1. Expectations

: $< f(x) >= \int f(x) PDF(x) dx$ where PDF is the probability density of x. If x can only take on discrete values, $< f(x) >= \sum f(x) P(x)$. These are correct when the PDF/probabilities are properly normalized, i.e. $\int PDF(x) dx = 1$ if continuous, and $\sum P(x) = 1$ if discrete. If we only have relative probabilities $\tilde{P}$, then we would have to normalize: $P(x) = \tilde{P} / \sum \tilde{P}$. For black body, statistical mechanics told us the relative probabilities of states, $\tilde{P} = e^{-nh\nu/kT}$. To get a proper expectation, we had to normalize to get true probabilities, hence

$$\langle E \rangle = \frac{\sum nh\nu \exp -nh\nu/kT}{\sum \exp -nh\nu/kT}$$

Very important - if I take a particular state and measure its energy in one black body, I'll get some multiple of $h\nu$. If I measure it again, *I will get the same value* unless something has changed. When we talk about expectations, especially in quantum, we mean that if we prepared thousands of copies of a black body (by which we mean they are statistically identical, but are different realizations), pick out a single state, and measure the energy of that state in *each* black body, then the average of all those measurements gives us the expectation value.

2. Magnetic Moments/Angular Momentum

In SI units, the magnetic moment μ of a current loop is the current times the area IA. If we have a particle with charge q moving in a circle of radius r with velocity v then the area is just πr^2. The current is slightly more complicated - the particle goes around the loop $v/2\pi r$ times per second. Since the current is the charge per second, if we sit and watch, we'll see that the charge per second is the charge per particle times particles per second. For our setup, we have $I = qv/2\pi r$, so the magnetic moment is

$$\mu = IA = \frac{qv}{2\pi r} \pi r^2 = \frac{qvr}{2}$$

The angular momentum is $L = mr \times v$, and since for circular motion, v is always perpendicular to r, then the magnitude of L is mrv. If we take the ratio of the magnetic moment to the angular momentum we have:

$$\frac{\mu}{L} = \frac{qvr/2}{mrv} = \frac{q}{2m}$$

That is, the only thing that matters is the ratio of the charge to the mass. If I had a macroscopic spinning ball of electrons and I change the distribution of the elctrons or mess with how they're moving, I'll get different values for L and for μ, but the *ratio* would always be the same. In a classical world, if I had a ball with fixed total charge and mass, the only way to change μ/L is to move some of the charges/mass around relative to each other. For instance, if I moved all the charge (but not the mass) to the center of my ball, then μ would be zero but I'd still have angular momentum.

Of course, in the quantum world, this classical picture is just wrong, and any classical picture we tried to make would just fall over, especially for an electron. However, decent theories of particle physics do need to be able to predict the ratio. Protons/neutrons and electrons have very different ratios, which tells us that there's something fundamentally different about protons/neutrons. In fact, you might even wonder why a neutron has a magnetic moment at all, given it has no net charge!

3. Polarizers vs. Stern-Gerlach

We think of light as a wave phenomenon and electrons/atoms as particles, but we'll see that light can act like particles (which we call photons), and electrons/atoms act like waves. While this situation is strange, happily for us, nature treats electrons and photons (mostly) the same. That means we can reproduce a lot of the Stern-Gerlach results using polarized filters and light. An important difference is that electrons are spin-1/2 particles, while photons are spin-1. Essentially, the "opposite" of an electron with spin pointing up is an electron with spin pointing down, but the electric field in light oscillates up and down, the "opposite" of up-and-down (vertically polarized) light is left-and-right (horizontally polarized) light. The equivalent of $|+z\rangle$ and $|-z\rangle$ for electrons is $|+Q\rangle$ and $|-Q\rangle$ for light, where $|+Q\rangle$ has the electron vertically polarized, and $|-Q\rangle$ is horizontally polarized. The equivalent of $|+x\rangle$ and $|-x\rangle$ is $|+U\rangle$ and $|-U\rangle$ where the $E-$field is rotated 45° relative to either vertical or horizontal.

With this in mind, we can reproduce many of the Stern-Gerlach experiments with light. We just have to remember to divide all the angles by 2. A polarizing sheet passes a single polarization of light, and so is equivalent to a Stern-Gerlach machine splits the silver atoms into $|+z\rangle$ and $|-z\rangle$, but then blocks the $|-z\rangle$ atoms, so we're left with pure $|+z\rangle$ states. If we hold two polarizers at 90° to each other, then that's equivalent to a Stern-Gerlach machine that passes $|+z\rangle$ followed by one that passes $|-z\rangle$, and we would not expect anything to make it through. Indeed, that is what we see. If we put a $|+x\rangle$ after a $|+z\rangle$, we'd expect half the particles coming out of $|+z\rangle$ to make it through $|+x\rangle$. The light equivalent is to rotate a second polarizer by 45°. Try that, and you'll see things get a bit darker, but not too much.

