Worksheet 4 FMMC

Nicolás Rojo Esteban

February 2025

Universidad Autónoma de Madrid

1 Exercise 1:

Consider the addition of two angular momentum $J_1 = \frac{3}{2}$ and $J_2 = \frac{1}{2}$.

If $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$, construct the eigenvectors $|J, M\rangle$ of J^2 and J_z in terms of the eigenvalues of $J_{1,z}$, J_1, $J_{2,z}$, J_2, that is:

$$\{|J_1, m_{J_1}\rangle|J_2, m_{J_2}\rangle\},$$

a. Use the step-down operators to build the eigenvectors.

b. Verify the results using Clebsch-Gordan Tables.

Section a)

Given

$$J_1 = \tfrac{3}{2}, \quad J_2 = \tfrac{1}{2},$$

the possible values of the total angular momentum $J = J_1 + J_2, J_1 + J_2 - 1, \ldots, |J_1 - J_2|$ are

$$J = 2 \quad \text{and} \quad J = 1.$$

Hence we have:

- For $J = 2$, there are $2J + 1 = 5$ states: $M = 2,\ 1,\ 0,\ -1,\ -2$.
- For $J = 1$, there are $2J + 1 = 3$ states: $M = 1,\ 0,\ -1$.

To construct the eigenstates of total angular momentum $|J, M\rangle$ from the basis states $|J_1, m_{J_1}\rangle|J_2, m_{J_2}\rangle$, we start with the highest projection state along the z-axis.

Since we are adding two angular momenta, $J_1 = \frac{3}{2}$ and $J_2 = \frac{1}{2}$, the maximum total angular momentum is:

$$J = J_1 + J_2 = 2.$$

The corresponding state for the maximum projection along z satisfies:

$$M = J = 2.$$

The quantum number M is obtained as the sum of the individual projections:

$$M = m_{J_1} + m_{J_2}.$$

The only combination of m_{J_1} and m_{J_2} that satisfies $M = 2$ is:

$$m_{J_1} = \frac{3}{2}, \quad m_{J_2} = \frac{1}{2}.$$

Since this is the highest projection state, it does not mix with other states. Therefore, the state $|2,2\rangle$ is given by:

$$\boxed{|2,2\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle \left|\frac{1}{2},\frac{1}{2}\right\rangle.}$$

This result shows that when both individual angular momenta are in their maximum projection states, the total angular momentum also reaches its maximum value. From

To obtain the other states the step-down operator $J_- = J_{1,-} + J_{2,-}$ needs to be used, this operator acts on the state $|J,M\rangle$ to produce $|J,M-1\rangle$. The action of J_- is given by:

$$J_-|J,M\rangle = \sqrt{J(J+1) - M(M-1)}|J,M-1\rangle.$$

For $J=2$ and $M=2$, this gives:

$$J_-|2,2\rangle = \sqrt{2(2+1) - 2(2-1)}|2,1\rangle.$$

$$J_-|2,2\rangle = \sqrt{6}|2,1\rangle.$$

Now, using the definition $J_- = J_{1,-} + J_{2,-}$:

$$J_-|2,2\rangle = (J_{1,-} + J_{2,-}) \left|\frac{3}{2},\frac{3}{2}\right\rangle \left|\frac{1}{2},\frac{1}{2}\right\rangle.$$

Then:

$$J_{1,-}\left|\frac{3}{2},\frac{3}{2}\right\rangle = \sqrt{\frac{3}{2}\cdot\frac{5}{2} - \frac{3}{2}\cdot\frac{1}{2}}\left|\frac{3}{2},\frac{1}{2}\right\rangle = \sqrt{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle.$$

$$J_{2,-}\left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{1}{2}\cdot\frac{3}{2} - \frac{1}{2}\cdot(-\frac{1}{2})}\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

Thus:

$$J_-|2,2\rangle = \sqrt{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle \left|\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{3}{2},\frac{3}{2}\right\rangle \left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

