{R}$ s.t. $\quad \forall n \in \mathbb{N}, \quad S_n \leq M$.
3. $\exists M > 0$ s.t. $\quad \forall n \in \mathbb{N}, \quad a_n \leq M$.
4. $\exists M > 0$ s.t. $\quad \forall n \in \mathbb{N}, \quad a_n \geq M$.

Harmonic series

For each $n > 0$ we define

$$r_n = \text{ smallest power of 2 that is greater than or equal to } n$$

Consider the series $S = \displaystyle\sum_{n=1}^{\infty} \frac{1}{r_n}$

1. Compute r_1 through r_8
2. Compute the partial sums S_1, S_2, S_4, S_8 for the series S.
3. Calculate $S = \displaystyle\sum_{n=1}^{\infty} \frac{1}{r_n}$.
4. Calculate $H = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. *Hint:* "Compare" H and S.

True or False – Definition of series

Let $\sum_{n=0}^{\infty} a_n$ be a series. Let $\{S_n\}_{n=0}^{\infty}$ be its partial-sum sequence.

1. IF the series $\sum_{n=0}^{\infty} a_n$ is convergent,

   THEN the sequence $\{S_n\}_{n=0}^{\infty}$ is bounded.

2. IF the series $\sum_{n=0}^{\infty} a_n$ is convergent,

   THEN the sequence $\{S_n\}_{n=0}^{\infty}$ is eventually monotonic.

3. IF the sequence $\{S_n\}_{n=0}^{\infty}$ is bounded and eventually monotonic,

   THEN the series $\sum_{n=0}^{\infty} a_n$ is convergent.

True or False – Definition of series

Let $\sum_{n=0}^{\infty} a_n$ be a series. Let $\{S_n\}_{n=0}^{\infty}$ be its partial-sum sequence.

4. IF $\forall n > 0,\ a_n > 0$,
   THEN the sequence $\{S_n\}_{n=0}^{\infty}$ is increasing.

5. IF the sequence $\{S_n\}_{n=0}^{\infty}$ is increasing,
   THEN $\forall n > 0,\ a_n > 0$.

6. IF $\forall n > 0,\ a_n \geq 0$,
   THEN the sequence $\{S_n\}_{n=0}^{\infty}$ is non-decreasing.

7. IF the sequence $\{S_n\}_{n=0}^{\infty}$ is non-decreasing,
   THEN $\forall n > 0,\ a_n \geq 0$

Rapid questions: geometric series

Convergent or divergent?

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n/2}}$

4. $\displaystyle\sum_{n=5}^{\infty} \frac{3^n}{2^{2n+1}}$

5. $\displaystyle\sum_{n=3}^{\infty} \frac{3^n}{1000 \cdot 2^{n+2}}$

6. $\displaystyle\sum_{n=0}^{\infty} (-1)^n$

Geometric series

Calculate the value of the following series:

1. $1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \dfrac{1}{81} + \dots$

2. $\dfrac{1}{2} - \dfrac{1}{4} + \dfrac{1}{8} - \dfrac{1}{16} + \dfrac{1}{32} - \dots$

3. $\dfrac{3}{2} - \dfrac{9}{4} + \dfrac{27}{8} - \dfrac{81}{16} + \dots$

4. $1 + \dfrac{1}{2^{0.5}} + \dfrac{1}{2} + \dfrac{1}{2^{1.5}} + \dfrac{1}{2^{2}} + \dfrac{1}{2^{2.5}} + \dots$

5. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{2^{2n+1}}$

6. $\displaystyle\sum_{n=k}^{\infty} x^n$

Is $0.999999\ldots = 1$?

1. Write the number $0.9999999\ldots$ as a series
   Hint: $427 = 400 + 20 + 7$.

2. Compute the first few partial sums

3. Add up the series.
   Hint: it is geometric.

Decimal expansions of rational numbers

We can interpret any finite decimal expansion as a finite sum. For example:

$$2.13096 = 2 + \frac{1}{10} + \frac{3}{10^2} + \frac{0}{10^3} + \frac{9}{10^4} + \frac{6}{10^5}$$

Similarly, we can interpret any infinite decimal expansion as an infinite series.

