**Phys 357 Problem Set 1 Answers**

Q1: Since $|\pm X\rangle = \frac{1}{\sqrt{2}}(|+Z\rangle \pm |-Z\rangle)$, the amplitude to be in $|+Z\rangle$ is just the first coefficient, or $\frac{1}{\sqrt{2}}$. Similarly, the amplitude to be in $|-Z\rangle$ is the absolute value of the second coefficient, which is again $\frac{1}{\sqrt{2}}$. We have $|\pm Y\rangle = \frac{1}{\sqrt{2}}(|+Z\rangle \pm i\,|-Z\rangle)$, and again the coefficients of both states have an absolute value of $\frac{1}{\sqrt{2}}$ to both $|\pm Z\rangle$. Finally, if we take the bras of $X$ (so there are no conjugates to worry about), then $\langle \pm X| \pm Y\rangle$ will give us $\frac{1}{2}(1 \pm i)$. The absolute value of $1 \pm i$ is $\sqrt{2}$, so the absolute value of the amplitude is again $\frac{1}{\sqrt{2}}$.

For the opposite states, we accept that $\langle +Z| - Z\rangle = 0$, so $\langle +X| - X\rangle = \frac{1}{2}(\langle +Z| + \langle -Z|)(|+Z\rangle - |-Z\rangle) = \frac{1}{2}(\langle +Z| + Z\rangle - \langle -Z| - Z\rangle + \langle -Z| + Z\rangle - \langle +Z| - Z\rangle)$. The first two terms are one, so they cancel because of the opposite sign, and we already know the second two terms are zero. Similarly, for $\langle -Y| + Y\rangle$, and not forgetting to take the conjugate in the bra, we have $\frac{1}{2}(\langle +Z| + i\,\langle -Z|)(|+Z\rangle + i\,|-Z\rangle) = \frac{1}{2}(\langle +Z| + Z\rangle - \langle -Z| - Z\rangle + i\,\langle +Z| - Z\rangle + i\,\langle -Z| + Z\rangle)$. Again the first two terms cancel because of opposite sign, and the second two terms are zero.

Q2 (Townsend 1.3): a) +x will have $\theta = \pi/2$ and $\phi = 0$, which gives $|+X\rangle = \cos(\pi/4)\,|+Z\rangle + \exp(0)\sin(\pi/4)\,|-Z\rangle = \frac{1}{\sqrt{2}}(|+Z\rangle + |-Z\rangle)$, as expected. To get $|+Y\rangle$ we want to rotate from $|+X\rangle$ by 90 degrees, so $\phi = \pi/2$. That gives the same state, but instead of exp(0), we have $\exp(i\pi/2) = i$, so our $|+Y\rangle = \frac{1}{\sqrt{2}}(|+Z\rangle + i\,|-Z\rangle)$, as expected.
b) The amplitude to be in $|+Z\rangle$ is $\cos(\theta/2)$, and the probability is the amplitude squared, so $\cos(\theta/2)^2$. The amplitude to be in $|-Z\rangle$ is $\exp(i\phi)\sin(\theta/2)$. When we take the absolute value, the $\exp(i\phi)$ goes away, so once we square we're left with $\sin(\theta/2)^2$ for the probability to be in $|-Z\rangle$. The sum of the probabilities is just $\cos(\theta/2)^2 + \sin(\theta/2)^2 = 1$, as it must be. c) We always measure $\pm\hbar/2$ for the spin, so $< S^2 >= \hbar^2/4$. We want $< S >= \hbar/2(P_+ - P_-) = \hbar/2(\cos(\theta/2)^2 - \sin(\theta/2)^2$. We need to square that, which gives $\cos^2 + \sin^2 - 2\cos\sin = 1 - \sin(2\theta/2) = 1 - \sin(\theta)$ for the trig term. The uncertainty is then $\frac{\hbar^2}{4}(1 - (1 - \sin(\theta))) = \frac{hbar^2}{4}\sin(\theta)$. The uncertainty is the square root of this, or $\frac{\hbar}{2}\sqrt{(}\sin(\theta))$. This makes general sense - if $\theta$ is 0 or $\pi$, then the uncertainty goes to zero since we're in a pure $|+Z\rangle$ or $|-Z\rangle$ state. At the equator, $\sin(\theta) = 1$ and the uncertainty reaches a maximum of $\hbar/2$.

Q3 (Townsend 1.6): We take the bra of $|-n\rangle$, which is:

$$\langle -n| = \sin(\theta/2)\,\langle +Z| - \exp(-i\phi)\cos(\theta/2)\,\langle -Z|$$

If we apply this to $|+n\rangle$, we have

$$\sin(\theta/2)\cos(\theta/2)\,\langle +Z| + Z\rangle - \exp(-i\phi)\exp(i\phi)\cos(\theta/2)\sin(\theta/2)\,\langle -Z| - Z\rangle + ...$$