Since:

$$J_-|2,2\rangle = \sqrt{6}|2,1\rangle,$$

we divide by $\sqrt{6}$ to obtain:

$$|2,1\rangle = \frac{\sqrt{3}}{\sqrt{6}}\left|\frac{3}{2},\frac{1}{2}\right\rangle \left|\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{\sqrt{6}}\left|\frac{3}{2},\frac{3}{2}\right\rangle \left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

$$= \sqrt{\frac{3}{4}}\left|\frac{3}{2},\frac{1}{2}\right\rangle \left|\frac{1}{2},\frac{1}{2}\right\rangle + \sqrt{\frac{1}{4}}\left|\frac{3}{2},\frac{3}{2}\right\rangle \left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

Thus, the final normalized state is:

$$\boxed{|2,1\rangle = \sqrt{\frac{3}{4}}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{2}\left|\frac{3}{2},\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.}$$

Following the same process the J_- operator is applied to the state with $J = 2$ and $M = 1$:

$$J_-|2,1\rangle = \sqrt{2(2+1) - 1(1-1)}|2,0\rangle.$$

$$J_-|2,1\rangle = \sqrt{6}|2,0\rangle.$$

Now, using the definition $J_- = J_{1,-} + J_{2,-}$:

$$J_-|2,1\rangle = (J_{1,-} + J_{2,-})\left(\sqrt{\frac{3}{4}}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{2}\left|\frac{3}{2},\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right).$$

Applying the operator to the first term:

$$J_{1,-}\left|\frac{3}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{3}{2}\times\frac{5}{2} - \frac{1}{2}\times(-\frac{1}{2})}\left|\frac{3}{2},-\frac{1}{2}\right\rangle = 2\left|\frac{3}{2},-\frac{1}{2}\right\rangle.$$

$$J_{2,-}\left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{1}{2}\times\frac{3}{2} - \frac{1}{2}\times(-\frac{1}{2})}\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

Summing both terms:

$$(J_{1,-} + J_{2,-})\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle = 2\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

Since this term had a factor $\sqrt{\frac{3}{4}}$, the first term will be:

$$\sqrt{3}\left(\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{2}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right).$$

Applying the operator to the second term:

$$J_{1,-}\left|\frac{3}{2},\frac{3}{2}\right\rangle = \sqrt{\frac{3}{2}\times\frac{5}{2} - \frac{3}{2}\times\frac{1}{2}}\left|\frac{3}{2},\frac{1}{2}\right\rangle = \sqrt{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle.$$

For $j = \frac{1}{2}$, the minimum value of m is $-\frac{1}{2}$. If we apply J_- to that state, we would attempt to lower it to $m = -\frac{3}{2}$, which does not exist for $j = \frac{1}{2}$. Therefore,

$$J_{2,-}|\tfrac{1}{2},-\tfrac{1}{2}\rangle = 0.$$

Therefore, the second term is:

$$(J_{1,-} + J_{2,-})\left|\frac{3}{2},\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

Since this term had a factor $\frac{1}{2}$, the first term will be:

$$\sqrt{\frac{3}{4}}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

Summing all the terms:

$$\begin{aligned}J_-|2,1\rangle &= \sqrt{3}\left(\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle+\frac{1}{2}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right)+\sqrt{\frac{3}{4}}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\\
&= \sqrt{3}\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle+\sqrt{\frac{3}{4}}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle+\sqrt{\frac{3}{4}}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\\
&= \sqrt{3}\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle+\sqrt{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.\end{aligned}$$

Since:

$$J_-|2,1\rangle = \sqrt{6}|2,0\rangle,$$

Dividing by $\sqrt{6}$ to obtain:

$$\boxed{|2,0\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle+\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right)}$$

Following the same style for the state with $J=2$ and $M=0$:

$$J_-\left|2,0\right\rangle = \sqrt{2(2+1)-0(0-1)}\left|2,-1\right\rangle = \sqrt{6}\left|2,-1\right\rangle.$$

We also use $J_- = J_{1-}+J_{2-}$, and the starting state is:

$$|2,0\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle+\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right).$$

The first term:

$$J_{1-}\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right)-\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}\left|\frac{3}{2},-\frac{3}{2}\right\rangle = \sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle,$$

$$J_{2-}\left|\frac{1}{2},\frac{1}{2}\right\rangle = 1\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