Interpret the following numbers as series, and add up the series to calculate their value as fractions:

1. $0.99999\ldots$
2. $0.11111\ldots$
3. $0.252525\ldots$
4. $0.3121212\ldots$

Hint: Use geometric series

Functions as series

You know that when $|x| < 1$:

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Find similar ways to write the following functions as series:

1. $g(x) = \dfrac{1}{1+x}$
2. $h(x) = \dfrac{1}{1-x^2}$
3. $A(x) = \dfrac{1}{2-x}$
4. $G(x) = \ln(1+x)$

Hint: For the last one, compute G'.

Challenge

We want to calculate the value of

$$A = \sum_{n=0}^{\infty} \frac{1}{2^n}, \qquad B = \sum_{n=1}^{\infty} \frac{n}{2^n}, \qquad C = \sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

Let $f(x) = \dfrac{1}{1-x}$.

---

1. Recall that $f(x) = \displaystyle\sum_{n=0}^{\infty} x^n$ for $|x| < 1$. Use it to compute A.
2. Pretend you can take derivatives of series the way you take them of finite sums. Write $f'(x)$ as a series.
3. Use it to compute B.
4. Do something similar to compute C.

Challenge - 2

We want to calculate the value of

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,3^n}$$

---

1. Compute $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$
2. Compute $\dfrac{d}{dx}\left[\arctan x\right]$
3. Pretend you can take derivatives and antiderivatives of series the way you can take them of finite sums. Which series adds up to $\arctan x$?
4. Now calculate the value of the original series.

Examples

1. A series $\displaystyle\sum_{n=0}^{\infty} a_n$ may be

$$\begin{cases} \text{convergent (a number)} \\ \text{divergent} \begin{cases} \text{to } \infty \\ \text{to } -\infty \\ \text{“oscillating”} \end{cases} \end{cases}$$

   Give one example of each of the four results.

2. Now assume $\forall n \in \mathbb{N},\ a_n \geq 0$.
   Which of the four outcomes is still possible?

True or False – The tail of a series

1. IF the series $\sum_{n=0}^{\infty} a_n$ converges,

   THEN the series $\sum_{n=7}^{\infty} a_n$ converges

2. IF the series $\sum_{n=7}^{\infty} a_n$ converges,

   THEN the series $\sum_{n=0}^{\infty} a_n$ converges

3. IF the series $\sum_{n=0}^{\infty} a_n$ converges,

   THEN the series $\sum_{n=7}^{\infty} a_n$ converges to a smaller number.

True or False – The Necessary Condition

1. IF $\lim_{n\to\infty} a_n = 0$, THEN $\sum_n^{\infty} a_n$ is convergent.
2. IF $\lim_{n\to\infty} a_n \neq 0$, THEN $\sum_n^{\infty} a_n$ is divergent.
3. IF $\sum_n^{\infty} a_n$ is convergent THEN $\lim_{n\to\infty} a_n = 0$.
4. IF $\sum_n^{\infty} a_n$ is divergent THEN $\lim_{n\to\infty} a_n \neq 0$.

True or False – Harder questions

1. IF $\sum_{n=0}^{\infty} a_n$ is convergent, THEN $\lim_{k\to\infty} \left[\sum_{n=k}^{\infty} a_n \right] = 0.$

2. IF $\lim_{k\to\infty} \left[\sum_{n=k}^{\infty} a_n \right] = 0,$ THEN $\sum_{n=0}^{\infty} a_n$ is convergent.

3. IF $\sum_{n=1}^{\infty} a_{2n}$ and $\sum_{n=1}^{\infty} a_{2n+1}$ are convergent,

   THEN $\sum_{n=1}^{\infty} a_n$ is convergent.

4. IF $\sum_{n=1}^{\infty} a_n$ is convergent,

   THEN $\sum_{n=1}^{\infty} a_{2n}$ and $\sum_{n=1}^{\infty} a_{2n+1}$ are convergent.

Series are linear

Let $\sum_{n=0}^{\infty} a_n$ be a series. Let $c \in \mathbb{R}$. Prove that

- IF $\sum_{n=0}^{\infty} a_n$ is convergent.
- THEN $\sum_{n=0}^{\infty} (ca_n)$ is convergent and $\sum_{n=0}^{\infty} (ca_n) = c \left[\sum_{n=0}^{\infty} a_n \right]$.

Write a proof directly from the definition of series.

Rapid questions: improper integrals

Convergent or divergent?

1. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
3. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$
4. $\displaystyle\int_1^\infty \frac{x+1}{x^3+2}\,dx$
5. $\displaystyle\int_1^\infty \frac{\sqrt{x^2+5}}{x^2+6}\,dx$
6. $\displaystyle\int_1^\infty \frac{x^2+3}{\sqrt{x^5+2}}\,dx$

For which values of $a \in \mathbb{R}$ are these series convergent?

1. $\displaystyle\sum_{n}^{\infty} \frac{1}{a^n}$

2. $\displaystyle\sum_{n}^{\infty} \frac{1}{n^a}$

3. $\displaystyle\sum_{n}^{\infty} a^n$

4. $\displaystyle\sum_{n}^{\infty} n^a$

Quick comparisons: convergent or divergent?

1. $$\sum_{n}^{\infty} \frac{n+1}{n^2+1}$$

2. $$\sum_{n}^{\infty} \frac{n^2+3n}{n^4+5n+1}$$

3. $$\sum_{n}^{\infty} \frac{\sqrt{n}+1}{n^2+1}$$

4. $$\sum_{n}^{\infty} \frac{\sqrt[3]{n^2+1}+1}{\sqrt{n^3+n}+n+1}$$

Slow comparisons: convergent or divergent?

1. $\displaystyle\sum_{n}^{\infty} \frac{2^n - 40}{3^n - 20}$

2. $\displaystyle\sum_{n}^{\infty} \frac{(\ln n)^{20}}{n^2}$

3. $\displaystyle\sum_{n}^{\infty} \sin^2 \frac{1}{n}$

4. $\displaystyle\sum_{n}^{\infty} \frac{1}{n\,(\ln n)^3}$

5. $\displaystyle\sum_{n}^{\infty} \frac{1}{n \ln n}$

6. $\displaystyle\sum_{n}^{\infty} e^{-n^2}$

Convergence tests: ninja level

We know

- $\forall n \in \mathbb{N},\ a_n > 0.$
- the series $\displaystyle\sum_n^{\infty} a_n$ is convergent

Determine whether the following series are convergent, divergent, or we do not have enough information to decide:

1. $\displaystyle\sum_n^{\infty} \sin a_n$
2. $\displaystyle\sum_n^{\infty} \cos a_n$
3. $\displaystyle\sum_n^{\infty} \sqrt{a_n}$
4. $\displaystyle\sum_n^{\infty} (a_n)^2$

Are all decimal expansions well-defined?

We had defined a real number as "any number with a decimal expansion". Now we understand what it means to write a number with an infinite decimal expansion. It is a series!

$$0.a_1 a_2 a_3 a_4 a_5 \cdots = \frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{1000} + \dots$$

for any "digits" a_1, a_2, a_3, ...
But this raises a question: are these series always convergent, no matter which infinite string of digits we choose?
Yes, they are! Prove it.
(Hint: all the terms in the series are positive.)

Rapid questions: alternating series test

Convergent or divergent?

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.5}}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3}}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sin n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.5}}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3}}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sin n}$

True or False - Odd and even partial sums

Let $\sum_{n=0}^{\infty} a_n$ be a series. Let $\{S_n\}_{n=0}^{\infty}$ be its partial-sum sequence.

1. IF $\lim_{n\to\infty} S_{2n}$ exists, THEN $\sum_{n=0}^{\infty} a_n$ is convergent.

2. IF $\lim_{n\to\infty} S_{2n}$ exists and $\lim_{n\to\infty} S_{2n+1}$ exists,

   THEN $\sum_{n=0}^{\infty} a_n$ is convergent.

3. IF $\lim_{n\to\infty} S_{2n}$ exists and $\lim_{n\to\infty} a_n = 0$,

   THEN $\sum_{n=0}^{\infty} a_n$ is convergent.

An Alternating Series Test example

Verify carefully the 3 hypotheses of the Alternating Series Test for

$$\sum_{n=0}^{\infty} (-1)^n \frac{n - \pi}{e^n}$$

Can we conclude it is convergent?

Estimation

Estimate the sum

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

with an error smaller than 0.001. Write your final answer as a rational number (i.e. as a quotient of two integers).

Not exactly alternating

Are these series convergent or divergent?

$$A = 1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11} - \frac{1}{12} + \dots$$

$$B = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} - \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} - \dots$$

Suggestion: Draw the partial sums on the real line.

