Twisted $2k$th moments of primitive Dirichlet L-functions: beyond the diagonal

Siegfred Baluyot
American Institute of Mathematics (AIM)

joint work with Caroline Turnage-Butterbaugh

research supported by NSF DMS-1854398 FRG

Number Theory Seminar
UC Irvine
1 Dec 2022

Why study moments of L-functions?

- Moments contain information about the size of the L-functions and the distribution of their values.

- We can use variants of moments to deduce properties of L-functions, such as the distribution of their zeros. Moments are also useful in studying the behavior of arithmetic functions like the divisor function.

- The *Katz-Sarnak philosophy* is the notion that the behavior of families of L-functions is governed by some underlying symmetry groups.

- Studying different families may allow us to compare them and investigate how their (conjectured) symmetry types affect their behavior.

$$\sum_{\chi \bmod q}^{*} |L(\tfrac{1}{2},\chi)|^{2k} \qquad \sum_{d\le X}^{\flat} L(\tfrac{1}{2},\chi_d)^k \qquad \sum_{f\in\mathcal{H}_r} \frac{1}{\omega_r} L(\tfrac{1}{2},f)^k$$

The family of primitive Dirichlet L-functions

- Paley (1931): For some explicit constant C,

$$\sum_{\chi \bmod q} |L(\tfrac{1}{2},\chi)|^2 = \frac{\phi^2(q)}{q}\log q + \frac{\phi^2(q)}{q}\left(\sum_{p|q}\frac{\log p}{p-1} + C\right) + O\big(q^{\frac{1}{2}+\varepsilon}\big).$$

- Heath-Brown (1981): If $\phi^*(q) = \sum^*_{\chi \bmod q} 1$, then

$$\sideset{}{^*}\sum_{\chi \bmod q} |L(\tfrac{1}{2},\chi)|^4 = \frac{\phi^*(q)}{2\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}(\log q)^4 + O\big(2^{\omega(q)}q(\log q)^3\big).$$

- Soundararajan (2007):

$$\sideset{}{^*}\sum_{\chi \bmod q} |L(\tfrac{1}{2},\chi)|^4 = \frac{\phi^*(q)}{2\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}(\log q)^4\left(1 + O\left(\frac{\omega(q)}{\log q}\sqrt{\frac{q}{\phi(q)}}\right)\right) + O\big(q(\log q)^{7/2}\big).$$

- Young (2011): If $p > 2$ is a prime, then for some constants c_i,

$$\sideset{}{^*}\sum_{\chi \bmod p} |L(\tfrac{1}{2},\chi)|^4 = \sum_{i=0}^{4} c_i p(\log p)^i + O\big(p^{1-\frac{5}{512}+\varepsilon}\big).$$

Upper and lower bounds for the prime modulus case

$$p(\log p)^{k^2} \ll_k \sum_{\chi \bmod p}^{*} |L(\tfrac{1}{2},\chi)|^{2k} \ll_k p(\log p)^{k^2}$$

Lower bound:

- Rudnick and Soundararajan (2005): all rational $k \geq 1$
- Chandee and Li (2012): all rational $0 < k < 1$
- Radziwiłł and Soundararajan (2013): all real $k \geq 1$
- Heap and Soundararajan (2022): all real $k > 0$

Upper bound (conditional on GRH):

- Soundararajan (2008): $\ll_k p(\log p)^{k^2+\varepsilon}$ for all real $k > 0$
- Heath-Brown (2009): $\ll_k p(\log p)^{k^2}$ for all real $0 < k < 2$
- Harper (2013): $\ll_k p(\log p)^{k^2}$ for all real $k > 0$

Additional averaging and the large sieve

If we also average over q, then we can leverage the large sieve

- Huxley (1970):

$$\sum_{q\leq Q}\ \sum_{\chi \bmod q}^{*} |L(\tfrac{1}{2},\chi)|^6 \ll Q^2(\log Q)^9$$

$$\sum_{q\leq Q}\ \sum_{\chi \bmod q}^{*} |L(\tfrac{1}{2},\chi)|^8 \ll Q^2(\log Q)^{16}$$