Hence,

$$(J_{1-}+J_{2-})\left(\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle\right) = \sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle+\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

The second term:

$$J_{1-}\left|\frac{3}{2},\frac{1}{2}\right\rangle = 2\left|\frac{3}{2},-\frac{1}{2}\right\rangle.$$

Since $\left|\frac{1}{2},-\frac{1}{2}\right\rangle$ is the lowest m for $j=\frac{1}{2}$, we have:

$$J_{2-}\left|\frac{1}{2},-\frac{1}{2}\right\rangle = 0.$$

Thus, the total second term is:

$$(J_{1-}+J_{2-})\left(\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right) = 2\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

Summing everything:

The factor in front of $|2,0\rangle$ is $\frac{1}{\sqrt{2}}$:

$$J_-\left|2,0\right\rangle = \frac{1}{\sqrt{2}}\left[\sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle + 2\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right].$$

$$= \frac{1}{\sqrt{2}}\left[\sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + 3\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right].$$

Since,

$$J_-\left|2,0\right\rangle = \sqrt{6}\left|2,-1\right\rangle.$$

Dividing both sides by $\sqrt{6}$:

$$\boxed{\left|2,-1\right\rangle = \frac{1}{2}\left[\left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle\right] + \sqrt{3}\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.}$$

For the state with $J=2$ and $M=-1$:

$$J_-\left|2,-1\right\rangle = \sqrt{2(2+1)-(-1)(-1-1)}\left|2,-2\right\rangle = \sqrt{4}\left|2,-2\right\rangle = 2\left|2,-2\right\rangle.$$

We also use $J_- = J_{1-}+J_{2-}$, and the starting state is:

$$\left|2,-1\right\rangle = \frac{1}{2}\left(\sqrt{3}\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle\right).$$

The first term:

$$J_{1-}\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right)-\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}\left|\frac{3}{2},-\frac{3}{2}\right\rangle = \sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle.$$

$$J_{2-}\left|\frac{1}{2},-\frac{1}{2}\right\rangle = 0.$$

Since in the initial expression there is a $\sqrt{3}$:

$$(J_{1-}+J_{2-})\left(\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right) = \sqrt{3}\left[\sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right] = 3\left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

The second term:

$$J_{1-}\left|\frac{3}{2},-\frac{3}{2}\right\rangle = 0.$$

$$J_{2-}\left|\frac{1}{2},\frac{1}{2}\right\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

Thus,

$$(J_{1-}+J_{2-})\left(\left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle\right) = \left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

Summing everything:

$$\begin{aligned} J_-\left|2,-1\right\rangle &= \frac{1}{2}\left[\sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right]. \\ &= \frac{1}{2}\left[\left(\sqrt{3}+1\right)\left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right]. \end{aligned}$$

Since,

$$J_-\left|2,-1\right\rangle = 2\left|2,-2\right\rangle.$$

Dividing both sides by 2:

$$\boxed{\left|2,-2\right\rangle = \frac{1}{\sqrt{2}}\left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.}$$

To construct the states with total angular momentum $J=1$, we must find linear combinations of the product states that are orthogonal to the previously determined states with $J=2$.

State $|1,1\rangle$

For $J=1, M=1$, we seek a linear combination:

$$\left|1,1\right\rangle = a\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + b\left|\frac{3}{2},\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

This state must be orthogonal to the $J=2, M=1$ state:

$$|2,1\rangle = \sqrt{\frac{3}{4}}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{2}\left|\frac{3}{2},\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

Imposing the orthogonality condition $\langle 2,1|1,1\rangle = 0$:

$$\sqrt{\frac{3}{4}}a + \frac{1}{2}b = 0.$$

Solving for a and b, we choose $a = \frac{1}{2}$ and $b = -\frac{\sqrt{3}}{2}$, leading to:

$$|1,1\rangle = \frac{1}{2}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle - \frac{\sqrt{3}}{2}\left|\frac{3}{2},\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

State $|1,0\rangle$
For $J = 1, M = 0$, we seek:

$$|1,0\rangle = c\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + d\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

