Worksheet 3 FMMC

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Exercise 1

At $t = 0$, a linear harmonic oscillator is in a state described by the normalized wave function:

$$|\psi(x,0)\rangle = \frac{1}{\sqrt{5}}|u_0(x)\rangle + \frac{1}{\sqrt{2}}|u_2(x)\rangle + c_3|u_3(x)\rangle$$

where $u_n(x)$ is the n-th eigenfunction of the corresponding Hamiltonian.

a) Determine the value of c_3 assuming that it is real and positive.

The wave function $|\psi(x,0)\rangle$ is normalized. Therefore the sum of the squares of the coefficients must equal 1:

$$\left|\frac{1}{\sqrt{5}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 + |c_3|^2 = 1$$

Adding these contributions:

$$\frac{1}{5} + \frac{1}{2} + |c_3|^2 = 1$$

Then:

$$|c_3|^2 = 1 - \frac{7}{10} = \frac{3}{10}$$

Finally:

$$\boxed{c_3 = \sqrt{\frac{3}{10}} = \frac{\sqrt{3}}{\sqrt{10}} = \frac{\sqrt{30}}{10}} \tag{1}$$

b) Write the wave function for t > 0

For $t > 0$, the wave function evolves in time according to the Schrödinger equation. The general form of the time-dependent wave function is:

$$\psi(x,t) = \sum_n c_n u_n(x) e^{-iE_n t/\hbar}$$

-c_n are the initial coefficients associated with each eigenstate $u_n(x)$, -$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ are the energy levels of the harmonic oscillator, -ω is the angular frequency of the oscillator, -$u_n(x)$ are the eigenfunctions of the harmonic oscillator.

Given the initial wave function:

$$\psi(x,0) = \frac{1}{\sqrt{5}}u_0(x) + \frac{1}{\sqrt{2}}u_2(x) + c_3 u_3(x),$$

with $c_3 = \frac{\sqrt{30}}{10}$, the time-dependent wave function is:

$$\psi(x,t) = \frac{1}{\sqrt{5}}u_0(x)e^{-iE_0t/\hbar} + \frac{1}{\sqrt{2}}u_2(x)e^{-iE_2t/\hbar} + \frac{\sqrt{30}}{10}u_3(x)e^{-iE_3t/\hbar}.$$

The energy levels for the harmonic oscillator are:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

Thus:

$$E_0 = \frac{1}{2}\hbar\omega, \quad E_2 = \frac{5}{2}\hbar\omega, \quad E_3 = \frac{7}{2}\hbar\omega.$$

Substituting these into the equation, the explicit time-dependent wave function is:

$$\boxed{\psi(x,t) = \frac{1}{\sqrt{5}}u_0(x)e^{-i\frac{\omega t}{2}} + \frac{1}{\sqrt{2}}u_2(x)e^{-i\frac{5\omega t}{2}} + \frac{\sqrt{30}}{10}u_3(x)e^{-i\frac{7\omega t}{2}}.} \tag{2}$$

c) What is the expected value of the oscillator energy at $t = 0$? And at t = 1 s?

The expected value of the oscillator energy is given by:

$$\langle E \rangle = \sum_n |c_n|^2 E_n,$$

where $|c_n|^2$ are the probabilities and $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ are the energy levels. The coefficients are:

$$|c_0|^2 = \frac{1}{5}, \quad |c_2|^2 = \frac{1}{2}, \quad |c_3|^2 = \frac{3}{10},$$

and the energy levels:

$$E_0 = \frac{1}{2}\hbar\omega, \quad E_2 = \frac{5}{2}\hbar\omega, \quad E_3 = \frac{7}{2}\hbar\omega,$$

Substituting:

$$\langle E \rangle = \frac{1}{5}\cdot\frac{1}{2}\hbar\omega + \frac{1}{2}\cdot\frac{5}{2}\hbar\omega + \frac{3}{10}\cdot\frac{7}{2}\hbar\omega.$$

Therefore, the expected energy at $t = 0$ is:

$$\boxed{\langle E \rangle = \frac{12}{5}\hbar\omega.} \tag{3}$$

The expected energy $\langle E \rangle$ at $t = 1s$ remains the same as at $t = 0$. This is because the energy levels E_n and the coefficients $|c_n|^2$, which represent the probabilities, do not change with time, even though the wave function evolves with time.

The wave function for $t > 0$ is:

$$\psi(x,t) = \frac{1}{\sqrt{5}}u_0(x)e^{-iE_0t/\hbar} + \frac{1}{\sqrt{2}}u_2(x)e^{-iE_2t/\hbar} + \frac{\sqrt{30}}{10}u_3(x)e^{-iE_3t/\hbar}.$$

The expected energy is calculated as:

$$\langle E \rangle = \sum_n |c_n|^2 E_n,$$

where the probabilities $|c_n|^2$ remain unaffected by the time-dependent factors $e^{-iE_nt/\hbar}$. Therefore, the result is independent of t.
At $t = 1s$, the expected energy is:

$$\boxed{\langle E \rangle = \frac{12}{5}\hbar\omega.} \tag{4}$$

d) What is the dispersion of the mean value

The dispersion of the mean value, or the variance $(\Delta E)^2$ is:

$$(\Delta E)^2 = \langle E^2 \rangle - \langle E \rangle^2.$$

Computing $\langle E^2 \rangle$:

$$\langle E^2 \rangle = \sum_n |c_n|^2 E_n^2.$$

Using the coefficients:

$$|c_0|^2 = \frac{1}{5}, \quad |c_2|^2 = \frac{1}{2}, \quad |c_3|^2 = \frac{3}{10},$$

and the energy levels:

$$E_0 = \frac{1}{2}\hbar\omega, \quad E_2 = \frac{5}{2}\hbar\omega, \quad E_3 = \frac{7}{2}\hbar\omega,$$

we calculate E_n^2:

$$E_0^2 = \left(\frac{1}{2}\hbar\omega\right)^2 = \frac{1}{4}(\hbar\omega)^2,$$

$$E_2^2 = \left(\frac{5}{2}\hbar\omega\right)^2 = \frac{25}{4}(\hbar\omega)^2,$$

$$E_3^2 = \left(\frac{7}{2}\hbar\omega\right)^2 = \frac{49}{4}(\hbar\omega)^2.$$

Substituting into $\langle E^2 \rangle$:

$$\begin{aligned}\langle E^2 \rangle &= \frac{1}{5} \cdot \frac{1}{4}(\hbar\omega)^2 + \frac{1}{2} \cdot \frac{25}{4}(\hbar\omega)^2 + \frac{3}{10} \cdot \frac{49}{4}(\hbar\omega)^2 \\ &= \frac{1}{20}(\hbar\omega)^2 + \frac{25}{8}(\hbar\omega)^2 + \frac{147}{40}(\hbar\omega)^2.\end{aligned}$$

