PHYS 357 Pset 4. Due 11:59 PM Thursday Oct. 17

In some of these problems, we'll see that interesting behavior can emerge when we work with states with larger angular momentum. It's a pain to do this by hand, so feel free to carry out calculations on a computer. If you do this, do please explain how your codes work so the TAs can understand what you did when it comes time to mark.

1. Townsend 3.24. Please do this one by hand - Townsend helpfully supplies all the relevant matrices for you.

A) $\langle\Psi|\,\Psi\rangle = 1+4+9+16 = 30$, so normalziation is sqrt(30), and $N = 1/\sqrt{30}$.
B) $S_x\,|\Psi\rangle = [2\sqrt{3}, 6+\sqrt{3}i, 4+4\sqrt{3}i, 3\sqrt{3}i]/\sqrt{120}$. Multiplying out, we see that $\langle\Psi|\,S_x\,|\Psi\rangle = 0.4\hbar$. As always, don't forget the conjugate when you convert $|\Psi\rangle$ into a bra.
C) The $S_x = \hbar/2$ eigenstate is $(\sqrt{3}, 1, -1, -\sqrt{3})/\sqrt{8}$, and when we take the inner product of that with Ψ, we get a $(\sqrt{3}i + 2 - 3 - 4\sqrt{3}i)/\sqrt{240} = (-1 - 3\sqrt{3}J)/\sqrt{240}$. That magnitude squared is $(1+27)/240 = 7/60 = .116666....$

2. Consider a ping pong ball (mass 2.7 grams and radius 20 mm) rotating at 10 radians per second (in real play, the spins can be 100 times higher). What is the approximate angular momentum (order of mangitude is fine, so you can use MR^2 for the moment of inertia)? If we say the ping pong ball is in a state $|j, j\rangle$ rotating about the $z-$axis, what is the approximate value of j? Given that value of j, what is the total angular momentum in the $x-y$ plane, $\sqrt{J_x^2 + J_y^2}$? We can express this as an uncertainty in the rotation axis of the ping pong ball, with $\sigma(\tan(\theta)) \sim \frac{\sqrt{J_x^2+J_y^2}}{J_z}$. What is your approximate value for $\sigma(\tan(\theta))$? For a macroscopic object, talking about "the" axis of rotation is a very good approximation!

M=2.7e-3, R=0.02, so $MR^2 = 1.08e-6$. We have $j\hbar = 1.08e-6$, so $j \sim 10^{28}$. Since $J^2 = j(j+1)$ and $J_z = j$, we know that $J_x^2 + J_y^2 = (j^2+j) - j^2 = j$, so $\sqrt{J_x^2 + J_y^2} = \sqrt{j}$. That gives us that the uncertainty in $\tan\theta$ is about $\sqrt{10^{28}}/10^{28} = 10^{-14}$. Given the radius of the ball is 2cm, then the uncertainty in the physical location of the rotation axis at the surface of the ping pong ball is $0.02 * 10^{-14} = 2 \times 10^{-16}$ metres. This is about a millionth of the size of a typical atom - the quantum world can be very hard to notice on human scales!

3. **Part A)** Consider a spin-1 Stern-Gerlach experiment where you send a beam of particles from the oven down a modified SGz machine, where you block the $J_z = 0$ beam before recombining. You might think that would leave your particles in the state $J_z = (1, 0, 1)/\sqrt{2}$. If that were the case, what would you see (in terms of what fraction of particles go up, go down, or aren't deflected at all) if you then send the beam through an SGx machine? Through an SGy machine?

The $\Psi = (1, 0, 1)/\sqrt{2}$ in the $z-$basis state is a pure $J_y = 0$ state, as you can verify by looking at the eigenvectors of J_y. If you look at the eigenvectors of J_x, you'll see that the $J_x = 0$ state is orthogonal to Ψ, and that it has equal amplitudes to be in the $J_x = +\hbar$ and $J_x = -\hbar$ states. So if we send this through an SGx machine, we see 50% go one way, 50% go the other, and none go straight. If we send through an SGy, since it's a pure $J_y = 0$ state, we get all the particles going straight through and none are deflected.