This state must be orthogonal to $|2,0\rangle$:

$$|2,0\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right).$$

Imposing orthogonality:

$$\frac{1}{\sqrt{2}}c + \frac{1}{\sqrt{2}}d = 0.$$

Choosing $c = \frac{1}{\sqrt{2}}$ and $d = -\frac{1}{\sqrt{2}}$, the normalized state is:

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle - \left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right).$$

State $|1,-1\rangle$
For $J = 1, M = -1$, we seek:

$$|1,-1\rangle = e\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle + f\left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle.$$

This state must be orthogonal to $|2,-1\rangle$:

$$|2,-1\rangle = \frac{1}{2}\left(\sqrt{3}\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle\right).$$

Imposing orthogonality:

$$\frac{\sqrt{3}}{2}e + \frac{1}{2}f = 0.$$

Choosing $e = \frac{1}{2}$ and $f = -\frac{\sqrt{3}}{2}$, the final state is:

$$|1,-1\rangle = \frac{1}{2}\left(\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle - \sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle\right).$$

Therefore the eigenvectors are:

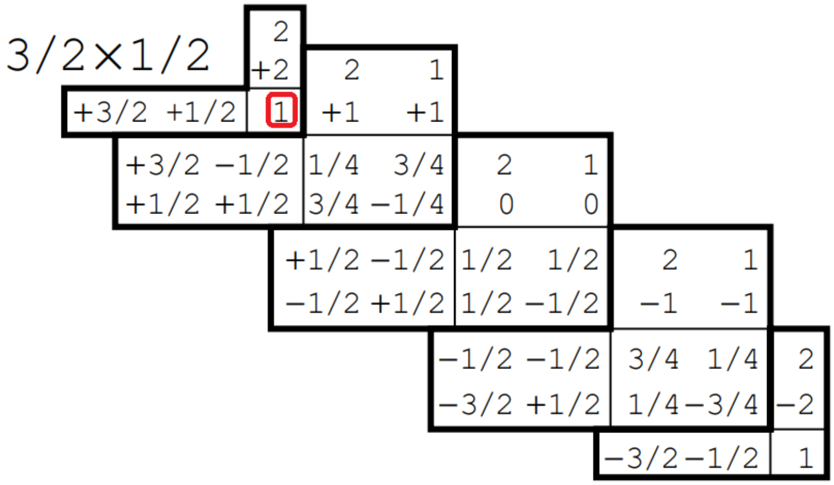
$$\boxed{\begin{aligned}
|2,2\rangle &= \left|\frac{3}{2},\frac{3}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle,\\
|2,1\rangle &= \sqrt{\frac{3}{4}}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{2}\left|\frac{3}{2},\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle,\\
|2,0\rangle &= \frac{1}{\sqrt{2}}\left(\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right),\\
|2,-1\rangle &= \frac{1}{2}\left(\sqrt{3}\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle\right),\\
|2,-2\rangle &= \left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle,\\
|1,1\rangle &= \frac{1}{2}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle - \frac{\sqrt{3}}{2}\left|\frac{3}{2},\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle,\\
|1,0\rangle &= \frac{1}{\sqrt{2}}\left(\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle - \left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right),\\
|1,-1\rangle &= \frac{1}{2}\left(\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle - \sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle\right).
\end{aligned}}$$

Section b)

To verify the correctness of the constructed states, we compare them with the Clebsch-Gordan coefficients from the standard table. The table images contain the coefficients $|j_1 j_2 m_1 m_2|\, JM\rangle$, which must match those used in the expansion of the states.

