Geometric Incidence Theory and Uniform Distribution

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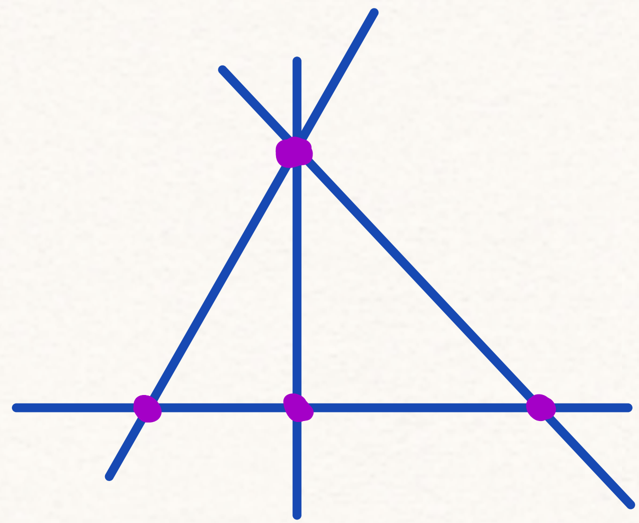
Point-Line Incidences

$\mathcal{P}$ = set of m points in $\mathbb{R}^2$

$\mathcal{L}$ = set of n lines in $\mathbb{R}^2$

$$I(\mathcal{P}, \mathcal{L}) = \#\{(p, \ell) \in \mathcal{P} \times \mathcal{L} : p \in \ell\}$$



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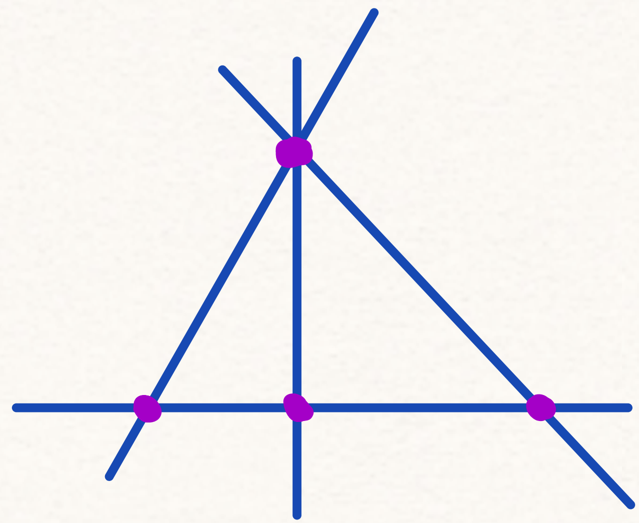

$|\mathcal{P}| = |\mathcal{L}| = 4$

$I(\mathcal{P}, \mathcal{L}) = 9$

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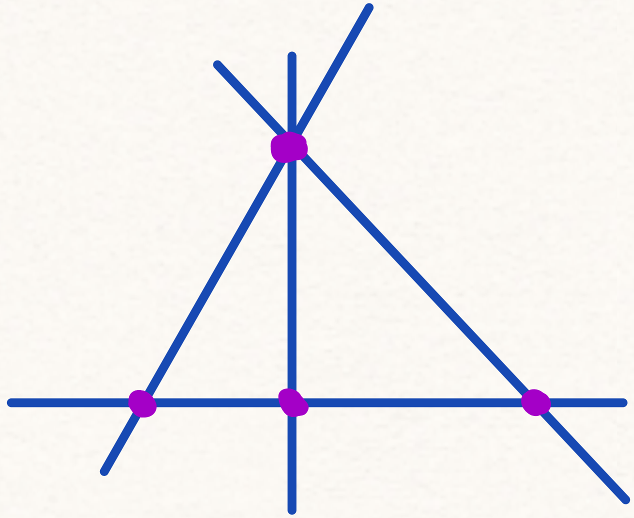

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Erdös : $I(m,n) \gg (m + n + (mn)^{2/3})$