Converting to a common denominator of 40:

$$\langle E^2 \rangle = \frac{137}{20}(\hbar\omega)^2$$

Computing $\langle E \rangle^2$:

From the earlier result:

$$\langle E \rangle = \frac{12}{5}\hbar\omega,$$

$$\langle E\rangle^2 = \left(\frac{12}{5}\hbar\omega\right)^2 = \frac{144}{25}(\hbar\omega)^2.$$

Computing $(\Delta E)^2$:

Substituting into the formula for $(\Delta E)^2$:

$$(\Delta E)^2 = \langle E^2\rangle - \langle E\rangle^2,$$

$$(\Delta E)^2 = \frac{137}{20}(\hbar\omega)^2 - \frac{144}{25}(\hbar\omega)^2.$$

Thus:

$$(\Delta E)^2 = \frac{109}{100}(\hbar\omega)^2$$

Finally:

$$\boxed{\Delta E = \sqrt{\frac{109}{100}}\hbar\omega = \frac{\sqrt{109}}{10}\hbar\omega.} \tag{5}$$

e) At $t = t_0$, the energy of the system is measured. Which are the possible outcomes of this measurement and what is the probability of obtaining them?

For a measurement of $\hat{H}$ at any time t_0, the system is in:

$$|\psi(t_0)\rangle = \frac{1}{\sqrt{5}}e^{-iE_0t_0/\hbar}|u_0\rangle + \frac{1}{\sqrt{2}}e^{-iE_2t_0/\hbar}|u_2\rangle + \sqrt{\frac{3}{10}}e^{-iE_3t_0/\hbar}|u_3\rangle$$

When the energy of the system is measured at $t = t_0$, the possible outcomes are the energy levels of the harmonic oscillator, and the probabilities correspond to the square of the eigenvalues of the Hamiltonian. Therefore:

- $E_0 = \frac{1}{2}\hbar\omega$ with probability $\left|\frac{1}{\sqrt{5}}\right|^2 = \frac{1}{5}$.
- $E_2 = \frac{5}{2}\hbar\omega$ with probability $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$.
- $E_3 = \frac{7}{2}\hbar\omega$ with probability $\left|\sqrt{\frac{3}{10}}\right|^2 = \frac{3}{10}$.

f) Suppose that $t = t_0$ at, the result of the measurement is $\frac{1}{2}\hbar\omega$. What is the state of the system after this measurement?

If at $t = t_0$, the result of the energy measurement is $\frac{\hbar\omega}{2}$ (corresponding to the ground state E_0), the wave function collapses to the eigenstate associated with this energy, which is $|u_0\rangle$. After the measurement, the state of the system becomes:

$$|\psi(t > t_0)\rangle = |u_0\rangle,$$

This state does not evolve in time, the state remains $|u_0\rangle$ after the measurement.

g) The energy is measured later at $t = 2t_0$. Which are the possible outcomes of this measurement and the corresponding probabilities?

After the first measurement at $t = t_0$, where the energy was found to be $\frac{\hbar\omega}{2}$, the system collapsed into the eigenstate $|u_0\rangle$, which corresponds to the ground state energy. Since the system is now in $|u_0\rangle$, any subsequent measurement of energy at $t = 2t_0$ will give the same energy value.

$$E_0 = \frac{\hbar\omega}{2}.$$

The probability of measuring this energy is 100% because the system remains in the $|u_0\rangle$ state after the first measurement.

This conclusion holds regardless of the time t, as the state $|u_0\rangle$ does not change with time.

Exercise 2

Let us assume that a physical system can be completely described in the three-dimensional space defined by the orthonormal basis $|u_1\rangle, |u_2\rangle, |u_3\rangle$. On this basis, the Hamiltonian $\hat{H}$ and the two observables $\hat{A}$ and $\hat{B}$ are represented by the matrices:

$$\hat{H} = E\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}, \quad \hat{A} = a\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad \hat{B} = b\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},$$

where E, a, and b are real and positive constants. At time zero, the state of the system is given by the state vector:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle.$$

a) Choose among the three previous operators a complete set of commuting operators (CSCO).

The commutator $[H, A]$ is:

$$[H, A] = HA - AH \tag{6}$$

Calculating HA:

$$HA = E\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix} \cdot a\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} = aE\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 2\\ 0 & 2 & 0\end{pmatrix} \tag{7}$$

Calculating AH:

$$AH = a\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \cdot E\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix} = aE\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 2\\ 0 & 2 & 0\end{pmatrix} \tag{8}$$

Since $HA = AH$, it follows that:

$$[H, A] = HA - AH = 0$$

Therefore H and A commute and they can form a CSCO.

b) Find a common vector basis of the CSCO.

To form a CSCO, we need a set of observables that commute with each other and whose simultaneous eigenbasis spans the entire Hilbert space.

Eigenvectors of H:

$$H = E \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The eigenvectors are:

$$H\mathbf{e} = \mu \mathbf{e} \tag{9}$$

where $\mathbf{e} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and μ is the eigenvalue.