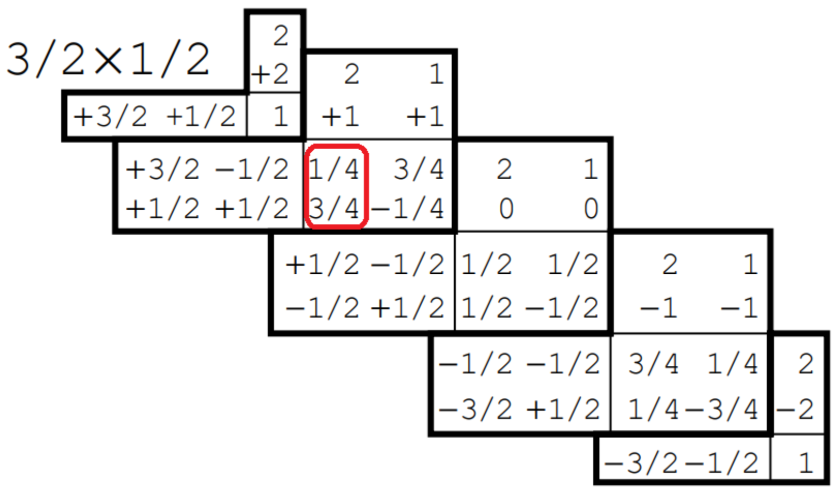
State $|2,2\rangle$

$$|2,2\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle.$$



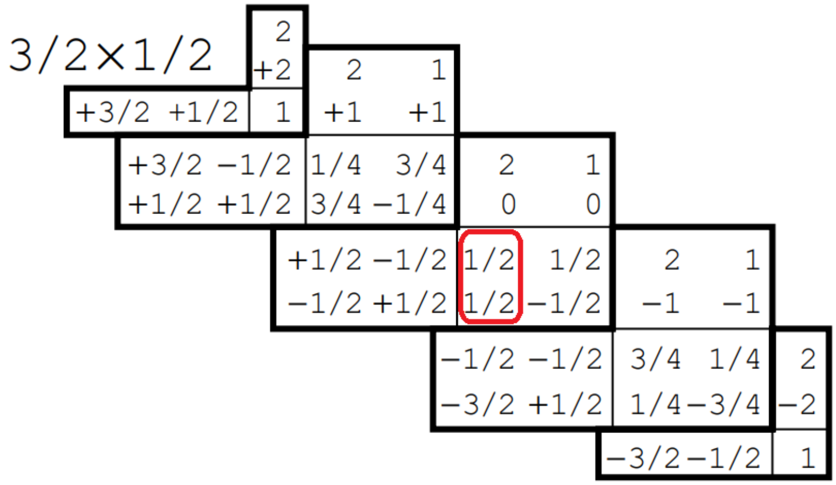
State $|2,1\rangle$

$$|2,1\rangle = \sqrt{\frac{3}{4}}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{2}\left|\frac{3}{2},\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$