Szemerédi-Trotter (1983) : $I(m,n) \ll (m + n + (mn)^{2/3})$

Point-Hyperplane Incidences

P = set of m points in $\mathbb{R}^d$

$\mathcal{H}$ = set of n codimension 1 hyperplanes in $\mathbb{R}^d$

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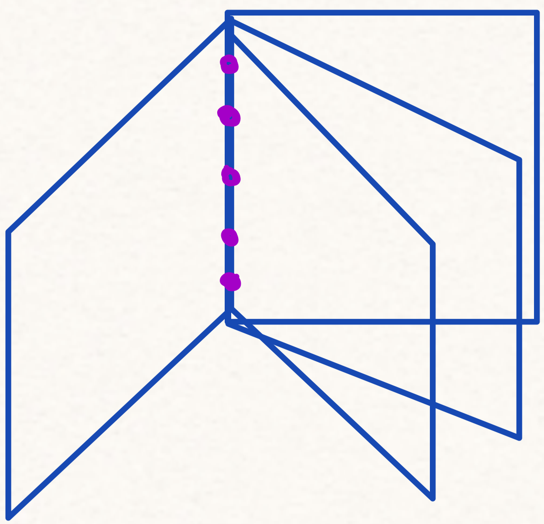
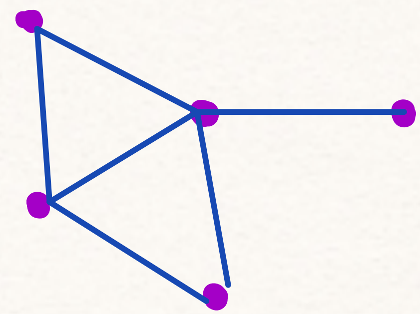


$I(m, n) = mn =$ trivial bound

silly :(

Unit Distance Problem

P = set of n points in $\mathbb{R}^2$



How many pairs of points can have fixed distance t?

$$U(P) = \#\{(x,y) \in P \times P : |x-y| = t\}$$

$$U(n) = \max\{U(P) : |P| = n\}$$

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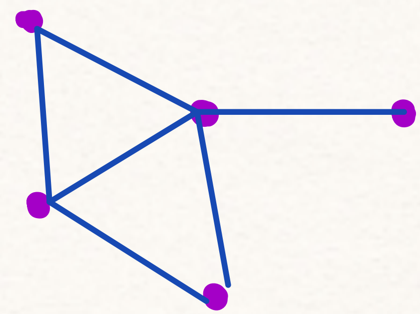

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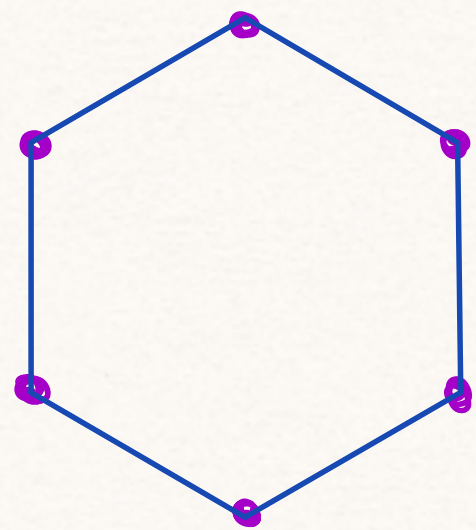
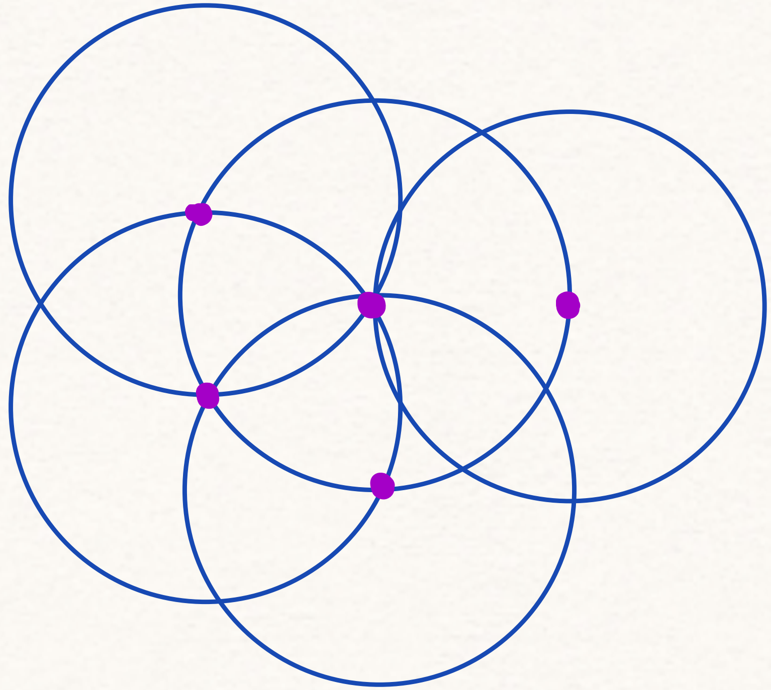
Regular polygon $\Rightarrow U(n) \geq 2n$.



