Twisted $2k$th moments of primitive Dirichlet L-functions: beyond the diagonal

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Why study moments of L-functions?

- Moments contain information about the size of the L-functions and the distribution of their values.

- We can use variants of moments to deduce properties of L-functions, such as the distribution of their zeros. Moments are also useful in studying the behavior of arithmetic functions like the divisor function.

- The *Katz-Sarnak philosophy* is the notion that the behavior of families of L-functions is governed by some underlying symmetry groups.

- Studying different families may allow us to compare them and investigate how their (conjectured) symmetry types affect their behavior.

$$\sum_{\chi \bmod q}^{*} |L(\tfrac{1}{2}, \chi)|^{2k} \qquad \sum_{d \le X}^{\flat} L(\tfrac{1}{2}, \chi_d)^k \qquad \sum_{f \in \mathcal{H}_r} \frac{1}{\omega_r} L(\tfrac{1}{2}, f)^k$$

The family of primitive Dirichlet L-functions: known facts

- Paley (1931): $\displaystyle\sum_{\chi \bmod q} |L(\tfrac{1}{2},\chi)|^2 \sim \frac{\phi^2(q)}{q}\log q$
- Heath-Brown (1981) and Soundararajan (2007):
$$\sideset{}{^*}\sum_{\chi \bmod q} |L(\tfrac{1}{2},\chi)|^4 \sim \frac{\phi^*(q)}{2\pi^2}(\log q)^4 \prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})},$$
where $\phi^*(q) := \sum^*_{\chi \bmod q} 1$.
- Young (2011): improved error term for $q = p$
- Heap and Soundararajan (2022): For all real $k > 0$, we have
$$\sideset{}{^*}\sum_{\chi \bmod p} |L(\tfrac{1}{2},\chi)|^{2k} \gg p(\log p)^{k^2}$$
- Harper (2013): Assuming GRH, if $k > 0$ then
$$\sideset{}{^*}\sum_{\chi \bmod p} |L(\tfrac{1}{2},\chi)|^{2k} \ll p(\log p)^{k^2}$$

Additional averaging and the large sieve

- Huxley (1970) If we also average over q, then we can leverage the large sieve:

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q} |L(\tfrac{1}{2},\chi)|^6 \ll Q^2(\log Q)^9 \quad\text{and}\quad \sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q} |L(\tfrac{1}{2},\chi)|^8 \ll Q^2(\log Q)^{16}$$

- Conrey, Iwaniec, and Soundararajan (2012): For some explicit constant C_1,

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q} \int_{-\infty}^{\infty} |\Lambda(\tfrac{1}{2}+iy,\chi)|^6\,dy \sim C_1 \sum_{q\le Q} \phi^*(q)(\log q)^9 \prod_{p\mid q} \frac{(1-\frac{1}{p})^5}{(1+\frac{4}{p}+\frac{1}{p^2})},$$

where $\Lambda(s,\chi) := \left(\frac{q}{\pi}\right)^{s/2}\Gamma(\frac{s}{2})L(s,\chi)$ is the completed L-function. Their main new idea is a technique called the asymptotic large sieve.

- Chandee, Li, Matomäki, and Radziwiłł (2020): For some explicit C_2,

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q} |L(\tfrac{1}{2},\chi)|^6 \sim C_2 \sum_{q\le Q} \phi^*(q)(\log q)^9 \prod_{p\mid q} \frac{(1-\frac{1}{p})^5}{(1+\frac{4}{p}+\frac{1}{p^2})}.$$

- Chandee and Li (2014): Assuming GRH, for some explicit C_3 we have

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q} \int_{-\infty}^{\infty} |\Lambda(\tfrac{1}{2}+iy,\chi)|^8\,dy \sim C_3 \sum_{q\le Q} \phi^*(q)(\log q)^{16} \prod_{p\mid q} \frac{(1-\frac{1}{p})^7}{(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3})}.$$

- Chandee, Li, Matomäki, and Radziwiłł (2022): This holds *unconditionally*.

The CFKRS recipe (guided by random matrix theory)

Conjecture (Conrey, Farmer, Keating, Rubinstein, & Snaith, 2005):

$$\sum_{\chi \bmod q}^{*} \prod_{\alpha\in A} L(\tfrac{1}{2}+\alpha,\chi) \prod_{\beta\in B} L(\tfrac{1}{2}+\beta,\overline{\chi})$$

$$\sim \sum_{\substack{U\subseteq A, V\subseteq B\\ |U|=|V|}} \sum_{\chi \bmod q}^{*} \left(\frac{q}{\pi}\right)^{-\sum\limits_{\alpha\in U}\alpha - \sum\limits_{\beta\in V}\beta} \sum_{\substack{1\le m,n<\infty\\ m=n\\ (mn,q)=1}} \frac{\tau_{A\smallsetminus U\cup V^-}(m)\tau_{B\smallsetminus V\cup U^-}(n)}{\sqrt{mn}}$$

- Here, the "shifts" α,β are small complex numbers (say $\ll 1/\log q$)
- Notation: $\displaystyle\prod_{\alpha\in A}\zeta(\alpha+s) =: \sum_{n=1}^{\infty}\frac{\tau_A(n)}{n^s}$ and $U^- := \{-\alpha : \alpha\in U\}$
- We call the cardinality $|U|=|V|$ the number of *swaps*