Eigenvector Calculation

$$\begin{aligned} H\mathbf{e} &= E \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} && (10) \\ &= \begin{pmatrix} E \cdot x \\ 2E \cdot y \\ 2E \cdot z \end{pmatrix} && (11) \end{aligned}$$

Eigenvalues and Eigenvectors

1. For $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$:

$$\begin{aligned} H\mathbf{e}_1 &= E \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} && (12) \\ &= \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix} && (13) \\ &= E \cdot \mathbf{e}_1 && (14) \end{aligned}$$

2. For $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$:

$$H\mathbf{e}_2 = E \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{15}$$

$$= \begin{pmatrix} 0 \\ 2E \\ 0 \end{pmatrix} \tag{16}$$

$$= 2E \cdot \mathbf{e}_2 \tag{17}$$

3. For $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$:

$$H\mathbf{e}_3 = E \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{18}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 2E \end{pmatrix} \tag{19}$$

$$= 2E \cdot \mathbf{e}_3 \tag{20}$$

Eigenvectors of A

$$A = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The eigenvectors are:

$$A\mathbf{v} = \lambda\mathbf{v} \tag{21}$$

where $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and λ is the eigenvalue.

Characteristic Polynomial

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{pmatrix}\right) \tag{22}$$

$$= (1-\lambda)\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \tag{23}$$

$$= (1-\lambda)(\lambda^2 - 1) \tag{24}$$

$$= (1-\lambda)(\lambda - 1)(\lambda + 1) \tag{25}$$

The eigenvalues are $\lambda_1 = 1$ (twice degenerate) and $\lambda_2 = -1$.

Eigenvectors

1. For $\lambda_1 = 1$:

$$(A - I)\mathbf{v} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{v} = 0 \tag{26}$$

The eigenvectors for $\lambda_1 = 1$ are:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

2. For $\lambda_2 = -1$:

$$(A + I)\mathbf{v} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{v} = 0 \tag{27}$$

To solve this system, we find that the eigenvectors corresponding to $\lambda_2 = -1$ are:

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Summary

1. Eigenvectors and Eigenvalues of H:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mu_1 = E \tag{28}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mu_2 = 2E \tag{29}$$

$$\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mu_3 = 2E \tag{30}$$

2. Eigenvectors and Eigenvalues of A:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda_1 = a \tag{31}$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \lambda_2 = -a \tag{32}$$

$$\mathbf{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = a \tag{33}$$

Since $\mathbf{e}_1$ is an eigenvector of both H and A, it is a common eigenvector. For the other two eigenvectors:

$$H\mathbf{v}_3 = \frac{1}{\sqrt{2}}H\begin{pmatrix}0\\1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\2E\\2E\end{pmatrix} = \frac{2E}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} = 2E\,\mathbf{v}_3.$$

Similarly,

$$H\mathbf{v}_2 = \frac{1}{\sqrt{2}}H\begin{pmatrix}0\\1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\2E\\-2E\end{pmatrix} = \frac{2E}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix} = 2E\,\mathbf{v}_2.$$

Thus, the common eigenvectors of H and A are:

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}.$$

Therefore a suitable CSCO is:

$$\{H, A\}.$$

With a common eigenbasis

$$\{\mathbf{e}_1, \frac{\mathbf{e}_2+\mathbf{e}_3}{\sqrt{2}}, \frac{\mathbf{e}_2-\mathbf{e}_3}{\sqrt{2}}\}$$

c) If at $t = 0$ the energy of the system is measured, which are the values that could be obtained and with what probability? What if we measured $\hat{A}$ instead of $\hat{H}$?

Measurement of Operator ($\hat{H}$)

The possible energies correspond to the eigenvalues of the operator $\hat{H}$ which are $\mu_1 = E$ and $\mu_2 = 2E$, with the corresponding eigenvectors:

$$\mathbf{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

The initial state is:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}v_1 + \frac{1}{2}v_2 + \frac{1}{2}v_3.$$

Projecting $|\psi(0)\rangle$ onto the eigenbasis of $\hat{H}$:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2 + \frac{1}{2}\mathbf{e}_3.$$

The probabilities of obtaining each eigenvalue are:

- For $\mu_1 = E$ (eigenvector $\mathbf{e}_1$):

$$P(\mu_1) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

- For $\mu_2 = 2E$ (eigenvectors $\mathbf{e}_2$ and $\mathbf{e}_3$):

$$P(\mu_2) = \left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Measurement of Operator $\hat{A}$

The operator $\hat{A}$ has the following eigenvalues and eigenvectors:

$$\lambda_1 = a, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\lambda_2 = a, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

$$\lambda_3 = -a, \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

The initial state is given as:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} v_1 + \frac{1}{2} v_2 + \frac{1}{2} v_3,$$

where v_1, v_2, v_3 can be expressed in terms of $\hat{A}$'s eigenvectors as:

$$v_1 = \mathbf{e}_1, \quad v_2 = \frac{1}{\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3), \quad v_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_2 - \mathbf{e}_3).$$

Substituting this into $|\psi(0)\rangle$:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{2}\left(\frac{1}{\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3)\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}(\mathbf{e}_2 - \mathbf{e}_3)\right).$$

Simplifying:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{2\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3) + \frac{1}{2\sqrt{2}}(\mathbf{e}_2 - \mathbf{e}_3).$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2.$$

The probability of measuring an eigenvalue λ_i is given by:

$$P(\lambda_i) = |\langle \mathbf{e}_i | \psi(0) \rangle|^2.$$

1. Probability for $\lambda_1 = a$: The coefficient of $\mathbf{e}_1$ in $|\psi(0)\rangle$ is $\frac{1}{\sqrt{2}}$. Thus:

$$P(\lambda_1) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

2. Probability for $\lambda_2 = a$: The coefficient of $\mathbf{e}_2$ in $|\psi(0)\rangle$ is $\frac{1}{\sqrt{2}}$. Thus:

$$P(\lambda_2) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

3. Probability for $\lambda_3 = -a$: The coefficient of $\mathbf{e}_3$ in $|\psi(0)\rangle$ is 0. Thus:

$$P(\lambda_3) = 0.$$

The probabilities are:

$$P(\lambda_1 = a) = \frac{1}{2}, \quad P(\lambda_2 = a) = \frac{1}{2}, \quad P(\lambda_3 = -a) = 0.$$

Therefore, the total probability of measuring a is:

$$P(a) = P(\lambda_1) + P(\lambda_2) = \frac{1}{2} + \frac{1}{2} = 1.$$

The system will always return a when measuring $\hat{A}$.

