Old and new conjectures for moments of the Riemann zeta-function

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The Lindelöf hypothesis

- Let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ $(\mathrm{Re}(s) > 1)$ be the Riemann zeta-function.

- The yet unproved Lindelöf hypothesis (LH) asserts that for any (small) $\varepsilon > 0$, we have $\zeta(\frac{1}{2} + it) = O(t^{\varepsilon})$ as $t \to \infty$.

- If the Riemann hypothesis (RH) is true, then LH is true. LH has many consequences that are almost as strong as those implied by RH.

- For example, LH implies that if $p_1 < p_2 < p_3 < \dots$ is the sequence of primes, then $p_{n+1} - p_n = O(p_n^{\frac{1}{2}+\varepsilon})$. This is almost as strong as the bound $p_{n+1} - p_n = O(p_n^{1/2} \log p_n)$, which is implied by RH.

- The current best bound for $\zeta(\frac{1}{2} + it)$ is $O(t^{\frac{13}{84}+\varepsilon})$, where $\varepsilon > 0$ is arbitrarily small (Bourgain, 2017).

Moments of $\zeta(s)$

- Denote $\mathcal{M}_k(T) := \int_T^{2T} |\zeta(\frac{1}{2}+it)|^{2k}\, dt$.

This is called the $2k$th moment of zeta

- LH is equivalent to the statement that for any (small) $\varepsilon > 0$ and every positive integer k, we have $\mathcal{M}_k(T) = O(T^{1+\varepsilon})$ as $T \to \infty$

- It is known that

  $\mathcal{M}_1(T) \sim T \log T$ as $T \to \infty$ (Hardy and Littlewood, 1918)

  and

  $\mathcal{M}_2(T) \sim \dfrac{T}{2\pi^2}(\log T)^4$ as $T \to \infty$ (Ingham, 1926)

- Asymptotic formulas are *not* known for all other k, but by extrapolation we expect the following.

Conjecture (Folklore)

$\mathcal{M}_k(T) \sim c_k T(\log T)^{k^2}$ as $T \to \infty$, where c_k is some (unspecified) constant

Conjecture (Folklore)

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \sim c_k T(\log T)^{k^2} \quad \textit{as } T \to \infty$$

for some (unspecified) constant c_k

- Up until recently, there has been no procedural guess for the exact value of c_k.

- It is known that (here, $f \ll g$ means $f = O(g)$)

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \gg T(\log T)^{k^2} \quad \text{for all real } k \geq 0$$

(Radziwiłł and Soundararajan, 2013 for $k \geq 1$;
Heap and Soundararajan, 2022 for $k \geq 0$)

- The Riemann hypothesis implies that

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \ll T(\log T)^{k^2} \quad \text{for all real } k \geq 0$$

(Harper, 2013, by improving a method of Soundararajan, 2009)

Starting point: the approximate functional equation

- To evaluate $\int_T^{2T} |\zeta(\frac{1}{2}+it)|^{2k}\,dt$, we can plug-in the approximate functional equation
$$\zeta^k(\tfrac{1}{2}+it) \sim \sum_{n\le(\frac{t}{2\pi})^{k/2}} \frac{d_k(n)}{n^{\frac{1}{2}+it}} + \left(\frac{t}{2\pi e}\right)^{-ikt} \sum_{n\le(\frac{t}{2\pi})^{k/2}} \frac{d_k(n)}{n^{\frac{1}{2}-it}}$$
and the corresponding expression for $\zeta^k(\frac{1}{2}-it)$ (take the complex conjugate). Here, $d_k(n)$ is the k-fold divisor function.

- Then, we multiply out the terms. In the resulting expression, we expect that terms with $\left(\frac{t}{2\pi e}\right)^{-it}$ are negligible. This can be proved for $k=1$ and $k=2$.

- The main problem, therefore, is to evaluate
$$\int_T^{2T} \sum_{m\le(\frac{t}{2\pi})^{k/2}} \sum_{n\le(\frac{t}{2\pi})^{k/2}} \frac{d_k(m)d_k(n)}{\sqrt{mn}} \left(\frac{m}{n}\right)^{it}\,dt.$$
We treat the terms with $m=n$ ("diagonal terms") and those with $m\ne n$ ("off-diagonal terms") separately.

- The contribution of the diagonal terms is more-or-less straightforward to evaluate using Perron's formula and Euler products:

$$\approx T \sum_{n \leq (\frac{T}{2\pi})^{k/2}} \frac{d_k^2(m)}{m} \approx T \int_{2-i\infty}^{2+i\infty} \frac{1}{s} \left(\frac{T}{2\pi} \right)^{sk/2} \zeta^{k^2}(1+s)\, ds$$

The key point here is that $\zeta^{k^2}(1+s)$ has a pole of order k^2 at $s=0$. This is why we get $T(\log T)^{k^2}$ in the theorems/conjecture.

- If $k=1$ or 2, then the off-diagonal terms do *not* contribute to the main term in the asymptotic formula because

$$\int_T^{2T} \left(\frac{m}{n} \right)^{it} dt \ll \frac{1}{|\log(m/n)|},$$

which is $O(1)$ for most m, n. This is known (i.e., proved).

- On the other hand, all predictions indicate that if $k \geq 3$, then the off-diagonal terms do contribute to the main term.

The heuristic of Conrey and Gonek (1998)

- Conrey and Gonek carry out the evaluation of the off-diagonal terms under the assumption of an asymptotic formula for

$$\sum_{n\leq x} d_k(n)d_k(n+h)$$

(think of $m = n + h$). This yet unproved asymptotic formula results from applying the delta method of Duke, Friedlander, and Iwaniec (1994).

- With this and a method of Goldston and Gonek (1998), Conrey and Gonek are led to conjecture (6th moment agrees with Conrey-Ghosh, '93)

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^6\,dt \sim \frac{42}{9!}\prod_p \left\{ \left(1-\frac{1}{p}\right)^4 \left(1+\frac{4}{p}+\frac{1}{p^2}\right) \right\} T\log^9 T$$

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^8\,dt \sim \frac{24024}{16!}\prod_p \left\{ \left(1-\frac{1}{p}\right)^9 \left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right) \right\} T\log^{16} T$$

- The heuristic fails to give a correct prediction for the 10th moment because in that case the resulting main term is negative. (Later, we will see a possible explanation for why it fails.)

The heuristic of Keating and Snaith (1998)

- This heuristic is based on the following (still unproved) idea, discovered by Montgomery and Dyson in the 1970's:

The limiting distribution of the non-trivial zeros of $\zeta(s)$ might be the same as that of the eigenphases of matrices in the group $U(N)$ of $N \times N$ unitary matrices, equipped with Haar measure.

