In these notes, we'll concentrate on drawing analogies to electromagnetic (EM) fields to hopefully develop a little intuition about why some of the things in quantum mechanics (QM) work the way they do. If we're careful, the analogies work very well - after all, if we take quantum electrodynamics in the suitable classical limit, we have to end up with Maxwell's equations. There's no way to stare at Maxwell's equations hard enough and say “gee, I guess the world must really be quantum, and QM must behave this way.” Instead, we'll say “here's how this particular thing works in QM, and here's how it works in EM. Now look - the math is exactly the same!”

1. Waves vs. States

Before we start, picture a violinist playing a note. How would you describe the motion of the string? Well, you've got two choices. You could specify the position and velocity at every point along the string, then use Newton's laws to work out what will happen as the string vibrates. Alternatively, since a string can only vibrate with (half-)integer periods, you could instead specify the amplitude and phase of each harmonic. The first way requires specifying two continuous functions, while the second way just requires specifying a handful of numbers. It doesn't matter which way you pick because both descriptions carry the full information about the motion of the string. QM is very similar. We could specify the probability amplitude of a particle's amplitude to be in a particular location, then evolve that function using the Schrodinger equation - this is the Schrodinger wave mechanics path. Equivalently, we could specify the amplitude for the particle to be in a range of states (equivalent to the different harmonics of the violin string), and watch how those states evolve - this is the Heisenberg matrix mechanics path. Although they were developed separately, Schrodinger eventually showed that the two paths are indeed mathematically equivalent. In this class we start with matrix mechanics, because the math is a lot easier. As we draw analogies with electromagnetic waves, you may have more intuition for the wave mechanics description, but keep in mind that what is true for one path must also be true for the other.

2. Quantum States through an Electromagnetic Lens

With that out of the way, let's go through some of the properties of EM fields, and see how the relate to QM. For starters, the energy density of an EM field is proportional to $E^2 + B^2$, the *square* of the electric/magnetic fields. Since photons are quantized in units of energy, that means the probability of finding a photon in a location has to be proportional to the field strengh squared. It is most decidedly *not* proportional to the field strength. In EM we generally try to work out the fields - it is the fields after all that add linearly together, and have phases. After that, we can square the fields to see where the energy is, which has to be where the photons are. The analog of the fields in QM is the wave function, or the *amplitude* to be in a state. We will put all our work into figuring out the amplitudes to be in different states. It is the amplitudes that add when we combine states *without measuring along the way.* Then once we've worked out all the amplitudes, we square the final amplitude to figure out where we're likely to find a particle/what value(s) of an observable we're likely to measure.

Now consider a EM plane wave propagating in the $+z$ direction in a vacuum. The most general way of writing the associated electric field for a wave at a single frequency $\nu = 2\pi\omega$ uniform in space is:

$$E(\vec{x}, t) = Re\left[(c_x\hat{x} + c_y\hat{y})e^{i\omega(t-z/c)}\right] \tag{1}$$

where both c_x and c_y can be complex. If I set $c_x = 1$ and $c_y = 0$ then we get

$$E(\vec{x}, t) = \cos\left(\omega(t - z/c)\right)\hat{x} \tag{2}$$

If you recall from the last set of lecture notes, this corresponds to a pure $+Q$ EM wave. Also note that if I made c_x complex, nothing would really change. I would pick up a phase shift in the wave, which is equivalent to shifting our zero point in time, or our reference point in space where $z = 0$. These aren't really fundamental, so if I sit and watch the wave go by, all I would say is "I have a $+Q$ EM wave." Similarly, I could set $c_x = 0$ and $c_y = 1$. Now I would have a wave with E purely in the y direction - I would call this a $-Q$ EM wave.

Life gets more interesting when I let both c_x and c_y be non-zero. If I have $c_x = c_y = \frac{1}{\sqrt{2}}$, then my E-field is

$$E(\vec{x}, t) = \frac{1}{\sqrt{2}}\left(\hat{x} + \hat{y}\right)\cos\left(\omega(t - z/c)\right) \tag{3}$$

We would call this a $+U$ wave, since the $E-$field is oscillating at 45° to the $x-$axis. Once again, an overall phase change doesn't affect the big picture physically - I would still have a $+U$ wave. If I flip the sign of c_y, I get the same thing but now instead of the $E-$field going at 45° to the *right* of our $+x$-axis, we have if oriented at 45° to the *left*. Once again, an overall phase doesn't change anything. I'd also like to point out that we get the same answer if we flip the sign of c_x, since that is equal to flipping the sign of c_y, then introducing an overall phase of -1. There's nothing special about flipping the sign of the $+Q/x-$component or the sign of the $-Q/y-$component. As long as we flip only one, we can go from $+U$ to $-U$. If we flip *both*, we have just put in an overall phase, and we stay in a $+U$ state.