This is an incidence problem

$U(P)$ = # of incidences between points of P and the circles of radius t centered at points of P



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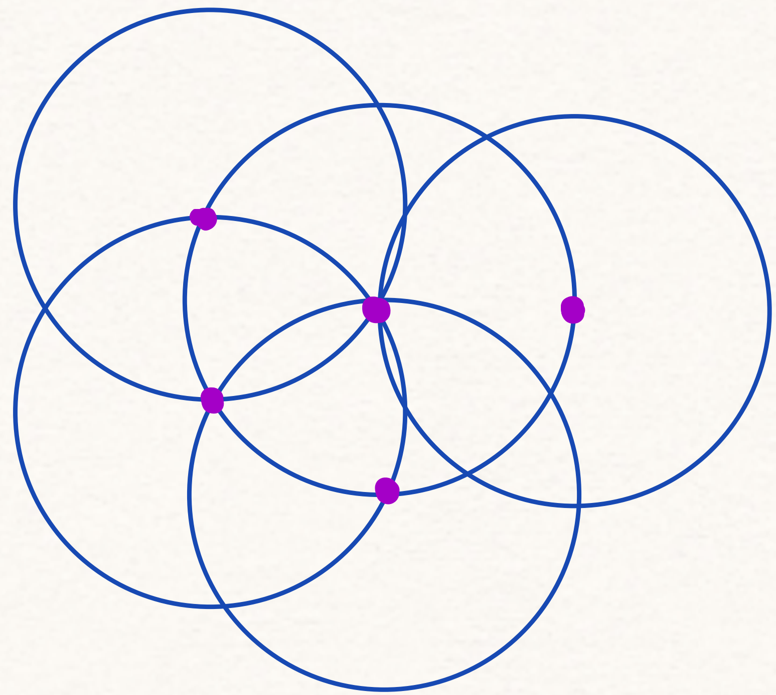

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Erdös (1946):