(Definition: If θ is an *eigenphase*, this means $e^{i\theta}$ is an eigenvalue.)

- Thus it seems reasonable to expect that the values of $\zeta(\frac{1}{2} + it)$, averaged over $t \in [T, 2T]$, are related to the values of

$$\det(1 - M),$$

averaged over $M \in U(N)$. Keating and Snaith then use

$$\int_{U(N)} \det(1 - M)^k \det(1 - M^{-1})^k \, dM$$

to model $\int_T^{2T} |\zeta(\frac{1}{2} + it)|^{2k} \, dt$ in the limit as $N, T \to \infty$

The heuristic of Keating and Snaith (1998)

Theorem (Keating and Snaith, 1998)

If k is a positive integer, then

$$\lim_{N\to\infty} \frac{1}{N^{k^2}} \int_{U(N)} \det(1-M)^k \det(1-M^{-1})^k \, dM = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

(They actually prove a more general theorem that holds for any complex k with $\mathrm{Re}(k) > -1/2$.) This leads them to conjecture that

$$\lim_{T\to\infty} \frac{1}{T(\log T)^{k^2}} \int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k} \, dt = \frac{a_k g_k}{(k^2)!},$$

where

$$a_k := \prod_p \left\{ \left(1-\frac{1}{p}\right)^{k^2} \sum_{j=0}^{\infty} \frac{d_k^2(p^j)}{p^j} \right\}$$

and

$$g_k := (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

- A key ingredient of their proof is Selberg's integral formula (1944).

The heuristic of Keating and Snaith (1998)

Conjecture (Keating and Snaith, 1998)

$$\lim_{T\to\infty} \frac{1}{T(\log T)^{k^2}} \int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt = \frac{a_k g_k}{(k^2)!},$$

For example, direct calculations give

$$\begin{aligned} g_1 &= 1 \\ g_2 &= 2 \\ g_3 &= 42 \\ g_4 &= 24024 \end{aligned}$$

- g_1 agrees with the theorem of Hardy and Littlewood, while g_2 agrees with the theorem of Ingham (Note: you can show $a_1 = 1$ and $a_2 = 6/\pi^2$)

- Remarkably, g_3 agrees with the conjecture of Conrey and Ghosh, while g_4 agrees with the conjecture of Conrey and Gonek!

- Define $Z(s_1, \ldots, s_{2k})$ by

$$Z(s_1, \ldots, s_{2k}) := \int_1^\infty \zeta(s_1 + it) \cdots \zeta(s_k + it)\zeta(s_{k+1} - it) \cdots \zeta(s_{2k} - it) \, \frac{dt}{t^w}.$$

This is holomorphic in each of $s_1, \ldots, s_{2k}, w$ when all real parts are > 1. Diaconu, Goldfeld, and Hoffstein show that if $\mathrm{Re}(s_j) > 1$ for all j, then Z can be analytically continued to $\mathrm{Re}(w) > 0$, with a simple pole at $w = 1$ and no other poles.

- By the functional equation

$$\zeta(s + it) \approx e^{i\frac{\pi}{4}} \left(\frac{2\pi}{t}\right)^{s-\frac{1}{2}} \left(\frac{2\pi e}{t}\right)^{it} \zeta(1 - s - it),$$

we have

$$\begin{aligned} Z(s_1, \ldots, s_{2k}) \approx &(2\pi)^{s_1-\frac{1}{2}} e^{i\frac{\pi}{4}} \int_1^\infty \left(\frac{2\pi e}{t}\right)^{it} \zeta(1 - s_1 - it) \\ &\times \cdots \zeta(s_k + it)\zeta(s_{k+1} - it) \cdots \zeta(s_{2k} - it) \, \frac{dt}{t^{w+s_1-\frac{1}{2}}} \end{aligned}$$

The heuristic of Diaconu, Goldfeld, and Hoffstein (2000)

$$
\begin{aligned}
Z(s_1,\ldots,s_{2k}) &:= \int_1^\infty \zeta(s_1+it)\cdots\zeta(s_k+it)\zeta(s_{k+1}-it)\cdots\zeta(s_{2k}-it)\,\frac{dt}{t^w}\\
&\approx (2\pi)^{s_1-\frac12} e^{i\frac{\pi}{4}} \int_1^\infty \left(\frac{2\pi e}{t}\right)^{it} \zeta(1-s_1-it)\\
&\quad \times \cdots\zeta(s_k+it)\zeta(s_{k+1}-it)\cdots\zeta(s_{2k}-it)\,\frac{dt}{t^{w+s_1-\frac12}}
\end{aligned}
$$

- The latter essentially says

$$Z(s_1,\ldots,s_{2k}) \approx Z(1-s_1,s_2,\ldots,s_{2k},w+s_1-\tfrac12).$$

This, with the known region of meromorphy of Z, allows us to further extend the region of meromorphy.

- We may also apply the functional equation in the other s_j variables. We may compose any of these transformations to get new ones.

Theorem (Bochner's Tube Theorem)

If a function of several complex variables is holomorphic in a tube domain $\mathbf{T}$*, then it can be analytically continued to the convex hull of* $\mathbf{T}$*.*

A tube domain is a region in $\mathbb{C}^n$ that is of the form $\{\mathbf{x}+i\mathbf{y} : \mathbf{x}\in V\}$ for some open subset V of $\mathbb{R}^n$

The heuristic of Diaconu, Goldfeld, and Hoffstein (2000)

- Applying the functional equation and Bochner's Tube Theorem allows us to significantly extend the region of meromorphy of Z. However, it doesn't reach the point needed to evaluate moments of zeta.

Conjecture (Diaconu, Goldfeld, and Hoffstein (2000))

$Z(s_1, \ldots, s_{2k}, w)$ has a meromorphic continuation to a tube domain that contains that point $(\frac{1}{2}, \ldots, \frac{1}{2}, 1)$. In this domain, the only poles are those resulting from $w = 1$ and the functional equations.

- They show that this conjecture leads to the same predictions as Keating and Snaith for the moments of zeta. The key lemma is a Tauberian theorem due to Stark that relates the analytic properties of the function

$$w \longmapsto \int_1^\infty f(t) \frac{dt}{t^{1+w}}$$

to the asymptotic behavior of $f(x)$ as $x \to \infty$.

