Supplementary Notes for lectures 6 (and possibly more) for McGill Phys 357.

1. Exponentials as Limits

We're going to show that large powers of $1+\delta$ turn into exponentials, but perhaps via a somewhat roundabout path. First, let's go back to the derivative of an exponential with some base a, a^x. Going back to the fundamental definition of a derivative, we have:

$$\frac{d(a^x)}{dx} = \lim_{dx\to 0} \frac{a^{x+dx} - a^x}{dx} = a^x \lim_{dx\to 0} \frac{a^{dx}-1}{dx} \tag{1}$$

By factoring out the a^x, we're left with a number that is a function of a in the rightmost term. If that limit equals one for some value of a, which we'll call e, then we have a function that is equal to its own derivative. For that to be true, we need:

$$\lim_{dx\to 0} \frac{e^{dx}-1}{dx} = 1 \tag{2}$$

$$\lim_{dx\to 0} e^{dx} = 1 + dx \tag{3}$$

$$e = \lim_{dx\to 0} (1+dx)^{1/dx} \tag{4}$$

The more usual way of writing this is taking $n \equiv \frac{1}{dx}$, in which case we have:

$$e = \lim_{n\to\infty} (1+\frac{1}{n})^n \tag{5}$$

We can of course raise this number to some power x:

$$e^x = \left(\lim_{n\to\infty} (1+\frac{1}{n})^n\right)^x = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{nx} \tag{6}$$

We can clean this up by making the substitution[1] $nx \to n$, leaving us with

$$e^x = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n \tag{7}$$

From the book, we have that a rotation $\hat{R}(d\phi)$ by a small angle $d\phi$ about the $z-$axis is

$$\hat{R}(d\phi\mathbf{k}) = 1 - \frac{i}{\hbar}\hat{J}_z d\phi \tag{8}$$

(Townsend eqn. 2.29). When we write a macroscopic rotation by angle ϕ as a sequence of N rotations by angle $d\phi = \phi/N$, then we can write the infinitessimal rotation

$$\hat{R}(\phi/N\mathbf{k}) = 1 - \frac{i}{\hbar}\phi/N \tag{9}$$

[1] Yes, I should really say $n' = nx$, write out in terms of n' and then drop the primes to leave the expression we get, but that's a lot of steps to accomplish the same thing.

Since we can get our macroscopic rotation by taking a sequence of N small rotations as N goes to infinity, we have

$$\hat{R}(\phi \mathbf{k}) = \lim_{N\to\infty} \left(1 - \frac{i}{\hbar}\hat{J}_z\phi/N\right)^N = e^{-\frac{i}{\hbar}\hat{J}_z\phi} \tag{10}$$

where we invoke Equation 7 to carry out the final step. I find this a much more natural way of seeing why the limit turns into an exponential. Yes, Townsend's approach of expanding the power using the binomial theorem, taking the terms as the number of steps goes to infinity and matching them with the Taylor series for e^x will *work*, but it doesn't really explain *why*. Forms like Equation7 show up pretty frequently in physics, so it's worth having that expression in the back of your mind for the day you need it.

2. Simple Rotation Derivation

The book goes through a relatively abstract derivation of rotation operators, but it may also be useful to see a more direct derivation in a particular coordinate system. We'll start with the observation that a 90-degree rotation about the $z-$axis rotates $|+x\rangle$ into $|+y\rangle$, and $|+y\rangle$ into $|-x\rangle$. If we're working in the $|\pm z\rangle$ basis, we can write the states as:

$$|+x\rangle = \frac{1}{\sqrt{2}}[1, 1]^T \tag{11}$$

$$|+y\rangle = \frac{1}{\sqrt{2}}[1, i]^T \tag{12}$$

$$|-x\rangle = \frac{1}{\sqrt{2}}[1, -1]^T \tag{13}$$

$$|-y\rangle = \frac{1}{\sqrt{2}}[1, -i]^T \tag{14}$$

We know the same rotation matrix has to take $|+x\rangle$ to $|+y\rangle$, so we can write

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = R \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{15}$$

Similarly, since we also have to take $|+y\rangle$ to $|-x\rangle$, we have:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = R \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} \tag{16}$$

The laws of linear algebra let us merge these equations to get:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -1 \end{bmatrix} = R \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & i \end{bmatrix} \tag{17}$$

We multiply on the right by the inverse of the 2x2 matrix to get:

$$R(\pi/2) = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \tag{18}$$

We can write this in a slightly more useful way by expressing terms as complex exponentials:

$$R(\pi/2) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{bmatrix} \tag{19}$$

We've derived this by ensuring that $|+x\rangle$ goes into $|+y\rangle$, and $|+y\rangle$ goes into $|-x\rangle$, but it's a good exercise to check that this same matrix also takes $|-x\rangle$ into $|-y\rangle$, and $|-y\rangle$ into $|+x\rangle$. If that doesn't work, we've made a mistake somewhere.