$$u(n) = \Omega\left(n^{1+\frac{1}{\log\log n}}\right)$$

$$u(n) \leq cn^{3/2} .$$

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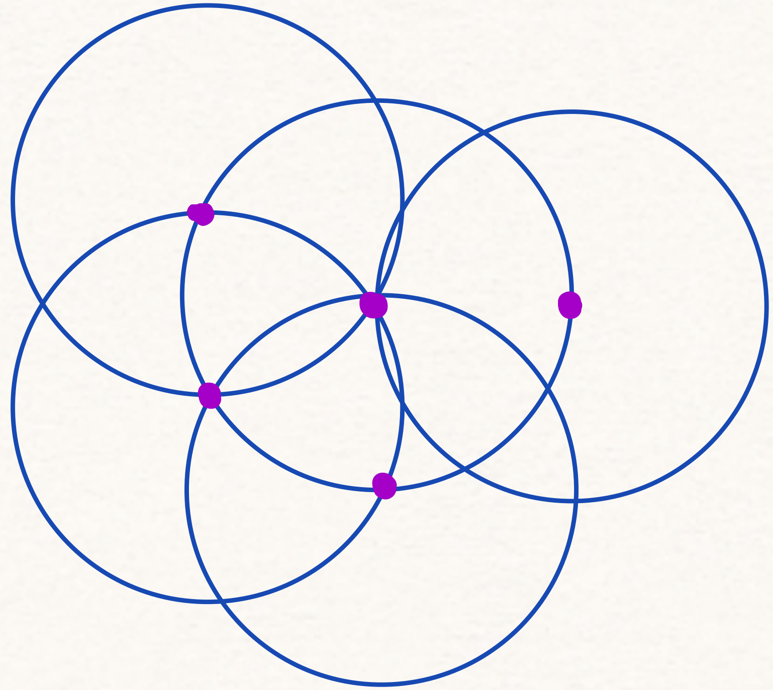

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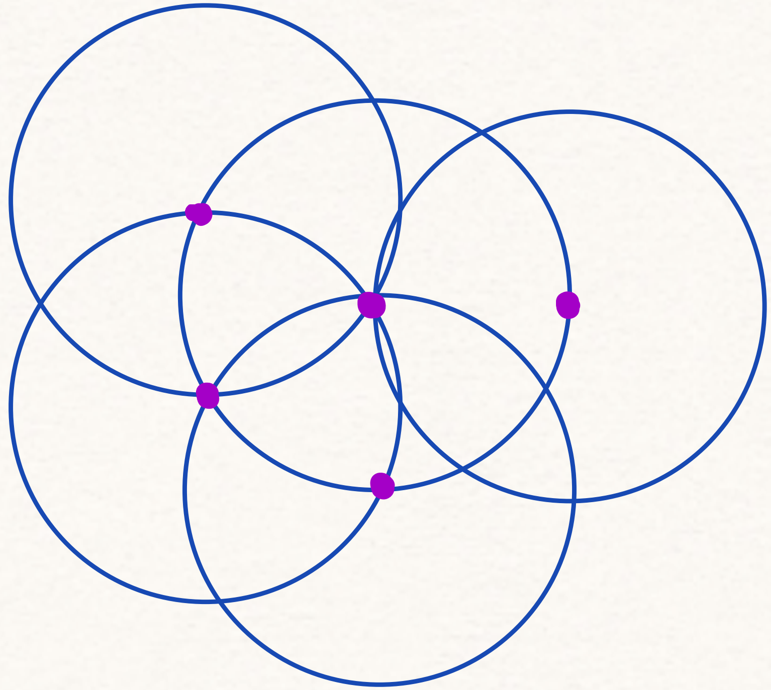

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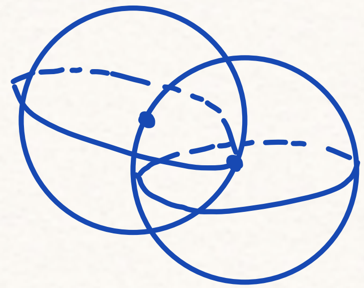
Expected true upper bound: $O(n^{1+\varepsilon})$

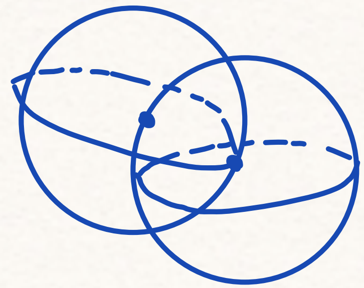
Unit Distance Problem in $\mathbb{R}^d$



$d=3$ current record:

Zahl (2019) : $U_3(n) \leq c\, n^{\frac{295}{197}}$

Unit Distance Problem in $\mathbb{R}^d$



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$d \ge 4$: $P_1 = \frac{n}{2}$ points on circle $(\cos\theta, \sin\theta, 0, 0)$

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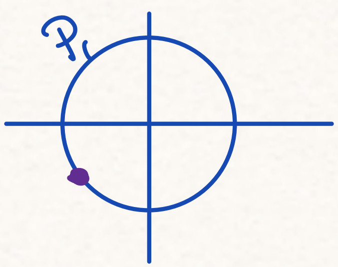

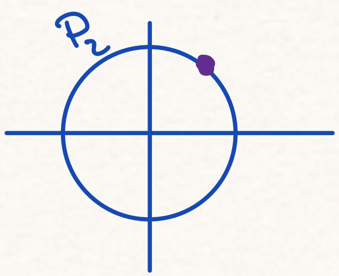

$x \in P_1$

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$|x-y| = \sqrt{2}$

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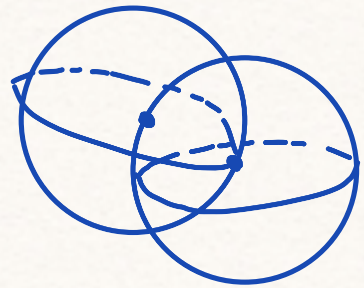

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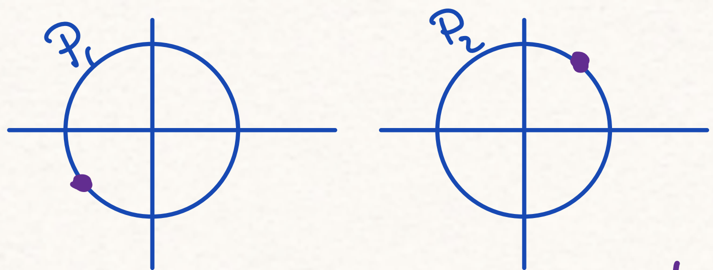

$x \in P_1$
$y \in P_2$

$|x-y| = \sqrt{2}$

$U_d(n) \geq \frac{1}{4} n^2$ = trivial bound

silly $\ddot{\frown}$

Uniform distribution

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in $[0,1]^d$

$\{x_n\}$ is uniformly distributed in $[0,1]^d$ if

for all boxes $I \subset [0,1]^d$

$$\frac{1}{N}\#\{n \le N : x_n \in I\} \longrightarrow \operatorname{vol} I \quad \text{as } N \to \infty$$

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Weyl's Criterion: $\{x_n\}$ is u.d. in $[0,1]^d$ iff

$$\sum_{n=1}^{N} \exp(2\pi i\, k \cdot x_n) = o(N) \quad \text{for each } k \in \mathbb{Z}^d \setminus \{0\}$$

Quantitative Weyl Criterion

For $\gamma \in (0, \frac{1}{2}]$, $\{x_n\}_{n=1}^{\infty}$ is γ-uniformly distributed in $[0,1]^d$ if for every $\varepsilon > 0$ there is a constant C_ε s.t. for all $k \in \mathbb{Z}^d - \{0\}$,

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Remarks:

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2. If $\{x_n\}$ is a uniform random sequence and $\gamma < \frac{1}{2}$, then $\{x_n\}$ is γ-ud with probability 1.

Incidences between points and annuli

Thm 1 (G - Iosevich - Wyman, 2022)

If $\{v_n\}_{n=1}^{\infty}$ is γ-ud on $[0,1]^d$, then

$$\#\{(n,m) \in [1,N]^2 : a \leq |v_n - v_m| \leq b\} = N^2 \, vol(\Omega) + R(a,b,N)$$

where $\Omega = \{(x,y) \in ([0,1]^d)^2 : a \leq |x-y| \leq b\}$

$$|R(a,b,N)| \leq C N^{2-2\gamma+\varepsilon} (b-a)^{-\left(\frac{d-1}{2}\right)}$$