The heuristic of Conrey, Farmer, Keating, Rubinstein, and Snaith

- Let $\mathcal{M}_{A,B}(T) := \int_T^{2T} \prod_{\alpha\in A} \zeta(\frac{1}{2}+\alpha+it) \prod_{\beta\in B} \zeta(\frac{1}{2}+\beta-it)\,dt,$ where the "shifts" α, β are small complex numbers.
- basic steps in recipe: Use the approximate functional equation, ignore any oscillating terms, ignore any "off-diagonal" terms
- The expectation is that these ignored terms somehow cancel

Conjecture (CFKRS, 2000)

$$\mathcal{M}_{A,B}(T) \sim \sum_{\substack{U\subseteq A, V\subseteq B\\ |U|=|V|}} \int_T^{2T} \left(\frac{t}{2\pi}\right)^{-\sum_{\alpha\in U}\alpha - \sum_{\beta\in V}\beta} \sum_{n=1}^{\infty} \frac{\tau_{A\smallsetminus U\cup V^-}(n)\tau_{B\smallsetminus V\cup U^-}(n)}{n}\,dt$$

- Notation: $\prod_{\alpha\in A} \zeta(\alpha+s) =: \sum_{n=1}^{\infty} \frac{\tau_A(n)}{n^s}$ and $U^- := \{-\alpha : \alpha\in U\}$
- We call the cardinality $|U| = |V|$ the number of *swaps*
- Letting each $\alpha, \beta \to 0$ recovers the Keating-Snaith prediction

Analogous theorem in random matrix theory

Conjecture (CFKRS, 2000)

$$\int_T^{2T} \prod_{\alpha \in A} \zeta(\tfrac{1}{2} + \alpha + it) \prod_{\beta \in B} \zeta(\tfrac{1}{2} + \beta - it)\, dt \sim$$

$$\sum_{\substack{U \subseteq A, V \subseteq B \\ |U| = |V|}} \int_T^{2T} \left(\frac{t}{2\pi}\right)^{-\sum_{\alpha \in U} \alpha - \sum_{\beta \in V} \beta} \sum_{n=1}^{\infty} \frac{\tau_{A \setminus U \cup V^-}(n) \tau_{B \setminus V \cup U^-}(n)}{n}\, dt$$

Theorem (CFKRS, 2000)

Let $U(N)$ be the group of $N \times N$ unitary matrices. Then integrating with respect to the Haar measure gives

$$\int_{U(N)} \prod_{\alpha \in A} \det\left(1 - e^{-\alpha} M\right) \prod_{\beta \in B} \det\left(1 - e^{-\beta} M^{-1}\right) dM$$

$$= \sum_{\substack{U \subseteq A, V \subseteq B \\ |U| = |V|}} \left(e^N\right)^{-\sum_{\alpha \in U} \alpha - \sum_{\beta \in V} \beta} Z(A \setminus U \cup V^-, B \setminus V \cup U^-),$$

where $Z(A, B) := \prod_{\alpha \in A, \beta \in B} (1 - e^{-\alpha - \beta})^{-1}$.

The heuristic of Conrey and Keating (2017)

- Conrey and Keating examine Dirichlet polynomial approximations to $\zeta(s)$. Instead of

$$\int_T^{2T} \prod_{\alpha \in A} \zeta(\tfrac{1}{2} + \alpha + it) \prod_{\beta \in B} \zeta(\tfrac{1}{2} + \beta - it)\, dt,$$

they consider

$$\mathcal{M}_{A,B}(T,X) := \int_T^{2T} \sum_{m \leq X} \frac{\tau_A(m)}{m^{\frac{1}{2}+it}} \sum_{n \leq X} \frac{\tau_A(n)}{n^{\frac{1}{2}-it}}\, dt.$$

- We can use Perron's formula and the CFKRS heuristic to predict the asymptotic behavior of this. The point of the CK heuristic is to match this prediction by extending the ideas of Conrey and Gonek to the general $2k$th moment of zeta.

- Given an integer ℓ, we may partition $A = A_1 \cup \cdots \cup A_\ell$ and thus write the Dirichlet convolution $\tau_A = \tau_{A_1} * \cdots * \tau_{A_\ell}$. We can do the same for τ_B and write

$$\mathcal{M}_{A,B}(T,X) = \int_T^{2T} \sum_{m_1 \cdots m_\ell \leq X} \frac{\tau_{A_1}(m_1) \cdots \tau_{A_\ell}(m_\ell)}{(m_1 \cdots m_\ell)^{\frac{1}{2}+it}} \sum_{n_1 \cdots n_\ell \leq X} \frac{\tau_{B_1}(n_1) \cdots \tau_{B_\ell}(n_\ell)}{(n_1 \cdots n_\ell)^{\frac{1}{2}-it}}\, dt$$

$$\mathcal{M}_{A,B}(T,X) = \int_T^{2T} \sum_{m_1\cdots m_\ell \le X} \frac{\tau_{A_1}(m_1)\cdots\tau_{A_\ell}(m_\ell)}{(m_1\cdots m_\ell)^{\frac{1}{2}+it}} \sum_{n_1\cdots n_\ell \le X} \frac{\tau_{B_1}(n_1)\cdots\tau_{B_\ell}(n_\ell)}{(n_1\cdots n_\ell)^{\frac{1}{2}-it}}\, dt$$

- The idea behind the heuristic of Conrey and Keating is to group together the terms with m_j/n_j "close" to the "rational direction" M_j/N_j, and then sum over all possible directions (subject to some natural conditions). This was inspired by previous work of Bogomolny and Keating (1995).

- And so instead of the above expression, we examine

$$\mathcal{S}_\ell := \frac{1}{(\ell!)^2} \sum_{\substack{M_1\cdots M_\ell = N_1\cdots N_\ell \\ (M_j,N_j)=1\ \forall j}} \sum_{\substack{h_1,\ldots,h_\ell\in\mathbb{Z} \\ h_1\cdots h_\ell \ne 0}} \times \int_T^{2T} \sum_{\substack{1\le m_1,\ldots,m_\ell\le X \\ 1\le n_1,\ldots,n_\ell\le X \\ m_jN_j-n_jM_j=h_j\ \forall j}} \frac{\tau_{A_1}(m_1)\cdots\tau_{A_\ell}(m_\ell)\tau_{B_1}(n_1)\cdots\tau_{B_\ell}(n_\ell)}{(m_1\cdots m_\ell)^{\frac{1}{2}+it}(n_1\cdots n_\ell)^{\frac{1}{2}-it}}\, dt.$$