Of course, we could put a $|-z\rangle$ SG after our $|+x\rangle$ one, and half of what makes it through $|+x\rangle$ should come out $|-z\rangle$. A $|+z\rangle\,|+x\rangle\,|-z\rangle$ SG sequence lets 25% of the particles that came through $|+z\rangle$ through, while if we pull the $|+x\rangle$ machine, nothing comes through. To see this in action, I suggest you hold a pair of polarizers together rotated at 90°, then hold a third polarizer at 45° and stick before, after, and between the two polarizers.

This behavior is easy to explain classically - the field *strength* projected onto an axis rotated by 45° is reduced by a sqrt(2). Since power goes like the field squared, the power is reduced by a factor of 2. Add another polarizer and the field is down by another sqrt(2). Where things get wacky is if we turn down our light source so only one photon at a time makes it through. I could set up a detector that puts out a sound proportional to the amount of energy coming in. What we'd hear is a bunch of clicks, where each click is the same loudness (since all photons of a given frequency have the same energy).

If I put a 45° polarizer after a vertical polarizer, I'd get half as many clicks, but they would remain the same loudness. When a photon comes out of the first polarizer, it happily says "yup, I'm vertically polarized". If we put a bunch of vertical polarizers together, one after another, once a photon makes it through the first one and has decided it's vertically polarized, it will stay vertically polarized and make it through all the polarizers. However, if it hits a 45° polarizer, the photon will flip a coin to decide if it's $+45°$ or $-45°$. If it's $+45°$ then the photon makes it through unscathed - no energy is lost, and the vertical field strength before the polarizer is *the same* as the 45° field strength after the polarizer. Of course, if the photon's coin flip comes up $-45°$, then *all* the energy is lost. Rather remarkably, after a photon makes it through the 45° polarizer, it has lost all memory that it used to be a $|+Q\rangle$ photon. Now the photon has happily decided it is $|+U\rangle$ *and no experiment will ever show it used to be* $|+Q\rangle$. Any memory it could have had has been completely lost. Now when the photon hits the $|-Q\rangle$ polarizer, all the photon knows is right now it's $|+U\rangle$. The photon flips another coin, and if that comes up heads, the photon decides it's now horizontally polarized/$|-Q\rangle$ and makes it through, again with no energy loss. There's absolutely no way to explain this classically - instead of losing sqrt(2) in field strength at each polarizer, the photon either makes it through at full strength, or gets absorbed.

We could (but won't) also do the analog of SG experiment 4 with light. There are polarized beam splitters/combiners. To do experiment 4, we would start with say a vertically polarized beam, send it through a splitter at 45° so $|+U\rangle$ goes one direction, and $|-U\rangle$ goes another. We could then use mirrors to send those beams down different paths, and then recombine them with a beam combiner, giving us once again a single beam. Since we haven't messed with the beam, what comes out remains vertically polarized (*i.e.* in the $|+Q\rangle$ state), so if we feed it through a horizontal polarizer, the whole beam is blocked. However, if we block *one* of the paths, then what comes out of the combiner will be either $|+U\rangle$ or $|-U\rangle$, depending on which path we blocked. Now some light *will* make it through the final horizontal polarizer. This remains true even when we go back to the single-photon limit. The probability the photon makes it through is *not* equal to the probability it goes through the $|+U\rangle$ path plus the probability it goes through the $|-U\rangle$ path. Somehow the photon is going through *both* paths, and what gets added at the end is not probability, but a complex amplitude. What *is* correct is that we can calculate the amplitudes of going through both paths, and we *can* add those amplitudes together. The final probability is the square of the absolute value of that amplitude. When a photon can go through both paths, the $|+Q\rangle$ amplitudes constructively interfere, while the $|-Q\rangle$ paths destructively interfere.

Of course, our photons (and our silver atoms) are very shy. If we watch them to find out *which* path they followed, the interference goes away, as shown in Figure 1. Photons and atoms behave identically in this way - we can send either through an experiment where and individual photon/particle can go down multiple paths, have it interfere with itself, and get an interference pattern out the end. How we do that splitting is different between photons and particles, but the *way* in which the paths interfere is identical and is governed by the same math. If we try to measure which path the particle/photon followed, then the interference goes away. We'll see this more quantitatively later on, but it is absolutely impossible to make *any* measurement that would tell us which path it actually followed without destroying the interference.

Fig. 1.— A scientifically accurate quantum particle. When sent through two paths, the particle interferes with itself. However, when we measure which of the two paths the particle took, the above represents the new interference pattern.