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Cor 2a: If $\{v_n\}$ is $\frac{1}{2}$-ud in $[0,1]^d$, and $0<t<\frac{1}{2}$, then the number of pairs (v_n, v_m), $1 \le n,m \le N$ with distance t is

$$\le C_\varepsilon N^{2-\frac{2}{d+1}+\varepsilon}$$

Incidences between points and thick hyperplanes

Thm 2 (G - Iosevich - Wyman, 2022)

If $\{v_n\}_{n=1}^{\infty}$, $\{w_m\}_{m=1}^{\infty}$ γ-ud on $[0,1]^d$, then

$$\#\{(n,m) \in [1,N]^2 : a \leq v_n \cdot w_m \leq b\} = N^2 \operatorname{vol}(\Omega) + R(a,b,N)$$

where $\Omega = \{(x,y) \in ([0,1]^d)^2 : a \leq x \cdot y \leq b\}$,

$$|R(a,b,N)| \leq CN^{2-\gamma+\varepsilon}$$

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Cor 2b If $\{v_n\}$ and $\{w_m\}$ are $\frac{1}{2}$-ud and $0 < t < d$ is fixed, then the number of pairs (v_n, w_m) with $1 \leq n, m \leq N$ and $v_n \cdot w_m = t$ is $\leq C_\varepsilon N^{3/2+\varepsilon}$

Sketch of the Proof

Let $\Omega = \{ (x,y) \in [0,1]^d \times [0,1]^d : a \leq |x-y| \leq b \}$

Note that Ω has smooth boundary and positive volume.

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Construct a smooth cutoff χ_δ which differs from $\mathbb{1}_\Omega$ only on δ-neighborhood of $\partial\Omega$

$$\sum_{n,m=1}^{N} \sum \mathbb{1}_{\Omega}(x_n, x_m) = \sum_{n,m=1}^{N} \sum \chi_{\delta}(x_n, x_m) + \sum_{n,m=1}^{N} \sum (\mathbb{1}_{\Omega} - \chi_{\delta})(x_n, x_m)$$

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$\tilde{\chi}_{\delta}$ smooth, supported on δ-nbhd of $\partial\Omega$, and $\tilde{\chi}_{\delta} \geq |\mathbb{1}_{\Omega} - \chi_{\delta}|$.

$$\sum_{n,m=1}^{N}\sum \mathbb{1}_{\Omega}(x_n,x_m) = \sum_{n,m=1}^{N}\sum \chi_\delta(x_n,x_m) + \underbrace{\sum_{n,m=1}^{N}\sum (\mathbb{1}_\Omega - \chi_\delta)(x_n,x_m)}_{\text{bound by } \tilde{\chi}_\delta}$$

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Now we need to estimate

① $\displaystyle\sum_{n,m=1}^{N}\sum \chi_\delta(x_n,x_m)$ ② $\displaystyle\sum_{n,m=1}^{N}\sum \tilde{\chi}_\delta(x_n,x_m)$

We approximate both using Fourier series

Main Term: 0^{th} coeff. of 1 and is $\approx \mathrm{vol}(\Omega)$
(by design)

Rest is bounded by

$$N^2 \iint \tilde{\chi}_\delta(x,y)\,dx\,dy + \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \left(|\hat{\tilde{\chi}}_\delta(k)| + |\hat{\chi}_\delta(k)| \right) \left| \sum_{n,m=1}^{N} e^{2\pi i k\cdot(x_n - x_m)} \right|$$

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The game is to find the right δ to make it all line up.

Closing Thoughts and Future Questions:

① The bound for points & annuli is $N^{2-\frac{2}{d+1}+\varepsilon}$
the bound for points & hyperplanes is $N^{3/2+\varepsilon}$
Can we find an example to justify the discrepancy?

② What is the "true" upper bound for $\frac{1}{2}$-ud. sets?

③ Apply this assumption to other incidence problems...
will we get interesting results?

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How does this property relate to
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Thank You!