Conjecture (Conrey and Keating, 2017; B. and Conrey, 2022)

$$\mathcal{M}_{A,B}(T,X) \sim \sum^{\min\{|A|,|B|\}} \mathcal{S}_\ell$$

We apply the delta method in a way similar to that of Conrey and Gonek to conjecture for each j that

$$\sum_{\substack{1\leq m,n<\infty \\ mN_j-nM_j=h_j}} \tau_{A_j}(m)\tau_{B_j}(n)f(mN_j,nM_j)$$

$$\sim \frac{1}{(2\pi i)^2}\oint_{|z|=\varepsilon}\oint_{|w|=\varepsilon}\frac{1}{N_j^{1+z}M_j^{1+w}}\prod_{\alpha\in A_j}\zeta(1+z+\alpha)\prod_{\beta\in B_j}\zeta(1+w+\beta)$$

$$\times\sum_{q=1}^{\infty}\frac{c_q(h_j)(q,N_j)^{1+z}(q,M_j)^{1+w}}{q^{2+z+w}}G_{A_j}\left(1+z,\frac{q}{(q,N_j)}\right)G_{B_j}\left(1+w,\frac{q}{(q,M_j)}\right)$$

$$\times\int_{-\infty}^{\infty}x^z(x-h_j)^w f(x,x-h_j)\,dx\,dw\,dz,$$

where $c_q(h_j)$ is the Ramanujan sum and G_E is defined for sets E by

$$G_E(s,q)=\sum_{d|q}\frac{\mu(d)d^s}{\phi(d)}\sum_{e|d}\frac{\mu(e)}{e^s}g_E\left(s,\frac{eq}{d}\right),$$

with $g_E(s,n)$ defined by

$$g_E(s,n)=\prod_{p|n}\left\{\prod_{\gamma\in E}(1-p^{-s-\gamma})\right\}\sum_{m=0}^{\infty}\frac{\tau_E(p^{m+\mathrm{ord}_p(n)})}{p^{ms}}$$

- Conrey and Keating do *not* write the delta method prediction as an integral but instead as a sum over elements of A and B.

- Conrey and I refine the heuristic by realizing the sum as a sum over residues and thus writing it as an integral in the previous slide.

- We then write sums as Euler products and then apply the identity (which is locally rigorous)

$$\sum_{\substack{M_1\cdots M_\ell = N_1\cdots N_\ell \\ (M_j,N_j)=1\ \forall j}} \prod_{j=1}^{\ell} \left(\sum_{\substack{1\le m,n<\infty \\ mN_j = nM_j}} \frac{\tau_{A_j}(m)\tau_{B_j}(n)}{m^s n^s} \right) = \sum_{\substack{1\le m,n<\infty \\ m=n}} \frac{\tau_A(m)\tau_B(n)}{m^s n^s}$$

- This leads to the prediction that $\mathcal{S}_\ell$ is asymptotic to a certain "Vandermonde integral"

Vandermonde integral prediction for zeta (B. and Conrey)

$$
\begin{aligned}
\mathcal{S}_\ell \sim \frac{1}{(\ell!)^2(2\pi i)^2} \int_{(2)}\int_{(2)} \frac{X^{\xi+\eta}}{\xi\eta} \int_T^{2T} \frac{1}{(2\pi i)^{2\ell}} \oint_{|z_1|=\varepsilon} \cdots \oint_{|z_\ell|=\varepsilon} \\
\times \oint_{|w_1|=\varepsilon} \cdots \oint_{|w_\ell|=\varepsilon} \left(\frac{t}{2\pi}\right)^{\sum_{j=1}^{\ell}(z_j+w_j-\xi-\eta)} \\
\times \prod_{\substack{\alpha\in A\\ \beta\in B}} \zeta(1+\alpha+\beta+\xi+\eta) \prod_{\substack{1\le j\le \ell\\ \alpha\in A}} \zeta(1+\alpha+z_j) \prod_{\substack{1\le j\le \ell\\ \beta\in B}} \zeta(1+\beta+w_j) \\
\times \prod_{\substack{1\le j\le \ell\\ \alpha\in A}} (1/\zeta)(1+\alpha+\xi+\eta-w_j) \prod_{\substack{1\le j\le \ell\\ \beta\in B}} (1/\zeta)(1+\beta+\xi+\eta-z_j) \\
\times \prod_{\substack{1\le i,j\le \ell\\ i\ne j}} (1/\zeta)(1-z_i+z_j) \prod_{\substack{1\le i,j\le \ell\\ i\ne j}} (1/\zeta)(1-w_i+w_j) \\
\times \prod_{1\le i,j\le \ell} \zeta(1+z_i+w_j-\xi-\eta)\zeta(1-z_i-w_j+\xi+\eta) \\
\times \mathcal{A}(A,B,Z,W,\xi+\eta)\, dw_\ell \cdots dw_1\, dz_\ell \cdots dz_1\, d\xi\, d\eta
\end{aligned}
$$

- Here, $\mathcal{A}$ is an absolutely convergent Euler product that has an explicit definition.
- The right-hand side may be evaluated to give exactly the ℓ-swap terms from the recipe prediction for moments of zeta.

Let $U(N)$ denote the group of $N \times N$ unitary matrices, with Haar measure. Then

$$\int_{U(N)} \prod_{\alpha \in A} \det\left(1 - e^{-\alpha} g\right) \prod_{\beta \in B} \det\left(1 - e^{-\beta} g^{-1}\right) dg = \sum_{\ell=0}^{\min\{|A|,|B|\}} J_\ell,$$

where J_ℓ is defined by

$$\begin{aligned} J_\ell = &\frac{1}{(\ell!)^2 (2\pi i)^{2\ell+1}} \oint_{|\xi|=1} \mathfrak{z}(\xi) \oint_{|z_1|=\varepsilon} \cdots \oint_{|z_\ell|=\varepsilon} \oint_{|w_1|=\varepsilon} \cdots \oint_{|w_\ell|=\varepsilon} \\ &\times \left(e^N\right)^{\sum_{j=1}^{\ell}(z_j + w_j - \xi)} \prod_{\substack{\alpha \in A \\ \beta \in B}} \mathfrak{z}(\alpha + \beta) \prod_{\substack{1 \le j \le \ell \\ \alpha \in A}} \mathfrak{z}(\alpha + z_j) \prod_{\substack{1 \le j \le \ell \\ \beta \in B}} \mathfrak{z}(\beta + w_j) \\ &\times \prod_{\substack{1 \le j \le \ell \\ \alpha \in A}} (1/\mathfrak{z})(\alpha + \xi - w_j) \prod_{\substack{1 \le j \le \ell \\ \beta \in B}} (1/\mathfrak{z})(\beta + \xi - z_j) \\ &\times \prod_{\substack{1 \le i,j \le \ell \\ i \ne j}} (1/\mathfrak{z})(z_i - z_j) \prod_{\substack{1 \le i,j \le \ell \\ i \ne j}} (1/\mathfrak{z})(w_i - w_j) \\ &\times \prod_{1 \le i,j \le \ell} \mathfrak{z}(z_i + w_j - \xi) \mathfrak{z}(-z_i - w_j + \xi) \\ &\times dw_\ell \cdots dw_1 \, dz_\ell \cdots dz_1 \, d\xi, \end{aligned}$$

with $\mathfrak{z}(x)$ defined by $\mathfrak{z}(x) := (1 - e^{-x})^{-1}$.