- Conrey, Iwaniec, and Soundararajan (2012): For some explicit constant C_3,

$$\sum_{q\leq Q}\ \sum_{\chi \bmod q}^{\flat} \int_{-\infty}^{\infty} |\Lambda(\tfrac{1}{2}+iy,\chi)|^6\,dy$$

$$\sim C_3 \sum_{q\leq Q} \prod_{p\mid q} \frac{(1-\frac{1}{p})^5}{(1+\frac{4}{p}+\frac{1}{p^2})} \phi^\flat(q)(\log q)^9 \int_{-\infty}^{\infty} |\Gamma(\tfrac{1}{4}+\tfrac{iy}{2})|^6\,dy,$$

where $\sum^\flat_{\chi \bmod q}$ denotes summation over even primitive characters, $\phi^\flat(q) = \sum^\flat_{\chi \bmod q} 1$, and $\Lambda(s,\chi)$ is the completed L-function.

The asymptotic large sieve

The main new idea of Conrey, Iwaniec, and Soundararajan is a technique called the asymptotic large sieve.

- Chandee, Li, Matomäki, and Radziwiłł (2020):

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q} |L(\tfrac{1}{2},\chi)|^6 \sim C_3 \sum_{q\le Q}\prod_{p|q}\frac{(1-\frac{1}{p})^5}{(1+\frac{4}{p}+\frac{1}{p^2})}\phi^*(q)(\log q)^9.$$

- Chandee and Li (2014): Assuming GRH, for some explicit constant C_4, we have

$$\sum_{q\le Q}\ \sideset{}{^\flat}\sum_{\chi \bmod q}\int_{-\infty}^{\infty} |\Lambda(\tfrac{1}{2}+iy,\chi)|^8\,dy$$
$$\sim C_4\sum_{q\le Q}\prod_{p|q}\frac{(1-\frac{1}{p})^7}{(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3})}\phi^\flat(q)(\log q)^{16}\int_{-\infty}^{\infty}|\Gamma(\tfrac{1}{4}+\tfrac{iy}{2})|^8\,dy.$$

- Chandee, Li, Matomäki, and Radziwiłł (2022): This asymptotic formula holds *unconditionally*.

The CFKRS recipe (guided by random matrix theory)

Conjecture (Conrey, Farmer, Keating, Rubinstein, & Snaith, 2005):

$$\sideset{}{^*}\sum_{\chi \bmod q} \prod_{\alpha\in A} L(\tfrac{1}{2}+\alpha,\chi) \prod_{\beta\in B} L(\tfrac{1}{2}+\beta,\overline{\chi})$$

$$\sim \sum_{\substack{U\subseteq A, V\subseteq B\\ |U|=|V|}} \sideset{}{^*}\sum_{\chi \bmod q} \left(\frac{q}{\pi}\right)^{-\sum\limits_{\alpha\in U}\alpha - \sum\limits_{\beta\in V}\beta} \sum_{\substack{1\le m,n<\infty\\ m=n\\ (mn,q)=1}} \frac{\tau_{A\smallsetminus U\cup V^-}(m)\tau_{B\smallsetminus V\cup U^-}(n)}{\sqrt{mn}}$$

- Here, the "shifts" α,β are small complex numbers (say $\ll 1/\log q$)
- Notation: $\displaystyle\prod_{\alpha\in A}\zeta(\alpha+s) =: \sum_{n=1}^{\infty}\frac{\tau_A(n)}{n^s}$ and $U^- := \{-\alpha : \alpha\in U\}$
- We call the cardinality $|U|=|V|$ the number of *swaps*

