PHYS 357 Pset 4. Due 11:59 PM Thursday Oct. 3

1. Townsend 3.1

A) A(B+C)=AB+AC, so associativity of commutators follows directly.

B) [A,BC]=ABC-BCA. B[A,C]+[A,B]C= BAC-BCA + ABC-BAC. The two BAC terms cancel, and so you're left with ABC-BCA, which is exactly [A,BC].

C) We can either repeat as above, or note that [AB,C]=-[C,AB], and then use the result from part B) to say that equals -(A[C,B]+[C,A]B)=A[B,C]+[A,C]B.

2. Townsend 3.2. If you choose, you may just verify that the states shown are eigenvectors rather than solve the full eigenvector problem by hand.

3. Townsend 3.7

As suggested, we have

$$(\langle\alpha| + \lambda^* \langle\beta|)(|\alpha\rangle + \lambda |\beta\rangle) \geq 0 \tag{1}$$

Expanding, we have

$$\langle\alpha|\,\alpha\rangle + \lambda^* \langle\beta|\,\alpha\rangle + \lambda \langle\alpha|\,\beta\rangle + \lambda^*\lambda \langle\beta|\,\beta\rangle \geq 0 \tag{2}$$

Let $\lambda = \lambda_r + i\lambda_i$, and to shorten the t y ping, we'll let $\langle\alpha|\,\alpha\rangle \equiv \alpha\alpha$ and $\langle\alpha|\,\beta\rangle \equiv \alpha\beta_r + i\alpha\beta_i$. Expanding out, we have (noting that imaginary terms cancel):

$$\alpha\alpha + \lambda_r\alpha\beta_r - \lambda_i\alpha\beta_i + \lambda_r\alpha\beta_r - \lambda_i\alpha\beta_i + (\lambda_r^2 + \lambda_i^2)\beta\beta \tag{3}$$

Differentiate w.r.t the different parts of λ and we have:

$$2\alpha\beta_r + 2\lambda_r\beta\beta = 0 \tag{4}$$

$$-2\alpha\beta_i + 2\lambda_i\beta\beta = 0 \tag{5}$$

and we have $\lambda_r = -\alpha\beta_r/\beta\beta$ and $\lambda_i = \alpha\beta_i\beta\beta$ or $\lambda = -\alpha\beta^*/\beta\beta$. Plug that, and we have

$$\alpha\alpha - \alpha\beta^*\alpha\beta/\beta\beta - \alpha\beta\alpha\beta^*/\beta\beta + \alpha\beta(\alpha\beta^*)\beta\beta \geq 0 \tag{6}$$

$$\alpha\alpha \geq (\alpha\beta)^*\alpha\beta/\beta\beta \tag{7}$$

$$\alpha\alpha\beta\beta \geq (\alpha\beta)^*\alpha\beta \tag{8}$$

and we're done.

4. For a 3-state spin-1 system, we know the raising/lowering operators need to look like

$$J_+ \propto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and J_- is the conjugate-transpose. We know that for J_z, $J_\pm = J_x \pm iJ_y$. Use these forms to solve for J_x and J_y in terms of J_+ and J_-. For a spin-1 system, the eigenvalues of $J_{x,y,z}$ must be $\hbar(1, 0, -1)$. Use this fact to find the coefficient of proportionality for J_x, J_y and write the properly weighted forms of J_x and J_y. If all has gone well, they should agree with Equation 3.28 in Townsend.

Since

$$J_+ \propto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{9}$$

and $J_- = J_+^\dagger$, we have $J_x \propto (J_+ + J_-)$ (we can let the $\propto$ eat the 2 since they'll still be proportional to each other), and from that we have

$$J_x \propto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{10}$$

We can ask numpy to get the eigenvalues for us (or you can do it by hand, we don't judge), which are $(\sqrt{2}, 0, -\sqrt{2})$. We know for a spin-1 particle they should be $(\hbar, 0, -\hbar)$, so we need to rescale J_x by $\hbar/\sqrt{2}$. That leaves us with

$$J_x = \hbar/\sqrt{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{11}$$

Repeat the process with $J_y = (J_+ - J_-)/2i$ to get

$$J_y = \hbar/\sqrt{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \tag{12}$$

If we take $J_x + iJ_y$, we get

$$J_+ = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \frac{i\hbar}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \tag{13}$$

The usual convention is to pull the 2 inside the matrix out front, leaving:

$$J_+ = \sqrt{2}\hbar\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

One can go through the same proces to work out J_- from J_x and J_y, or just note that $J_- = J_+^\dagger$, but since the matrices are real, they are just transposes of each other.

5. Show that the commutation relations we expect for angular momentum hold for the spin-1 basis you've just worked out. You may do this on a computer if you choose.

   see python code for this question and the next

6. What are the eigenstates of J_x and J_y in the J_z basis? What are the raising and lowering operators? Show that the raising and lowering operators for J_x behave as expected on the eigenstates of J_x. Do the same for J_y. Once again, you may do this on a computer.