Arithmetic stratification

- In an AIM Workshop in 2016, Trevor Wooley suggested that the Conrey-Keating heuristic has an interpretation in terms of the counting of rational points in algebraic varieties that is the subject of Manin's arithmetic stratification conjectures.

- The idea is that splitting the "higher" moments of zeta into "lower" twisted moments, as in the Conrey-Keating heuristic, is analogous to counting rational points on high dimensional varieties by stratification and counting points on subvarieties.

Higher swap terms: The Conrey-Keating heuristic

- Inspired by ideas of Bogomolny and Keating (1995), Conrey and Keating (2019) have developed a heuristic that shows how we may be able to split moments of zeta to get the ℓ-swap terms for general ℓ.
- In ongoing work, Brian Conrey and I refine the heuristic and adapt it to other families of L-functions.
- Starting observation: if we partition $A = A_1 \cup \cdots \cup A_\ell$, then $\tau_A = \tau_{A_1} * \cdots * \tau_{A_\ell}$, and similarly for B. Thus

$$\int_T^{2T} \sum_{m\leq X} \frac{\tau_A(m)}{m^{\frac{1}{2}+it}} \sum_{n\leq X} \frac{\tau_B(n)}{n^{\frac{1}{2}-it}}\, dt$$

$$= \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(2)} \frac{X^{\xi+\eta}}{\xi\eta} \int_T^{2T} \sum_{m=1}^{\infty} \frac{\tau_A(m)}{m^{\frac{1}{2}+\xi+it}} \sum_{n=1}^{\infty} \frac{\tau_B(n)}{n^{\frac{1}{2}+\eta-it}}\, dt\, d\xi\, d\eta$$

$$= \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(2)} \frac{X^{\xi+\eta}}{\xi\eta} \int_T^{2T} \prod_{j=1}^{\ell} \left\{ \sum_{m=1}^{\infty} \frac{\tau_{A_j}(m)}{m^{\frac{1}{2}+\xi+it}} \sum_{n=1}^{\infty} \frac{\tau_{B_j}(n)}{n^{\frac{1}{2}+\eta-it}} \right\} dt\, d\xi\, d\eta$$

- Instead of using this, we introduce some twisting and replace each factor by its average with respect to *only* the 1-swap terms

- By Perron's formula and writing τ_A, τ_B as Dirichlet convolutions (from previous slide):

$$\int_T^{2T} \sum_{m \le X} \frac{\tau_A(m)}{m^{\frac{1}{2}+it}} \sum_{n \le X} \frac{\tau_B(n)}{n^{\frac{1}{2}-it}} \, dt$$

$$= \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(2)} \frac{X^{\xi+\eta}}{\xi\eta} \int_T^{2T} \prod_{j=1}^{\ell} \left\{ \sum_{m=1}^{\infty} \frac{\tau_{A_j}(m)}{m^{\frac{1}{2}+\xi+it}} \sum_{n=1}^{\infty} \frac{\tau_{B_j}(n)}{n^{\frac{1}{2}+\eta-it}} \right\} dt \, d\xi \, d\eta$$

- We consider instead

$$\mathcal{S}_\ell := \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(2)} \frac{X^{\xi+\eta}}{\xi\eta}$$

$$\times \int_T^{2T} \frac{1}{(\ell!)^2} \sum_{\substack{M_1 \cdots M_\ell = N_1 \cdots N_\ell \\ (M_j, N_j) = 1 \ \forall j}} \prod_{j=1}^{\ell} \left\langle \left(\frac{M_j}{N_j} \right)^{it} \sum_{m=1}^{\infty} \frac{\tau_{A_j}(m)}{m^{\frac{1}{2}+\xi+it}} \sum_{n=1}^{\infty} \frac{\tau_{B_j}(n)}{n^{\frac{1}{2}+\eta-it}} \right\rangle_1 \times dt \, d\xi \, d\eta$$

- Here, the symbols $\langle \ \rangle_1$ mean that we are replacing the integrand of the twisted moment by its average with respect to *only* the 1-swap terms.

$$\mathcal{S}_\ell := \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(2)} \frac{X^{\xi+\eta}}{\xi\eta} \int_T^{2T} \frac{1}{(\ell!)^2} \sum_{\substack{M_1\cdots M_\ell = N_1\cdots N_\ell \\ (M_j,N_j)=1\ \forall j}} \prod_{j=1}^{\ell} \Bigg\{ \sum_{\substack{U\subseteq A_j, V\subseteq B_j \\ |U|=|V|=1}}$$

$$\times \left(\frac{t}{2\pi}\right)^{-\sum\limits_{\alpha\in U}(\alpha+\xi) - \sum\limits_{\beta\in V}(\beta+\eta)} \sum_{\substack{1\le m,n<\infty \\ mN_j = nM_j}} \frac{\tau_{(A_j)_\xi \smallsetminus U_\xi \cup V_\eta^-}(m)\tau_{(B_j)_\eta \smallsetminus V_\eta \cup U_\xi^-}(n)}{\sqrt{mn}} \Bigg\}$$

$$\times\, dt\, d\xi\, d\eta$$

- We write the sum over U, V as a multiple contour integral along small circles
- We then evaluate the sum over $M_1, \ldots, M_\ell, N_1, \ldots, N_\ell$ using the "unitary identity"

$$\sum_{\substack{M_1\cdots M_\ell = N_1\cdots N_\ell \\ (M_j,N_j)=1\ \forall j}} \prod_{j=1}^{\ell} \left(\sum_{\substack{1\le m,n<\infty \\ mN_j = nM_j}} \frac{\tau_{A_j}(m)\tau_{B_j}(n)}{m^s n^s} \right) = \sum_{\substack{1\le m,n<\infty \\ m=n}} \frac{\tau_A(m)\tau_B(n)}{m^s n^s}$$

- This leads to the prediction that $\mathcal{S}_\ell$ is asymptotic to a certain "Vandermonde integral"

