**PHYS 357 Pset 2. Due 11:59 PM Thursday Sep. 19**

1. Solve for the eigenvalues of a 2x2 matrix A in terms of the trace and determinant of A. Use this expression to show that the trace is the sum of the eigenvalues.

   If you need a hint to get started, we know that

$$\mathrm{A}v = \lambda v$$

   so

$$\mathrm{A} - \lambda \mathrm{I} v = 0$$

   For that to be true for non-zero $v$, $\mathrm{A} - \lambda\mathrm{I}$ must be singular, so its determinant must be zero.

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   We need $|A - \lambda I| = 0$. Determinant of 2x2 matrix is $ad - bc$, so we have

$$(a - l)(d - l) - (bc) = 0 \tag{1}$$

$$ad - ld - la + l^2 - bc = 0 \tag{2}$$

$$l^2 - (a + d)l + (ad - bc) = 0 \tag{3}$$

$$l^2 - (Tr)l + (det) = 0 \tag{4}$$

$$l = \frac{Tr \pm \sqrt{Tr^2 - 4det)}}{2} \tag{5}$$

   In particular, the sum of the eigenvalues cancels out the square root part, so we're left that the trace of the matrix is the sum of the eigenvalues. This remains true in general.

2. Explicitly show using the definition of matrix multiplication that $(\mathrm{AB})^\dagger = \mathrm{B}^\dagger\mathrm{A}^\dagger$

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   We have $(AB)_{ij} = \sum A_i k B_k j$. We know that $(\mathrm{AB}^\dagger)_{ij} = (\mathrm{AB}_{ji})^\dagger$ so we have

$$(\mathrm{AB}^\dagger)_{ij} = \sum A_j k^\dagger B_k i^\dagger \tag{6}$$

   We now plug in that $(\mathrm{A}^\dagger)_{kj} = (\mathrm{A_{jk}})^\dagger)$ *etc.* so

$$(\mathrm{AB}^\dagger)_{ij} = \sum (\mathrm{A}^\dagger)_k j (\mathrm{B}^\dagger)_i k = \mathrm{B}^\dagger \mathrm{A}^\dagger \tag{7}$$

3. Townsend 2.8

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The braket is:

$$\frac{1}{\sqrt{5}}\begin{bmatrix}-i & 2\end{bmatrix} / \begin{bmatrix} i \\ 2 \end{bmatrix}\frac{1}{\sqrt{5}} = \frac{1}{5}(1+4) = 1 \tag{8}$$

The bra for $|{+x}\rangle$ is $(1,1)\sqrt{2}$ so we can find the amplitude to be in $|{+x}\rangle$:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix} i \\ 2 \end{bmatrix}\frac{1}{\sqrt{5}} = \frac{i+2}{\sqrt{10}} \tag{9}$$

The probability is the absolute value of that squared, or $(2+i)(2-i)/10 = 5/10 = 0.5$. So, we find ourselves in $|{+x}\rangle$ 50% of the time. The ket for +y is $(1,i)/\sqrt{2}$, so the bra $\langle +y| = (1,-i)/\sqrt(2)$. The amplitude is then

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\end{bmatrix}\begin{bmatrix} i \\ 2 \end{bmatrix}\frac{1}{\sqrt{5}} = (i-2i)/\sqrt{10} = -i/\sqrt{10} \tag{10}$$

The probability is the absolute value of that squared, or 10%.

4. A) Working in the $z-$basis, express the projection operators $|{+y}\rangle\langle{+y}|$ and $|{-y}\rangle\langle{-y}|$ as 2x2 matrices.

   B) Show that the $+y$ projection matrix times an arbitrary vector $(a,b)$ outputs a vector that is proportional to $|{+y}\rangle$ (*i.e.* it comes out as $(c, ic)$ for some value $c$). Show the same for the $-y$.

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A) We have $|{+y}\rangle = (1,i)/\sqrt{2}$, $\langle{+y}| = (1,-i)/\sqrt{2}$. The outer product of that is

$$\frac{1}{2}\begin{bmatrix}1 & -i \\ i & 1\end{bmatrix} \tag{11}$$

Since the bras/kets for $|{-y}\rangle$ are just the conjugates of $|{+y}\rangle$, then $|{-y}\rangle\langle{-y}|$ is the conjugate of $|{+y}\rangle\langle{+y}|$, or

$$\frac{1}{2}\begin{bmatrix}1 & i \\ -i & 1\end{bmatrix} \tag{12}$$

As expected, they sum to the identity matrix.

