PHYS 357 Pset 10. Due 11:59 PM Thursday Nov. 28

1. Show that feeding the probability current (6.115) into the continuity equation (6.114) indeed gives $\frac{\partial}{\partial t}(\Psi^*\Psi)$ as expressed in 6.113.

The continuity equation is $\frac{\partial \rho}{\partial t} + \partial \rho v \partial x = 0$. We're asserting that ρv corresponds to the probability current, so let's differentiate w.r.t x:

$$\frac{\partial}{\partial x}(\frac{\hbar}{2mi}(\Psi^*\Psi' - \Psi(\Psi^*)') = \frac{\hbar}{2mi}((\Psi^*)'\Psi' + \Psi^*\Psi'' - \Psi'(\Psi^*)' - \Psi(\Psi^*)''$$

The first order terms cancel, so we're left with

$$\frac{\hbar}{2mi}\left(\Psi^*\Psi'' - \Psi(\Psi^*)''\right)$$

Now take $\frac{\partial \rho}{\partial t} = \frac{\partial(\Psi^*\Psi)}{\partial t}$. That equals

$$\frac{\partial(\Psi^*\Psi)}{\partial t} = \Psi\frac{\partial \Psi^*}{\partial t} + \Psi^*\frac{\partial \Psi}{\partial t}$$

From the Schrodinger equation, we know that

$$\frac{\partial \Psi}{\partial t} = H\Psi/i\hbar = -\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi/i\hbar$$

The V term goes away because it flips sign under conjugation, so we're left with the first term. Plug that into the previous equation, and you're left with the one before, so indeed the probability current satisfies the continuity equation.

2. A gaussian wave packet has $\Psi(x, t = 0) = c_0 \exp(ikx)\exp(-x^2/2\sigma^2)$. What is the probability current? We know from previous problem sets that a Gaussian wave packet spreads with time. Can you explain how that is consistent with the $k = 0$ result from the probability current? ——— Take the derivative:

$$\frac{d\Psi}{dx} = c_0 ik\exp(ikx)\exp(-x^2/2\sigma^2) - c_0\exp(ikx)\exp(-x^2/2\sigma^2)x/\sigma^2 = (ik - x/\sigma^2)\Psi(x, t = 0)$$

So the first term in the current $\Psi^*\frac{\partial \Psi}{\partial x} = (ik - x/sigma^2)\Psi^2$. The second term gives

$$\frac{d\Psi^*}{dx} = c_0^*(-ik\exp(-ikx\exp(-x^2/2\sigma^2) - c_0\exp(-ikx)\exp(-x^2/2\sigma^2)x/\sigma^2 = (-ik - x/\sigma^2)\Psi^*$$

And with the Ψ, we have:

$$(-ik - x\sigma^2)\Psi^2$$

. When we subtract them, the second term goes away because it has the same sign, but the first term adds, so we get

$$\frac{\hbar}{2mi}(2ik\Psi^2) = \frac{\hbar k}{m}\Psi^2$$

Since $\hbar k$ is the momentum, $\hbar k/m$ is like the classical velocity, so we get the very sensible result that the current is just the velocity times the density. For $k = 0$, the probability current is zero. What's going on is that initally, the wave function doesn't spread, but the phase of the wave function does evolve. When the phase evolves, now we do pick up a probability gradient and the wave function starts to spread. A wrong but possibly useful way of envisioning this semi-classically is that at the begining, the system is balanced so there's as many right-moving as left-moving bits. However, at time evolves, the right-moving bits move right, and the left-moving bits move left, so then we do get a net flow, but only after things have a chance to move a little.

3. Townsend 6.24

4. A way to estimate tunneling numerically is to start with a square well with a barrier in it, and add a wave packet on one side of the barrier. If you then solve Schrodinger's equation, you can calculate the probability as a function of time to find the particle on the right side of the barrier. If b is the width of the region to the left of the barrier, then particle hits the barrier something like every $2b/v$ seconds, where $v = p/m$. From that, you can estimate $T(k)$. Do this for a square barrier with a transmission probability of $\sim 1e-6$ and show you get a roughly sensible result. Then switch to a Gaussian barrier where Townsend 6.149 would predict the same tunneling probability. Do this for a) a Gaussian whose height is the same as the barrier and adjust the width, and b) for a gaussian with $\sigma = a/2$ (where a is the width of the square barrier) and adjust the height. How do you feel about the approximation in 6.149?

5. Quantum Snell's Law. Consider a 2-dimensional space where $V(x, y) = 0$ for $y < 0$, and $V(x, y) = V_0$ for $y > 0$ (note that V_0 could well be negative, but you may assume $E > V_0$). The momentum eigenstates in 2-d are $\exp(i\vec{k} \cdot \vec{x}) = \exp(i(k_x x + k_y y))$. The de Broglie

relation carries straight over: $\vec{p} = \hbar\vec{k}$, and so the magnitude of k is just what we expect from 1D quantum mechanics. In particular $E - V = p^2/2m = \hbar^2(k_x^2 + k_y^2)/2m$.

Now let's say we have an incoming wave $a\exp(i(k_x x + k_y y))$ for $y < 0$, we'll get a reflected wave $b\exp(i(k_x x - k_y y))$ for $y < 0$, and a transmitted wave $c\exp(i(k'_x x + k'_y y))$. Derive the three boundary conditions on b, c given that the combined wave function must be continuous and have continuous derivatves in both x and y at the $y = 0$ interface for all x.

a) Show that these conditions require $k'_x = k_x$

b) Solve for k'^2_y in terms of k_y, m and V_0.

c) If $k'^2_y < 0$, the wave can't penetrate the barrier since it will decay exponentially in y, even if $E > V_0$. This is the quantum version of total internal reflection. What is the condition for total internal reflection to happen?

d) Show that we recover Snell's law in quantum mechanics.

6. Bonus If you aren't sick of algebra yet, work out the reflection and transmission probabilities for Snell's law, and show that probability is conserved.