Finally, let's have[1] $c_x = 1/\sqrt{2}$ and $c_y = i/\sqrt{2}$. Now our wave changes quite a lot. We get

$$E(\vec{x}, t) = \cos\left(\omega(t - x/c)\right)\hat{x} + \sin\left(\omega(t - x/c)\right)/\sqrt{2}\hat{y} \tag{4}$$

Instead of the electric field oscillating in a plane, now we have the electric field sweeping clockwise around in a circle as the wave propagates. If I flip the sign of c_y (or c_x - once again it's equivalent), now I still have a field going around a circle, but this time it's in a counter-clockwise direction. These circular modes are known as $+V$ and $-V$. Not surprisingly, once again an overall phase does not change the big picture - we'll still have the $E-$field going in circles.

Let's look and see where we're at. I can specify the coefficients c_x and c_y, and that in turn specifies the behavior of our EM wave. An *overall* phase change doesn't affect the qualitative behavior of our wave, but a *relative* phase change between c_x and c_y can change the picture

[1] We don't want to change the average energy in the field with these rotations. For the linearly polarized states, the E-field oscillates between -1 and 1, so the average *squared* field is 1/2. For this circular mode, the field strength doesn't change, so to keep the average squared field to be 1/2, we need to have a $1/\sqrt{2}$ in the field strength.

dramatically. To make the analogy to our spin-1/2 states even clearer, we'll relabel c_x and c_y as c_{+Q} and c_{-Q}. I get complete information about my wave by specifying the *relative* amplitudes and phases of c_{+Q} and c_{-Q}. Once I have these two numbers, I have everything, so I don't need to specify anything else. Similarly, I could have instead told you $c_{+U} = \left(c_{+Q} + c_{-Q}\right)/\sqrt{2}$ and $c_{-U} = \left(c_{+Q} - c_{-Q}\right)/\sqrt{2}$. In both cases, all I need to do is tell you the relative amplitudes and phases - if I tell you that in the $\pm U$ basis, you can work out what it is in the $\pm Q$ basis, since $c_{+Q} = \left(c_{+U} + C_{-U}\right)/\sqrt{2}$ and $c_{-Q} = \left(c_{+U} - c_{-U}\right)/\sqrt{2}$. I could even have told you the coefficients in the $\pm V$ basis where $c_{+V} = \left(c_{+Q} + ic_{-Q}\right)/\sqrt{2}$ and $c_{-V} = \left(c_{+Q} - ic_{-Q}\right)/\sqrt{2}$. As long as I tell you the relative amplitudes and phases of the two components in any of these basis sets, you have complete information about what the wave is doing. Sometimes a particular basis is easier to work in than others - if so, by all means pick that one! However, when you ask yourself any physical question, you'll get the same answer no matter which basis set you use.

I hope this analogy helps develop the intuition for spin-1/2 systems. We can let $+Q$ correspond to $|+Z\rangle$ and $-Q$ correspond to $|-Z\rangle$. I find it more intuitive to think of the amplitudes along the $+x$ and $+y$ directions as independent. For spin-1/2 systems, it's the $|\pm z\rangle$ (or equivalently $|\pm x\rangle$ or $|\pm y\rangle$) amplitudes that are independent. I can pick out a pure state in any direction just by specifying the relative amplitudes and phases of the $|\pm Z\rangle$ states. Again, there's nothing special about the $z-$direction. I could just as well have specified the $|\pm Y\rangle$ states instead, or any other basis I care to choose.

Finally, now that we know we can write down the amplitudes in a particular basis, we can see how to calculate probabilities. The *strength* of an $E-$field in a particular direction corresponds to finding the amplitude of a state along a basis vector. The *energy* goes like the field squared, so if I grab a linear polarizer, orient it some direction, and ask "what fraction of the energy makes it through?", the answer is going to be set by the field in that direction *squared*. For simplicity, let's say we have a linearly polarized EM wave once again propagating in the $+z$ direction, but at some random orientation in the $xy-$plane. We find the $x-$component by taking $\vec{E}\cdot\hat{x}$, and the $y-$component by taking $\vec{E}\cdot\hat{y}$. We can then express the field in by adding the unit vectors back in:

$$\vec{E} = \left(\vec{E}\cdot\hat{x}\right)\hat{x} + \left(\vec{E}\cdot\hat{y}\right)\hat{y} \tag{5}$$

Similarly, if I have a silver atom in some spin state $|S\rangle$, then the amplitude c_{+Z} to be in $|+Z\rangle$ is $\langle +Z|S\rangle$. If I send that atom through a Stern-Gerlach machine, the probability I measure its spin to be $|+Z\rangle$ is $c^*_{+Z}c_{+Z}$. Similarly, the amplitude to be in $|-Z\rangle$ is $c_{-Z} = \langle -Z|S\rangle$, and we can write our state in the same way we wrote the field in Equation 5:

$$|S\rangle = |+Z\rangle\langle +Z|S\rangle + |-Z\rangle\langle -Z|S\rangle \tag{6}$$

Yet again, there is nothing special about the $|\pm Z\rangle$ basis, so we could just as well have used $|\pm X\rangle$ or $|\pm Y\rangle$ instead of $|\pm Z\rangle$. We'd be writing down the state in a different basis, so the components would change, but the state itself doesn't care what basis you use.