We're always free to rotate states by an overall phase. If we want to make the $|+z\rangle$ and $|-z\rangle$ states more symmetric, we can add in an extra phase of $-\pi/4$, which gives us:

$$R(\pi/2) = \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \tag{20}$$

We can generalize this to

$$R(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix} \tag{21}$$

To see this, picture splitting our $\pi/2$ rotation into N smaller rotations. Each of those rotations must be the same, so the product of N of the tiny rotations must equal Equation 20. Since the rotation matrices are diagonal, the product of N matrices is just a diagonal matrix with entries raised to the N^{th} power. If we let $N\theta = \pi/2$ then the only way to get Equation 20 for an arbitrary value of N is if the rotation by θ is Equation 21.

Once we have the rotation matrix in the form of Equation 21, we can see why the generator of rotations is written the way it is. Since we're working in the $|\pm z\rangle$ basis, we know that the angular momentum of a $|+z\rangle$ state must be $\frac{\hbar}{2}$, and the angular momentum of a $|-z\rangle$ state must be $-\frac{\hbar}{2}$. If we have an operator that gives us the angular momentum along the $z-$axis, we can produce Equation 21 just by cancelling the $\hbar$. That gives:

$$R_z(\theta) = \begin{bmatrix} e^{-i\hat{J}_z/\hbar} & 0 \\ 0 & e^{-i\hat{J}_z/\hbar} \end{bmatrix} \tag{22}$$

3. Hermitianness and Rotation Operators

We've seemingly randomly introduced a factor of $\frac{i}{\hbar}$ in our definition of rotations. Of course, it's not random, but you only know that after you've arrived at your destination. What we can say robustly, though, is that once you have a rotation matrix of the form of Equation 10, we can say something important about $\hat{J}_z$. Since our operations obey the rules of linear algebra, an operator has to behave the same way when operating to the right on a ket/column vector as when it operates to the left on a bra/row vector. To conserve probability, we absolutely need that

$$\langle\Psi|\,\hat{R}^\dagger(\phi)\hat{R}(\phi)\,|\Psi\rangle = 1 \tag{23}$$

since the state $\hat{R}(\phi)\,|\Psi\rangle$ has to be normalized. That means

$$\hat{R}^\dagger(\phi)\hat{R}(\phi) = \mathrm{I} \tag{24}$$

We know that if we rotate through an angle ϕ about an axis, to undo the rotation we must rotate through an angle of $-\phi$ about the same axis. That means that our $\hat{R}^\dagger(\phi)$ must be equal to $\hat{R}(-\phi)$. We can plug that into Equation 10 where $\phi \to -\phi$, so

$$\hat{R}^\dagger(\phi) = e^{\frac{i}{\hbar}\hat{J}_z\phi} \tag{25}$$

Of course, we know that we need to take the complex conjugate when going from a ket to a bra, so we also know that

$$\hat{R}^\dagger(\phi) = e^{\frac{i}{\hbar}\hat{J}_z^\dagger\phi} \tag{26}$$

The only way that Equations 25 and 26 can be true is if

$$\hat{J}_z = \hat{J}_z^\dagger \tag{27}$$

In other words, the operator $\hat{J}_z$ is its own conjugate-transpose. We'll see that this is a general property of any physical observable. Any operator that gives us a physical observable *must* equal its own conjugate-transpose when written as a matrix. We call such operators/matrices *Hermitian.* In this case, $\hat{J}_z$ gives us the angular momentum about the $z-$axis. The rotation operator does not produce a physical quantity, and so it does not have to be Hermitian. The entire point of the rather odd way we wrote the generator of rotations was to express it in terms of a Hermitian operator.