Vandermonde integral prediction for zeta (B. and Conrey)

$$
\begin{aligned}
\mathcal{S}_\ell \sim \frac{1}{(\ell!)^2(2\pi i)^2} \int_{(2)} \int_{(2)} \frac{X^{\xi+\eta}}{\xi\eta} \int_T^{2T} \frac{1}{(2\pi i)^{2\ell}} \oint_{|z_1|=\varepsilon} \cdots \oint_{|z_\ell|=\varepsilon} \\
\times \oint_{|w_1|=\varepsilon} \cdots \oint_{|w_\ell|=\varepsilon} \left(\frac{t}{2\pi}\right)^{\sum_{j=1}^{\ell}(z_j+w_j-\xi-\eta)} \\
\times \prod_{\substack{\alpha\in A\\ \beta\in B}} \zeta(1+\alpha+\beta+\xi+\eta) \prod_{\substack{1\le j\le \ell\\ \alpha\in A}} \zeta(1+\alpha+z_j) \prod_{\substack{1\le j\le \ell\\ \beta\in B}} \zeta(1+\beta+w_j) \\
\times \prod_{\substack{1\le j\le \ell\\ \alpha\in A}} (1/\zeta)(1+\alpha+\xi+\eta-w_j) \prod_{\substack{1\le j\le \ell\\ \beta\in B}} (1/\zeta)(1+\beta+\xi+\eta-z_j) \\
\times \prod_{\substack{1\le i,j\le \ell\\ i\ne j}} (1/\zeta)(1-z_i+z_j) \prod_{\substack{1\le i,j\le \ell\\ i\ne j}} (1/\zeta)(1-w_i+w_j) \\
\times \prod_{1\le i,j\le \ell} \zeta(1+z_i+w_j-\xi-\eta)\zeta(1-z_i-w_j+\xi+\eta) \\
\times \mathcal{A}(A,B,Z,W,\xi+\eta)\, dw_\ell \cdots dw_1\, dz_\ell \cdots dz_1\, d\xi\, d\eta
\end{aligned}
$$

- Here, $\mathcal{A}$ is an absolutely convergent Euler product that has an explicit definition.
- The right-hand side may be evaluated to give exactly the ℓ-swap terms from the recipe prediction for moments of zeta.

Let $U(N)$ denote the group of $N \times N$ unitary matrices, with Haar measure. Then

$$\int_{U(N)} \prod_{\alpha \in A} \det\left(1 - e^{-\alpha} g\right) \prod_{\beta \in B} \det\left(1 - e^{-\beta} g^{-1}\right) dg = \sum_{\ell=0}^{\min\{|A|,|B|\}} J_\ell,$$

where J_ℓ is defined by

$$\begin{aligned} J_\ell = & \frac{1}{(\ell!)^2 (2\pi i)^{2\ell+1}} \oint_{|\xi|=1} \mathfrak{z}(\xi) \oint_{|z_1|=\varepsilon} \cdots \oint_{|z_\ell|=\varepsilon} \oint_{|w_1|=\varepsilon} \cdots \oint_{|w_\ell|=\varepsilon} \\ & \times \left(e^N\right)^{\sum_{j=1}^{\ell}(z_j + w_j - \xi)} \prod_{\substack{\alpha \in A \\ \beta \in B}} \mathfrak{z}(\alpha+\beta) \prod_{\substack{1 \le j \le \ell \\ \alpha \in A}} \mathfrak{z}(\alpha + z_j) \prod_{\substack{1 \le j \le \ell \\ \beta \in B}} \mathfrak{z}(\beta + w_j) \\ & \times \prod_{\substack{1 \le j \le \ell \\ \alpha \in A}} (1/\mathfrak{z})(\alpha + \xi - w_j) \prod_{\substack{1 \le j \le \ell \\ \beta \in B}} (1/\mathfrak{z})(\beta + \xi - z_j) \\ & \times \prod_{\substack{1 \le i,j \le \ell \\ i \ne j}} (1/\mathfrak{z})(z_i - z_j) \prod_{\substack{1 \le i,j \le \ell \\ i \ne j}} (1/\mathfrak{z})(w_i - w_j) \\ & \times \prod_{1 \le i,j \le \ell} \mathfrak{z}(z_i + w_j - \xi)\mathfrak{z}(-z_i - w_j + \xi) \\ & \times dw_\ell \cdots dw_1 \, dz_\ell \cdots dz_1 \, d\xi, \end{aligned}$$

with $\mathfrak{z}(x)$ defined by $\mathfrak{z}(x) := (1 - e^{-x})^{-1}$.

- Let

$$\mathcal{S}_\ell := \frac{1}{(2\pi i)^2}\int_{(2)}\int_{(2)} \frac{X^{\xi+\eta}}{\xi\eta} \sum_{Q<q<2Q} \sideset{}{^*}\sum_{\chi \bmod q} \frac{1}{(\ell!)^2}$$

$$\times \sum_{\substack{M_1\cdots M_\ell = N_1\cdots N_\ell \\ (M_j,N_j)=1\ \forall j}} \prod_{j=1}^{\ell} \left\langle \overline{\chi}(M_j)\chi(N_j) \sum_{m=1}^\infty \frac{\tau_{A_j}(m)\chi(m)}{m^{\frac12+\xi}} \sum_{n=1}^\infty \frac{\tau_{B_j}(n)\overline{\chi}(n)}{n^{\frac12+\eta}} \right\rangle_1 \times d\xi\, d\eta$$

- Again, $A = A_1 \cup \cdots \cup A_\ell$ and similarly for B, and the symbols $\langle\ \rangle_1$ mean that we are replacing the integrand of the twisted moment by its average with respect to *only* the 1-swap terms.
- As before, we evaluate the M_j, N_j-sum using the unitary identity

$$\sum_{\substack{M_1\cdots M_\ell = N_1\cdots N_\ell \\ (M_j,N_j)=1\ \forall j \\ (M_jN_j,q)=1\ \forall j}} \prod_{j=1}^{\ell} \left(\sum_{\substack{1\le m,n<\infty \\ mN_j = nM_j \\ (mn,q)=1}} \frac{\tau_{A_j}(m)\tau_{B_j}(n)}{m^s n^s} \right) = \sum_{\substack{1\le m,n<\infty \\ m=n \\ (mn,q)=1}} \frac{\tau_A(m)\tau_B(n)}{m^s n^s}$$