- To understand how the off-diagonal terms cancel, Conrey and Keating (2015) examined Dirichlet polynomial approximations of zeta.
- Using Perron's formula and the recipe, we expect that

$$\sum_{\chi \bmod q}^{*} \sum_{m \leq X} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} \sum_{n \leq X} \frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}}$$

$$\sim \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \frac{X^{s_1+s_2}}{s_1 s_2} \sum_{\chi \bmod q}^{*} \prod_{\alpha \in A} L(\tfrac{1}{2}+\alpha+s_1,\chi) \prod_{\beta \in B} L(\tfrac{1}{2}+\beta+s_2,\overline{\chi})\, ds_2\, ds_1$$

$$\sim \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \frac{X^{s_1+s_2}}{s_1 s_2} \sum_{\substack{U \subseteq A, V \subseteq B \\ |U|=|V|}} \sum_{\chi \bmod q}^{*} \left(\frac{q}{\pi}\right)^{-\sum\limits_{\alpha \in U}(\alpha+s_1) - \sum\limits_{\beta \in V}(\beta+s_2)}$$

$$\times \sum_{\substack{1 \leq m,n < \infty \\ m=n \\ (mn,q)=1}} \frac{\tau_{A_{s_1} \smallsetminus U_{s_1} \cup (V_{s_2})^-}(m)\,\tau_{B_{s_2} \smallsetminus V_{s_2} \cup (U_{s_1})^-}(n)}{\sqrt{mn}}\, ds_2\, ds_1$$

- If $|U| = |V| = \ell$, then we have $\left(\dfrac{X}{q^\ell}\right)^{s_1+s_2}$ here. Thus, intuitively, the ℓ-swap terms contribute to the main term only if $X \gg q^\ell$

The twisted recipe prediction, averaged over q

- Here we use smooth cutoffs and use Mellin inversion instead of Perron.
- Let W and V be smooth cutoff functions, with W supported on $[1,2]$ and V on $[0,1]$, say. Then the recipe predicts

$$\sum_{\substack{1\le q<\infty\\(q,hk)=1}} W\left(\frac{q}{Q}\right)\sideset{}{^\flat}\sum_{\chi \bmod q}\chi(h)\overline{\chi}(k)\sum_{m=1}^{\infty}\frac{\tau_A(m)\chi(m)}{\sqrt{m}}V\left(\frac{m}{X}\right)\sum_{n=1}^{\infty}\frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}}V\left(\frac{n}{X}\right)$$

$$\sim \frac{1}{(2\pi i)^2}\int_{(\varepsilon)}\int_{(\varepsilon)} X^{s_1+s_2}\widetilde{V}(s_1)\widetilde{V}(s_2)\sum_{\substack{U\subseteq A,V\subseteq B\\|U|=|V|}}\sum_{\substack{1\le q<\infty\\(q,hk)=1}} W\left(\frac{q}{Q}\right)$$

$$\times \sideset{}{^\flat}\sum_{\chi \bmod q}\left(\frac{q}{\pi}\right)^{-\sum\limits_{\alpha\in U}(\alpha+s_1)-\sum\limits_{\beta\in V}(\beta+s_2)}$$

$$\times \sum_{\substack{1\le m,n<\infty\\mh=nk\\(mn,q)=1}}\frac{\tau_{A_{s_1}\smallsetminus U_{s_1}\cup(V_{s_2})^-}(m)\tau_{B_{s_2}\smallsetminus V_{s_2}\cup(U_{s_1})^-}(n)}{\sqrt{mn}}\,ds_2\,ds_1.$$

- Denote the right-hand side by $\sum_{\ell=0}^{\min\{|A|,|B|\}}\mathcal{I}_\ell(h,k)$, where $\mathcal{I}_\ell(h,k)$ is the sum of the ℓ-swap terms, i.e., the terms that have $|U|=|V|=\ell$.

Main result

Theorem (B. and Turnage-Butterbaugh, 2022)

Assume the Generalized Lindelöf Hypothesis. If $X = Q^\eta$ *with* $1 < \eta < 2$, *then*

$$\sum_{\substack{1\le q<\infty \\ (q,hk)=1}} W\left(\frac{q}{Q}\right) \sideset{}{^\flat}\sum_{\chi \bmod q} \chi(h)\overline{\chi}(k) \sum_{m=1}^{\infty} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} V\left(\frac{m}{X}\right) \sum_{n=1}^{\infty} \frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}} V\left(\frac{n}{X}\right)$$

$$= \mathcal{I}_0(h,k) + \mathcal{I}_1(h,k) + \mathcal{E}(h,k),$$

where $\mathcal{E}(h,k)$ *satisfies*

$$\sum_{h,k\le Q^\vartheta} \frac{\lambda_h \overline{\lambda_k}}{\sqrt{hk}} \mathcal{E}(h,k) \ll Q^{1+\frac{\vartheta}{2}+\frac{\eta}{2}+\varepsilon} + Q^{\frac{5}{2}-\frac{\eta}{2}+\vartheta+\varepsilon}$$

uniformly for all $0 < \vartheta < 2-\eta$ *and arbitrary complex numbers* λ_h *such that* $\lambda_h \ll h^\varepsilon$.

- One may interpret our theorem as saying that the 1-swap terms predicted by CFKRS are correct for this family (that is averaged over q).

Evaluating the character sum, and initial splitting

- We start with

$$\mathcal{S}(h,k) := \sum_{\substack{1\leq q<\infty \\ (q,hk)=1}} W\left(\frac{q}{Q}\right) \sideset{}{^\flat}\sum_{\chi \bmod q} \chi(h)\overline{\chi}(k)$$

$$\times \sum_{m=1}^{\infty} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} V\left(\frac{m}{X}\right) \sum_{n=1}^{\infty} \frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}} V\left(\frac{n}{X}\right).$$

- We bring in the sum over χ and use the standard lemma

$$\sideset{}{^\flat}\sum_{\chi \bmod q} \chi(mh)\overline{\chi}(nk) = \frac{1}{2}\left(\sum_{\substack{cd=q \\ d|(mh+nk)}} \phi(d)\mu(c) \;+\; \sum_{\substack{cd=q \\ d|(mh-nk)}} \phi(d)\mu(c) \right)$$

