PHYS 357 Pset 10. Due 11:59 PM Thursday Nov. 28

1. Show that feeding the probability currnent (6.115) into the continuity equation (6.114) indeed gives $\frac{\partial^2}{\partial t^2}(\Psi^*\Psi)$ as expressed in 6.113.

2. A gaussian wave packet has $\Psi(x, t = 0) = c_0 \exp(ikx) \exp(-x^2/2\sigma^2)$. What is the probability current? We know from previous problem sets that a Gaussian wave packet spreads with time. Can you explain how that is consistent with the $k = 0$ result from the probability current?

3. Townsend 6.24

4. A way to estimate tunneling numerically is to start with a square well with a barrier in it, and add a wave packet on one side of the barrier. If you then solve Schrodinger's equation, you can calculate the probability as a function of time to find the particle on the right side of the barrier. If b is the width of the region to the left of the barrier, then the rate at which the particle hits the barrier is something like $2b/v$ where $v = p/m$. From that, you can estimate $T(k)$. Do this for a square barrier with a transmission probability of $\sim 1e-6$ and show you get a roughly sensible result. Then switch to a Gaussian barrier where Townsend 6.149 would predict the same tunneling probability. Do this for a) a Gaussian whose height is the same as the barrier and adjust the width, and b) for a gaussian with $\sigma = a/2$ (where a is the width of the square barrier) and adjust the height. How do you feel about the approximation in 6.149?

5. Quantum Snell's Law. Consider a 2-dimensional space where $V(x, y) = 0$ for $y < 0$, and $V(x, y) = V_0$ for $y > 0$ (note that V_0 could well be negative, but you may assume $E > V_0$). The momentum eigenstates in 2-d are $\exp(i\vec{k} \cdot \vec{x}) = \exp(i(k_x x + k_y y))$. The de Broglie relation carries straight over: $\vec{p} = \hbar\vec{k}$, and so the magnitude of k is just what we expect from 1D quantum mechanics. In particular $E - V = p^2/2m = \hbar(k_x^2 + k_y^2)/2m$.

   Now let's say we have an incoming wave $a \exp(i(k_x x + k_y y))$ for $y < 0$, we'll get a reflected wave $b \exp(i(k_x x - k_y y))$ for $y < 0$, and a transmitted wave $c \exp(i(k'_x x + k'_y y))$. Derive the three boundary conditions on b, c given that the combined wave function must be continuous and have continuous derivatves in both x and y at the $y = 0$ interface for all x.

a) Show that these conditions require $k'_x = k_x$

b) Solve for k'^2_y in terms of k_y, m and V_0.

c) If $k'^2_y < 0$, the wave can't penetrate the barrier since it will decay exponentially in y, even if $E > V_0$. This is the quantum version of total internal reflection. What is the condition for total internal reflection to happen?

d) Show that we recover Snell's law in quantum mechanics.

6. Bonus If you aren't sick of algebra yet, work out the reflection and transmission probabilities for Snell's law, and show that probability is conserved.