Dirichlet polynomial approximations

- To understand how the off-diagonal terms cancel, Conrey and Keating (2015) examined Dirichlet polynomial approximations of zeta.
- Using Perron's formula and the recipe, we expect that

$$\sideset{}{^*}\sum_{\chi \bmod q} \sum_{m \leq X} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} \sum_{n \leq X} \frac{\tau_A(n)\chi(n)}{\sqrt{n}}$$

$$\sim \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \frac{X^{s_1+s_2}}{s_1 s_2} \sideset{}{^*}\sum_{\chi \bmod q} \prod_{\alpha \in A} L(\tfrac{1}{2}+\alpha+s_1,\chi) \prod_{\beta \in B} L(\tfrac{1}{2}+\beta+s_2,\overline{\chi})\, ds_2\, ds_1$$

$$\sim \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \frac{X^{s_1+s_2}}{s_1 s_2} \sum_{\substack{U \subseteq A, V \subseteq B \\ |U|=|V|}} \sideset{}{^*}\sum_{\chi \bmod q} \left(\frac{q}{\pi}\right)^{-\sum\limits_{\alpha \in U}(\alpha+s_1) - \sum\limits_{\beta \in V}(\beta+s_2)}$$

$$\times \sum_{\substack{1 \leq m,n < \infty \\ m=n \\ (mn,q)=1}} \frac{\tau_{A_{s_1} \smallsetminus U_{s_1} \cup (V_{s_2})^-}(m)\,\tau_{B_{s_2} \smallsetminus V_{s_2} \cup (U_{s_1})^-}(n)}{\sqrt{mn}}\, ds_2\, ds_1$$

- If $|U| = |V| = \ell$, then we have $\left(\dfrac{X}{q^\ell}\right)^{s_1+s_2}$ here. Thus, intuitively, the ℓ-swap terms contribute to the main term only if $X \gg q^\ell$

The twisted recipe prediction, averaged over q

- Here we use smooth cutoffs and use Mellin inversion instead of Perron.
- Let W and V be smooth cutoff functions, with W supported on $[1,2]$ and V on $[0,1]$, say. Then the recipe predicts

$$\sum_{\substack{1\leq q<\infty\\ (q,hk)=1}} W\left(\frac{q}{Q}\right) \sideset{}{^\flat}\sum_{\chi \bmod q} \chi(h)\overline{\chi}(k) \sum_{m=1}^{\infty} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} V\left(\frac{m}{X}\right) \sum_{n=1}^{\infty} \frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}} V\left(\frac{n}{X}\right)$$

$$\sim \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} X^{s_1+s_2} \widetilde{V}(s_1)\widetilde{V}(s_2) \sum_{\substack{U\subseteq A, V\subseteq B\\ |U|=|V|}} \sum_{\substack{1\leq q<\infty\\ (q,hk)=1}} W\left(\frac{q}{Q}\right)$$

$$\times \sideset{}{^\flat}\sum_{\chi \bmod q} \left(\frac{q}{\pi}\right)^{-\sum_{\alpha\in U}(\alpha+s_1)-\sum_{\beta\in V}(\beta+s_2)}$$

$$\times \sum_{\substack{1\leq m,n<\infty\\ mh=nk\\ (mn,q)=1}} \frac{\tau_{A_{s_1}\smallsetminus U_{s_1}\cup(V_{s_2})^-}(m)\tau_{B_{s_2}\smallsetminus V_{s_2}\cup(U_{s_1})^-}(n)}{\sqrt{mn}}\, ds_2\, ds_1 .$$

- Denote the right-hand side by $\sum_{\ell=0}^{\min\{|A|,|B|\}} \mathcal{I}_\ell(h,k)$, where $\mathcal{I}_\ell(h,k)$ is the sum of the ℓ-swap terms, i.e., the terms that have $|U|=|V|=\ell$.

