PHYS 357 Pset 4. Due 11:59 PM Thursday Oct. 17

In some of these problems, we'll see that interesting behavior can emerge when we work with states with larger angular momentum. It's a pain to do this by hand, so feel free to carry out calculations on a computer. If you do this, do please explain how your codes work so the TAs can understand what you did when it comes time to mark.

1. Townsend 3.24. Please do this one by hand - Townsend helpfully supplies all the relevant matrices for you.

2. Consider a ping pong ball (mass 2.7 grams and radius 20 mm) rotating at 10 radians per second (in real play, the spins can be 100 times higher). What is the approximate angular momentum (order of mangitude is fine, so you can use MR^2 for the moment of inertia)? If we say the ping pong ball is in a state $|j, j\rangle$ rotating about the $z-$axis, what is the approximate value of j? Given that value of j, what is the total angular momentum in the $x - y$ plane, $\sqrt{J_x^2 + J_y^2}$? We can express this as an uncertainty in the rotation axis of the ping pong ball, with $\sigma(\tan(\theta)) \sim \frac{\sqrt{J_x^2+J_y^2}}{J_z}$. What is your approximate value for $\sigma(\tan(\theta))$? For a macroscopic object, talking about "the" axis of rotation is a very good approximation!

3. **Part A)** Consider a spin-1 Stern-Gerlach experiment where you send a beam of particles from the oven down a modified SGz machine, where you block the $J_z = 0$ beam before recombining. You might think that would leave your particles in the state $J_z = (1, 0, 1)/\sqrt{2}$. If that were the case, what would you see (in terms of what fraction of particles go up, go down, or aren't deflected at all) if you then send the beam through an SGx machine? Through an SGy machine?

   Part B) Explain why these results can't be correct (hint - what happens if I take my SGz into SGx apparatus, and rotate the whole setup by 90 degrees about the $z-$axis).

   Part C) If you were to actually set this experiment up and run it, what you would see is that 50% of the particles show up with $J_{x,y} = 0$, 25% show up with $J_{x,y} = +\hbar$, and 25% show up with $J_{x,y} = -\hbar$ (where $J_{x,y}$ means I measure *either* J_x or J_y), so indeed the apparatus behaves the same if I rotate my coordinate system by 90 degrees, as it must. Can you quantitatively explain the actual observed results? You may use a computer if you like.

4. **Part A)** We saw that J_z commutes with $J_x^2 + J_y^2$. Does J_x commute with $J_x^2 + J_y^2$? If not, what is the commutator $[J_x, J_x^2 + J_y^2]$? Feel free to express as an anti-commutator where $\{\mathrm{A}, \mathrm{B}\} = \mathrm{AB} + \mathrm{BA}$.

   Part B) Let's say I take a set of spin-2 particles, all of which have $J_z = 2\hbar$, which we write as $|2,2\rangle_z$ (note the z subscript, which says this state is an eigenstate of J_z). If I were to measure $J_x^2+J_y^2$, what would I observe, and what would the uncertainty in that measurement be?

   Part C) I now send my beam of $|2,2\rangle_z$ particles down an SGx machine. Show that the expectation of J_x^2 that you see is one half of $< J_x^2 + J_y^2 >$ you calculated in part B).

   Part D) What is the probability that you measure $J_x = +2\hbar$? If this is non-zero, can you explain the apparent contradiction between finding a particle witih $J_x = 2\hbar$ (and hence $J_x^2 = 4\hbar^2$) and the maximum value you found for $J_x^2 + J_y^2$ from part B?

5. **Part A)** Sticking with spin-2 particles, express the pure state $|2,2\rangle_y$ in both the z-basis and the y-basis. In the $z-$basis, please pick the phase that makes the amplitude of $J_z = +2\hbar$ purely real and positive.

   Part B) For this same $|2,2\rangle_y$ pure state, what are the uncertainty in J_x and J_z? Show that the uncertainty relation is satisfied.

6. **Part A)** Calculate the matrix that rotates a state about the $y-$axis by $+90$ degrees for a spin-2 particle. Print the absolute value of the matrix.

   Part B) Show that the above matrix rotates the state $|2,2\rangle_z$ to the pure state $|2,2\rangle_x$, and $|2,2\rangle_x$ to $|2,-2\rangle_z$.

7. Bonus: Our end goal is to derive the angular momentum commutation relation $[J_a, J_b] = i\hbar\epsilon_{abc}J_c$ but to do that we'll first need the canonical commutation relation $[x,p] = i\hbar$. In the first bonus we'll work that out, in the second bonus, we'll use it to derive the angular

momentum commutation relations. You are more than welcome to do the second bonus only, and just use the canonical commutation relation as supplied.

In 1924, Louis De Broglie hypothesized (in his PhD thesis!) that all matter behaved like waves, with wavelength set by their momentum. This was the key insight that opened the door to modern quantum mechanics, and within a couple of years both wave and matrix mechanics were basically fleshed out in their modern forms. The De Broglie relation is

$$p = \hbar k \tag{1}$$

where $k = 2\pi/\lambda$ and p is the momentum. If I have a wave function of a particle with wave vector k, I can write that down as a function of x as follows: $\Psi = \exp(ikx)$.

Part A:Show that the operator $-i\hbar\frac{\partial}{\partial x}$ operating on Ψ returns $p\Psi$. In other words, the plane wave is an eigenstate of momentum, and the operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ returns the wave function times the momentum.

Part B: Now the position operator $\hat{x}$ in position space (where $\Psi = \Psi(x)$) is, not surprisingly, just x. You can now derive the canonical commutation relation $[x, p]$ by applying the operators to a wave function $(\hat{x}\hat{p} - \hat{p}\hat{x})\Psi$. Show that when you do this, you get $i\hbar\Psi$. That is, when you apply $[x, p]$ to a wave function, you get $i\hbar$ times the wave function, so $[x, p] = i\hbar$.

8. Bonus 2. Now we'll work out the angular momentum commutators, using the fact that $J = r \times p$. If you expand that out, you can see that $J_x = r_y p_z - r_z p_y$, which will all get applied to a wave function.

   Position and momentum on the same axis do not commute, but they do commute if you measure position along one axis, and momentum along another, perpendicular axis. More formally, $[x_i, p_j] = i\hbar\delta_{ij}$. Use this commutation relation and the expressions for Jx, Jy, Jz you get from $J = r \times p$ to show that $[Jx, Jy] = i\hbar Jz$. The commutators for other pairs of axes follow from cyclic permutations of the cross product, so you only need to do it once.

   Note - technically we've only shown this for orbital angular momentum and not spin. We did show that spin-1/2 particles obey the same commutation relation, so I hope it's not a surprise that the relation holds for higher spins as well.