3. Spin Basis States

I'm not a fan of how the book derives the $|\pm Y\rangle$ states in the $|\pm Z\rangle$ basis. If this isn't making sense to you, go back and look through the book Section 1.5. We'll accept that the $|+X\rangle$ state in the $Z-$basis is $\frac{1}{\sqrt{2}}|+Z\rangle + \frac{1}{\sqrt{2}}|-Z\rangle$. If we have a $|+Y\rangle$ state, we know the probability of measuring its angular moment along the $z-$axis will give us $+\hbar/2$ half the time, and $-\hbar/2$ half the time. Similarly, if we measure along the $x-$axis, we'll once again find a 50-50 split. We know the amplitude of a $|+Y\rangle$ state in the $|+Z\rangle$ direction has to be $\frac{1}{\sqrt{2}}$ times some phase, since we have a 50% change of measuring $+\hbar/2$ when we go measure the angular momentum in the $z-$direction. Similarly, we know the amplitude in the $|-Z\rangle$ must also be $\frac{1}{\sqrt{2}}$, again up to a phase factor. Now, any *overall* phase can't matter, so we can freely pick the phase that makes the $|+Z\rangle$ term be $\frac{1}{\sqrt{2}}$. That leaves us with

$$|+Y\rangle = \frac{1}{\sqrt{2}}|+Z\rangle + \frac{\alpha}{\sqrt{2}}|-Z\rangle \tag{7}$$

where α is a complex number with magnitude 1, but phase to be determined. We also know the amplitude squared to be in the $|+X\rangle$ state must be $\frac{1}{2}$, so we can write:

$$\langle +X|+Y\rangle^* \langle +X|+Y\rangle = \frac{1}{2} \tag{8}$$

Since we're working in the $|\pm Z\rangle$ basis, we can sub in our $z-$basis representations of $\langle +X|$ and $|+Y\rangle$ to get the amplitude:

$$\langle +X|+Y\rangle = \left(\langle +Z|/\sqrt{2} + \langle -Z|/\sqrt{2}\right)\left(|+Z\rangle/\sqrt{2} + \alpha|-Z\rangle/\sqrt{2}\right) \tag{9}$$

If we multiply these terms out, recalling that $\langle +Z|+Z\rangle = 1$ and $\langle +Z|-Z\rangle = 0$, then we have

$$\langle +X|+Y\rangle = \frac{1+\alpha}{2} \tag{10}$$

We know that thing squared had better be equal to $\frac{1}{2}$, and that the magnitude of α is 1. Let's see where that leaves us:

$$\langle +X|+Y\rangle^* \langle +X|+Y\rangle = \left(\frac{1+\alpha^*}{2}\right)\left(\frac{1+\alpha}{2}\right) = \frac{1+\alpha^*\alpha+\alpha+\alpha^*}{4} \tag{11}$$

Since the magnitude of α is one, we're left with

$$\frac{2+\alpha+\alpha^*}{4} = \frac{1}{2} \tag{12}$$

The only way that is true is if $\alpha + \alpha^* = 0$. For that to happen, the real part of α must be zero, and so α must be either $+i$ or $-i$ (since those are the only pure imaginary numbers with amplitude 1). In fact these two solutions correspond to the $|+Y\rangle$ and $|-Y\rangle$ states. The fact that we use right-handed coordinates (*i.e.* we have $\hat{x}\times\hat{y} = \hat{z}$ and not $-\hat{z}$) means that the $|+Y\rangle$ state corresponds to $\alpha = +i$. We can now write down the $|\pm Y\rangle$ states in the $|\pm Z\rangle$ basis:

$$|\pm Y\rangle = \frac{1}{\sqrt{2}}|+Z\rangle \pm \frac{i}{\sqrt{2}}|-Z\rangle \tag{13}$$

Incidentally, even this simple example shows why we absolutely have to use complex numbers in quantum mechanics. If we say that a state measured to have angular momentum along one axis has a 50% change of getting either $+\hbar/2$ or $-\hbar/2$ along either of the other two axes, and we accept that specifying a pair of amplitudes in the positive and negative directions is a complete description, then the only way that can work is if (at least) one of the sets of basis vectors has complex amplitudes. We just can't explain even very simple behaviors without complex amplitudes.