- This leads to the prediction that $\mathcal{S}_\ell$ is asymptotic to the following Vandermonde integral

$$\mathcal{S}_\ell \sim \frac{1}{(\ell!)^2(2\pi i)^2}\int_{(2)}\int_{(2)}\frac{X^{\xi+\eta}}{\xi\eta}\sum_{Q<q<2Q}\sideset{}{^*}\sum_{\chi \bmod q}\frac{1}{(2\pi i)^{2\ell}}\oint_{|z_1|=\varepsilon}\cdots\oint_{|z_\ell|=\varepsilon}$$
$$\times\oint_{|w_1|=\varepsilon}\cdots\oint_{|w_\ell|=\varepsilon}\left(\frac{q}{\pi}\right)^{\sum_{j=1}^{\ell}(z_j+w_j-\xi-\eta)}$$
$$\times\prod_{\substack{\alpha\in A\\ \beta\in B}}\zeta(1+\alpha+\beta+\xi+\eta)\prod_{\substack{1\le j\le\ell\\ \alpha\in A}}\zeta(1+\alpha+z_j)\prod_{\substack{1\le j\le\ell\\ \beta\in B}}\zeta(1+\beta+w_j)$$
$$\times\prod_{\substack{1\le j\le\ell\\ \alpha\in A}}(1/\zeta)(1+\alpha+\xi+\eta-w_j)\prod_{\substack{1\le j\le\ell\\ \beta\in B}}(1/\zeta)(1+\beta+\xi+\eta-z_j)$$
$$\times\prod_{\substack{1\le i,j\le\ell\\ i\ne j}}(1/\zeta)(1-z_i+z_j)\prod_{\substack{1\le i,j\le\ell\\ i\ne j}}(1/\zeta)(1-w_i+w_j)$$
$$\times\prod_{1\le i,j\le\ell}\zeta(1+z_i+w_j-\xi-\eta)\zeta(1-z_i-w_j+\xi+\eta)$$
$$\times\mathcal{A}_q(A,B,Z,W,\xi+\eta)\,dw_\ell\cdots dw_1\,dz_\ell\cdots dz_1\,d\xi\,d\eta$$

- Here, $\mathcal{A}_q$ is an absolutely convergent Euler product that has an explicit definition.
- The right-hand side may be evaluated to give exactly the ℓ-swap terms from the recipe prediction for moments of primitive Dirichlet L-functions.

- Let

$$\mathcal{S}_\ell := \frac{1}{2\pi i}\int_{(2)} \frac{X^\xi}{\xi} \sum_{D<d<2D} \mu^2(2d)\,\frac{1}{\ell!}$$

$$\times \sum_{M_1\cdots M_\ell=\square} \mu^2(2M_1)\cdots\mu^2(2M_\ell)\prod_{j=1}^{\ell}\left\langle (8d|M_j)\sum_{n=1}^{\infty}\frac{\tau_{A_j}(n)(8d|n)}{n^{\frac{1}{2}+\xi}}\right\rangle_1 d\xi$$

- Again, $A = A_1 \cup \cdots \cup A_\ell$, and the symbols $\langle\ \rangle_1$ mean that we are replacing the integrand of the twisted moment by its average with respect to *only* the 1-swap terms.
- We then evaluate the M_j-sums using the "symplectic identity"

$$\sum_{\substack{M_1\cdots M_\ell=\square\\(M_1\cdots M_\ell,d)=1}} \mu^2(2M_1)\cdots\mu^2(2M_\ell)\prod_{j=1}^{\ell}\sum_{\substack{1\le n<\infty\\ nM_j=\square\\(n,2d)=1}}\frac{\tau_{A_j}(n)}{n^s} = \sum_{\substack{1\le n<\infty\\ n=\square\\(n,2d)=1}}\frac{\tau_A(n)}{n^s}.$$

- This leads to the prediction that $\mathcal{S}_\ell$ is asymptotic to the following Vandermonde integral

$$\mathcal{S}_\ell \sim \frac{1}{2\pi i(\ell!)} \int_{(2)} \frac{X^\xi}{\xi} \sum_{D<d<2D} \mu^2(2d) \frac{1}{(2\pi i)^\ell} \oint_{|z_1|=\varepsilon} \cdots \oint_{|z_\ell|=\varepsilon}$$

$$\times \left(\frac{8d}{\pi}\right)^{\sum_{j=1}^{\ell}(z_j-\xi)} \prod_{\alpha\in A} \zeta(1+2\alpha+2\xi) \prod_{\substack{\{\alpha,\hat\alpha\}\subseteq A \\ \alpha\neq\hat\alpha}} \zeta(1+\alpha+\hat\alpha+2\xi)$$

$$\times \prod_{j=1}^{\ell} \zeta(1+2z_j-2\xi) \prod_{\substack{1\le j\le \ell \\ \alpha\in A}} \zeta(1+\alpha+z_j)(1/\zeta)(1+\alpha-z_j+2\xi)$$

$$\times \prod_{1\le i<j\le \ell} \zeta(1-z_i-z_j+2\xi)\zeta(1+z_i+z_j-2\xi)$$

$$\times \prod_{1\le i\neq j\le \ell} (1/\zeta)(1+z_i-z_j)\ \mathcal{A}_d(A,Z,\xi)\ dz_\ell\cdots dz_1\, d\xi$$

- Here, $\mathcal{A}_d$ is an absolutely convergent Euler product that has an explicit definition.
- The right-hand side may be evaluated to give exactly the ℓ-swap terms from the recipe prediction for moments of quadratic Dirichlet L-functions.

Vandermonde integral theorem for $USp(2N)$ (B. and Conrey)

Let $USp(2N)$ denote the group of $2N \times 2N$ symplectic unitary matrices, with Haar measure. Then

$$\int_{USp(2N)} \prod_{\alpha \in A} \det\left(1 - e^{-\alpha} g\right) dg = \sum_{\ell=0}^{|A|} J_\ell,$$

where J_ℓ is defined by

$$\begin{aligned}
J_\ell = & \frac{1}{(\ell!)(2\pi i)^{\ell+1}} \oint_{|\xi|=1} \mathfrak{z}(\xi) \oint_{|z_1|=\varepsilon} \cdots \oint_{|z_\ell|=\varepsilon} \left(e^{2N}\right)^{-\sum\limits_{\alpha \in A} \alpha - \sum\limits_{j=1}^{\ell} (z_j - \xi)} \\
& \times \prod_{\alpha \in A} \mathfrak{z}(2\alpha) \prod_{\substack{\{\alpha, \hat{\alpha}\} \subseteq A \\ \alpha \neq \hat{\alpha}}} \mathfrak{z}(\alpha + \hat{\alpha}) \prod_{j=1}^{\ell} \mathfrak{z}(2z_j - 2\xi) \\
& \times \prod_{\substack{1 \le j \le \ell \\ \alpha \in A}} \mathfrak{z}(\alpha + z_j)(1/\mathfrak{z})(\alpha - z_j + 2\xi) \\
& \times \prod_{1 \le i < j \le \ell} \mathfrak{z}(-z_i - z_j + 2\xi)\mathfrak{z}(z_i + z_j - 2\xi) \\
& \times \prod_{1 \le i \neq j \le \ell} (1/\mathfrak{z})(z_i - z_j)\, dz_\ell \cdots dz_1\, d\xi,
\end{aligned}$$

with $\mathfrak{z}(x)$ defined by $\mathfrak{z}(x) := (1 - e^{-x})^{-1}$.

