1. Black-body Radiation and the Quantization of Light

One of the first indications that nature is quantized came from looking at the thermal emission from so-called "black bodies". If you make a box and hold the walls at fixed temperature, then the intensity of light as a function of frequency inside the box will be a universal function of temperature, independent of what the walls are made of. This is very different from our usual life experience, where different materials have different colors because how they interact with light depends on the frequency of the light. An ideal black body would absorb all light falling on it, and the light it emits would match the spectrum of our cavity.

A basic challenge of late 19th century physics was to understand this black-body radiation. At low frequencies, everything made sense, with brightness increasing with frequency, and higher temperatures emitting more light. However, a classical model would predict that the brightness would just keep increasing with frequency, resulting in an infinite amount of power getting radiated. That obviously can't happen in real life, and the observed spectrum rolls over and starts to drop at high frequencies. Classical physics could not explain this, and the fact that the energy diverges at high frequencies was known as the "ultraviolet catastrophe", so physicists of the time were well aware there was a serious problem.

Before we see how quantization solves the UV catastrophe, let's look at the classical prediction. The key fact that we'll need is that if we observe the energy in a state at some temperature T, the *relative* probability of observing either energy E_1 or E_2 is $\exp(-(E_1 - E_2)/kT)$ where k is Boltzmann's constant, and T is the temperature in Kelvin. If you look at this, you'll see that for energy differences much less than kT, the two values are roughly equally likely, while for large energy differences, the high-energy state is exponentially disfavored. For those of you, like me, who can't jump very well, it's *possible* for nature to give you enough of a boost to dunk a basketball, it's just *exceedingly* unlikely. Say you need to jump a metre, g is 10, and you weigh 100 kg. Then your potential energy at dunking height is 1*100*10=1 kJ. How likely are you to get this "for free"? Well, assuming you're playing at something like room temperature, T is around 300K, and $k = 1.38e-23$, so the typical thermal energy is around $300 * 1.38e-23 \sim 4e-23$ Joules. The odds that the universe conspires to boost you is then $\exp(-1e3/4e-23) = \exp(-2.5e25)$, or converting to base 10, $10 * *(-2.5e25 * \log_{10}(e)) = 10^{-1.1e25}$. Just to be clear, your odds of dunking are not one in 10^{25}, they are one in $10^{10^{25}}$.

Now let's ask the question "What *is* our typical heigh boost?" The relative probability of observing energy $E = mgz$ (assuming $E = 0$ is the ground and we're at height z) is $P_{rel}(E) = \exp(-E/kT)$. These are relative probabilities, so to get an actual probability, we need to normalize by our total integrated probability, $\int_0^\infty \exp(-E/kT)dE$. We can then calculate the expectation of the energy:

$$E_{typ} = \frac{\int_0^\infty E\exp(-E/kT)dE}{\int_0^\infty \exp(-E/kT)dE}$$

These integrals aren't hard to do, and the answer is that the average energy we'll see is just $< E >= kT$. Note that the mass of whatever is getting bounced around doesn't matter, it's a constant energy of kT. Our typical height is then kT/mg, which for a person is tiny (4e-26 metres). It's *not* tiny for say a nitrogen molecule, where the typical height will be around 8 km. If you've ever wondered why the air pressure is lower at the top of a mountain, but not zero, this is why. Note also that there's nothing special about height - any way we have of putting energy into a system is going to give us around kT of energy, so if we have n ways of putting energy in, then we'll have nkT Joules on average.

Now that we've worked out the typical energy, we can apply that to a box at fixed temperature, and ask how many ways we can put energy into the light. First, consider a string of length L, tied down at both ends. Since the ends can't move, I can't make the string vibrate with arbitrary wavelengths - only modes that don't move at the ends of the string will work. The allowed wavelengths are then $2L/\lambda$, or if we use frequency instead of wavelength, we have $2L\nu/c$, where c is the speed at which vibrations move in the string. Random vibrations of the string can be broken into terms that have wavelengths of $\lambda/2$, λ, $3\lambda/2$..., and classical physics tells us we can expect *each* of these vibrations to have energy $\sim kT$.