- We split the resulting expression into three parts and write

$$\mathcal{S}(h,k) = \mathcal{L}(h,k) + \mathcal{D}(h,k) + \mathcal{U}(h,k),$$

where, for some parameter C that we choose to be $C = Q^{\varepsilon}$,

  - $\mathcal{L}(h,k)$ = sum of the terms with $c > C$
  - $\mathcal{D}(h,k)$ = sum of the "diagonal" terms with $c \leq C$ and $mh = nk$
  - $\mathcal{U}(h,k)$ = sum of the "off-diagonal" terms with $c \leq C$ and $mh \neq nk$

Summary of key steps in the proof

- For $\mathcal{L}(h,k)$, we use orthogonality of characters to detect the conditions $d|mh \pm nk$. We bound the contribution $\mathcal{L}^r(h,k)$ of the non-principal characters via the large sieve. Since $d \asymp Q/c$, we have $d \ll Q^{1-\varepsilon}$ if $c \geq Q^{\varepsilon}$, and the large sieve leads to

$$\sum_{h,k \leq Q^{\vartheta}} \frac{\lambda_h \overline{\lambda_k}}{\sqrt{hk}} \mathcal{L}^r(h,k) \ll Q^{2-\varepsilon}.$$

- For $\mathcal{U}(h,k)$, we first switch to the "complementary modulus"

$$d' = \frac{|mh \pm nk|}{d} \asymp \frac{c|mh \pm nk|}{Q} \ll XCQ^{\vartheta-1}$$

This bound is $\ll Q^{1-\varepsilon}$ provided that $X \ll Q^{2-\varepsilon}$ and we choose C and Q^{ϑ} to each be $\ll Q^{\varepsilon}$. We then again use orthogonality of characters to detect divisibility, and then apply the large sieve to bound the contribution of the non-principal characters.

- To estimate the contributions of the principal characters, we shift contours and find that the main contributions come from the residues. To show that these residues equal the 1-swap terms, we prove Euler product identities through some divisor function identities, including one identity discovered by Conrey and Keating in 2015.

The family of primitive Dirichlet L-functions

- Paley (1931): For some explicit constant C,

$$\sum_{\chi \bmod q} |L(\tfrac{1}{2},\chi)|^2 = \frac{\phi^2(q)}{q}\log q + \frac{\phi^2(q)}{q}\left(\sum_{p|q}\frac{\log p}{p-1}+C\right)+O\big(q^{\frac{1}{2}+\varepsilon}\big).$$

- Heath-Brown (1981): If $\phi^*(q) = \sum^*_{\chi \bmod q} 1$, then

$$\sideset{}{^*}\sum_{\chi \bmod q} |L(\tfrac{1}{2},\chi)|^4 = \frac{\phi^*(q)}{2\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}(\log q)^4 + O\big(2^{\omega(q)}q(\log q)^3\big).$$

- Soundararajan (2007):

$$\sideset{}{^*}\sum_{\chi \bmod q} |L(\tfrac{1}{2},\chi)|^4 = \frac{\phi^*(q)}{2\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}(\log q)^4\left(1+O\left(\frac{\omega(q)}{\log q}\sqrt{\frac{q}{\phi(q)}}\right)\right) + O\big(q(\log q)^{7/2}\big).$$

- Young (2011): If $p>2$ is a prime, then for some constants c_i,

$$\sideset{}{^*}\sum_{\chi \bmod p} |L(\tfrac{1}{2},\chi)|^4 = \sum_{i=0}^{4} c_i p(\log p)^i + O\big(p^{1-\frac{5}{512}+\varepsilon}\big).$$

Upper and lower bounds for the prime modulus case

$$p(\log p)^{k^2} \ll_k \sum_{\chi \bmod p}^{*} |L(\tfrac{1}{2},\chi)|^{2k} \ll_k p(\log p)^{k^2}$$

Lower bound:

- Rudnick and Soundararajan (2005): all rational $k \geq 1$
- Chandee and Li (2012): all rational $0 < k < 1$
- Radziwiłł and Soundararajan (2013): all real $k \geq 1$
- Heap and Soundararajan (2022): all real $k > 0$

Upper bound (conditional on GRH):

- Soundararajan (2008): $\ll_k p(\log p)^{k^2+\varepsilon}$ for all real $k > 0$
- Heath-Brown (2009): $\ll_k p(\log p)^{k^2}$ for all real $0 < k < 2$
- Harper (2013): $\ll_k p(\log p)^{k^2}$ for all real $k > 0$

Additional averaging and the large sieve

If we also average over q, then we can leverage the large sieve

- Huxley (1970):

$$\sum_{q\leq Q}\ \sideset{}{^*}\sum_{\chi \bmod q} |L(\tfrac{1}{2},\chi)|^6 \ll Q^2(\log Q)^9$$

$$\sum_{q\leq Q}\ \sideset{}{^*}\sum_{\chi \bmod q} |L(\tfrac{1}{2},\chi)|^8 \ll Q^2(\log Q)^{16}$$

- Conrey, Iwaniec, and Soundararajan (2012): For some explicit constant C_3,

$$\sum_{q\leq Q}\ \sideset{}{^\flat}\sum_{\chi \bmod q} \int_{-\infty}^{\infty} |\Lambda(\tfrac{1}{2}+iy,\chi)|^6\,dy$$

$$\sim C_3 \sum_{q\leq Q} \prod_{p|q} \frac{(1-\frac{1}{p})^5}{(1+\frac{4}{p}+\frac{1}{p^2})} \phi^\flat(q)(\log q)^9 \int_{-\infty}^{\infty} |\Gamma(\tfrac{1}{4}+\tfrac{iy}{2})|^6\,dy,$$

where $\sum^\flat_{\chi \bmod q}$ denotes summation over even primitive characters, $\phi^\flat(q)=\sum^\flat_{\chi \bmod q} 1$, and $\Lambda(s,\chi)$ is the completed L-function.