Summary of Probabilities

1. Measurement of $\hat{H}$: - $\mu_1 = E$: $P(\mu_1) = \frac{1}{2}$, - $\mu_2 = 2E$: $P(\mu_2) = \frac{1}{2}$.
2. Measurement of $\hat{A}$: - $\lambda_1 = a$: $P(\lambda_1) = 1$, - $\lambda_3 = -a$: $P(\lambda_3) = 0$.

d) Calculate $|\psi(t)\rangle$, $\langle\hat{A}\rangle(t)$ and $\langle\hat{B}\rangle(t)$.

The time evolution of a quantum state is given by:

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle.$$

The initial state is:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2 + \frac{1}{2}\mathbf{e}_3.$$

Since the Hamiltonian is diagonal in this basis, the time evolution of each basis state is:

$$e^{-\frac{i}{\hbar}\hat{H}t}\mathbf{e}_1 = e^{-\frac{i}{\hbar}Et}\mathbf{e}_1,$$

$$e^{-\frac{i}{\hbar}\hat{H}t}\mathbf{e}_2 = e^{-\frac{i}{\hbar}2Et}\mathbf{e}_2,$$

$$e^{-\frac{i}{\hbar}\hat{H}t}\mathbf{e}_3 = e^{-\frac{i}{\hbar}2Et}\mathbf{e}_3.$$

Thus, the time-evolved state is:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}Et}\mathbf{e}_1 + \frac{1}{2}e^{-\frac{i}{\hbar}2Et}\mathbf{e}_2 + \frac{1}{2}e^{-\frac{i}{\hbar}2Et}\mathbf{e}_3.$$

The expectation value of $\hat{A}$ is given by:

$$\langle \hat{A} \rangle(t) = \langle \psi(t) | \hat{A} | \psi(t) \rangle.$$

The operator $\hat{A}$ is:

$$\hat{A} = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The matrix elements of $\hat{A}$ are:

$$\langle \mathbf{e}_1 | \hat{A} | \mathbf{e}_1 \rangle = a, \quad \langle \mathbf{e}_2 | \hat{A} | \mathbf{e}_3 \rangle = a, \quad \langle \mathbf{e}_3 | \hat{A} | \mathbf{e}_2 \rangle = a, \quad \text{all others } = 0.$$

Substituting $|\psi(t)\rangle$, the expectation value becomes:

$$\langle \hat{A} \rangle(t) = a \left(|c_1(t)|^2 + c_2^*(t) c_3(t) + c_3^*(t) c_2(t) \right),$$

where:

$$c_1(t) = \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar} E t}, \quad c_2(t) = \frac{1}{2} e^{-\frac{i}{\hbar} 2 E t}, \quad c_3(t) = \frac{1}{2} e^{-\frac{i}{\hbar} 2 E t}.$$

Calculating each term:

Computing $|c_1(t)|^2$:

$$|c_1(t)|^2 = \left| \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar} E t} \right|^2 = \frac{1}{2}.$$

Computing $c_2^*(t) c_3(t) + c_3^*(t) c_2(t)$:

$$c_2^*(t) c_3(t) = \left(\frac{1}{2} e^{\frac{i}{\hbar} 2 E t} \right) \left(\frac{1}{2} e^{-\frac{i}{\hbar} 2 E t} \right) = \frac{1}{4}.$$

$$c_3^*(t) c_2(t) = \frac{1}{4}.$$

Adding the terms:

$$c_2^*(t) c_3(t) + c_3^*(t) c_2(t) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Thus:

$$\langle \hat{A} \rangle(t) = a \left(\frac{1}{2} + \frac{1}{2} \right) = a.$$

The expectation value of $\hat{B}$ is given by:

$$\langle \hat{B} \rangle(t) = \langle \psi(t) | \hat{B} | \psi(t) \rangle.$$

The time-evolved state is:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar} E t} \mathbf{e}_1 + \frac{1}{2} e^{-\frac{i}{\hbar} 2 E t} \mathbf{e}_2 + \frac{1}{2} e^{-\frac{i}{\hbar} 2 E t} \mathbf{e}_3.$$

The operator $\hat{B}$ is:

$$\hat{B} = b \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix elements of $\hat{B}$ are:

$$\langle \mathbf{e}_1 | \hat{B} | \mathbf{e}_2 \rangle = b, \quad \langle \mathbf{e}_2 | \hat{B} | \mathbf{e}_1 \rangle = b, \quad \langle \mathbf{e}_3 | \hat{B} | \mathbf{e}_3 \rangle = b.$$

Substituting $|\psi(t)\rangle$, the expectation value becomes:

$$\langle \hat{B} \rangle(t) = b \left(c_1^*(t) c_2(t) + c_2^*(t) c_1(t) + |c_3(t)|^2 \right),$$

where:

$$c_1(t) = \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar} E t}, \quad c_2(t) = \frac{1}{2} e^{-\frac{i}{\hbar} 2 E t}, \quad c_3(t) = \frac{1}{2} e^{-\frac{i}{\hbar} 2 E t}.$$

Calculating each term

Computing $c_1^*(t) c_2(t)$:

$$c_1^*(t) c_2(t) = \frac{1}{2\sqrt{2}} e^{-\frac{i}{\hbar} E t}.$$

Computing $c_2^*(t) c_1(t)$:

$$c_2^*(t) c_1(t) = \frac{1}{2\sqrt{2}} e^{\frac{i}{\hbar} E t}.$$

Computing $c_1^*(t) c_2(t) + c_2^*(t) c_1(t)$:

$$c_1^*(t) c_2(t) + c_2^*(t) c_1(t) = \frac{1}{\sqrt{2}} \cos\left(\frac{E t}{\hbar} \right).$$

Computing $|c_3(t)|^2$:

$$|c_3(t)|^2 = \frac{1}{4}.$$

Substituting all terms:

$$\langle \hat{B} \rangle(t) = b \left(\frac{1}{\sqrt{2}} \cos\left(\frac{E t}{\hbar} \right) + \frac{1}{4} \right).$$

e) Which are the results that could be obtained and with what probabilities in case of measuring $\hat{A}$ and $\hat{B}$ at an instant t?