A counterexample to Alternating Series Test?

Construct a series of the form $\sum_{n=1}^{\infty} (-1)^n b_n$ such that

- $b_n > 0$ for all $n \geq 1$
- $\lim_{n \to \infty} b_n = 0$
- the series $\sum_{n=1}^{\infty} (-1)^n b_n$ is divergent.

Absolutely convergent or conditionally convergent?

1. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.5}}$$

2. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1.5}}$$

3. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sin n}$$

True or False - Absolute Values

1. IF $\{a_n\}_{n=1}^{\infty}$ is convergent, THEN $\{\,|a_n|\,\}_{n=1}^{\infty}$ is convergent.

2. IF $\{\,|a_n|\,\}_{n=1}^{\infty}$ is convergent, THEN $\{a_n\}_{n=1}^{\infty}$ is convergent.

3. IF $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, THEN $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is convergent.

4. IF $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is convergent, THEN $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

Positive and negative terms - 1

- Let $\sum a_n$ be a series.
- Call $\sum$(P.T.) the sum of only the positive terms of the same series.
- Call $\sum$(N.T.) the sum of only the negative terms of the same series.

IF $\sum$(P.T.) is...	AND $\sum$(N.T.) is...	THEN $\sum a_n$ may be..
CONV	CONV	
∞	CONV	
CONV	$-\infty$	
∞	$-\infty$	

Positive and negative terms - 2

- Let $\sum a_n$ be a series.
- $\sum (\text{P.T.}) \;=\;$ sum of only the positive terms of the same series.
- $\sum (\text{N.T.}) \;=\;$ sum of only the negative terms of the same series.

	$\sum (\text{P.T.})$ may be...	$\sum (\text{N.T.})$ may be...
If $\sum a_n$ is CONV		
If $\sum \lvert a_n \rvert$ is CONV		
If $\sum a_n$ is ABS CONV		
If $\sum a_n$ is COND CONV		
If $\sum a_n = \infty$		
If $\sum a_n$ is DIV oscillating		

Quick review: Convergent or divergent?

1. $\displaystyle\sum_{n}^{\infty}(1.1)^n$

2. $\displaystyle\sum_{n}^{\infty}(0.9)^n$

3. $\displaystyle\sum_{n}^{\infty}\frac{1}{n^{1.1}}$

4. $\displaystyle\sum_{n}^{\infty}\frac{1}{n^{0.9}}$

5. $\displaystyle\sum_{n}^{\infty}\frac{(-1)^n}{\ln n}$

6. $\displaystyle\sum_{n}^{\infty}\frac{(-1)^n}{e^{1/n}}$

7. $\displaystyle\sum_{n}^{\infty}\frac{n^3+n^2+11}{n^4+2n-3}$

8. $\displaystyle\sum_{n}^{\infty}\frac{\sqrt{n^5+2n+16}}{n^4-11n+7}$

Ratio Test: Convergent or divergent?

Use Ratio Test to decide which series are convergent.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2 \; 3^{n+1}}$

3. $\displaystyle\sum_{n=2}^{\infty} \frac{n!}{n^n}$

4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

Root test

Here is a new convergence test

Theorem

Let $\sum_n a_n$ be a series. Assume the limit $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists.

- IF $0 \leq L < 1$ THEN the series is ???
- IF $L > 1$ THEN the series is ???

Without writing an actual proof, guess the conclusion of the theorem and argue why it makes sense.
Hint: Imitate the argument on Video 13.18 for the Ratio Test. For large values of n, what is $|a_n|$ approximately?

Interval of convergence

Find the interval of convergence of each power series:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2\, 2^{2n+1}}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{42^n} x^n$

4. (Hard!) $\displaystyle\sum_{n=0}^{\infty} \frac{(3n)!}{n!(2n)!}\, x^n$

What can you conclude?

Think of the power series $\sum_{n}^{\infty} a_n x^n$. Do not assume $a_n \geq 0$.

In each case, may the given series be absolutely convergent (AC)? conditionally convergent (CC)? divergent (D)? all of them?

IF	$\sum_{n}^{\infty} a_n 3^n$ is ...	AC	CC	D
THEN	$\sum_{n}^{\infty} a_n 2^n$ may be ...	???	???	???
	$\sum_{n}^{\infty} a_n (-3)^n$ may be ...	???	???	???
	$\sum_{n}^{\infty} a_n 4^n$ may be ...	???	???	???