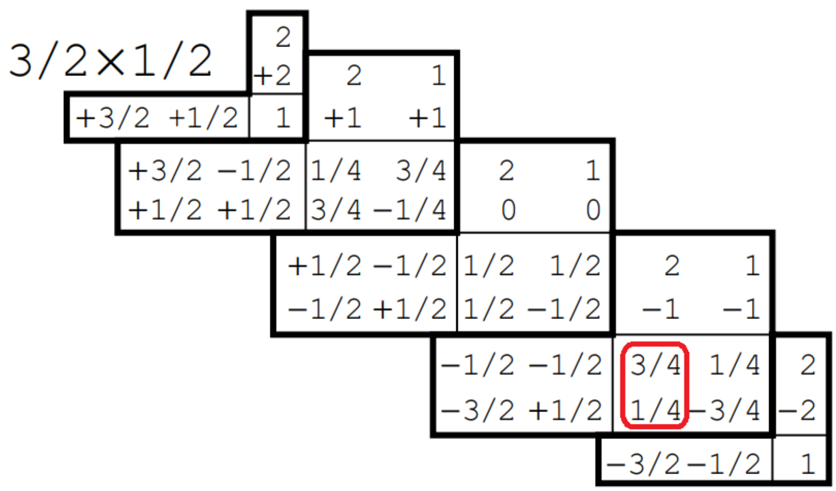
State $|2,0\rangle$

$$|2,0\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right).$$



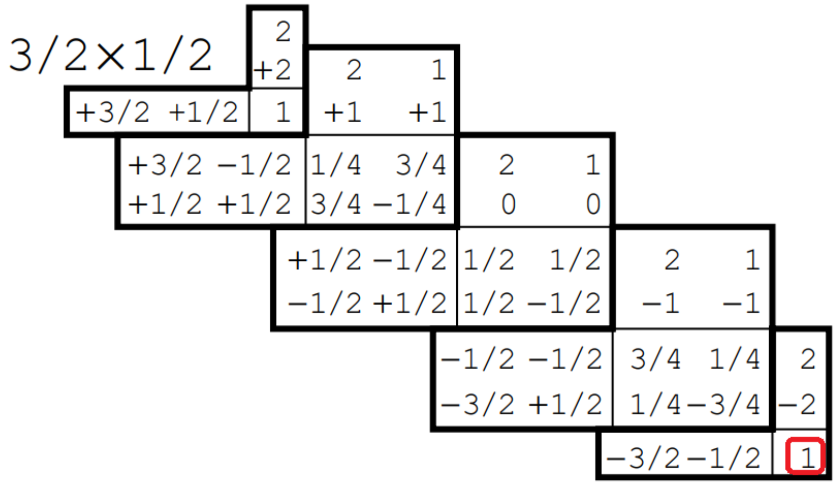
State $|2,-1\rangle$

$$|2,-1\rangle = \frac{1}{2}\left(\sqrt{3}\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle\right).$$



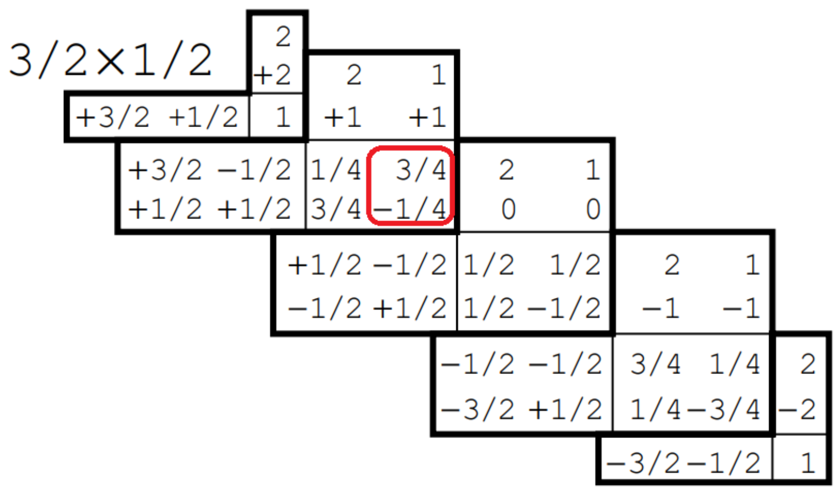
State $|2,-2\rangle$

$$|2,-2\rangle = \left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$



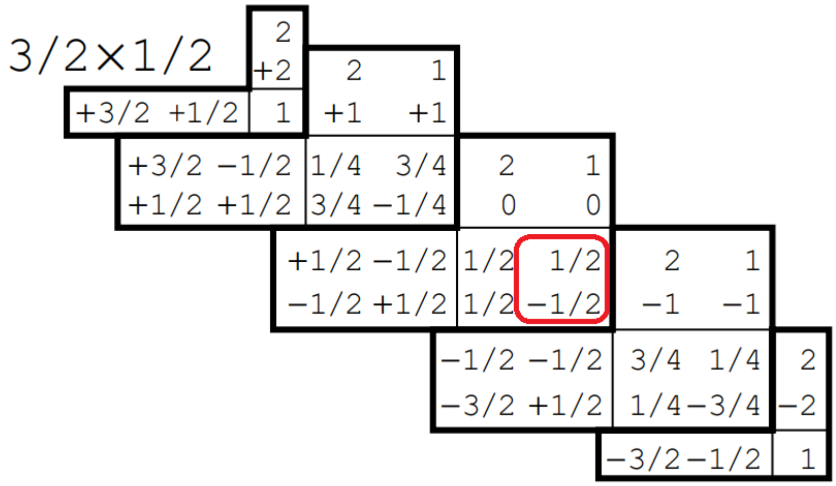
State $|1,1\rangle$

$$|1,1\rangle = \frac{1}{2}\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle - \frac{\sqrt{3}}{2}\left|\frac{3}{2},\frac{3}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$



