Supplementary Notes for lecture 9 for McGill Phys 357.

1. Orthogonal Eigenvectors

We know that operators that produce an observable must be Hermitian (equal to their own conjugate-transpose). The eigenvalues must be real since for eigenvector v

$$v^\dagger \mathrm{A} v = v^\dagger \lambda v = \lambda \tag{1}$$

but if I take the conjugate transpose, I still get λ since $\mathrm{A}^\dagger$=A. Before moving to the general case, take the case of a symmetric matrix, which is just a Hermitian matrix with no imaginary part. If we have eigenvectors v_1 and v_2 with eigenvalues λ_1 and λ_2, then starting with the symmetmry of A, we know

$$v_1^T \mathrm{A} v_2 = v_2^T \mathrm{A} v_1 \tag{2}$$

$$v_1^T \lambda_2 v_2 = v_2^T \lambda_1 v_1 \tag{3}$$

$$(\lambda_2 - \lambda_1)(v_2^T v_1) = 0 \tag{4}$$

where we have used that the transpose of a number is itself, so $v_2^T v_1 = v_1^T v_2$. The only way this is true if the eigenvalues are different is if $v_2^T v_1 = 0$, so the eigenvectors of a symmetric matrix are orthogonal.

We need to be slightly (but only slightly) more careful. Keeping track of daggers, and using the fact that the eigenvalues are real, we end up with the very similar equation

$$\lambda_2 v_1^\dagger v_2 = \lambda_1 v_2^\dagger v_1 = \lambda_1 (v_1^\dagger v_2)^\dagger \tag{5}$$

The magnitude of $v_1^\dagger v_2$ doesn't change when you take its dagger, so the only way Equation refeqn:evecdag can hold when $\lambda_1 \neq \lambda_2$ is if $v_1^\dagger v_2 = 0$. Therefore all the eigenvectors of a Hermitian matrix must be orthogonal, just like the eigenvectors of a real symmetric matrix.

2. Expectation and Variance

Now we'll try to figure out the uncertainty on a measurement if our state is an eigenvector of an observable operator. Recall that

$$Var(x) = \langle x^2 \rangle - \langle x \rangle^2 \tag{6}$$

In quantum mechanics, if x is the observable we get by taking an operator $\hat{A}$ operating on wave function Ψ, then we have

$$\langle x \rangle \, \Psi^\dagger \hat{A} \Psi \tag{7}$$

We can always feed the output of an operator into another operator [1]. Since we want to get $\langle x^2 \rangle$, we can get that by operating by $\hat{A}$ twice:

$$\langle x^2 \rangle \, \Psi^\dagger \hat{A}\hat{A}\Psi \tag{8}$$

We can see this makes sense in the context of our familiar J_Z operator:

$$\mathrm{J}_z = \begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix} \tag{9}$$

$$\mathrm{J}_z^2 = \mathrm{J}_z\mathrm{J}_z = \begin{bmatrix} \hbar^2/4 & 0 \\ 0 & \hbar^2/4 \end{bmatrix} \tag{10}$$

$$\tag{11}$$

Since the operator for J_z^2 is just $\hbar/4$ times the identity matrix, we know the square of the angular momentum of any state will give us $\hbar^2/4$. Every measurement gives us either $\hbar/2$ or $-\hbar/2$, so the square of every measurement will give us $\hbar^2/4$, just like the J_z^2 operator tells us.

Now what happens if Ψ is an eigenvector of $\hat{A}$ with eigenvalue λ? We need to make

$$\langle \Psi | \hat{A} | \Psi \rangle = \langle \Psi | \Psi \rangle \lambda = \lambda \tag{12}$$

$$\langle \Psi | \hat{A}\hat{A} | \Psi \rangle = \lambda \langle \Psi | \hat{A} | \Psi \rangle = \lambda^2 \tag{13}$$

We have $\langle x \rangle = \lambda$ and $\langle x^2 \rangle = \lambda^2$, so we know that the uncertainty is $\langle x^2 \rangle - \langle x \rangle^2 = \lambda^2 - \lambda^2 = 0$. If we have a state that is an eigenvector of $\hat{A}$, then when we measure the value of the corresponding observable, the mean value is the eigenvalue, and the uncertainty is zero. Because of the zero uncertainty, we don't just get λ on average, we always get exactly λ.

In summary, if we have an observable operator $\hat{A}$ then the eigenvectors of $\hat{A}$ are *pure* states. Whenever we measure the value associated with $\hat{A}$, a pure state always returns its corresponding eigenvalue. Since the eigenvectors are orthogonal, we can always describe an arbitrary state as a combination of eigenvectors. That means when we make a measurement, the only possible results are the eigenvalues of $\hat{A}$. When we make a measurement of the observable, we collapse the wave function into the pure state corresponding to the value we measured.

3. Hermitian to Unitary

We'll often see objects of the form $exp(\hat{A})$ where $\hat{A}$ is a Hermitian operator. There are lots of ways to derive this, but let's just accept that by $\exp(\hat{A})$ we mean:

$$\exp(\hat{A}) = I + \hat{A} + \hat{A}^2/2! + \hat{A}^3/3! + ... \tag{14}$$

[1] After all, they are both just matrix multiplies. Townsend goes through a lot of effort to show explicitly that the result of a pair of operators is the product of their matrix representations, but once you know an operator is a matrix multiply, you do actually know the result of a sequence of operators has to be a sequence of matrix multiplies.

where $\hat{A}^2 = \hat{A}\hat{A}$. If we decompose $\hat{A}$ into eigenvalues and eigenvectors, then

$$\hat{A}^2 = \mathrm{V}\Lambda\mathrm{V}^{-1}\mathrm{V}\Lambda\mathrm{V}^{-1} = \mathrm{V}\Lambda^2\mathrm{V}^{-1} \tag{15}$$

Carrying out the sum, we're left with

$$\exp(\hat{A}) = \mathrm{V}\left(I + \Lambda + \Lambda^2/2! + \Lambda^3/3!\right)\mathrm{V}^{-1} = \mathrm{V}\exp(\Lambda)\mathrm{V}^{-1} \tag{16}$$

If I take $\hat{A}$ to be Hermitian, then

$$\exp(i\hat{A}) = \mathrm{V}\exp(i\Lambda)\mathrm{V}^{\dagger} \tag{17}$$

Since the eigenvalues of a Hermitian matrix are real, then the magnitude of the eigenvalues of $\hat{A}$ is one (this would *not* be true if there were an imaginary component to Λ), and we see that $\exp(i\hat{A})$ must be unitary. We can map every Hermitian operator to a unitary rotation matrix by taking $\exp(i\hat{A})$.