To calculate the possible results and probabilities when measuring $\hat{A}$, we use the eigenvalues and eigenvectors of $\hat{A}$ and the time-evolved state $|\psi(t)\rangle$.

The operator $\hat{A}$ is:

$$\hat{A} = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Its eigenvalues and eigenvectors are:

$$\lambda_1 = a, \quad \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\lambda_2 = a, \quad \mathbf{e}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

$$\lambda_3 = a, \quad \mathbf{e}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

The time-evolved state is:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}Et} \mathbf{e}_1 + \frac{1}{2} e^{-\frac{i}{\hbar}2Et} \mathbf{e}_2 + \frac{1}{2} e^{-\frac{i}{\hbar}2Et} \mathbf{e}_3,$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the standard basis vectors.

The state $|\psi(t)\rangle$ can be expressed in the eigenbasis of $\hat{A}$ as:

$$|\psi(t)\rangle = d_1(t)\mathbf{e}_1 + d_2(t)\mathbf{e}_2 + d_3(t)\mathbf{e}_3,$$

where:

$$d_i(t) = \langle \mathbf{e}_i | \psi(t) \rangle.$$

Calculating the Overlaps:

-$d_1(t)$: The eigenvector $\mathbf{e}_1$ is:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The overlap is:

$$d_1(t) = \langle \mathbf{e}_1 | \psi(t) \rangle = \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}Et}.$$

-$d_2(t)$: The eigenvector $\mathbf{e}_2$ is:

$$\mathbf{e}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

The overlap is:

$$d_2(t) = \langle \mathbf{e}_2 | \psi(t) \rangle = \frac{1}{\sqrt{2}} \left(\frac{1}{2} e^{-\frac{i}{\hbar}2Et} + \frac{1}{2} e^{-\frac{i}{\hbar}2Et} \right).$$

Simplifying:

$$d_2(t) = \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}2Et}.$$

-$d_3(t)$: The eigenvector $\mathbf{e}_3$ is:

$$\mathbf{e}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

The overlap is:

$$d_3(t) = \langle \mathbf{e}_3 | \psi(t) \rangle = \frac{1}{\sqrt{2}} \left(\frac{1}{2} e^{-\frac{i}{\hbar} 2Et} - \frac{1}{2} e^{-\frac{i}{\hbar} 2Et} \right).$$

Simplifying:

$$d_3(t) = 0.$$

The probabilities are:

$$P(\lambda_i) = |d_i(t)|^2.$$

(a) Probability $P(\lambda_1 = a)$:

$$P(\lambda_1 = a) = |d_1(t)|^2 = \left| \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar} Et} \right|^2.$$

Simplifying:

$$P(\lambda_1 = a) = \frac{1}{2}.$$

(b) Probability $P(\lambda_2 = a)$:

$$P(\lambda_2 = a) = |d_2(t)|^2 = \left| \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar} 2Et} \right|^2.$$

Simplifying:

$$P(\lambda_2 = a) = \frac{1}{2}.$$

(c) Probability $P(\lambda_3 = a)$:

$$P(\lambda_3 = a) = |d_3(t)|^2 = |0|^2.$$

Simplifying:

$$P(\lambda_3 = a) = 0.$$

Final Results: -Possible results:

$$\lambda_1 = a, \quad \lambda_2 = a, \quad \lambda_3 = -a$$

-Probabilities:

$$P(a) = 1.$$

$$P(-a) = 0.$$

To calculate the possible results and probabilities when measuring $\hat{B}$, we use the eigenvalues and eigenvectors of $\hat{B}$ and the time-evolved state $|\psi(t)\rangle$.

The operator $\hat{B}$ is:

$$\hat{B} = b \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues and eigenvectors are:

$$\mu_1 = b, \quad \mathbf{g}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

$$\mu_2 = b, \quad \mathbf{g}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\mu_3 = -b, \quad \mathbf{g}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

The time-evolved state is:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}Et} \mathbf{e}_1 + \frac{1}{2} e^{-\frac{i}{\hbar}2Et} \mathbf{e}_2 + \frac{1}{2} e^{-\frac{i}{\hbar}2Et} \mathbf{e}_3.$$

Where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Projection onto basis $\hat{B}$:

$$|\psi(t)\rangle = c_1(t)\mathbf{g}_1 + c_2(t)\mathbf{g}_2 + c_3(t)\mathbf{g}_3,$$

where:

$$c_1(t) = \langle \mathbf{g}_1 | \psi(t) \rangle, \quad c_2(t) = \langle \mathbf{g}_2 | \psi(t) \rangle, \quad c_3(t) = \langle \mathbf{g}_3 | \psi(t) \rangle.$$

Computing:

-$c_1(t)$:

$$\mathbf{g}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \psi(t) = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}Et} \\ \frac{1}{2} e^{-\frac{i}{\hbar}2Et} \\ \frac{1}{2} e^{-\frac{i}{\hbar}2Et} \end{pmatrix}.$$

$$c_1(t) = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}Et} + \frac{1}{2} e^{-\frac{i}{\hbar}2Et} \right).$$

Simplifying:

$$c_1(t) = \frac{1}{2}e^{-\frac{i}{\hbar}Et} + \frac{1}{2\sqrt{2}}e^{-\frac{i}{\hbar}2Et}.$$

