Supplementary Notes for lecture 30 for McGill Phys 357.

1. Alternate Derivation of the Probability Current

In general, if we want to talk about how much "stuff" is flowing past us per unit time, the answer is simply the density of the "stuff" times its velocty, or ρv. If we picture a bucket that stretches between x and $x + \delta x$, and ask how much stuff is flowing into the bucket, the net amount is the flow rate into the bucket at x minus the flow rate out of the bucket at $x + dx$. The net flow rate is $\rho v(x) - \rho v(x + \delta x)$. If we are neither creating nor destroying our stuff, then this net flow rate is the rate of change of stuff in our bucket, which is $\frac{\partial \rho}{\partial t}\delta x$. Expand the flow rate to first order and we have

$$\rho v(x) - \rho v(x + \delta x) = -\delta x \frac{\partial(\rho v)}{\partial x}$$

Set the two equal, cancel out a δx, and we have our standard conservation law

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v)}{\partial x}$$

It's more commonly written with both terms on the same side, which gives:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 \tag{1}$$

Equation 1 holds for any conserved quantity. In our case, we'll look at the probability for our conserved quantity, and the probability current j is the associated flow. One way to think about the probability current is if we start with some wave function Ψ which has *total* probability to have $x > x_0$ at sime time t. If we come back a short while dt later, the new total probability for $x > x_0$ is the old probability plus jdt, the amount of probability that has flown past x_0. The book derives the probability current by appealing to the conservation law, and using $\rho = \psi^*\psi$ for the density. In this note, we'll try to work out ρv directly. First, note that the probability current *must* be real, since ρ and v must be real. We're going to start with a janky guess, then because the current has to be real, we'll take the real part of that guess to get the correct answer.

If we were to integrate the probability current $\int_{-\infty}^{\infty} \rho v dx$ we get the expectation of the velocity, assuming that the integral of $\rho = \Psi^*\Psi = 1$. If someone asked you for the expectation of the velocity in Diract notation, it would be easy to answer: $\langle \Psi | \hat{p}/m | Psi \rangle$. If we want to ask "what's the current at point x_0?", it would be tempting to just take $\frac{1}{m}\Psi(x)^*\hat{p}\Psi(x)$ evaluated at $x = x_0$. We can put in the momentum operator to get

$$\frac{\hbar}{im}\Psi(x)^*\frac{\partial}{\partial x}\Psi \tag{2}$$

That answer is tempting, but wrong. If we picture this in the discretized case, we're using just one row of the matrix for the momentum operator since we're evaluating at a single point. Taken as a whole, the momentum operator is of course Hermitian, but one row of a Hermitian matrix is not Hermitian. In general, if we look at just a single point, we'll find Equation 2 is complex. The final integral of the probability current is real, so the contribution at each point has to be just the real part of Equation 2. I can get the real part of a number b by taking $(b+b^*)/2$, so we'll do that by using the dagger of Equation 2. As usual, we flip the sign of all i's, and reverse the order of any multiplies/operations. That gives

$$\frac{\hbar}{2im}\Psi^*(x)\frac{\partial}{\partial x}\Psi(x) - \frac{\hbar}{2im}\Psi(x)\frac{\partial}{\partial x}\Psi^*(x)$$

We can write this more compactly as

$$\frac{\hbar}{2im}\left(\Psi^*\frac{\partial}{\partial x}\Psi - \Psi\frac{\partial}{\partial x}\Psi^*\right) \tag{3}$$

We can look at Equation 3 for some simple cases. First, if Ψ is strictly real, then the current is zero because $\Psi^* = \Psi$, and the terms inside the parenthesis are identical. We already knew the net momentum for a real function is zero, but now we see that there's no net flow of probability anywhere. This makes sense because if we just cut a piece of the real wave function out, and renormalized to make another valid wave function, it would also have zero net momentum. We could think about our original real wave function as being made up of a bunch of little pieces, each of which has zero net momentum. The probability current tells us this is correct.

We can also look at the current for a piece of a wave function that looks at least locally like a complex exponential $\Psi(x) = \alpha\exp(ikx)$. The density is $\alpha^*\alpha$, and the velocty is $p/m = \hbar k/m$, so we expect the current to be $\frac{\hbar\alpha^*\alpha}{2m}$. If we plug Ψ into Equation 3, we get

$$\frac{\hbar}{2im}\left(\alpha^*\exp(-ikx)ik\alpha\exp(ikx) - \alpha\exp(ikx)(-ik)\alpha^*\exp(-ikx)\right) \tag{4}$$

$$= \frac{\hbar}{2im}\left(ik\alpha^*\alpha + ik\alpha^*\alpha\right) = \frac{\hbar k}{m} \tag{5}$$

exactly what we expected.