Supplementary Notes for lecture 22 for McGill Phys 357.

1. Some Properties of Fourier Transforms

The Fourier transform of a Dirac delta function $\delta(x)$ is

$$\int \delta(x)\exp(-ikx)dx = \exp(-ik0) = 1$$

so, equal to 1 everywhere. If we put the delta function not at zero, though, and instead take the FT of $\delta(x - x_0)$, we get:

$$\int \delta(x - x_0)\exp(-ikx)dx = \exp(-ikx_0)$$

This is a plane wave in k, with wave number x_0. The amplitude of this wave is still 1 everywhere, but now we have a phase ramp. So, if we have a particle localized to an infinitessimally small region, in momentum space, we have contributions from all momenta, but the relative phases of those momentum eigenstates are set by the x position of our particle. Similarly, if we are in a momentum eigenstate, we have an equal probability of finding the particle anywhere in space, but again the relative phases are set by the momentum. More momentum means fast phase wrapping in space.

We won't prove it here (see the dfts notes in the notes directory on github for a proof of the discrete version), but another hugely important use of Fourier transforms comes via the *convolution theorem*. A convolution is defined to be

$$h(x) \equiv \int f(t)g(x - t)dt \equiv f \circledast g$$

You can think of the convolution as taking function f and smearing every point in it out by g. The reverse is true as well, since $f \circledast g = g \circledast f$. The convolution theorem tells us that a convolution in real space is a multiplication in Fourier space, and vice-versa, so if $F = FT(f)$ *etc.*, then we have:

$$H = FG$$

We can now combine these two results to work out the shift theorem. That is, if I have $f(x)$ and the corresponding $F(k)$, what is the Fourier transform of $f(x - x_0)$? Well, note that I can write that as a convolution with a δ-function:

$$\int f(x)\delta(x_0 - x)dx = \int f(x' - x_0)\delta(x')dx' = f(x' - x_0)$$

where $x' = x - x_0$. Since the label we call our variable doesn't matter, we can drop the prime on x', and see that the convolution gives us $f(x - x_0)$. Convolving by a $\delta-$function shifted our function over by the argument of the $\delta-$function.

In Fourier space then, we know that the FT of $f(x - x_0$ which we'll call $H(k)$ is then the product of the Fourier transforms of $f(x)$ and the FT of the $\delta-$function, which is a phase ramp. So that leaves us with

$$H(k) = F(k)\exp(-ikx)$$

So we can shift a function in one space by multiplying it by a phase ramp in the other space.

We could also have proved this directly from the definition of the FT, since

$$\int f(x - x_0) \exp(-ikx) dx = \int f(x') \exp(-ik(x' + x_0)) dx' = \exp(-ikx_0) \int f(x') \exp(-ikx') dx'$$

for $x' = x - x_0$. Doing the shift via the convolution theorem, though, opens up a powerful set of tools when using Fourier transforms.

Finally, we can now see how we could start a particle with a non-zero velocity. Let's say we have a particle with some starting wave function (say a wave packet of some sort). We can get the momentum-space representation of that by taking the Fourier transform. If we want to give the particle some initial momentum, we can do that by centering the momentum-space representation of the wave function by some momentum value $k_0 = p_0/\hbar$. Since we want to *shift* the momentum-space wave function, we can think of it as convolving the initial wave function with a $\delta-$function $\delta(k - k_0)$. Since we convolved in Fourier space, we *multiplied* in real space, so

$$h(x) = f(x) \exp(-ik_0 x) = f(x) \exp(-ip_0 x/\hbar)$$

where $h(x)$ is our wave function with momentum shifted by p_0. If we start with say a Gaussian wave function, we can give it an initial momentum just by multiplying by $\exp(-ip_0 x/\hbar)$.

The convolution theorem is extremely powerful, so I'd suggest you try to think about it in different contexts. One example - the FT of a boxcar is a sinc function $\sin(x)/x$. So if I multiply in one space by a boxcar, I've convolved in the dual space by a sinc. For single-slit diffration, if I send in a pure plane wave, then what comes out is the multiplication of a plane wave by the boxcar that describes the slit. In momentum space, we've convolved the initial plane wave with a sinc, so our new momentum-space wave function is a sinc centered on the initial momentum, with width set by one over the slit width.