Writing functions as power series

You know that $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

Manipulate it to write the following functions as power series centered at 0:

1. $g(x) = \dfrac{1}{1+x}$
2. $A(x) = \dfrac{1}{2-x}$

   Hint: Factor $1/2$.
3. $h(x) = \dfrac{1}{1-x^2}$
4. $F(x) = \ln(1+x)$

   Hint: Compute F'

Challenge

Compute $\quad A = \sum_{n=1}^{\infty} \frac{n}{3^n}$

---

1. What is the value of the sum $\sum_{n=0}^{\infty} x^n$?
2. Use derivatives to relate $\sum_{n}^{\infty} x^n$ and $\sum_{n}^{\infty} nx^{n-1}$.
3. Compute $\sum_{n=1}^{\infty} nx^{n-1}$. Then compute $\sum_{n=1}^{\infty} nx^n$.
4. Compute the value of series A.

Challenge

Compute $\quad A = \displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n} \quad$ and $\quad B = \displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

---

1. What is the value of the sum $\displaystyle\sum_{n=0}^{\infty} x^n$?
2. Use derivatives to relate $\displaystyle\sum_{n}^{\infty} x^n$ and $\displaystyle\sum_{n}^{\infty} nx^{n-1}$.
3. Compute $\displaystyle\sum_{n=1}^{\infty} nx^{n-1}$. Then compute $\displaystyle\sum_{n=1}^{\infty} nx^n$.
4. Compute the value of series A.
5. Compute the value of series B.

Challenge - 2

We want to calculate the value of

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,3^n}$$

---

1. Write $F(x) = \arctan x$ as a power series.

   Hint: Compute $F'(x)$. Using the geometric series, write $F'(x)$ as a series. Then integrate.

2. Now calculate the original sum.

Warm up

Write down the (various equivalent) definitions of Taylor polynomial you have learned so far.

Tangent line

Let f be a C^1 function at $a \in \mathbb{R}$.
Then the tangent line of f at a is given by

$$y = L(x)$$

1. Recall the explicit formula for L
2. Prove that L is the 1-st Taylor polynomial for f at a using the 1st definition.
3. Prove that L is the 1-st Taylor polynomial for f at a using the 2nd definition.

Taylor polynomial of a polynomial

Let $f(x) = x^3$.
Let $Q_{n,a}$ be the n-th Taylor polynomial for f at a.

1. Using the 2nd definition, find $Q_{2,0}$.
   Then verify it also satisfies the 1st definition.
2. Repeat for $Q_{3,0}$
3. Repeat for $Q_{3,1}$
4. Repeat for $Q_{2,1}$.

True or False – Taylor polynomials

Let f be a function defined at and near $a \in \mathbb{R}$. Let $n \in \mathbb{N}$.
Let P_n be the n-th Taylor polynomial for f at a.
Which ones of these are true?

1. P_n is an approximation for f of order n near a.
2. f is an approximation for P_n of order n near a.
3. P_3 is an approximation for f of order 4 near a.
4. P_4 is an approximation for f of order 3 near a.
5. $\lim_{x \to a} [f(x) - P_n(x)] = 0$
6. $\lim_{x \to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0$
7. If x is close to a, then $f(x) = P_n(x)$.

True or False – smooth functions

Let f be a function. Let $a \in \mathbb{R}$. Let $m \in \mathbb{N}$.

1. IF f is continuous, THEN f is C^0.
2. IF f is C^0, THEN f is continuous.
3. IF f is differentiable, THEN f is C^1.
4. IF f is C^1, THEN f is differentiable.
5. IF f is C^∞, THEN $\forall n \in \mathbb{N}$, f is C^n.
6. IF $\forall n \in \mathbb{N}$, f is C^n, THEN f is C^∞.
7. IF f is C^m at a,
   THEN f is C^m on some interval centered at a.
8. IF f is C^m at a,
   THEN f is C^{m-1} on some interval centered at a.

True or False – Operations with smooth functions

Let f and g be two functions with domain $\mathbb{R}$. Let $n \in \mathbb{N}$.