The asymptotic large sieve

The main new idea of Conrey, Iwaniec, and Soundararajan is a technique called the asymptotic large sieve.

- Conrey, Iwaniec, and Soundararajan (2012): For some explicit constant C_3,

$$\sum_{q\le Q}\ \sideset{}{^\flat}\sum_{\chi \bmod q} \int_{-\infty}^{\infty} |\Lambda(\tfrac12+iy,\chi)|^6\,dy$$

$$\sim C_3 \sum_{q\le Q}\prod_{p|q}\frac{(1-\frac1p)^5}{(1+\frac4p+\frac1{p^2})}\phi^\flat(q)(\log q)^9\int_{-\infty}^{\infty}|\Gamma(\tfrac14+\tfrac{iy}{2})|^6\,dy,$$

- Chandee, Li, Matomäki, and Radziwiłł (2020):

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q} |L(\tfrac12,\chi)|^6 \sim C_3 \sum_{q\le Q}\prod_{p|q}\frac{(1-\frac1p)^5}{(1+\frac4p+\frac1{p^2})}\phi^*(q)(\log q)^9.$$

The asymptotic large sieve

- Chandee and Li (2014): Assuming GRH, for some explicit constant C_4, we have

$$\sum_{q\le Q}\ \sideset{}{^\flat}\sum_{\chi \bmod q}\int_{-\infty}^{\infty}|\Lambda(\tfrac12+iy,\chi)|^8\,dy$$

$$\sim C_4\sum_{q\le Q}\prod_{p\mid q}\frac{(1-\frac1p)^7}{(1+\frac9p+\frac9{p^2}+\frac1{p^3})}\phi^\flat(q)(\log q)^{16}\int_{-\infty}^{\infty}|\Gamma(\tfrac14+\tfrac{iy}{2})|^8\,dy.$$

- Chandee, Li, Matomäki, and Radziwiłł (2022):

This asymptotic formula holds *unconditionally*.

The CFKRS recipe (guided by random matrix theory)

- Conjecture (Conrey, Farmer, Keating, Rubinstein, and Snaith, 2005):

$$\sum_{\chi \bmod q}^{*} \prod_{\alpha\in A} L(\tfrac{1}{2}+\alpha,\chi) \prod_{\beta\in B} L(\tfrac{1}{2}+\beta,\overline{\chi})$$

$$\sim \sum_{\substack{U\subseteq A, V\subseteq B\\ |U|=|V|}} \sum_{\chi \bmod q}^{*} \left(\frac{q}{\pi}\right)^{-\sum\limits_{\alpha\in U}\alpha-\sum\limits_{\beta\in V}\beta} \sum_{\substack{1\le m,n<\infty\\ m=n\\ (mn,q)=1}} \frac{\tau_{A\smallsetminus U\cup V^-}(m)\tau_{B\smallsetminus V\cup U^-}(n)}{\sqrt{mn}}$$

- Here, the "shifts" α,β are small complex numbers (say $\ll 1/\log q$)
- basic steps in the recipe: Use the approximate functional equation, ignore any oscillating terms, ignore any "off-diagonal" terms
- The expectation is that these ignored terms somehow cancel
- Notation: $\prod\limits_{\alpha\in A}\zeta(\alpha+s) =: \sum\limits_{n=1}^{\infty}\frac{\tau_A(n)}{n^s}$ and $U^- := \{-\alpha : \alpha\in U\}$
- We call the cardinality $|U|=|V|$ the number of *swaps*
- Let $\mathscr{X}(s)$ be the factor in $\zeta(s)=\mathscr{X}(s)\zeta(1-s)$. More precise version has

$$\prod_{\alpha\in U}\frac{\mathscr{X}(\frac{1}{2}+\alpha)}{q^{\alpha}} \prod_{\beta\in V}\frac{\mathscr{X}(\frac{1}{2}+\beta)}{q^{\beta}} \quad \text{in place of} \quad \left(\frac{q}{\pi}\right)^{-\sum\limits_{\alpha\in U}\alpha-\sum\limits_{\beta\in V}\beta}$$

Dirichlet polynomial approximations

- To understand how the off-diagonal terms cancel, Conrey and Keating (2015) examined Dirichlet polynomial approximations of zeta.
- Using Perron's formula and the recipe, we expect that

$$\sum_{\chi \bmod q}^{*} \sum_{m\le X} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} \sum_{n\le X} \frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}}$$

$$\sim \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \frac{X^{s_1+s_2}}{s_1 s_2} \sum_{\chi \bmod q}^{*} \prod_{\alpha\in A} L(\tfrac12+\alpha+s_1,\chi) \prod_{\beta\in B} L(\tfrac12+\beta+s_2,\overline{\chi})\, ds_2\, ds_1$$

$$\sim \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \frac{X^{s_1+s_2}}{s_1 s_2} \sum_{\substack{U\subseteq A, V\subseteq B\\ |U|=|V|}} \sum_{\chi \bmod q}^{*} \left(\frac{q}{\pi}\right)^{-\sum\limits_{\alpha\in U}(\alpha+s_1)-\sum\limits_{\beta\in V}(\beta+s_2)}$$

$$\times \sum_{\substack{1\le m,n<\infty\\ m=n\\ (mn,q)=1}} \frac{\tau_{A_{s_1}\smallsetminus U_{s_1}\cup(V_{s_2})^-}(m)\,\tau_{B_{s_2}\smallsetminus V_{s_2}\cup(U_{s_1})^-}(n)}{\sqrt{mn}}\, ds_2\, ds_1$$

- If $|U|=|V|=\ell$, then we have $\left(\dfrac{X}{q^\ell}\right)^{s_1+s_2}$ here. Thus, intuitively, the ℓ-swap terms contribute to the main term only if $X \gg q^\ell$

