Supplementary notes for intro to wave functions for McGill Phys 357.

1. Generator of Translations

Let's say we have $\Psi(x)$ and we want to shift it over by dx. If we, say, pick $dx = 2$, that means we want to shift the wave function to the right by 2. Let our original wave function be $\Psi(x)$ and our new wave function be $\Psi'(x)$. If we're shifting to the right by 2, then we know that $\Psi'(2) = \Psi(0)$ *etc.*. If $\Psi(x)$ has a spike at 0, then $\Psi'(x)$ will have a spike at 2 - this is exactly what we wanted to happen if we wanted to shift Ψ to the right by 2. More generally, we know that $\Psi'(x+2) = \Psi(x)$, and if we let the shift be dx, then we have $\Psi'(x+dx) = \Psi(x)$. Let $x' = x + dx$, then we have $\Psi'(x') = \Psi(x' - dx)$. There's nothing special about what we call x of x', so drop the prime on x, and we have:

$$\Psi'(x) = \Psi(x - dx) \tag{1}$$

. We'll call the operator that shifts Ψ by dx to the right the translation operator, $\hat{\mathrm{T}}$, and we know that $\Psi' = \hat{\mathrm{T}}(dx)\Psi$. If we expand Equation 1, we have

$$\Psi'(x) = \hat{\mathrm{T}}(dx)\Psi = \Psi(x - dx) = \Psi(x) - dx\frac{\partial \Psi}{\partial x} \tag{2}$$

As an operator, then,

$$\hat{\mathrm{T}}(dx) = 1 - dx\frac{\partial}{\partial x} \tag{3}$$

. Looking at this expression, it isn't obvious that $\hat{\mathrm{T}}$ is unitary. It does have to be unitary, though, because if I just shift a wave function, the total probability can't change.

We know we can write any unitary operator as $\exp(i\mathrm{A})$ where A is Hermitian. We also know that if I want to shift by $2dx$, that's the same as shifting by dx twice, so as usual we know that $\hat{\mathrm{T}}(dx) = \exp(-i\mathrm{B}dx)$ where B is once again Hermitian, but now does not depend on dx. We've thrown in a minus sign that will turn out to be convenient as well. This is perfectly fine, since if a matrix is Hermitian, -1 times that matrix will also be Hermitian. Expand to first order and put in our usual $\hbar$, and we have

$$\hat{\mathrm{T}}(dx) = 1 - i\hat{\mathrm{P}}dx/\hbar \tag{4}$$

where we've labeled the scaled-by-$\hbar$ Hermitian operator P. We already know from Equation 3 what $\hat{\mathrm{T}}(dx)$ is, so we can set them equal to get:

$$1 - i\hat{\mathrm{P}}dx/\hbar = 1 - dx\frac{\partial}{\partial x} \tag{5}$$

or

$$\hat{\mathrm{P}} = -i\hbar\frac{\partial}{\partial x} \tag{6}$$

Written this way, we have a Hermitian operator with units of momentum. The de Broglie hypothesis reduces to claiming this operator *is* the momentum, and as I mentioned in an earlier problem set, this was the key insight that opened the floodgates for the development of modern quantum mechanics.

2. Eigenstates of P

I've made the claim that de Broglie reduces to saying the generator of translations *is* the momentum operator. Let's see how this plays out by looking at the eigenstates of P.

To solve for the eigenstates, we want our usual eigenvalue equation:

$$\hat{\mathrm{P}} |\Psi\rangle = p |\Psi\rangle \tag{7}$$

for some scalar p when the wave function is an eigenvector. If we plug in the operator for P we have

$$-i\hbar \frac{d}{dx}\Psi = p\Psi \tag{8}$$

$$\frac{d\Psi}{dx} = i\hbar p/\hbar\Psi \tag{9}$$

which has solution

$$\Psi = c\exp(ipx/\hbar) \tag{10}$$

This is just a plane wave $\exp(ikx)$ as long as we have $k = p/\hbar$ or $p = \hbar k$. We now know that the eigenstates of the generator of translations are plane waves with wavevector k and corresponding eigenvalue $p = \hbar k$. This is exactly what the de Broglie hypothesis is - matter is waves with momentum proportional to the wavevector, or alternatively $p = h/\lambda$.

We've now really closed the loop. Relying on the de Broglie hypothesis and the rules of matrix mechanics, we've shown that the generator of translations is the momentum operator. Once we have the momentum operator, it's easy to show the cannonical commutation relation $[x, p] = i\hbar$. Once we have that, we can derive the angular momentum commutation relations (as shown in the problem set bonus), which led to the quantization of angular momentum and the angular momentum raising/lowering operators. That in turn let us write down the actual angular momentum operators and show that they are the generators of rotations. We've now got an unbroken thread from the de Broglie hypothesis through everything we've seen in this course.