1. IF f and g are C^n, THEN $f + g$ is C^n.
2. IF f and g are C^n, THEN $f \cdot g$ is C^n.
3. IF f and g are C^n, THEN $f \circ g$ is C^n.
4. IF f and g are C^∞, THEN $f + g$ is C^∞.
5. IF f and g are C^∞, THEN $f \cdot g$ is C^∞.
6. IF f and g are C^∞, THEN $f \circ g$ is C^∞.

Approximating functions

Which one of the following functions is a better approximation for $F(x) = \sin x + \cos x$ near 0?

1. $f(x) = 1 + x - \dfrac{x^2}{2}$
2. $g(x) = e^x - x^2$
3. $h(x) = 1 + \ln(1 + x)$

♣

A polynomial given its derivatives

1. Consider the polynomial $P(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$. Find values of the coefficients that satisfy

$$P(0) = 1, \quad P'(0) = 5, \quad P''(0) = 3, \quad P'''(0) = -7$$

2. Find *all* polynomials P (of any degree) that satisfy

$$P(0) = 1, \quad P'(0) = 5, \quad P''(0) = 3, \quad P'''(0) = -7$$

3. Find a polynomial P of smallest possible degree that satisfies

$$P(0) = A, \quad P'(0) = B, \quad P''(0) = C, \quad P'''(0) = D$$

Competition!

- Do you prefer cats or dogs? You MUST choose one. Now you are in the C-team or the D-team.
- Copy only one polynomial (C or D):

$$C(x) = -\frac{293}{8} + 29x + \frac{13}{4}x^2 - 3x^3 + \frac{3}{8}x^4$$

$$D(x) = 29 + 8(x-3) - \frac{7}{2}(x-3)^2 + \frac{9}{6}(x-3)^3 + \frac{9}{24}(x-3)^4$$

- I will ask you questions.
  Answer only about your polynomial (C or D).
  No calculators!

Competition!

$$C(x) = -\frac{293}{8} + 29x + \frac{13}{4}x^2 - 3x^3 + \frac{3}{8}x^4$$

$$D(x) = 29 + 8(x-3) - \frac{7}{2}(x-3)^2 + \frac{9}{6}(x-3)^3 + \frac{9}{24}(x-3)^4$$

C-team compute...

- $C(3)$
- $C'(3)$
- $C''(3)$
- $C'''(3)$
- $C^{(4)}(3)$

D-team compute...

- $D(3)$
- $D'(3)$
- $D''(3)$
- $D'''(3)$
- $D^{(4)}(3)$

Simplify your answers (write them as rational numbers)

I spy a polynomial with my little eye

I'm thinking of a cubic polynomial P. It satisfies

$$P(1) = 8, \quad P'(1) = -\pi, \quad P''(1) = 4, \quad P'''(1) = \sqrt{7}$$

What is $P(x)$?

cosine

Obtain the Maclaurin series for $h(x) = \cos x$.
There are at least two ways to do this:

1. Use the general formula for Maclaurin series.
2. Use the Maclaurin series for sin to compute
   $\cos x = \dfrac{d}{dx} \sin x$.

Interval of convergence of Maclaurin series

1. (Recall) Write down the Maclaurin series for the following functions

$$f(x) = e^x, \qquad g(x) = \sin x, \qquad h(x) = \cos x$$

2. Compute the interval of convergence for each one of them.

Warm up

1. Write down the Maclaurin series for $f(x) = \sin x$. (Just recall it.)
2. Compute the interval of convergence of this power series.

3. Write down the statement of Lagrange's Remainder Theorem. (Just recall it. Look it up if needed.)

sin is analytic

Let $f(x) = \sin x$. You know its Maclaurin series is

$$S(x) \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

As you know, to prove that $\sin x = S(x)$ we need to show that

$$\forall x \in \mathbb{R}, \qquad \lim_{n\to\infty} R_n(x) = 0$$

Use Lagrange's Remainder Theorem to prove it!

Reminder: Lagrange's Remainder Theorem says that given f, a, x, and n with certain conditions,

$$\exists \xi \text{ between } a \text{ and } x \text{ s.t.} \qquad R_n(x) \;=\; \frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x-a)^{n+1}$$

Generalize your proof

Theorem

Let I be an open interval. Let $a \in I$. Let f be a C^∞ function on I.
Let $S(x)$ be the Taylor series for f centered at a.