The twisted recipe prediction, averaged over q

- Here we use smooth cutoffs and use Mellin inversion instead of Perron.
- Let W and V be smooth cutoff functions, with W supported on $[1,2]$ and V on $[0,1]$, say. Then the recipe predicts

$$\sum_{\substack{1\le q<\infty\\(q,hk)=1}} W\left(\frac{q}{Q}\right) \sum_{\chi \bmod q}^{\flat} \chi(h)\overline{\chi}(k) \sum_{m=1}^{\infty} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} V\left(\frac{m}{X}\right) \sum_{n=1}^{\infty} \frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}} V\left(\frac{n}{X}\right)$$

$$\sim \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} X^{s_1+s_2} \widetilde{V}(s_1)\widetilde{V}(s_2) \sum_{\substack{U\subseteq A, V\subseteq B\\|U|=|V|}} \sum_{\substack{1\le q<\infty\\(q,hk)=1}} W\left(\frac{q}{Q}\right)$$

$$\times \sum_{\chi \bmod q}^{\flat} \prod_{\alpha\in U} \frac{\mathscr{X}\left(\frac12+\alpha+s_1\right)}{q^{\alpha+s_1}} \prod_{\beta\in V} \frac{\mathscr{X}\left(\frac12+\beta+s_2\right)}{q^{\beta+s_2}}$$

$$\times \sum_{\substack{1\le m,n<\infty\\mh=nk\\(mn,q)=1}} \frac{\tau_{A_{s_1}\smallsetminus U_{s_1}\cup(V_{s_2})^-}(m)\,\tau_{B_{s_2}\smallsetminus V_{s_2}\cup(U_{s_1})^-}(n)}{\sqrt{mn}}\, ds_2\, ds_1.$$

- Denote the right-hand side by $\sum_{\ell=0}^{\min\{|A|,|B|\}} \mathcal{I}_\ell(h,k)$, where $\mathcal{I}_\ell(h,k)$ is the sum of the ℓ-swap terms, i.e., the terms that have $|U|=|V|=\ell$.

Main result

Theorem (B. and Turnage-Butterbaugh, 2022)

Assume the Generalized Lindelöf Hypothesis. If $X = Q^\eta$ with $1 < \eta < 2$, then

$$\sum_{\substack{1\le q<\infty\\(q,hk)=1}} W\left(\frac{q}{Q}\right) \sideset{}{^\flat}\sum_{\chi \bmod q} \chi(h)\overline{\chi}(k) \sum_{m=1}^{\infty} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} V\left(\frac{m}{X}\right) \sum_{n=1}^{\infty} \frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}} V\left(\frac{n}{X}\right)$$

$$= \mathcal{I}_0(h,k) + \mathcal{I}_1(h,k) + \mathcal{E}(h,k),$$

where $\mathcal{E}(h,k)$ satisfies

$$\sum_{h,k\le Q^\vartheta} \frac{\lambda_h \overline{\lambda}_k}{\sqrt{hk}} \mathcal{E}(h,k) \ll Q^{1+\frac{\vartheta}{2}+\frac{\eta}{2}+\varepsilon} + Q^{\frac{5}{2}-\frac{\eta}{2}+\vartheta+\varepsilon}$$

uniformly for all $0 < \vartheta < 2-\eta$ and arbitrary complex numbers λ_h such that $\lambda_h \ll h^\varepsilon$.

- One may interpret our theorem as saying that the 1-swap terms predicted by CFKRS are correct for this family (that is averaged over q).

Remarks on main result

$$\sum_{h,k\le Q^{\vartheta}} \frac{\lambda_h\overline{\lambda}_k}{\sqrt{hk}}\mathcal{E}(h,k) \ll Q^{1+\frac{\vartheta}{2}+\frac{\eta}{2}+\varepsilon} + Q^{\frac{5}{2}-\frac{\eta}{2}+\vartheta+\varepsilon}$$

uniformly for all $0 < \vartheta < 2-\eta$ and arbitrary complex numbers λ_h such that $\lambda_h \ll h^{\varepsilon}$.

- If $X = Q^{\eta}$ with $1 < \eta < 2$, then $\mathcal{I}_0(h,k)$ and $\mathcal{I}_1(h,k)$ are each of size about Q^2. If we also assume that $\vartheta < (\eta-1)/2$, then our average bound for $\mathcal{E}(h,k)$ is $\ll Q^{2-\varepsilon}$.

- The same method applies to odd characters.

- We look at twisted moments for applications. One of these applications is the generalization of the Conrey-Keating heuristic of combining the 1-swap terms to build the general ℓ-swap terms.

- The 1-swap terms have also been found (conditionally) for other specific families of L-functions.

Initial setup of proof

- We start with
$$\mathcal{S}(h,k) := \sum_{\substack{1\le q<\infty\\(q,hk)=1}} W\left(\frac{q}{Q}\right) \sideset{}{^\flat}\sum_{\chi \bmod q} \chi(h)\overline{\chi}(k)$$
$$\times \sum_{m=1}^{\infty} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} V\left(\frac{m}{X}\right) \sum_{n=1}^{\infty} \frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}} V\left(\frac{n}{X}\right).$$
- We bring in the sum over χ and use the standard lemma
$$\sideset{}{^\flat}\sum_{\chi \bmod q} \chi(mh)\overline{\chi}(nk) = \frac{1}{2}\left(\sum_{\substack{d|q\\ d|(mh+nk)}} \phi(d)\mu\left(\frac{q}{d}\right) + \sum_{\substack{d|q\\ d|(mh-nk)}} \phi(d)\mu\left(\frac{q}{d}\right)\right)$$
- We denote $c = q/d$ and split $\mathcal{S}(h,k)$ into
$$\mathcal{S}(h,k) = \mathcal{L}(h,k) + \mathcal{D}(h,k) + \mathcal{U}(h,k),$$
where
  - $\mathcal{L}(h,k) =$ sum of the terms with $c > C$
  - $\mathcal{D}(h,k) =$ sum of the "diagonal" terms with $c \le C$ and $mh = nk$
  - $\mathcal{U}(h,k) =$ sum of the "off-diagonal" terms with $c \le C$ and $mh \ne nk$