Main result

Theorem (B. and Turnage-Butterbaugh, 2022)

Assume the Generalized Lindelöf Hypothesis. If $X = Q^\eta$ *with* $1 < \eta < 2$*, then*

$$\sum_{\substack{1\le q<\infty \\ (q,hk)=1}} W\left(\frac{q}{Q}\right) \sideset{}{^\flat}\sum_{\chi \bmod q} \chi(h)\overline{\chi}(k) \sum_{m=1}^{\infty} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} V\left(\frac{m}{X}\right) \sum_{n=1}^{\infty} \frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}} V\left(\frac{n}{X}\right)$$

$$= \mathcal{I}_0(h,k) + \mathcal{I}_1(h,k) + \mathcal{E}(h,k),$$

where $\mathcal{E}(h,k)$ *satisfies*

$$\sum_{h,k \le Q^\vartheta} \frac{\lambda_h \overline{\lambda_k}}{\sqrt{hk}} \mathcal{E}(h,k) \ll Q^{1+\frac{\vartheta}{2}+\frac{\eta}{2}+\varepsilon} + Q^{\frac{5}{2}-\frac{\eta}{2}+\vartheta+\varepsilon}$$

uniformly for all $0 < \vartheta < 2-\eta$ *and arbitrary complex numbers* λ_h *such that* $\lambda_h \ll h^\varepsilon$*.*

- One may interpret our theorem as saying that the 1-swap terms predicted by CFKRS are correct for this family (that is averaged over q).

Remarks on main result

$$\sum_{h,k\le Q^\vartheta} \frac{\lambda_h \overline{\lambda}_k}{\sqrt{hk}} \mathcal{E}(h,k) \ll Q^{1+\frac{\vartheta}{2}+\frac{\eta}{2}+\varepsilon} + Q^{\frac{5}{2}-\frac{\eta}{2}+\vartheta+\varepsilon}$$

uniformly for all $0 < \vartheta < 2-\eta$ and arbitrary complex numbers λ_h such that $\lambda_h \ll h^\varepsilon$.

- If $X = Q^\eta$ with $1 < \eta < 2$, then $\mathcal{I}_0(h,k)$ and $\mathcal{I}_1(h,k)$ are each of size about Q^2. If we also assume that $\vartheta < (\eta-1)/2$, then our average bound for $\mathcal{E}(h,k)$ is $\ll Q^{2-\varepsilon}$.

- The same method applies to odd characters.

- We look at twisted moments for applications. One of these applications is the generalization of the Conrey-Keating heuristic of combining the 1-swap terms to build the general ℓ-swap terms.

- The 1-swap terms have also been found (conditionally) for other specific families of L-functions.

Evaluating the character sum, and initial splitting

- We start with

$$\mathcal{S}(h,k) := \sum_{\substack{1\le q<\infty \\ (q,hk)=1}} W\left(\frac{q}{Q}\right) \sideset{}{^\flat}\sum_{\chi \bmod q} \chi(h)\overline{\chi}(k)$$
$$\times \sum_{m=1}^{\infty} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} V\left(\frac{m}{X}\right) \sum_{n=1}^{\infty} \frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}} V\left(\frac{n}{X}\right).$$

- We bring in the sum over χ and use the standard lemma

$$\sideset{}{^\flat}\sum_{\chi \bmod q} \chi(mh)\overline{\chi}(nk) = \frac{1}{2}\left(\sum_{\substack{cd=q \\ d|(mh+nk)}} \phi(d)\mu(c) \quad + \sum_{\substack{cd=q \\ d|(mh-nk)}} \phi(d)\mu(c) \right)$$

- We split the resulting expression into three parts and write

$$\mathcal{S}(h,k) = \mathcal{L}(h,k) + \mathcal{D}(h,k) + \mathcal{U}(h,k),$$

where, for some parameter C,

  - $\mathcal{L}(h,k) =$ sum of the terms with $c > C$
  - $\mathcal{D}(h,k) =$ sum of the "diagonal" terms with $c \le C$ and $mh = nk$
  - $\mathcal{U}(h,k) =$ sum of the "off-diagonal" terms with $c \le C$ and $mh \neq nk$

Analysis of $\mathcal{L}(h,k)$ (the sum of the terms with $c > C$)

- We use orthogonality of characters to detect the conditions $d | mh \pm nk$

- The contribution of the principal character cancels with some term from the analysis of $\mathcal{U}(h,k)$