- IF $\boxed{???}$
- THEN $\forall x \in I, \ f(x) = S(x)$

Which condition can you write instead of "$\boxed{???}$" to make the theorem true?

If you are thinking "the derivatives must be bounded", then you are on the right track, but you need to be much more precise. Which derivatives? On which domain? There are a lot of variables here; can the bounds depend on any variable?

Generalize your proof (continued)

Which one or ones of the following conditions can be written instead of "$\boxed{???}$" to make the theorem true?

1. $\forall n \in \mathbb{N}, \quad f^{(n)}$ is bounded on I
2. $\forall n \in \mathbb{N}, \quad \forall x \in I, \quad f^{(n)}$ is bounded on $J_{x,a}$
3. $\forall n \in \mathbb{N}, \quad \forall x \in I, \quad \exists A, B \in \mathbb{R}, \quad \forall \xi \in J_{x,a}, \quad A \leq f^{(n)}(\xi) \leq B$
4. $\forall x \in I, \quad \exists A, B \in \mathbb{R}, \quad \forall n \in \mathbb{N}, \quad \forall \xi \in J_{x,a}, \quad A \leq f^{(n)}(\xi) \leq B$
5. $\forall x \in I, \quad \exists M \geq 0, \quad \forall n \in \mathbb{N}, \quad \forall \xi \in J_{x,a}, \quad \left|f^{(n)}(\xi)\right| \leq M$
6. $\exists A, B \in \mathbb{R}, \quad \forall x \in I, \quad \forall n \in \mathbb{N}, \quad \forall \xi \in J_{x,a}, \quad A \leq f^{(n)}(\xi) \leq B$
7. $\exists A, B \in \mathbb{R}, \quad \forall x \in I, \quad \forall n \in \mathbb{N}, \quad A \leq f^{(n)}(x) \leq B$

Notation: $J_{x,a}$ is the interval between x and a

A C^∞ but not analytic function

Consider the function $F(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$

1. Prove that, for every $n \in \mathbb{N}$, $\lim_{t \to \infty} t^n e^{-t} = 0$.
2. Prove that, for every $n \in \mathbb{N}$, $\lim_{x \to 0^+} \dfrac{e^{-1/x}}{x^n} = 0$.
3. Calculate $F'(x)$ for $x > 0$.
4. Calculate $F'(x)$ for $x < 0$.
5. Calculate $F'(0)$ from the definition.
6. Calculate $F''(0)$ from the definition.
7. Prove that for every $n \in \mathbb{N}$, $F^{(n)}(0) = 0$.
8. Write the Maclaurin series for F at 0.
9. Is F analytic? Is it C^∞?

Taylor series gymnastics

Write the following functions as power series centered at 0. Write them first with sigma notation, and then write out the first few terms. Indicate the domain where each expansion is valid.

1. $f(x) = e^{-x}$
2. $f(x) = x^2 \cos x$
3. $f(x) = \dfrac{1}{1+x}$
4. $f(x) = \dfrac{1}{1-x^2}$
5. $f(x) = \dfrac{x}{3+2x}$
6. $f(x) = \sin\left(2x^3\right)$
7. $f(x) = \dfrac{e^x + e^{-x}}{2}$
8. $f(x) = \ln \dfrac{1+x}{1-x}$

Note: You do not need to take any derivatives. You can reduce them all to other Maclaurin series you know.

Taylor series not at 0

Write the Taylor series...

1. for $f(x) = e^x$ at $a = -1$
2. for $g(x) = \sin x$ at $a = \pi/4$
3. for $H(x) = 1/x$ at $a = 3$

You can do these problems in two ways:

1. Compute first few derivatives, guess the pattern, use general formula
2. Use substitution $u = x - a$, use known Maclaurin series (without computing any derivative).

arctan

1. Write the Maclaurin series for $G(x) = \arctan x$

   Hint: Compute the first derivative. Then use the geometric series. Then integrate.

2. What is $G^{(137)}(0)$?

3. Use this previous results to compute

$$A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,3^n}$$

arcsin

Let $f(x) = \dfrac{1}{\sqrt{1+x}}$.

1. Find a formula for its derivatives $f^{(n)}(x)$.
2. Write its Maclaurin series at 0. Call it $S(x)$.
3. What is the radius of convergence of series $S(x)$?
   Note: Use without proof that $f(x) = S(x)$ inside the interval of convergence.
4. Use this result to write $h(x) = \text{arcsin}$ as a power series centered at 0.
   Hint: Compute $h'(x)$.
5. What is $h^{(7)}(0)$?

