PHYS 357 Pset 6. Due 11:59 PM Thursday Oct. 31

1. Townsend 4.3

2. Townsend 4.11

3. Let a spin-1/2 electron in the $|1/2, 1/2\rangle_x$ state be placed in a time-varying magnetic field in the $z-$direction: $B = B(t)\hat{z}$. Solve for the spin "direction" as a function of time. By direction I mean the direction in which we always measure $+\hbar/2$ (equivalently, that are θ, ϕ if we write the state as $|+n\rangle$). You may leave your answer in the form of an integral, and assume that the constants work out so that the Hamiltonian can be written $\omega(t)S_z$ where $\omega(t)$ is just $B(t)$ times constants.

   Now let the applied field be such that $H(t) = (\omega_0 + \omega_1 \cos(\omega_f t))S_z$. Work out the solution analytically, and compare to the numerical code I have helpfully worked out for you in magres_z.py in the codes directory on github.

4. Consider a 2x2 matrix
$$E_0 \begin{bmatrix} 1 & \epsilon \\ \epsilon & 1 \end{bmatrix}$$
   Work out the eigenvalues and eigenvectors of this matrix.

   ———————————————————————————————— By inspection, it's easy to see that $(1, 1)$ is an eigenvector, because the sum across rows is constant for this matrix. If we multiply $(1, 1)$ by the matrix on the left, we get $E_0(1+\epsilon, 1+\epsilon)$, so the eigenvalue is $E_0(1+\epsilon)$. Since the matrix is Hermitian, the other eigenvector must be orthogonal, so $(1, -1)$ is also an eigenvector. Multiply and we get $E_0(1 - \epsilon, -1 + \epsilon) = E_0(1 - \epsilon)(1, -1)$, so the eigenvalue is $E_0(1 - \epsilon)$.

5. Take the previous matrix to be the Hamiltonian of a 2-state system. If we start in the (1,0) state, what is the probability as a function of time to be in the (1,0)/(0,1) states? Ammonia is seen to emit radiation at 24 GHz, when it transitions between the higher and lower-energy eigenstates, so $2E_0\epsilon = h\nu$ (note that we're using $h = 2\pi\hbar$ here). Given this,

what is the length of time it takes for an ammonia molecule that starts in the in the (1,0) state to transition to the (0,1) state?

Note - in the interests of time and clarity, I'm going to skip the book's section on ammonia for now, and revisit it after we've done more work with continuous wave functions in chapters 6 and 7. Once we're there, we'll be able to work out why the Hamiltonian looks the way it does instead of it being plucked from thin air as it seems here. Even if we don't yet know how to work out the Hamiltonian, undestanding how to work with a split-energy state is still useful for understanding the energy-time uncertainty relationship.

———————————————————————————————— If we start in $(1,0)$ then the wave function at $t = 0$ is $\frac{1}{2}(1,1) + \frac{1}{2}(1,-1)$. At time t, the wave function is

$$\frac{1}{2}(1,1)\exp(iE_0(1+\epsilon)t/\hbar) + \frac{1}{2}(1,-1)\exp(iE_0(1-\epsilon)t/hbar)$$

We can get rid of an overall phase factor $\exp(iE_0t/hbar)$, leaving us with

$$\frac{1}{2}(1,1)\exp(iE_0\epsilon t/hbar) + \frac{1}{2}(1,-1)\exp(-iE_0\epsilon t/hbar)$$

We can combine the $(1,0)$ and $(0,1)$ bits and note that the sum/difference of the exponentials gives us sine/cosine terms (which also pull in the factors of $1/2$ out front) to get:

$$(1,0)\cos(E_0\epsilon t/\hbar) + (0,1)i\sin(E_0\epsilon t/\hbar)$$

This is the amplitude, so the probabilities are these things squared, which gives:

$$P(1,0) = \cos^2(E_0\epsilon t/\hbar) P(0,1) = \sin^2(E_0\epsilon t/hbar) \tag{1}$$

In particular, we'll transition from $(1,0)$ to $(0,1)$ when $E_0\epsilon t/\hbar = \pi/2$, or

$$t = \frac{\pi\hbar}{2E_0\epsilon}$$

We know from the previous problem that $2E_0\epsilon$ is the energy difference between the two states - if we call that dE, then the time to transition is $t = \frac{\pi\hbar}{dE}$.