Part B) Explain why these results can't be correct (hint - what happens if I take my SGz into SGx apparatus, and rotate the whole setup by 90 degrees about the $z-$axis).

These results can't be correct because I can rotate my apparatus through 90 degrees about the $z-$axis and I should still get the same answer. That's equivalent to switching x for y, so the two results must be the same. Since they aren't, we have done something wrong.

Part C) If you were to actually set this experiment up and run it, what you would see is that 50% of the particles show up with $J_{x,y} = 0$, 25% show up with $J_{x,y} = +\hbar$, and 25% show up with $J_{x,y} = -\hbar$ (where $J_{x,y}$ means I measure *either* J_x or J_y), so indeed the apparatus behaves the same if I rotate my coordinate system by 90 degrees, as it must. Can you quantitatively explain the actual observed results? You may use a computer if you like.

Coming out of the machine, what really happens is that the phases of the $|1, 1\rangle$ and $|1, -1\rangle$

components are uncorrelated - their phases are random relative to each other. We can always pick the phase of the $|1,1\rangle$ component to be one, which means the phase of the $|1,-1\rangle$ component is randomly distributed between 0 and 2π. If you pick a bunch of finely spaced values for θ for $\Psi = (|1,1\rangle + \exp(i\theta)\,|1,-1\rangle)/\sqrt{2}$ and average those results, you'll get the correct answer that 50% go straight, and 25% get deflected either up or down for both SGx and SGy machines. Incidentally, we can write this as

$$< |(\langle 1, m|_x\, |1, 0, \exp(i\theta)\rangle\, /\sqrt{2})|^2 >$$

where $\langle 1, m|_x$ means we're looking at the state with some value of m along the $x-$axis. We can rewrite as

$$< \langle 1, m|_x\, |1, 0, \exp(i\theta)\rangle\, \langle 1, 0, \exp(i\theta)|\, |1, m\rangle_x\, /2 >$$

The inner two components are now an outer product, so we end up averaging over

$$|1, 0, \exp(i\theta)\rangle\, \langle 1, 0, \exp(i\theta)| = \frac{1}{2}\begin{bmatrix} 1 & 0 & \exp(-i\theta) \\ 0 & 0 & 0 \\ \exp(i\theta) & 0 & 1 \end{bmatrix}$$

When we average over θ, the off-diagonal corners average to zero, and we're left with

$$\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is a *density matrix*, and all of our averaging is captured by this matrix. If we want to find the probability of being in some state, we take the bra and ket for that state, and apply them to the density matrix to get the probability. Since this density matrix is also the average of $|1,1\rangle_z \langle 1,1|_z$ and $|1,-1\rangle_z \langle 1,-1|_z$ then you can also answer any of these questions correctly by taking the average of the $m_z = \pm 1$ states, but you would not have known that was correct until looking at the density matrix. We'll see more of this in chapter 5.

4. **Part A)** We saw that J_z commutes with $J_x^2 + J_y^2$. Does J_x commute with $J_x^2 + J_y^2$? If not, what is the commutator $[J_x, J_x^2 + J_y^2]$? Feel free to express as an anti-commutator where $\{\mathrm{A}, \mathrm{B}\} = \mathrm{AB} + \mathrm{BA}$.

J_x does not commute with $J_x^2 + J_y^2$ because J_x does not commute with J_y. J_x does of course commute with J_x^2, so we just need to work out $[J_x, J_y^2]$. We've already done this, but let's do it again:

$$[J_x, J_y^2] = J_y[J_x, J_y] + [J_x, J_y]J_y$$

Since $[J_x, J_y] = i\hbar J_z$, we have

$$[J_x, J_y^2] = i\hbar J_y J_z + i\hbar J_z J_y = i\hbar\{J_y, J_z\}$$

Part B) Let's say I take a set of spin-2 particles, all of which have $J_z = 2\hbar$, which we write as $|2,2\rangle_z$ (note the z subscript, which says this state is an eigenstate of J_z). If I were to measure $J_x^2 + J_y^2$, what would I observe, and what would the uncertainty in that measurement be?