Using orthogonality of characters to detect divisibility

- $\mathcal{L}(h,k)$ is defined by

$$\mathcal{L}(h,k) := \frac{1}{2} \sum_{\substack{1\le q<\infty \\ (q,hk)=1}} W\left(\frac{q}{Q}\right) \sum_{\substack{1\le m,n<\infty \\ (mn,q)=1}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} V\left(\frac{m}{X}\right) V\left(\frac{n}{X}\right)$$

$$\times \left(\sum_{\substack{c>C, d\ge 1 \\ cd=q \\ d|mh+nk}} \phi(d)\mu(c) + \sum_{\substack{c>C, d\ge 1 \\ cd=q \\ d|mh-nk}} \phi(d)\mu(c) \right).$$

- We use orthogonality of characters to detect divisibility:

$$\frac{1}{\phi(d)} \sum_{\psi \bmod d} \psi(mh)\overline{\psi}(\pm nk) = \begin{cases} 1 & \text{if } d|mh \mp nk \\ 0 & \text{else} \end{cases}$$

- The result is

$$\mathcal{L}(h,k) = \sum_{\substack{1\le q<\infty \\ (q,hk)=1}} W\left(\frac{q}{Q}\right) \sum_{\substack{1\le m,n<\infty \\ (mn,q)=1}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} V\left(\frac{m}{X}\right) V\left(\frac{n}{X}\right)$$

$$\times \sum_{\substack{c>C, d\ge 1 \\ cd=q}} \mu(c) \sum_{\substack{\psi \bmod d \\ \psi \text{ even}}} \psi(mh)\overline{\psi}(nk).$$

Applying the large sieve

$$\mathcal{L}(h,k) = \sum_{\substack{1\le q<\infty\\(q,hk)=1}} W\left(\frac{q}{Q}\right) \sum_{\substack{1\le m,n<\infty\\(mn,q)=1}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} V\left(\frac{m}{X}\right) V\left(\frac{n}{X}\right)$$
$$\times \sum_{\substack{c>C,d\ge1\\cd=q}} \mu(c) \sum_{\substack{\psi \bmod d\\ \psi \text{ even}}} \psi(mh)\overline{\psi}(nk).$$

- Write this as $\mathcal{L}(h,k) = \mathcal{L}^0(h,k) + \mathcal{L}^r(h,k)$, where $\mathcal{L}^0$ is the contribution of the principal character mod d.
- $\mathcal{L}^0(h,k)$ cancels with a term from the analysis of $\mathcal{U}(h,k)$
- Use Mellin inversion on $V(m/X)V(n/X)$ and then write the m,n-sum as an Euler product and use GLH. Then, apply the large sieve

$$\sum_{\substack{1\le m,n<\infty\\(mn,q)=1}} \frac{\tau_A(m)\tau_B(n)\psi(m)\overline{\psi}(n)}{m^{\frac12+s_1}n^{\frac12+s_2}} \approx \prod_{\alpha\in A} L(\tfrac12+s_1+\alpha,\psi) \prod_{\beta\in B} L(\tfrac12+s_2+\beta,\overline{\psi})$$

$$\sum_{h,k\le Q^\vartheta} \frac{\lambda_h\overline{\lambda_k}}{\sqrt{hk}}\mathcal{L}^r(h,k) \ll X^\varepsilon \sum_{1\le q<\infty} W\left(\frac{q}{Q}\right) \sum_{\substack{c>C,d\ge1\\cd=q}} (cd)^\varepsilon \sum_{\substack{\psi \bmod d\\ \psi \text{ even}\\ \psi\ne\psi_0}} \left| \sum_{\substack{h\le Q^\vartheta\\(h,q)=1}} \frac{\lambda_h\psi(h)}{\sqrt{h}} \right|^2$$

Switching to the complementary divisor

- In this way, we get a bound that is essentially $\ll Q^{2+\varepsilon}/C^2$, which is admissible for $C = Q^{\varepsilon}$, say.
- We next look at $\mathcal{U}(h,k)$, which has $1 \leq c \leq C$.
- The key step in the asymptotic large sieve is to switch to the "complementary modulus" d' given by

$$d' = \frac{|mh \pm nk|}{d}$$

- The difficulty with using d is that d has size $\asymp Q/c$ because of the factor $W(cd/Q)$. Thus, if we use d as the modulus of character sums and apply the large sieve, then we get a bound of $Q^{2+\varepsilon}/c^2$, which is too big for small c.
- On the other hand, if $1 \leq c \leq C$, then d' has size

$$d' \asymp \frac{c|mh \pm nk|}{Q} \ll XCQ^{\vartheta-1}$$

for $m, n \ll X$ and $h, k \ll Q^{\vartheta}$. This bound is $\ll Q^{1-\varepsilon}$ provided $X \ll Q^{2-\varepsilon}$ and we choose C and Q^{ϑ} to each be $\ll Q^{\varepsilon}$.