Let's look at a 3D box now. Once again, we need to fit integer numbers of wavelengths into our box. Unlike the 1D string, though, we can pick different numbers of wavelengths to put in the x, y, and z directions, so for some frequency ν, the *total* number of ways I can pick a wave is something like $(2L\nu/c)^3$. If I divide by L^3 to get the energy per unit volume, we now have something like $8\nu^3/c^3$. If you want to ask about the number of possible modes with frequency between ν and $\nu + \delta\nu$, then we can differentiate w.r.t ν, to get $24\nu^2/c^3$. That's the number of distinct modes per unit volume, so if each of those modes has energy kT, then the energy per unit volume per unit frequency should be something like $24kT\nu^2/c^3$. We've been pretty sloppy about counting our modes, so we wouldn't expect the 24 to be exactly right, but it is actually pretty close - the true value is 8π[1]. Our final prediction for the energy density of black-body radiation is then:

$$\rho E_{BB,classical} = 8\pi kT\nu^2/c^3 \tag{1}$$

This works great at low frequencies, where it's known as the Rayleigh-Jeans law. However, you can see that the energy density just keeps increasing with frequency, so this picture can't be correct across all frequencies. Lots of people tried lots of things to try to fix this problem, but it was Planck who figured it out.

At high frequencies, the black body curve is experimentally seen to follow the Wien approximation law, an exponential decay of $\nu^2 \exp(-h\nu/kT)$ for some constant h. People tried several different ways to match the (well motivated) Rayleigh-Jeans law with the (seemingly arbitrary) Wien tail. Planck's big insight was that these could agree if each mode could only have certain allowed energies, instead of the continuum we expect from classical physics. You might start by guessing that the allowed energies for each mode are the same, but if you work through the math (a great exercise for the reader!), you'll find that the UV catastrophe doesn't go away. Planck tried the next obvious thing, which is that the allowed energies in each state are proportional to the frequency, so $E = nh\nu$ for integer n and constant h. If we re-do the energy per state caculation,

[1] We were lucky to get this close. Instead of $8L^3$ we should have used the volume of a sphere: $4\pi/3L^3$, but there are also two polarizations of light. Instead of $3 * 8$, we should have had $2 * 3 * 4\pi/3 = 8\pi$.

we now have

$$\frac{\int_0^\infty E\exp(-E/kT)}{\int_0^\infty \exp(-E/kT)} \rightarrow \frac{\sum_{n=0}^\infty nh\nu\exp(-nh\nu/kT)}{\sum_{n=0}^\infty \exp(-nh\nu/kT)}$$

We've just replaced E in the integral by $nh\nu$, and now we sum over integer values of n. Let's start with the sum on the bottom. We can re-write it as a geometric series $\sum(\exp(-h\nu/kT))^n$. Since the sum of x^n is $\frac{1}{1-x}$[2], the denominator sums to $\frac{1}{1-\exp(-h\nu/kT)}$. We can sum the numerator as well, using the fact[3] that $\sum_{n=0}^\infty nx^n = x/(1-x)^2$. Plugging everything in, we get that the energy per mode is now:

$$\frac{h\nu\exp(-h\nu/kT)/(1-\exp(-h\nu/kT))^2}{1/(1-\exp(-h\nu/kT))} = \frac{h\nu}{\exp(h\nu/kT)-1}$$

For $h\nu << kT$, this expression reduces to the classical limit of kT. However, for $h\nu >> kT$, the energy-per-mode gets exponentially suppressed. Qualitatively, when kT is much less than $h\nu$, then you have a hard time exciting a mode and so a given mode usually has zero energy, instead of kT. We can replace our classical energy-per-mode kT in our classical black-body expression, Equation 1, to get our new, quantum expression:

$$\rho E_{BB} = \frac{8\pi h\nu^3/c^3}{\exp(-h\nu/kT)-1} \tag{2}$$

This is the energy density inside our black-body cavity, but a more common expression is the brightness - how much power per unit solid angle per unit frequency does a black body radiate? To get that, we'll need to multiply by c (since the energy density is travelling at the speed of light) and divide by 4π since there are 4π steradians in a sphere, leaving us with:

$$I_\nu(\nu,T) = \frac{2h\nu^3}{c^2(\exp(h\nu/kT)-1)} \tag{3}$$

Planck's Law (Equation 3) is one of the fundamental results of physics. It established that light was quantized into packets of energy - what we now know are photons. When Planck worked this out, he measured not one but two fundamental constants of nature for the first time - both h and k. Statistical mechanics also that $R = N_A k$ where R is the ideal gas constant in moles and N_A is Avogadro's number. Avogadro's number in turn gives you the mass of atoms. I find it amazing that some of (if not the) first measurements of atomic weights were derived by looking at the properties of light!

Entertainingly, Planck himself did not immediately realize the fundamental importance of his work. In his words, it was "a purely formal assumption ... actually I did not think much about it ...". If you ever feel like you're struggling to understand quantum mechanics, I hope you can take comfort in the fact that Planck himself also didn't fully realize what he was doing.