We know that $J_x^2 + J_y^2 = J^2 - J_z^2$. We know that J^2 on $|2,2\rangle_z$ will give use $j(j+1)\hbar^2 = 6\hbar^2$, while J_z^2 gives us $4\hbar^2$, so $J_x^2 + J_y^2 = 2\hbar^2$. The uncertainty is zero because a pure state of J_z (which $|2,2\rangle_z$ certainly is) is an eigenstate of both J_z^2 and J^2. If you actualy form $J_{xy} \equiv J_x^2 + J_y^2$ and from that form J_{xy}^2, you'll see that for a pure J_z state, the uncertainty in J_{xy} is indeed zero if you calculate *e.g.* $\langle 2,2|_z J_{xy}^2 |2,2\rangle_z - (\langle 2,2|_z J_{xy} |2,2\rangle_z)^2$.

Part C) I now send my beam of $|2,2\rangle_z$ particles down an SGx machine. Show that the expectation of J_x^2 that you see is one half of $< J_x^2 + J_y^2 >$ you calculated in part B).

If we apply J_x to $|2,2\rangle_z$, we get $|2,1\rangle_z$ because the top row of J_r has a 2 in the (0,1) position, and since $J_x = (J_l + J_r)/2$, then J_x has a 1 in the (0,1) position, and zeros everywhere else in the first row. We can write $< J_x^2 >$ as $(J_x\Psi)^\dagger(J_x\Psi)$, so since $J_x |2,2\rangle_z$ is $|2,1\rangle_z$ with no other numerical factors (besides, of course, the obligatory $\hbar$), then that thing squared is one, and $< J_x^2 >= 1\hbar$, which is exactly half of $< J_x^2 + J_y^2 >= 2\hbar^2$.

Part D) What is the probability that you measure $J_x = +2\hbar$? If this is non-zero, can you explain the apparent contradiction between finding a particle witih $J_x = 2\hbar$ (and hence $J_x^2 = 4\hbar^2$) and the maximum value you found for $J_x^2 + J_y^2$ from part B?

If we crunch the numbers after taking the eigenvectors of J_x, we find that the amplitude to be in $|2,2\rangle_x$ is $\frac{1}{4}$, so the probability we measure $J_x = 2\hbar$ is $\frac{1}{16}$. This is a larger number than had we measured $J_x^2 + J_y^2$, but that's OK because J_x doesn't commute with $J_x^2 + J_y^2$, so by measuring J_x, we have messed up our formerly perfect measurement of $J_x^2 + J_y^2$. Our average value still comes out correct, though. Quantum mechanics can be very strange!

5. **Part A)** Sticking with spin-2 particles, express the pure state $|2,2\rangle_y$ in both the z-basis and the y-basis. In the $z-$basis, please pick the phase that makes the amplitude of $J_z = +2\hbar$ purely real and positive.

The $|2,2\rangle_y$ state in the $y-$basis is by definition $(1,0,0,0,0)$. If we take J_y in the $z-$basis (again, from calculating J_r and J_l), we see the eigenvector corresponding to $J_y = 2$ is $(\frac{1}{4}, \frac{1}{2}i, \sqrt{\frac{3}{8}}, \frac{1}{2}i, \frac{1}{4})$.