Parity

1. Write down the definition of odd function and even function. (Assume the domain is $\mathbb{R}$.)
2. Let f be an odd, C^∞ function. What can you say about its Maclaurin series? What if f is even?

   Hint: Think of $\sin$ and $\cos$.
3. Prove it.

   Hints:
   - Use the general formula for the Maclaurin series.
   - If h is odd then what is $h(0)$?
   - The derivative of an even function is ...?
   - The derivative of an odd function is ...?

Product of Taylor series

Let $f(x) = e^x \ln(1+x)$

1. Write the 4-th Taylor polynomial for f at $a = 0$.

   Hint: Write the first few terms of the Maclaurin series for each factor and multiply them.

2. What is $f^{(4)}(0)$?

3. Use it to calculate the limit

$$\lim_{x \to 0} \frac{e^x \ln(1+x) + \ln(1-x)}{x^4}$$

Composition of Taylor series

Let $g(x) = e^{\sin x}$.

1. Write the 4-th Taylor polynomial for g at $a = 0$.

   Hint: First use the Maclaurin series for the exponential. Then use the Maclaurin series for sin and treat it like a polynomial. You only need to keep the first few terms.

2. What is $g^{(4)}(0)$?

3. Find a value of $a \in \mathbb{R}$ such that the limit

$$\lim_{x \to 0} \frac{e^{\sin x} - e^x + ax^3}{x^4}$$

   exists and is not 0. Then compute the limit.

Tangent

There is no nice, compact formula for the Maclaurin series of tan, but we can obtain the first few terms. Set

$$\tan x = c_1 x + c_3 x^3 + c_5 x^5 + \dots$$

By definition of tan, we have:

$$\sin x = (\cos x)(\tan x)$$

Thus

$$\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right] = \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right] \cdot \left[c_1 x + c_3 x^3 + c_5 x^5 + \dots\right]$$

Multiply the two series on the right. Obtain equations for the coefficients c_n and solve for the first few ones.

Secant

I want to obtain the first few terms of the Maclaurin series of $f(x) = \sec x$. Notice that

$$\sec x = \frac{1}{\cos x} = \frac{1}{1 - [1 - \cos x]} = \frac{1}{1-u} \tag{3}$$

where I have called $u = 1 - \cos x$. Notice that as $x \to 0$, $u \to 0$.

Use the geometric series in (**??**). Then write u as a power series centered at 0. Then expand and regroups terms.

1. Use the above to obtain the 6-th Maclaurin polynomial for f.
2. Without taking any derivative, what is $f^{(6)}(0)$?

Integrals

I want to calculate

$$A = \int_0^1 t^{10} \sin t \; dt.$$

There are two ways to do it. Choose your favourite one:

1. Use integration by parts 10 times.
2. Use power series.

Estimate A with an error smaller than 0.001.

Add these series

1. $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^n}{(2n+1)!}$ *Hint:* Think of sin

2. $\displaystyle\sum_{n=0}^{\infty} (4n+1)x^{4n+2}$ *Hint:* $\dfrac{d}{dx}\left[x^{4n+1}\right] =???$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n)!}$ *Hint:* Write first few terms. Combine e^1 and e^{-1}.

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!(n+1)}$ *Hint:* Integrate

Add more series

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+1)(n+2)}$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n+2)n!}$

9. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)}{(2n)!} 2^n$

10. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$

Hint: Take derivatives or antiderivatives of series whose values you know.

Limits

Use Maclaurin series to compute these limits:

1. $$\lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$$

2. $$\lim_{x \to 0} \frac{\cos(2x) - e^{-2x^2}}{x^4}$$

3. $$\lim_{x \to 0} \frac{[\sin x - x]^3 \, x}{[\cos x - 1]^4 \, [e^x - 1]^2}$$

Estimations

I want to estimate these two numbers

$$A = \sin 1, \qquad B = \ln 0.9.$$

1. Use Taylor series to write A and B as infinite sums.
2. If you want to estimate A or B with a small error using a partial sum, the fastest way is to use different theorems for A and B. What are they?
3. Estimate B with an error smaller than 0.001.