-$c_2(t)$:

$$\mathbf{g}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \psi(t) = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}Et} \\ \frac{1}{2}e^{-\frac{i}{\hbar}2Et} \\ \frac{1}{2}e^{-\frac{i}{\hbar}2Et} \end{pmatrix}.$$

$$c_2(t) = \frac{1}{2}e^{-\frac{i}{\hbar}2Et}.$$

-$c_3(t)$:

$$\mathbf{g}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \psi(t) = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}Et} \\ \frac{1}{2}e^{-\frac{i}{\hbar}2Et} \\ \frac{1}{2}e^{-\frac{i}{\hbar}2Et} \end{pmatrix}.$$

$$c_3(t) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}Et} - \frac{1}{2}e^{-\frac{i}{\hbar}2Et}\right).$$

Simplifying:

$$c_3(t) = \frac{1}{2}e^{-\frac{i}{\hbar}Et} - \frac{1}{2\sqrt{2}}e^{-\frac{i}{\hbar}2Et}.$$

The probabilities are:

(a) Probability $P(\mu_1 = b)$:

$$P(\mu_1 = b) = |c_1(t)|^2 = \left|\frac{1}{2}e^{-\frac{i}{\hbar}Et} + \frac{1}{2\sqrt{2}}e^{-\frac{i}{\hbar}2Et}\right|^2.$$

Expanding the modulus square:

$$P(\mu_1 = b) = \frac{1}{4} + \frac{1}{4\sqrt{2}}\cos\left(\frac{Et}{\hbar}\right) + \frac{1}{8}.$$

Combining terms:

$$P(\mu_1 = b) = \frac{3}{8} + \frac{1}{2\sqrt{2}}\cos\left(\frac{Et}{\hbar}\right).$$

(b) Probability $P(\mu_2 = b)$:

$$P(\mu_2 = b) = |c_2(t)|^2 = \left|\frac{1}{2}e^{-\frac{i}{\hbar}2Et}\right|^2.$$

Simplifying:

$$P(\mu_2 = b) = \frac{1}{4}.$$

(c) Probability $P(\mu_3 = -b)$:

$$P(\mu_3 = -b) = |c_3(t)|^2 = \left| \frac{1}{2} e^{-\frac{i}{\hbar}Et} - \frac{1}{2\sqrt{2}} e^{-\frac{i}{\hbar}2Et} \right|^2 .$$

Expanding the modulus square:

$$P(\mu_3 = -b) = \frac{1}{4} - \frac{1}{4\sqrt{2}} \cos\left(\frac{Et}{\hbar}\right) + \frac{1}{8}.$$

Combining terms:

$$P(\mu_3 = -b) = \frac{3}{8} - \frac{1}{2\sqrt{2}} \cos\left(\frac{Et}{\hbar}\right).$$

Final Results for $\hat{B}$
-Possible results:

$$\mu_1 = b, \quad \mu_2 = b, \quad \mu_3 = -b.$$

-Probabilities:

$$P(b) = \frac{5}{8} + \frac{1}{2\sqrt{2}} \cos\left(\frac{Et}{\hbar}\right),$$

$$P(-b) = \frac{3}{8} - \frac{1}{2\sqrt{2}} \cos\left(\frac{Et}{\hbar}\right).$$

f) Suppose that at an instant $t = t_1$, the result of measuring the energy is E. Can the result of measuring $\hat{A}$ immediately after $t = t_1$ be predicted? If yes, what would that result be? What would the state of the system be after having measured $\hat{A}$? And if after measuring $\hat{A}$, we measure $\hat{B}$, what would the result of the measurement and the state of the system be after the measurement? With what probability?

At $t = t_1$, the result of measuring the energy is E. The corresponding eigenvector of the Hamiltonian $\hat{H}$ is:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, after measurement the system collapses into the eigenstate:

$$|\psi(t_1)\rangle = \mathbf{e}_1.$$

Measurement of $\hat{A}$ Immediately After $t = t_1$:
The operator $\hat{A}$ has eigenvalues $+a$ and $-a$, with eigenvectors:

$$\lambda_1 = +a, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\lambda_2 = +a, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

$$\lambda_3 = -a, \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Since the state $|\psi(t_1)\rangle = \mathbf{e}_1$ is already an eigenvector of $\hat{A}$ with eigenvalue $+a$, the result of measuring $\hat{A}$ will be:

$$\lambda_1 = +a.$$

The state of the system remains unchanged:

$$|\psi_{\hat{A}}(t_1)\rangle = \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Measurement of $\hat{B}$ After Measuring $\hat{A}$:
The operator $\hat{B}$ has eigenvalues $+b$ and $-b$, with eigenvectors:

$$\mu_1 = +b, \quad \mathbf{g}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

$$\mu_2 = +b, \quad \mathbf{g}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\mu_3 = -b, \quad \mathbf{g}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

The state $|\psi_{\hat{A}}(t_1)\rangle = \mathbf{e}_1$ can be expressed in the eigenbasis of $\hat{B}$:

$$|\psi_{\hat{A}}(t_1)\rangle = \mathbf{e}_1 = \frac{1}{\sqrt{2}}\mathbf{g}_1 + \frac{1}{\sqrt{2}}\mathbf{g}_3.$$

The probabilities of measuring $\hat{B}$ are:

$$P(+b) = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2},$$

$$P(-b) = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}.$$

If the result is $+b$, the system collapses to:

$$|\psi_{\hat{B}}(t_1)\rangle = \mathbf{g}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

If the result is $-b$, the system collapses to:

$$|\psi_{\hat{B}}(t_1)\rangle = \mathbf{g}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

g) Same as case (f), but, after having obtained the energy E, we measure $\hat{B}$ first and then $\hat{A}$. What would the result of the measurement and the state of the system be after the measurement of $\hat{A}$ and then $\hat{B}$? With what probabilities?