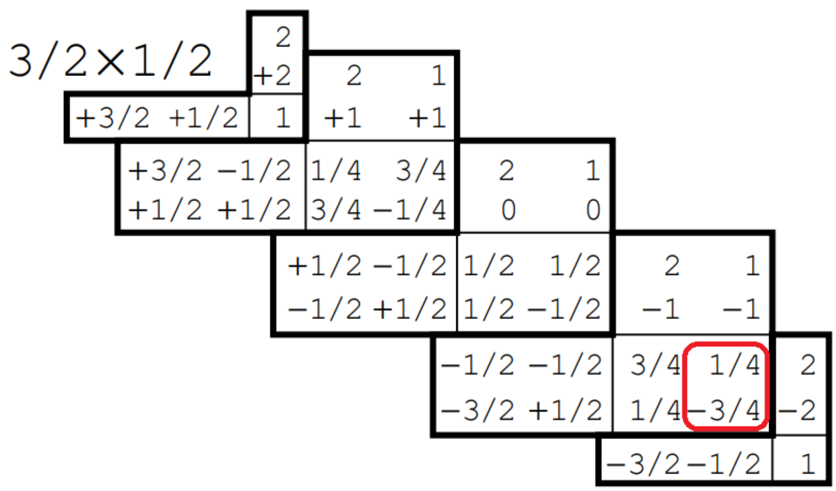
State $|1,0\rangle$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle - \left|\frac{3}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right).$$



State $|1,-1\rangle$

$$|1,-1\rangle = \frac{1}{2}\left(\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle - \sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle\right).$$



All Clebsch-Gordan coefficients obtained from the table match the ones used in the state construction, confirming that our derivation is correct.

2 Exercise 2:

For a system with three electrons, we can combine the spin function of each particular electron $|\alpha(i)\rangle$ and $|\beta(i)\rangle$ to build the eigenfunctions of

$$\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$$

For $S = 3/2$, the four eigenfunctions can be obtained by applying the step down operator

$$\hat{S}_- = \hat{S}_{1-} + \hat{S}_{2-} + \hat{S}_{3-}$$

and are given by:

$$|3/2, 3/2\rangle = |\alpha(1)\alpha(2)\alpha(3)\rangle$$

$$|3/2, 1/2\rangle = \frac{1}{\sqrt{3}}\big(|\beta(1)\alpha(2)\alpha(3)\rangle + |\alpha(1)\beta(2)\alpha(3)\rangle + |\alpha(1)\alpha(2)\beta(3)\rangle\big)$$

$$|3/2, -1/2\rangle = \frac{1}{\sqrt{3}}\big(|\beta(1)\beta(2)\alpha(3)\rangle + |\beta(1)\alpha(2)\beta(3)\rangle + |\alpha(1)\beta(2)\beta(3)\rangle\big)$$

$$|3/2, -3/2\rangle = |\beta(1)\beta(2)\beta(3)\rangle$$

(a) Justify that there exist in total 8 eigenfunctions of S^2 and S_z.

(b) Calculate the other 4 eigenfunctions (all correspond to $S = 1/2$).

Section a)

Each electron has two possible spin states:

$$|\uparrow\rangle = |\alpha\rangle \quad \text{(spin up)}, \quad |\downarrow\rangle = |\beta\rangle \quad \text{(spin down)}$$

Since we have three electrons, the total number of possible spin configurations is:

$$2^3 = 8.$$

This means that we can construct 8 orthogonal states, which must be eigenfunctions of S^2 and S_z.

When adding three spin-1/2 particles, we use the rules of angular momentum addition. First, combining two spin-1/2 particles

$$\frac{1}{2} \oplus \frac{1}{2} = 1 \oplus 0.$$

- This gives: - A triplet with $S = 1$ (three states: $S_z = 1, 0, -1$). - A singlet with $S = 0$ (one state: $S_z = 0$).

