Soft Thresholding Proof → by Simon Lucey

Take the following problem,

$$\arg\min \|x-b\|_2^2 + \lambda\|x\|_1$$

We know that this has $\underline{\underline{3}}$ unique solutions

① $\arg\min_x \|x-b\|^2 + \lambda x \Rightarrow$ "assuming $x \geq 0$"

② $\arg\min_x \|x-b\|^2 - \lambda x \Rightarrow$ "assuming $x < 0$"

③ $\arg\min_x \|x-b\|^2 \Rightarrow$ "assuming $x = 0$"

For ① $\frac{\partial}{\partial x}\left(\|x-b\|^2 + \lambda x\right) = 0$

$$\frac{\partial}{\partial x}\left(x^2 - 2xb + b^2 + \lambda x\right) = 0$$

$$2x - 2b + \lambda = 0$$

$$\therefore x = b - \frac{\lambda}{2}$$

Similarly for ②

$$x = b + \frac{\lambda}{2}$$

By deduction ③

$$x = 0$$

① only holds if $x > 0$

$\therefore b - \frac{\lambda}{2} > 0$

$b > \frac{\lambda}{2}$

② only holds if $x < 0$

$b + \frac{\lambda}{2} < 0$

$b < -\frac{\lambda}{2}$

③ only holds if

$\|b\| < \frac{\lambda}{2}$ due to ① & ②

Soft thresholding operator becomes,

$$S_\lambda(b) = \begin{cases} b - \frac{\lambda}{2}, & \text{if } b > \frac{\lambda}{2} \\ b + \frac{\lambda}{2}, & \text{if } b < -\frac{\lambda}{2} \\ b = 0, & \text{otherwise} \end{cases}$$