At $t = t_1$, the energy is measured to be E. The corresponding eigenvector of the Hamiltonian $\hat{H}$ is:

$$\mathbf{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}.$$

After the measurement, the system collapses into:

$$|\psi(t_1)\rangle = \mathbf{e}_1.$$

Measurement of $\hat{B}$ after measuring energy:

The operator $\hat{B}$ has eigenvalues $+b$ and $-b$, with eigenvectors:

$$\mathbf{g}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \mathbf{g}_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \quad \mathbf{g}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

Expressing $|\psi(t_1)\rangle$ in the eigenbasis of $\hat{B}$:

$$|\psi(t_1)\rangle = \mathbf{e}_1 = \frac{1}{\sqrt{2}}\mathbf{g}_1 + \frac{1}{\sqrt{2}}\mathbf{g}_3.$$

Probabilities of Measuring $\hat{B}$: -For $+b$ (corresponding to $\mathbf{g}_1$):

$$P(+b) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

-For $-b$ (corresponding to $\mathbf{g}_3$):

$$P(-b) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

State After Measuring $\hat{B}$: -If $+b$, the system collapses to:

$$|\psi_{\hat{B}}(t_1)\rangle = \mathbf{g}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}.$$

-If $-b$, the system collapses to:

$$|\psi_{\hat{B}}(t_1)\rangle = \mathbf{g}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Measurement of $\hat{A}$ After $\hat{B}$:
The operator $\hat{A}$ has eigenvalues $+a$ and $-a$, with eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

(a) If the Result of $\hat{B}$ is $+b$: State after measuring $\hat{B}$:

$$|\psi_{\hat{B}}(t_1)\rangle = \mathbf{g}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Expand $\mathbf{g}_1$ in the eigenbasis of $\hat{A}$:

$$\mathbf{g}_1 = \frac{1}{\sqrt{2}}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3.$$

Probabilities of measuring $\hat{A}$:

$$P(+a \text{ from } \mathbf{v}_1) = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2},$$

$$P(+a \text{ from } \mathbf{v}_2) = \left| \frac{1}{2} \right|^2 = \frac{1}{4},$$

$$P(-a \text{ from } \mathbf{v}_3) = \left| \frac{1}{2} \right|^2 = \frac{1}{4}.$$

(b) If the Result of $\hat{B}$ is $-b$: State after measuring $\hat{B}$:

$$|\psi_{\hat{B}}(t_1)\rangle = \mathbf{g}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Expanding $\mathbf{g}_3$ in the eigenbasis of $\hat{A}$:

$$\mathbf{g}_3 = \frac{1}{\sqrt{2}}\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2 - \frac{1}{2}\mathbf{v}_3.$$

Probabilities of measuring $\hat{A}$:

$$P(+a \text{ from } \mathbf{v}_1) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2},$$

$$P(+a \text{ from } \mathbf{v}_2) = \left|\frac{-1}{2}\right|^2 = \frac{1}{4},$$

$$P(-a \text{ from } \mathbf{v}_3) = \left|\frac{-1}{2}\right|^2 = \frac{1}{4}.$$

Final Results:
1. After Measuring $\hat{B}$:

$$P(+b) = \frac{1}{2}, \quad P(-b) = \frac{1}{2}.$$

2. After Measuring $\hat{A}$:

-For $\hat{B} = +b$:

$$P(+a) = \frac{1}{2}, \quad \text{state: } \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$P(+a) = \frac{1}{4}, \quad \text{state: } \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

$$P(-a) = \frac{1}{4}, \quad \text{state: } \mathbf{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

-For $\hat{B} = -b$:

$$P(+a) = \frac{1}{2}, \quad \text{state: } \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$P(+a) = \frac{1}{4}, \quad \text{state: } \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

$$P(-a) = \frac{1}{4}, \quad \text{state: } \mathbf{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

h) Same as (f), but the result of measuring the energy is $2E$ instead of E.

State After Energy Measurement

At $t = 0$, the initial state is:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2 + \frac{1}{2}\mathbf{e}_3.$$

The measurement of $\hat{H}$ with result $2E$ collapses the state onto the eigenspace spanned by $\mathbf{e}_2$ and $\mathbf{e}_3$. The component of $|\psi(0)\rangle$ in this subspace is:

$$|\psi_{\text{after}}\rangle = \frac{\frac{1}{2}\mathbf{e}_2 + \frac{1}{2}\mathbf{e}_3}{\sqrt{\left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2}} = \frac{1}{\sqrt{2}}\mathbf{e}_2 + \frac{1}{\sqrt{2}}\mathbf{e}_3.$$

After normalization:

$$|\psi_{\text{after}}\rangle = \frac{1}{\sqrt{2}}\mathbf{e}_2 + \frac{1}{\sqrt{2}}\mathbf{e}_3.$$

Measurement of $\hat{A}$ After Energy

The operator $\hat{A}$ has eigenvalues $+a$ and $-a$, with eigenvectors:

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3) \quad (\text{eigenvalue } + a),$$

$$\mathbf{v}_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_2 - \mathbf{e}_3) \quad (\text{eigenvalue } - a).$$

The post-energy measurement state:

$$|\psi_{\text{after}}\rangle = \frac{1}{\sqrt{2}}\mathbf{e}_2 + \frac{1}{\sqrt{2}}\mathbf{e}_3,$$

is **already aligned** with the eigenvector $\mathbf{v}_2$ of $\hat{A}$ with eigenvalue $+a$. Therefore after measuring $\hat{A}$ the state remains with 100% probability):

$$|\psi_{\hat{A}}\rangle = \mathbf{v}_2 = \frac{1}{\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3).$$

Measurement of $\hat{B}$ After $\hat{A}$

The operator $\hat{B}$ has eigenvalues $+b$ and $-b$, with eigenstates:

$$\mathbf{g}_1 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2), \quad \mathbf{g}_2 = \mathbf{e}_3, \quad \mathbf{g}_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2).$$

The state $|\psi_{\hat{A}}\rangle = \frac{1}{\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3)$ is now decomposed into the eigenbasis of $\hat{B}$:

$$|\psi_{\hat{A}}\rangle = c_1\mathbf{g}_1 + c_2\mathbf{g}_2 + c_3\mathbf{g}_3.$$

Computing the overlaps:
-For $\mathbf{g}_2 = \mathbf{e}_3$:

$$c_2 = \langle \mathbf{g}_2 | \psi_{\hat{A}} \rangle = \langle \mathbf{e}_3 | \frac{1}{\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3)\rangle = \frac{1}{\sqrt{2}}.$$