Next, adding the third spin-1/2 particle

$$(1 \oplus 0) \oplus \frac{1}{2}.$$

Combining the triplet ($S = 1$) with $S = 1/2$ results in:
- A quartet ($S = 3/2$, four states: $S_z = 3/2, 1/2, -1/2, -3/2$).
- A doublet ($S = 1/2$, two states: $S_z = 1/2, -1/2$).
Combining the singlet ($S = 0$) with $S = 1/2$ results in:
- Another doublet ($S = 1/2$, two states: $S_z = 1/2, -1/2$).

Section b)

Recall that m_s can be $+\frac{1}{2}$ or $-\frac{1}{2}$ for $S = \frac{1}{2}$. We therefore look separately at the two three-dimensional subspaces spanned by either:

$$m_s = +\tfrac{1}{2}: \quad \big|\beta(1)\alpha(2)\alpha(3)\big\rangle,\ \big|\alpha(1)\beta(2)\alpha(3)\big\rangle,\ \big|\alpha(1)\alpha(2)\beta(3)\big\rangle,$$

or

$$m_s = -\tfrac{1}{2}: \quad \big|\alpha(1)\beta(2)\beta(3)\big\rangle,\ \big|\beta(1)\alpha(2)\beta(3)\big\rangle,\ \big|\beta(1)\beta(2)\alpha(3)\big\rangle.$$

In each subspace, one normalized combination is the $S = \frac{3}{2}$ eigenstate; the remaining two orthogonal directions yield the $S = \frac{1}{2}$ states.

Case 1: $m_s = +\frac{1}{2}$**.** The $S = \frac{3}{2}$ combination for $m_s = +\frac{1}{2}$ is

$$\big|\tfrac{3}{2},\,\tfrac{1}{2}\big\rangle = \frac{1}{\sqrt{3}}\Big[\beta(1)\,\alpha(2)\,\alpha(3) \;+\; \alpha(1)\,\beta(2)\,\alpha(3) \;+\; \alpha(1)\,\alpha(2)\,\beta(3)\Big].$$

Any vector in the $m_s = +\frac{1}{2}$ subspace that is *orthogonal* to the above must satisfy:

$$\langle \tfrac{3}{2}, \tfrac{1}{2}|\psi\rangle = 0.$$

By Gram–Schmidt or direct inspection, one finds two independent, orthonormal combinations that are orthogonal to $\big|\frac{3}{2}, \frac{1}{2}\big\rangle$:

$$\boxed{\big|\tfrac{1}{2},\,\tfrac{1}{2}\big\rangle_1 \;=\; \frac{1}{\sqrt{6}}\Big[2\,\beta(1)\alpha(2)\alpha(3) \;-\; \alpha(1)\beta(2)\alpha(3) \;-\; \alpha(1)\alpha(2)\beta(3)\Big]}$$

$$\boxed{\big|\tfrac{1}{2},\,\tfrac{1}{2}\big\rangle_2 \;=\; \frac{1}{\sqrt{2}}\Big[\alpha(1)\beta(2)\alpha(3) \;-\; \alpha(1)\alpha(2)\beta(3)\Big]}$$

Case 2: $m_s = -\frac{1}{2}$**.** Similarly, the $S = \frac{3}{2}$ combination for $m_s = -\frac{1}{2}$ is

$$\big|\tfrac{3}{2},\,-\tfrac{1}{2}\big\rangle = \frac{1}{\sqrt{3}}\Big[\alpha(1)\,\beta(2)\,\beta(3) \;+\; \beta(1)\,\alpha(2)\,\beta(3) \;+\; \beta(1)\,\beta(2)\,\alpha(3)\Big].$$

We seek two other orthonormal states in this subspace, again orthogonal to that combination.

$$\boxed{\big|\tfrac{1}{2},\,-\tfrac{1}{2}\big\rangle_1 \;=\; \frac{1}{\sqrt{6}}\Big[2\,\alpha(1)\beta(2)\beta(3) \;-\; \beta(1)\alpha(2)\beta(3) \;-\; \beta(1)\beta(2)\alpha(3)\Big]}$$

$$\boxed{\big|\tfrac{1}{2},\,-\tfrac{1}{2}\big\rangle_2 \;=\; \frac{1}{\sqrt{2}}\Big[\beta(1)\alpha(2)\beta(3) \;-\; \beta(1)\beta(2)\alpha(3)\Big]}$$