- For the contribution $\mathcal{L}^r(h,k)$ of the non-principal characters, we apply Mellin inversion and use GLH to bound the m,n-sum to write

$$\sum_{h,k\le Q^\vartheta} \frac{\lambda_h\overline{\lambda_k}}{\sqrt{hk}}\mathcal{L}^r(h,k) \ll X^\varepsilon \sum_{1\le q<\infty} W\left(\frac{q}{Q}\right) \sum_{\substack{c>C,d\ge 1\\ cd=q}} (cd)^\varepsilon \sum_{\substack{\psi \bmod d\\ \psi \text{ even}\\ \psi\ne\psi_0}} \left| \sum_{\substack{h\le Q^\vartheta\\ (h,q)=1}} \frac{\lambda_h\psi(h)}{\sqrt{h}} \right|^2$$

- The character sum here has modulus $d \asymp \dfrac{Q}{c}$ because of the support of W. Thus the large sieve leads to (essentially)

$$\sum_{h,k\le Q^\vartheta} \frac{\lambda_h\overline{\lambda_k}}{\sqrt{hk}}\mathcal{L}^r(h,k) \ll \frac{Q^{2+\varepsilon}}{C}.$$

This is smaller than the main term (size about Q^2) if we choose $C = Q^{2\varepsilon}$.

Switching to the complementary divisor

- We next look at $\mathcal{U}(h,k)$, which has $1 \leq c \leq C$.

- The key step in the asymptotic large sieve of Conrey, Iwaniec, and Soundararajan is to switch to the "complementary modulus" d' given by
$$d' = \frac{|mh \pm nk|}{d}$$

- The difficulty with using d is that $d \asymp Q/c$ as we saw earlier, and thus the large sieve gives a bound of $Q^{2+\varepsilon}/c^2$, which is too big for small c.

- On the other hand, if $1 \leq c \leq C$, then d' has size
$$d' \asymp \frac{c|mh \pm nk|}{Q} \ll XCQ^{\vartheta-1}$$
for $m,n \ll X$ and $h,k \ll Q^{\vartheta}$. This bound is $\ll Q^{1-\varepsilon}$ provided $X \ll Q^{2-\varepsilon}$ and we choose C and Q^{ϑ} to each be $\ll Q^{\varepsilon}$.

Applying the asymptotic large sieve

- After switching to the complementary divisor, we again use orthogonality of characters to detect the divisibility condition.

- We split the resulting expression to write

$$\mathcal{U}(h,k) = \mathcal{U}^0(h,k) + \mathcal{U}^r(h,k),$$

where $\mathcal{U}^0$ is the contribution of the principal character.

- The treatment of $\mathcal{U}^r(h,k)$ is basically the same as $\mathcal{L}^r(h,k)$. However, the calculations are complicated by the switch to the complementary divisor. We closely follow the arguments in Conrey, Iwaniec, and Soundararajan (2019).

- This means the only place the 1-swap terms can be is in $\mathcal{U}^0(h,k)$.

Finding the 1-swap terms

$$\mathcal{U}^0(h,k) := \frac{1}{2} \sum_{\substack{1\le c\le C\\(c,hk)=1}} \mu(c) \sum_{\substack{1\le m,n<\infty\\(mn,c)=1\\mh\neq nk}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} V\left(\frac{m}{X}\right) V\left(\frac{n}{X}\right) \sum_{\substack{1\le e<\infty\\(e,g)=1}} \frac{\mu(e)}{e}$$

$$\times \sum_{a|g} \mu(a) \sum_{\substack{1\le \ell<\infty\\(ea\ell,\frac{mh}{g}\cdot\frac{nk}{g})=1}} \frac{|mh+nk|}{g\ell\phi(ea\ell)} W\left(\frac{c|mh+nk|}{g\ell Q}\right)$$

$$+ \quad \Big(\text{the same sum but with } |mh-nk| \text{ instead of } |mh+nk|\Big)$$

- We need to separate the variables in $|mh+nk|$ in order to write the m,n-sum as an Euler product. To do this, we use

Lemma (Conrey, Iwaniec, and Soundararajan, 2019)