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[2]As a reminder, $\sum x^n = 1 + x + x^2 + ... = 1 + x\sum x^n$. Since the sum has to be the same on both sides, you can solve for it to get $1/(1-x)$

[3]We have that $\sum_{0/1}^\infty nx^n = \sum_0^\infty (n+1)x^{n+1} = nx\sum(x^n) + x\sum(x^n) = x\sum nx^n + x/(1-x)$. We've used the fact that the first term in the series is zero when $n = 0$ so we can get away with starting the sum at either $n = 0$ or $n = 1$. Solving for the sum, we have $\sum_0^\infty nx^n = x/(1-x)^2$.

2. COBE-FIRAS Measurement of the Cosmic Microwave Background

It's actually really hard to make a good black body. If you want to see the spectrum of emitted radiation, you need to look inside the black body, which means you don't have a sealed cavity any more. Materials have all sorts of atomic transitions, which will alter the spectrum of emitted light for many setups. The best black body spectrum in nature comes not from the Earth, but from space.

Immediately after the big bang, the universe was hot and dense, and it has been slowly cooling and expanding ever since. When the universe was young, well before any stars or galaxies had formed, it was uniform, hot, and dense - ideal conditions to get nearly perfect black-body radiation. As the universe expands, black body radiation remains black body, just with increasingly lower temperatures. We can look at microwave frequencies and see the light left over from the big bang, which now has a temperature of only 2.73 degrees above absolute zero. This radiation is called the cosmic microwave background (CMB).

The best measurement[4] of the absolute temperature of the CMB was carried out by the FIRAS experiment on board the COBE satellite. The FIRAS measurement is shown in Figure 1, along with the theoretical prediction for a black body with temperature 2.725K. The agreement is exquisite - the error bars are too small to see on the plot. When these results were first presented at a meeting of the American Astronomical Society, the audience broke out into spontaneous applause. This work was recognized with a Nobel Prize in 2006.

3. Discussion

Planck's results on black-body radiation gave us the first indication that the microscopic world was fundamentally different from the world we're used to. Unlike in classical physics, where we can add arbitrary amounts of energy to a system, light comes in discrete packets of energy that depend on their frequency. We can only add/subtract multiples of that amount. This discrete nature, which Planck orignially thought of as a mathematical convenience, became crystal clear through studies of the photoelectric effect.

It would take another quarter-century of experimental and theoretical work before a clear picture of quantum physics came into view, but it all started with Planck's studies of black-body radiation.

[4]Canada led many of the first measurements of the CMB temperature. The first measurements of molecules at space were carried out at the Dominion Radio Astronomy Observatory (DRAO) in Penticton, BC. While analyzing various population levels, Andrew McKellar realized they only made sense if there was a background temperature of approsimately 2.3K. He didn't realize this was the CMB, but in fact his answer was closer than the original direct measurement by Penzias and Wilson, who found 3.5K. Herb Gush at UBC led a group using rockets to try to measure the spectrum of the CMB. They got the right answer and released their results around the same time as COBE, but the satellite data were orders of magnitude more precise, and so the rocket data languished in obscurity.

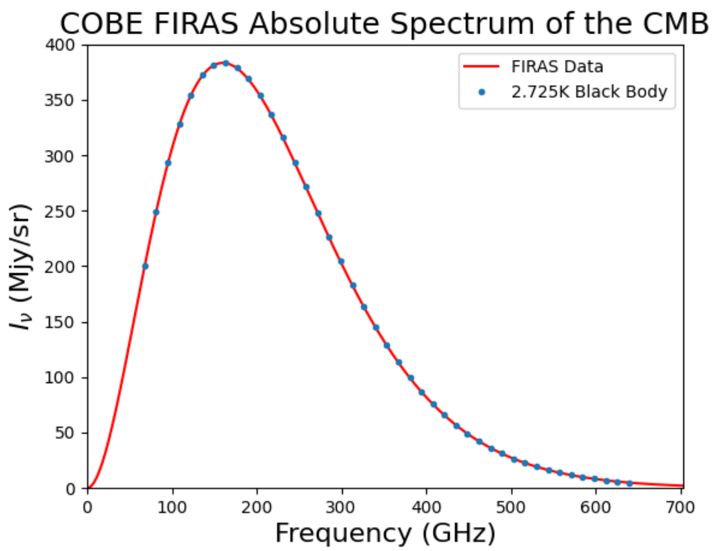

Fig. 1.— Results from the COBE-FIRAS measurement of the CMB absolute spectrum. $x-$axis is frequency in GHz, $y-$axis is specific intensity, in mega-Janksies per steradian. Blue points are the experimental data. The red curve is the theoretical prediction (including amplitude) for a black body with temperature 2.725K. The agreement is spectacular - the error bars are plotted, but they are much smaller than the points in the plot.