$$= \sin(\theta/2)\cos(\theta/2) - \cos(\theta/2)\sin(\theta/2) = 0$$

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Q4: a) If $\theta$ is the angle from the pole, the $\theta$ for the opposite direction is $\pi - \theta$. To see this, if I'm at the pole, my opposite point is at the other pole, while if I'm on the equator, my opposite point is on the equator on the other side of the sphere. Similarly, if I need to go to the opposite side of a sphere in $\phi$, I need to move by 180 degrees, or $\pi$. So, we have $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$. To check this, you can use $x = \sin(\theta)\cos(\phi)$, $y = \sin(\theta)\sin(phi)$ and $z = \cos(theta)$, plug in $(\theta, \phi)$ and $(\pi - \theta, \phi + \pi)$ and see that we have $(x, y, z)$ go to $-(x, y, z)$.
b) If $\theta \to \pi - \theta$, then $\cos(\theta/2) \to \cos(\pi/2 - \theta/2) = \sin(\theta/2)$. Similarly, $\sin(\theta/2) \to \cos(\theta/2)$. And with $\phi \to \phi + \pi$, then $\exp(i\phi) \to \exp(i\phi)\exp(i\pi) = -\exp(i\phi)$. So our new state is $\sin(\theta/2)\,|+Z\rangle - \exp(i\phi)\cos(\theta/2)\,|-Z\rangle$, which is exactly the state from 1.6.
c) We have a new state $|n'\rangle = \cos(\theta'/2)\,|+Z\rangle + \exp(i\phi')\sin(\theta'/2)\,|-Z\rangle$. I want to write this as

$$c_+ \left(\cos(\theta/2)\,|+Z\rangle + \exp(i\phi)\sin(\theta/2))\,|-Z\rangle\right) \tag{1}$$
$$+c_- \left(\sin(\theta/2)\,|+Z\rangle - \exp(i\phi)\cos(\theta/2)\,|-Z\rangle\right) \tag{2}$$

We can write this as a coupled set of linear equation:

$$\begin{bmatrix} \cos(\theta/2) & \exp(i\phi)\sin(\theta/2) \\ \sin(\theta/2) & -\exp(i\phi)\cos(\theta/2) \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} \cos(\theta'/2) \\ \exp(i\phi')\sin(\theta'/2) \end{bmatrix} \tag{3}$$

d) The determinant is top-left times bottom-right minus top-right times bottom left, which gives $-\exp(i\phi)\cos(\theta/2)^2 - \exp(i\phi)\sin(\theta/2)^2 = -\exp(i\phi)$. That clearly has a magnitude of one, so the length of an arbitrary state represented in an arbitrary basis is one, as it must be for quantum mechanics to hold.

Q5 (Townsend 1.13): The amplitude for $|\Psi\rangle$ to be in $|\Phi\rangle$ is $\langle\Phi|\Psi\rangle$. If we take $|\Psi\rangle \to \exp(i\theta)\,|\Psi\rangle$, then the amplitude becomes $\exp(i\theta)\,\langle\Phi|\Psi\rangle$. The probability is the amplitude times its conjugate, so we get

$$\exp(-i\theta)\,\langle\Phi|\Psi\rangle^* \exp(i\theta)\,\langle\Phi|\Psi\rangle = |\,\langle\Phi|\Psi\rangle\,|^2$$

This is unchanged from the unrotated state, so our probability didn't change. Since the expectation is $P_+\hbar/2 - P_-\hbar/2$ and our probabilities didn't change, then the expectation has to be unchanged as well.

Q6 (Townsend 1.15): Since we have a 90% change of measuring $+\hbar/2$, then $|c_+| = \sqrt{0.9}$. We can never know the phase, so let's set the phase of $c_+ = 0$. The amplitude to find $-\hbar/2$ must have absolute value $\sqrt{0.1}$, so the state can be written as

$$\exp(i\phi)\left(\sqrt{0.9}\,|+Z\rangle + \exp(i\theta)\sqrt{0.1}\,|-Z\rangle\right)$$

The amplitude to be in $|+Y\rangle$ is a phase times $\sqrt{0.2}$. If we take the amplitude by applying $\langle +Y|$ we have

$$\frac{\exp(i\phi)}{\sqrt{2}}\left(\langle +Z| - i\,\langle -Z|\right)\left(\sqrt{0.9}\,|+Z\rangle + \exp(i\theta)\sqrt{0.1}\,|-Z\rangle\right)$$

$$= \frac{\exp(i\phi)}{\sqrt{2}} \left(\sqrt{0.9} - i\exp(i\theta)\sqrt{0.1}\right)$$

That amplitude times its conjugate is:

$$\frac{1}{2}\left(0.9 + 0.1 + 2\sqrt{0.09}\sin(\theta)\right) = 0.5 + 0.3\sin(\theta)$$

Since that probability is 0.2, then $\sin(\theta) = -1$, and $\theta = -\pi/2$ so $\exp(i\theta) = -i$. Our final state is then

$$\exp(i\phi)\left(\sqrt{0.9}\,|{+Z}\rangle - i\sqrt{0.1}\,|{-Z}\rangle\right)$$

The amplitude to be in $|{+X}\rangle$ (ignoring the overall phase) is $\frac{1}{\sqrt{2}}\left(\sqrt{0.9} - i\sqrt{0.1}\right)$ and that starred with itself is

$$\frac{1}{2}(0.9 + 0.1) = \frac{1}{2}$$

We could have guessed that by looking at the state and noticing it is in the $Y - Z$ plane, and the probability of getting either $+\hbar/2$ or $-\hbar/2$ for $S_x$ is 50%.

Q7 (Townsend 1.9): If we're in $|{+X}\rangle$, we always measure $+\hbar/2$ when we measure the spin along the $x-$axis. That means $< S_x >= \hbar/2$ and $< S_x^2 >= \hbar^2/4$. The uncertainty squared is

$$< S_x^2 > - < S_x >^2 = \hbar^2/4 - \hbar^2/4 = 0$$

so we have zero uncertainty.