Part B) For this same $|2,2\rangle_y$ pure state, what are the uncertainty in J_x and J_z? Show that the uncertainty relation is satisfied. You can also see pset5_q5.py

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Taking our state in the $z-$basis, and using the J_x and J_z matrices we've already calculated, we have $\sigma(J_x) = \sigma(J_z) = \hbar$. We also have $< J_z >= 0$ and $\sigma(J_y) = 0$ (since it's a pure state of J_y), so the two uncertainty relations worth looking at are:

$$\sigma(J_x)\sigma(J_y) = (\hbar/2)(0) \geq < J_z > /2 = 0$$

so we're fine. The more interesting one is

$$\sigma(J_x)\sigma(J_z) \geq \frac{\hbar}{2} < J_y >$$

Plugging in the values we know gives

$$\hbar\hbar \geq \frac{\hbar}{2} 2\hbar$$

so in fact we have equality. Incidentally, if you crunch the numbers for the $|2,1\rangle_y$ state, you'll see that $\sigma(J_x) = \sigma(J_z) = \sqrt{5/2}\hbar$, so the uncertainty relation predicts that $5/2\hbar^2 \geq \frac{1}{2}\hbar^2$ - it's still satisfied, but no longer equal. Not surprisingly, there's more uncertainty about what J_x and J_z might be when J_y is not equal to its maximum value, and so we no longer have the minimum possible uncertainty.

6. **Part A)** Calculate the matrix that rotates a state about the $y-$axis by $+90$ degrees for a spin-2 particle. Print the absolute value of the matrix.

We take the eigenvalues/eigenvectors of the spin-2 J_y we've already calculated, and find $R_y(\pi/2) = v_y \exp(-i\pi/2e_y/\hbar)v_y^\dagger$ which has absolute value:

$$\begin{bmatrix} 0.25 & 0.5 & 0.6124 & 0.5 & 0.25 \\ 0.5 & 0.5 & 0. & 0.5 & 0.5 \\ 0.6124 & 0. & 0.5 & 0. & 0.6124 \\ 0.5 & 0.5 & 0. & 0.5 & 0.5 \\ 0.25 & 0.5 & 0.6124 & 0.5 & 0.25 \end{bmatrix}$$

Note - because it's kets first and bras last, and the eigenvectors are the kets, you need to make sure the dagger is in the right location. If you get it wrong, your matrix won't rotate correctly, but it turns out the absolute value is still the same. As is so often the case in quantum mechanics, the interesting bits are in the phases.

Part B) Show that the above matrix rotates the state $|2,2\rangle_z$ to the pure state $|2,2\rangle_x$, and $|2,2\rangle_x$ to $|2,-2\rangle_z$.

If we take $(1,0,0,0,0)$ and multiply on the left by the rotation matrix, we indee do indeed get something proportional to the $|2,2\rangle_x$ state, which is $(1/4, 1/2, \sqrt{3/8}, 1/2, 1/4)$. If we multiply R_y by that, we get $(0,0,0,0,1)$ (no minus sign), which is indeed $|2,-2\rangle_z$, as expected. Now that we have integer spins, if we rotate that, we go to $|2,-2\rangle_x$, and a final rotation brings us back to $|2,2\rangle_z$. Note that we don't have a minus sign this time - that only happens for spin-1/2 particles. You can also see pset5_q6.py

7. Bonus: Our end goal is to derive the angular momentum commutation relation $[J_a, J_b] = i\hbar\epsilon_{abc}J_c$ but to do that we'll first need the canonical commutation relation $[x,p] = i\hbar$. In

the first bonus we'll work that out, in the second bonus, we'll use it to derive the angular momentum commutation relations. You are more than welcome to do the second bonus only, and just use the canonical commutation relation as supplied.

In 1924, Louis De Broglie hypothesized (in his PhD thesis!) that all matter behaved like waves, with wavelength set by their momentum. This was the key insight that opened the door to modern quantum mechanics, and within a couple of years both wave and matrix mechanics were basically fleshed out in their modern forms. The De Broglie relation is

$$p = \hbar k \tag{1}$$

where $k = 2\pi/\lambda$ and p is the momentum. If I have a wave function of a particle with wave vector k, I can write that down as a function of x as follows: $\Psi = \exp(ikx)$.