-For $\mathbf{g}_1 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)$:

$$c_1 = \langle \mathbf{g}_1 | \psi_{\hat{A}} \rangle = \frac{1}{\sqrt{2}}\langle \mathbf{e}_1 + \mathbf{e}_2 | \frac{1}{\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3)\rangle.$$

Only $\mathbf{e}_2$ overlaps:

$$c_1 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \langle \mathbf{e}_2 | \mathbf{e}_2 \rangle = \frac{1}{2}.$$

-For $\mathbf{g}_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2)$:

$$c_3 = \langle \mathbf{g}_3 | \psi_{\hat{A}} \rangle = \frac{1}{\sqrt{2}} \langle \mathbf{e}_1 - \mathbf{e}_2 | \frac{1}{\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3) \rangle.$$

Only $\mathbf{e}_2$ overlaps:

$$c_3 = \frac{1}{\sqrt{2}} \cdot \frac{-1}{\sqrt{2}} \langle \mathbf{e}_2 | \mathbf{e}_2 \rangle = -\frac{1}{2}.$$

The probabilities are the square of the coefficients:

-For $+b$ (corresponding to $\mathbf{g}_1$ and $\mathbf{g}_2$):

$$P(+b) = |c_1|^2 + |c_2|^2 = \left| \frac{1}{2} \right|^2 + \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

-For $-b$ (corresponding to $\mathbf{g}_3$):

$$P(-b) = |c_3|^2 = \left| -\frac{1}{2} \right|^2 = \frac{1}{4}.$$

The Post-Measurement States are:

-If $+b$ is measured:

$$|\psi_{\hat{B},+b}\rangle = \frac{c_1 \mathbf{g}_1 + c_2 \mathbf{g}_2}{\sqrt{|c_1|^2 + |c_2|^2}} = \frac{\frac{1}{2}\mathbf{g}_1 + \frac{1}{\sqrt{2}}\mathbf{g}_2}{\sqrt{\frac{1}{4} + \frac{1}{2}}}.$$

-If $-b$ is measured:

$$|\psi_{\hat{B},-b}\rangle = \mathbf{g}_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2).$$

Final Results:

1. **After Measuring Energy $2E$:**

$$|\psi_{\text{after}}\rangle = \frac{1}{\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3).$$

2. **After Measuring $\hat{A}$:**

- Result: $+a$ with probability 100%.
- State:

$$|\psi_{\hat{A}}\rangle = \frac{1}{\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3).$$

3. **After Measuring $\hat{B}$:**

   - Result $+b$ with probability $\frac{3}{4}$, state:

$$|\psi_{\hat{B},+b}\rangle = \frac{\frac{1}{2}\mathbf{g}_1 + \frac{1}{\sqrt{2}}\mathbf{g}_2}{\sqrt{\frac{3}{4}}}.$$

   - Result $-b$ with probability $\frac{1}{4}$, state:

$$|\psi_{\hat{B},-b}\rangle = \mathbf{g}_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2).$$

i) Same as (g), but the result of measuring the energy is $2E$ instead of E.

Measurement of the Energy At $t = 0$, the initial state of the system is:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2 + \frac{1}{2}\mathbf{e}_3.$$

When the energy measurement yields $2E$, the state collapses to the eigenspace corresponding to the eigenvalue $2E$, which is spanned by $\mathbf{e}_2$ and $\mathbf{e}_3$. The projection of $|\psi(0)\rangle$ onto this subspace is:

$$|\psi_{\text{after}}\rangle = \frac{\frac{1}{2}\mathbf{e}_2 + \frac{1}{2}\mathbf{e}_3}{\sqrt{\left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2}} = \frac{1}{\sqrt{2}}\mathbf{e}_2 + \frac{1}{\sqrt{2}}\mathbf{e}_3.$$

Thus, after the energy measurement:

$$|\psi_{\text{after}}\rangle = \frac{1}{\sqrt{2}}\mathbf{e}_2 + \frac{1}{\sqrt{2}}\mathbf{e}_3.$$

Measurement of $\hat{B}$
The operator $\hat{B}$ has eigenvalues $+b$ and $-b$, with eigenvectors:

$$\mathbf{g}_1 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2), \quad \mathbf{g}_2 = \mathbf{e}_3, \quad \mathbf{g}_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2).$$

As it was deduced in the previous step: -For $+b$:

$$P(+b) = |c_1|^2 + |c_2|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

-For $-b$:

$$P(-b) = |c_3|^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Measurement of $\hat{A}$ The operator $\hat{A}$ has eigenvalues $+a$ and $-a$, with eigenvectors:
-$\mathbf{v}_1 = \mathbf{e}_1$, eigenvalue $+a$,
-$\mathbf{v}_2 = \frac{1}{\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3)$, eigenvalue $+a$,
-$\mathbf{v}_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_2 - \mathbf{e}_3)$, eigenvalue $-a$.

Case 1: After $\hat{B} = +b$ The post-$\hat{B}$ state is:

$$|\psi_{\hat{B}=+b}\rangle = \frac{\frac{1}{2}\mathbf{g}_1 + \frac{1}{\sqrt{2}}\mathbf{g}_2}{\sqrt{\frac{3}{4}}}.$$

Decomposing this state into the eigenbasis of $\hat{A}$, the probabilities are:

$$P(A = +a) = \frac{11}{12}, \quad P(A = -a) = \frac{1}{12}.$$

Case 2: After $\hat{B} = -b$ The post-$\hat{B}$ state is:

$$|\psi_{\hat{B}=-b}\rangle = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2).$$

Decomposing this state into the eigenbasis of $\hat{A}$, the probabilities are:

$$P(A = +a) = \frac{3}{4}, \quad P(A = -a) = \frac{1}{4}.$$

Final Results The overall probabilities for the sequence of measurements are:

- $P(B = +b, A = +a) = \frac{3}{4} \cdot \frac{11}{12} = \frac{11}{16}$,
- $P(B = +b, A = -a) = \frac{3}{4} \cdot \frac{1}{12} = \frac{1}{16}$,
- $P(B = -b, A = +a) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$,
- $P(B = -b, A = -a) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$.