B,C) One way to do this is look at the eigenvalues/vectors of the projection matrices. We actually know what they are by construction, but if you do the formal decomposition, you see that $|+y\rangle\langle+y|$ has eigenvectors $|+y\rangle, |-y\rangle$ with eigenvalues $(1, 0)$. Since we only have a single non-zero eigenvalue, everything that comes out of that matrix operating on a vector has to be proportional to the corresponding eigenvector, which is $|+y\rangle$. The same hold true for $|-y\rangle\langle-y|$, only now the eigenvalue is zero for $|+y\rangle$ and 1 for $|-y\rangle$.

5. A) For an arbitrary state $|+n\rangle, |-n\rangle$, write down the 2x2 projection operators in the $z-$basis. As a reminder, you can look at Townsend problem 1.3 for the state in an arbitrary direction. B) Show that the sum of these two matrices is the identity matrix. We expect this because the $|+n\rangle$ component of a state plus the $|-n\rangle$ component must give us the state we started with.

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The ket for

$$|+n\rangle = \cos(\theta/2)\,|+z\rangle + \exp(i\phi)\sin(\theta/2)\,|-z\rangle \tag{13}$$

The bra is then

$$\langle+n| = \cos(\theta/2)\,\langle+z| + \exp(-i\phi)\sin(\theta/2)\,\langle-z| \tag{14}$$

We can multiply them out to get:

$$\begin{bmatrix} \cos(\theta/2)^2 & \exp(-i\phi)\cos(\theta/2)\sin(\theta/2) \\ \exp(i\phi)\cos(\theta/2)\sin(\theta/2) & \sin(\theta/2)^2 \end{bmatrix} \tag{15}$$

The ket for $|-n\rangle$ is

$$|-n\rangle = \sin(\theta/2)\,|+z\rangle + \exp(i(\phi+\pi))\cos(\theta/2)\,|-z\rangle \tag{16}$$

the $i\pi$ in the exponential gives us a -1, so a slightly simper version is

$$|-n\rangle = \sin(\theta/2)\,|+z\rangle - \exp(i\phi)\cos(\theta/2)\,|-z\rangle \tag{17}$$

Taking the conjugate, we have the bra:

$$\langle-n| = \sin(\theta/2)\,\langle+z| - \exp(-i\phi)\cos(\theta/2)\,\langle-z| \tag{18}$$

Taking the outer product, we have:

$$\begin{bmatrix} \sin(\theta/2)^2 & -\exp(-i\phi)\sin(\theta/2)\cos(\theta/2) \\ -\exp(i\phi)\sin(\theta/2)\cos(\theta/2) & \cos(\theta/2)^2 \end{bmatrix} \tag{19}$$

B) If we add these two together, the off-diagonals are equal and opposite, so they cancel. The on-diagonals are $\cos(\theta/2)^2 + \sin(\theta/2)^2 = 1$ for both entries, so we have

$$|+n\rangle\langle+n| + |-n\rangle\langle-n| = \mathrm{I} \tag{20}$$

6. A) Work out the angular momentum operators $J_x, J_y$ in the $z-$basis. Verify that they are Hermitian. If you want to do this on a computer, that's fine, but include the (very short!) code you used to generate them, and comment what you are doing.

   B) Work out the angular momentum operators $J_x, J_y, J_z$ in the $|\pm y\rangle$ basis. Again, verify that they are Hermitian. ————————————————————-

   See code pset2_q6.py

7. Work our the $\pi/2$ rotation matrix about the $y-$axis in the $|\pm z\rangle$-basis. Do this two ways - first by writing down what this matrix has to do to the $|+x\rangle$ and $|+z\rangle$ states. Then by combining the matrices that turn a state represented in the $\pm z$-basis into the $\pm y$-basis, the matrix that rotates about its own axis (the rotation about $|+n\rangle$ represented in the $|\pm n\rangle$ basis can't depend on $|n\rangle$), and the matrix that converts states in the $|\pm y\rangle$ back into the $|\pm z\rangle$ basis. Show that these matrices are the same, possibly up to an overall phase factor.

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   See code pset2_q7.py