- Let $A = A_1 \cup \cdots \cup A_\ell$ and

$$\mathcal{S}_\ell := \frac{1}{2\pi i} \int_{(2)} \frac{X^\xi}{\xi} \sum_{f \in \mathcal{H}_k} \frac{1}{\omega_f(\ell!)}$$

$$\times \sum_{1 \le M_1, \dots M_\ell < \infty} g(M_1, \dots, M_\ell) \prod_{j=1}^{\ell} \left\langle \lambda_f(M_j) \prod_{\alpha \in A_j} \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{\frac{1}{2}+\alpha+\xi}} \right\rangle_1 d\xi,$$

where $g(M_1, \dots, M_\ell)$ is defined by

$$g(M_1, \dots, M_\ell) := \prod_p \frac{2}{\pi} \int_0^\pi U_{\mathrm{ord}_p(M_1)}(\cos\theta) \cdots U_{\mathrm{ord}_p(M_\ell)}(\cos\theta) \sin^2\theta \, d\theta.$$

- We evaluate the M_j-sums using the "orthogonal identity"

$$\sum_{1 \le M_1, \dots M_\ell < \infty} g(M_1, \dots, M_\ell) \prod_{j=1}^{\ell} G_{M_j}(A_j; s) = G_1(A; s),$$

where $G_M(A; s)$ is the function we saw in a previous slide: if $A = \{\alpha_1, \dots, \alpha_r\}$ then $G_M(A; s)$ is defined by

$$G_M(A; s) := \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{m_1 \cdots m_r = m} \frac{g(m_1, \dots, m_r, M)}{m_1^{\alpha_1} \cdots m_r^{\alpha_r}}$$

- This leads to the Vandermonde integral prediction

$$\mathcal{S}_\ell \sim \frac{1}{2\pi i(\ell!)} \int_{(2)} \frac{X^\xi}{\xi} \sum_{f\in\mathcal{H}_k} \frac{1}{\omega_f} \frac{1}{(2\pi i)^\ell} \oint_{|z_1|=\varepsilon} \cdots \oint_{|z_\ell|=\varepsilon}$$
$$\times \left(\frac{k^2}{4\pi^2}\right)^{\sum_{j=1}^{\ell}(z_j-\xi)} \prod_{\substack{\{\alpha,\hat\alpha\}\subseteq A\\ \alpha\neq\hat\alpha}} \zeta(1+\alpha+\hat\alpha+2\xi)$$
$$\times \prod_{j=1}^{\ell} \zeta(1-2z_j+2\xi) \prod_{\substack{1\le j\le\ell\\ \alpha\in A}} \zeta(1+\alpha+z_j)(1/\zeta)(1+\alpha-z_j+2\xi)$$
$$\times \prod_{1\le i<j\le\ell} \zeta(1-z_i-z_j+2\xi)\zeta(1+z_i+z_j-2\xi)$$
$$\times \prod_{1\le i\neq j\le\ell} (1/\zeta)(1+z_i-z_j)\ \mathcal{A}(A;z_1,\ldots,z_\ell;\xi)\ dz_\ell\cdots dz_1\, d\xi$$

- Here, $\mathcal{A}$ is an absolutely convergent Euler product that has an explicit definition.
- The right-hand side may be evaluated to give exactly the ℓ-swap terms from the recipe prediction for moments of the family of $L_f(s) := \sum_{n=1}^{\infty} \lambda_f(n)n^{-s}$ with $f \in \mathcal{H}_k$ (Hecke eigencuspforms of weight k)

Let $SO(2N)$ denote the group of $2N \times 2N$ orthogonal matrices with determinant $+1$. Then, integrating with respect to Haar measure, we have

$$\int_{SO(2N)} \prod_{\alpha\in A} \det\left(1 - e^{-\alpha} g\right) dg = \sum_{\ell=0}^{|A|} J_\ell,$$

where J_ℓ is defined by

$$\begin{aligned}
J_\ell = &\frac{1}{(\ell!)(2\pi i)^{\ell+1}} \oint_{|\xi|=1} \mathfrak{z}(\xi) \oint_{|z_1|=\varepsilon} \cdots \oint_{|z_\ell|=\varepsilon} \left(e^{2N}\right)^{-\sum_{\alpha\in A}\alpha - \sum_{j=1}^{\ell}(z_j-\xi)} \\
&\times \prod_{\substack{\{\alpha,\hat\alpha\}\subseteq A\\ \alpha\neq\hat\alpha}} \mathfrak{z}(\alpha+\hat\alpha) \prod_{j=1}^{\ell} \mathfrak{z}(-2z_j+2\xi) \\
&\times \prod_{\substack{1\le j\le \ell\\ \alpha\in A}} \mathfrak{z}(\alpha+z_j)(1/\mathfrak{z})(\alpha - z_j + 2\xi) \\
&\times \prod_{1\le i<j\le \ell} \mathfrak{z}(-z_i-z_j+2\xi)\mathfrak{z}(z_i+z_j-2\xi) \\
&\times \prod_{1\le i\neq j\le \ell} (1/\mathfrak{z})(z_i-z_j)\, dz_\ell \cdots dz_1\, d\xi,
\end{aligned}$$

with $\mathfrak{z}(x)$ defined by $\mathfrak{z}(x) := (1-e^{-x})^{-1}$.

Arithmetic stratification

- In an AIM Workshop in 2016, Trevor Wooley suggested that the Conrey-Keating heuristic has an interpretation in terms of the counting of rational points in algebraic varieties that is the subject of Manin's arithmetic stratification conjectures.

- The idea is that splitting the "higher" moments of zeta into "lower" twisted moments, as in the Conrey-Keating heuristic, is analogous to counting rational points on high dimensional varieties by stratification and counting points on subvarieties.

Two related open problems

- Find a random matrix theory analogue (theorem) of the Conrey-Keating heuristic.

- What is the "correct" analogue of twisted moments in random matrix theory?