Switching to the complementary divisor

- $\mathcal{U}(h,k)$ is defined by

$$\mathcal{U}(h,k) := \frac{1}{2} \sum_{\substack{1\le q<\infty \\ (q,hk)=1}} W\left(\frac{q}{Q}\right) \sum_{\substack{1\le m,n<\infty \\ (mn,q)=1 \\ mh\neq nk}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} V\left(\frac{m}{X}\right) V\left(\frac{n}{X}\right)$$

$$\times \left(\sum_{\substack{1\le c\le C, d\ge 1 \\ cd=q \\ d|mh+nk}} \phi(d)\mu(c) + \sum_{\substack{1\le c\le C, d\ge 1 \\ cd=q \\ d|mh-nk}} \phi(d)\mu(c) \right).$$

- We need to do preliminary transformations to facilitate later estimations: (1) factor out $g := (mh, nk)$ since we need coprimality to use character sums, (2) write $\phi(d) = \sum_{ef=d} \mu(e) f$, (3) apply Möbius inversion to detect $(f,g)=1$.
- Then make a change of variable by writing $\ell = |mh \pm nk|/efg$
- Then, use characters to detect divisibility as before. This results to

$$\mathcal{U}(h,k) = \frac{1}{2} \sum_{\substack{1\le c\le C \\ (c,hk)=1}} \mu(c) \sum_{\substack{1\le m,n<\infty \\ (mn,c)=1 \\ mh\neq nk}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} V\left(\frac{m}{X}\right) V\left(\frac{n}{X}\right) \sum_{\substack{1\le e<\infty \\ (e,g)=1}} \frac{\mu(e)}{e} \sum_{a|g} \mu(a)$$

$$\times \sum_{\substack{1\le \ell<\infty \\ (ea\ell, \frac{mh}{g}\cdot\frac{nk}{g})=1}} \frac{1}{\phi(ea\ell)} \sum_{\psi \bmod ea\ell} \psi\left(\frac{mh}{g}\right) \overline{\psi}\left(\mp\frac{nk}{g}\right) \frac{|mh\pm nk|}{g\ell} W\left(\frac{c|mh\pm nk|}{g\ell Q}\right).$$

Carrying out the method in the asymptotic large sieve

$$\mathcal{U}(h,k) = \frac{1}{2} \sum_{\substack{1\le c\le C \\ (c,hk)=1}} \mu(c) \sum_{\substack{1\le m,n<\infty \\ (mn,c)=1 \\ mh\neq nk}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} V\left(\frac{m}{X}\right) V\left(\frac{n}{X}\right) \sum_{\substack{1\le e<\infty \\ (e,g)=1}} \frac{\mu(e)}{e} \sum_{a|g} \mu(a)$$

$$\times \sum_{\substack{1\le \ell<\infty \\ (ea\ell, \frac{mh}{g}\cdot\frac{nk}{g})=1}} \frac{1}{\phi(ea\ell)} \sum_{\psi \bmod ea\ell} \psi\left(\frac{mh}{g}\right) \overline{\psi}\left(\mp\frac{nk}{g}\right) \frac{|mh\pm nk|}{g\ell} W\left(\frac{c|mh\pm nk|}{g\ell Q}\right)$$

(notation here: we are summing over both signs $\pm$).

- Write this as
$$\mathcal{U}(h,k) = \mathcal{U}^0(h,k) + \mathcal{U}^r(h,k),$$
where $\mathcal{U}^0$ is the contribution of the principal character mod $ea\ell$.

- The treatment of $\mathcal{U}^r(h,k)$ is, at its foundation, the same as the treatment of $\mathcal{L}^r(h,k)$ (i.e., use Mellin inversion, write m,n-sum as an Euler product, use GLH to bound L-functions, apply large sieve).

- However, the calculations are quite delicate because of the interdependence of the variables and the complexity of the multivariable Mellin transform. For this part, we closely follow the arguments in Conrey, Iwaniec, and Soundararajan (2019).

Finding the 1-swap terms

$$\mathcal{U}^0(h,k) := \frac{1}{2} \sum_{\substack{1\le c\le C\\(c,hk)=1}} \mu(c) \sum_{\substack{1\le m,n<\infty\\(mn,c)=1\\mh\ne nk}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} V\left(\frac{m}{X}\right) V\left(\frac{n}{X}\right) \sum_{\substack{1\le e<\infty\\(e,g)=1}} \frac{\mu(e)}{e}$$

$$\times \sum_{a|g} \mu(a) \sum_{\substack{1\le \ell<\infty\\(ea\ell,\frac{mh}{g}\cdot\frac{nk}{g})=1}} \frac{|mh\pm nk|}{g\ell\phi(ea\ell)} W\left(\frac{c|mh\pm nk|}{g\ell Q}\right)$$

- Apply Mellin inversion on W, and then write the ℓ-sum as an Euler product

$$\sum_{\substack{1\le \ell<\infty\\(ea\ell,\frac{mh}{g}\cdot\frac{nk}{g})=1}} \frac{1}{\ell^w\phi(ea\ell)} \approx \frac{1}{\phi(ea)}\zeta(1+w)$$

(up to an absolutely convergent Euler product factor)

- Move the line of integration to the left to get $\mathcal{U}^0(h,k)=\mathcal{U}^1(h,k)+\mathcal{U}^2(h,k)$, where $\mathcal{U}^1(h,k)$ is the residue from $\zeta(1+w)$ and

$$\mathcal{U}^2(h,k) \approx \frac{Q}{2} \sum_{\substack{1\le c\le C\\(c,hk)=1}} \frac{\mu(c)}{c} \sum_{\substack{1\le m,n<\infty\\(mn,c)=1\\mh\ne nk}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} V\left(\frac{m}{X}\right) V\left(\frac{n}{X}\right) \sum_{\substack{1\le e<\infty\\(e,g)=1}} \frac{\mu(e)}{e}$$

$$\times \sum_{a|g} \frac{\mu(a)}{\phi(ea)} \cdot \frac{1}{2\pi i} \int_{(-\varepsilon)} \left(\frac{c|mh\pm nk|}{gQ}\right)^w \widetilde{W}(1-w)\zeta(1+w)\,dw.$$