If $0<c<\mathrm{Re}(\omega)$ *and* $r>0$ *with* $r\neq 1$, *then*

$$|1+r|^{-\omega} + |1-r|^{-\omega} = \frac{1}{2\pi i}\int_{(c)} \sqrt{\pi}\,\frac{\Gamma(\frac{1-\omega}{2})\Gamma(\frac{z}{2})\Gamma(\frac{\omega-z}{2})}{\Gamma(\frac{\omega}{2})\Gamma(\frac{1-z}{2})\Gamma(\frac{1-\omega+z}{2})} r^{-z}\,dz.$$

- We then apply Mellin inversion to get a multiple contour integral. We move the lines of integration to the left one by one. In the process, we find several different residues. One of them cancels with $\mathcal{L}^0(h,k)$. Of the rest, five are main terms.

Finding the 1-swap terms

- We move the lines of integration to the left, one after the other. In the process, we find several different residues, of which five are main terms.
- We carry out the same kind of residue analysis on the predicted 1-swap terms from an earlier slide:

$$\frac{1}{(2\pi i)^2}\int_{(\varepsilon)}\int_{(\varepsilon)} X^{s_1+s_2}\widetilde{V}(s_1)\widetilde{V}(s_2)\sum_{\substack{\alpha\in A\\ \beta\in B}}\sum_{\substack{1\le q<\infty\\ (q,hk)=1}} W\left(\frac{q}{Q}\right)$$

$$\times\sum_{\chi \bmod q}^{\flat}\frac{\mathscr{X}\left(\frac{1}{2}+\alpha+s_1\right)\mathscr{X}\left(\frac{1}{2}+\beta+s_2\right)}{q^{s_1+s_2+\alpha+\beta}}$$

$$\times\sum_{\substack{1\le m,n<\infty\\ mh=nk\\ (mn,q)=1}}\frac{\tau_{A_{s_1}\smallsetminus\{\alpha+s_1\}\cup\{-\beta-s_2\}}(m)\tau_{B_{s_2}\smallsetminus\{\beta+s_2\}\cup\{-\alpha-s_1\}}(n)}{\sqrt{mn}}\,ds_2\,ds_1.$$

- We again find several different residues. Nine of these are main terms, of which four cancel each other.
- We match each of the five remaining residues with the five residues from our analysis of $\mathcal{U}^2(h,k)$ and show that corresponding residues are equal (up to negligible error term) via Euler product identities.

Finding the 1-swap terms

- We match each of the five remaining residues with the five residues from our analysis of $\mathcal{U}^2(h,k)$ and show that corresponding residues are equal (up to negligible error term) via Euler product identities.

- These Euler product identities are consequences of certain properties of the function τ_A, which we recall is defined by

$$\prod_{\alpha \in A} \zeta(\alpha + s) =: \sum_{n=1}^{\infty} \frac{\tau_A(n)}{n^s}.$$

- One of the key identities is the following

Lemma (Conrey and Keating, 2015)

$$\begin{aligned}
&\tau_{A\smallsetminus\{\alpha\}\cup\{-\beta\}}(p^j)\tau_{B\smallsetminus\{\beta\}}(p^\ell) \\
&\qquad + \tau_{A\smallsetminus\{\alpha\}}(p^j)\tau_{B\smallsetminus\{\beta\}\cup\{-\alpha\}}(p^\ell) - \tau_{A\smallsetminus\{\alpha\}}(p^j)\tau_{B\smallsetminus\{\beta\}}(p^\ell) \\
&= \tau_{A\smallsetminus\{\alpha\}\cup\{-\beta\}}(p^j)\tau_{B\smallsetminus\{\beta\}\cup\{-\alpha\}}(p^\ell) \\
&\qquad - p^{\alpha+\beta}\tau_{A\smallsetminus\{\alpha\}\cup\{-\beta\}}(p^{j-1})\tau_{B\smallsetminus\{\beta\}\cup\{-\alpha\}}(p^{\ell-1})
\end{aligned}$$