Part A:Show that the operator $-i\hbar\frac{\partial}{\partial x}$ operating on Ψ returns $p\Psi$. In other words, the plane wave is an eigenstate of momentum, and the operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ returns the wave function times the momentum.

Applying $-i\hbar\frac{\partial}{\partial x}$ on our plane wave gives $-i\hbar(ik)\exp(ikx) = \hbar k\Psi$, but from de Broglie, we know that $p = \hbar k$, so we have $-i\hbar\Psi = \hbar k\Psi = p\Psi$. At least for plane waves, this is indeed the momentum operator and we also now know that plane waves are its eigenstates.

Part B: Now the position operator $\hat{x}$ in position space (where $\Psi = \Psi(x)$) is, not surprisingly, just x. You can now derive the canonical commutation relation $[x, p]$ by applying the operators to a wave function $(\hat{x}\hat{p} - \hat{p}\hat{x})\Psi$. Show that when you do this, you get $i\hbar\Psi$. That is, when you apply $[x, p]$ to a wave function, you get $i\hbar$ times the wave function, so $[x, p] = i\hbar$.

We have

$$[x, p]\Psi = \left(x(-i\hbar\frac{\partial}{\partial x}) + i\hbar\frac{\partial}{\partial x}x \right)\Psi$$

The second term has $\frac{\partial}{\partial x}(x\Psi)$ which equals $x\frac{\partial}{\partial x}\Psi + \Psi$ when we use the product rule. The $\frac{\partial}{\partial x}\Psi$ terms cancel because they have opposite signs, so we're left with

$$[x, p]\Psi = i\hbar\Psi$$

which is the cannonical commutation relation we were after.

8. Bonus 2. Now we'll work out the angular momentum commutators, using the fact that $J = r \times p$. If you expand that out, you can see that $J_x = r_y p_z - r_z p_y$, which will all get applied to a wave function.

   Position and momentum on the same axis do not commute, but they do commute if you measure position along one axis, and momentum along another, perpendicular axis. More formally, $[x_i, p_j] = i\hbar\delta_{ij}$. Use this commutation relation and the expressions for Jx, Jy, Jz you get from $J = r \times p$ to show that $[Jx, Jy] = i\hbar Jz$. The commutators for other pairs of axes follow from cyclic permutations of the cross product, so you only need to do it once.

   Note - technically we've only shown this for orbital angular momentum and not spin. We did show that spin-1/2 particles obey the same commutation relation, so I hope it's not a surprise that the relation holds for higher spins as well.

   When we write down the cross product, we have $J_x = yp_z - zp_y$ and $J_y = zp_x - xp_z$. Now we take that commutator:

$$[J_x, J_y] = J_x J_y - J_y J_x = (yp_z - zp_y)(zp_x - xp_z) - (zp_x - xp_z)(yp_z - zp_y)$$

   Multiplying out we have:

$$yp_z zp_x - yp_z xp_z - zp_y zp_x + zp_y xp_z - (zp_x yp_z - zp_x zp_y - xp_z yp_z + xp_z zp_y)$$

   Grouping together, and noting that quantities along different axes do indeed commute, we have:

$$yp_x(p_z z - zp_z) + p_y x(zp_z - pz_z)$$

   since the $zp_y zp_x$ and $zp_x zp_y$ *etc.* terms commute and therefor cancel. We know $[z, p_z] = i\hbar$ so we now have

$$yp_x(-i\hbar) + p_y x(i\hbar) = i\hbar(p_y x - yp_x)$$

   But, we also know that $J_z = xp_y - yp_x$ (again from the cross product description of angular momentum), so the final term is just $i\hbar J_z$. We've now shown that as long as $J = r \times p$, that the cannonical commutation relation $[x, p] = i\hbar$ also predicts that $[J_x, J_y] = i\hbar J_z$. The same derivation holds for the other pairs of axes, so we have our angular momentum commutation relations.