Finding the 1-swap terms

- $\mathcal{U}^0(h,k) = \mathcal{U}^1(h,k) + \mathcal{U}^2(h,k)$
- $\mathcal{U}^1(h,k)$ cancels with $\mathcal{L}^0(h,k)$, while

$$\mathcal{U}^2(h,k) \approx \frac{Q}{2} \sum_{\substack{1\leq c\leq C \\ (c,hk)=1}} \frac{\mu(c)}{c} \sum_{\substack{1\leq m,n<\infty \\ (mn,c)=1 \\ mh\neq nk}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} V\left(\frac{m}{X}\right) V\left(\frac{n}{X}\right) \sum_{\substack{1\leq e<\infty \\ (e,g)=1}} \frac{\mu(e)}{e}$$
$$\times \sum_{a|g} \frac{\mu(a)}{\phi(ea)} \cdot \frac{1}{2\pi i} \int_{(-\varepsilon)} \left(\frac{c|mh\pm nk|}{gQ}\right)^w \widetilde{W}(1-w)\zeta(1+w)\,dw$$

- Next, to write the sum as an Euler product, we need to separate the variables in $|mh\pm nk|^w$

Lemma (Conrey, Iwaniec, and Soundararajan, 2019)

If $0 < c < \mathrm{Re}(\omega)$ *and* $r > 0$ *with* $r \neq 1$, *then*

$$|1+r|^{-\omega} + |1-r|^{-\omega} = \frac{1}{2\pi i} \int_{(c)} \sqrt{\pi} \frac{\Gamma(\frac{1-\omega}{2})\Gamma(\frac{z}{2})\Gamma(\frac{\omega-z}{2})}{\Gamma(\frac{\omega}{2})\Gamma(\frac{1-z}{2})\Gamma(\frac{1-\omega+z}{2})} r^{-z}\,dz.$$

- For technical convergence issues, we actually use an averaged form of this lemma, also due to the same authors.

Finding the 1-swap terms

$$\mathcal{U}^2(h,k) \approx \frac{Q}{2} \sum_{\substack{1\le c\le C\\(c,hk)=1}} \frac{\mu(c)}{c} \sum_{\substack{1\le m,n<\infty\\(mn,c)=1\\mh\neq nk}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} V\left(\frac{m}{X}\right) V\left(\frac{n}{X}\right) \sum_{\substack{1\le e<\infty\\(e,g)=1}} \frac{\mu(e)}{e}$$

$$\times \sum_{a|g} \frac{\mu(a)}{\phi(ea)} \cdot \frac{1}{2\pi i} \int_{(-\varepsilon)} \left(\frac{c|mh\pm nk|}{gQ}\right)^w \widetilde{W}(1-w)\zeta(1+w)\, dw$$

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- We then apply Mellin inversion on $V(m/X)V(n/X)$, and then express the sum as an Euler product. This leads to a quadruple integral expression for $\mathcal{U}^2(h,k)$.

- We move the lines of integration to the left, one after the other. In the process, we find several different residues, of which five are main terms.

Finding the 1-swap terms

- We move the lines of integration to the left, one after the other. In the process, we find several different residues, of which five are main terms.
- We carry out the same kind of residue analysis on the predicted 1-swap terms from an earlier slide:

$$\frac{1}{(2\pi i)^2}\int_{(\varepsilon)}\int_{(\varepsilon)} X^{s_1+s_2}\widetilde{V}(s_1)\widetilde{V}(s_2)\sum_{\substack{\alpha\in A\\ \beta\in B}}\sum_{\substack{1\le q<\infty\\ (q,hk)=1}} W\left(\frac{q}{Q}\right)$$
$$\times \sideset{}{^\flat}\sum_{\chi \bmod q}\frac{\mathscr{X}\left(\frac{1}{2}+\alpha+s_1\right)\mathscr{X}\left(\frac{1}{2}+\beta+s_2\right)}{q^{s_1+s_2+\alpha+\beta}}$$
$$\times \sum_{\substack{1\le m,n<\infty\\ mh=nk\\ (mn,q)=1}}\frac{\tau_{A_{s_1}\smallsetminus\{\alpha+s_1\}\cup\{-\beta-s_2\}}(m)\tau_{B_{s_2}\smallsetminus\{\beta+s_2\}\cup\{-\alpha-s_1\}}(n)}{\sqrt{mn}}\,ds_2\,ds_1.$$

- We again find several different residues. Nine of these are main terms, of which four cancel each other.
- We match each of the five remaining residues with the five residues from our analysis of $\mathcal{U}^2(h,k)$ and show that corresponding residues are equal (up to negligible error term) via Euler product identities.

Finding the 1-swap terms

- We match each of the five remaining residues with the five residues from our analysis of $\mathcal{U}^2(h,k)$ and show that corresponding residues are equal (up to negligible error term) via Euler product identities.

- These Euler product identities are consequences of certain properties of the function τ_A, which we recall is defined by

$$\prod_{\alpha \in A} \zeta(\alpha + s) =: \sum_{n=1}^{\infty} \frac{\tau_A(n)}{n^s}.$$

- One of the key identities is the following

Lemma (Conrey and Keating, 2015)

$$\begin{aligned}
&\tau_{A\smallsetminus\{\alpha\}\cup\{-\beta\}}(p^j)\tau_{B\smallsetminus\{\beta\}}(p^\ell) \\
&\qquad + \tau_{A\smallsetminus\{\alpha\}}(p^j)\tau_{B\smallsetminus\{\beta\}\cup\{-\alpha\}}(p^\ell) - \tau_{A\smallsetminus\{\alpha\}}(p^j)\tau_{B\smallsetminus\{\beta\}}(p^\ell) \\
&= \tau_{A\smallsetminus\{\alpha\}\cup\{-\beta\}}(p^j)\tau_{B\smallsetminus\{\beta\}\cup\{-\alpha\}}(p^\ell) \\
&\qquad - p^{\alpha+\beta}\tau_{A\smallsetminus\{\alpha\}\cup\{-\beta\}}(p^{j-1})\tau_{B\smallsetminus\{\beta\}\cup\{-\alpha\}}(p^{\ell-1})
\end{aligned}$$