

Solving Laplace's Equation

- Laplace's equation, $\nabla^2 V = -\rho$, is fundamental equation of electrostatics. Also steady-state heat flow, soap bubbles, fluid dynamics, gravity...
- How can we solve this when boundary conditions are given?
- $\nabla^2 V = d^2V/dx^2 + d^2V/dy^2 + \dots$ $dV/dx = (f(x+dx) - f(x))/dx$, $d(dV/dx) = (f'(x) - f'(x-dx))/dx = (f(x+dx) - 2f(x) + f(x-dx))/dx^2$. In 2D, $\nabla^2 f = f(r+dx) + f(r-dx) + f(r+dy) - f(r-dy) - 4f(r)$.
- Rewrite: $f(r) = (\rho + \sum f_{\text{neighbors}}) / 2n_{\text{dim}}$.

Method of Relaxation:

- $f(r) = (\rho + \sum f_{\text{neighbors}}) / 2n_{\text{dim}}$ suggests one possible way to solve.
- Take current $f(r)$, and replace by suitably averaged neighbors (plus ρ if charge is non-zero).
- This works, and is called method of relaxation (since we relax to solution).
- It does not, however, work very well. Convergence is slow.
- Another way to write: $f(\mathbf{x}, t+dt) = f(\mathbf{x}, t) + \alpha(f_{\text{neighb}}(\mathbf{x}, t) - f(\mathbf{x}, t))$.
- Relaxation iterations can be thought of as time evolution of this setup. We can increase α to speed convergence. Converges faster, but still slow. Need $0 < \alpha < 2$ (why?)

Conjugate Gradient

- We can also write as matrix equation
- In interior $[-4 \ 1 \ 1 \ 1 \ 1][f_{00} \ f_{10} \ f_{-10} \ f_{01} \ f_{0-1}] = \rho_{00}$
- If potential fixed on boundaries, we can just keep those function values fixed.
- This lets us use matrix solvers to solve $Ax=b$
- Standard tool is (preconditioned) conjugate gradient, works most naturally for positive-definite A . Happily, for Laplace/Poisson, A is already positive-definite.
- Conjugate gradient can be plugged in, converges much faster than standard relaxation - in particular gets to exact solution in finite # of iterations.
- Boundary conditions can be put on right, since we aren't solving for them.

CG 2

- In general, some cells in domain have fixed values. Rather than list all these cells, we'll use a mask. $\text{Mask}=1$ where we define boundary, 0 otherwise.
- Rather than keeping track, we can take $x \rightarrow (1-\text{mask})x$ to zero out boundaries. Then Ax doesn't need to know about mask.
- RHS we can get from using $\text{mask} * \text{boundary conditions}$.
- With this, we can do matrix operations by simple sums over domain.
- Let's write a few examples...

Multigrid

- Relaxation is fast on small scales, slow on large
- If grid is n cells across, takes n timesteps to propagate information.
- Trick: solve low-resolution problem. Gets large scales right quickly
- Gradually increase resolution - then only need a few steps to solve for corrections introduced by higher-res
- This can be very fast - low-res parts are extremely fast, so can solve full problem for the price of a few full-res iterations.

Fourier techniques

- For Laplace, we know Green's function is $1/r$.
- If we know charge, we could work out potential everywhere via convolution with $1/r$.
- We can again use CG to solve for ρ on the boundaries that gives V on the boundaries.
- We get V everywhere else through convolution.
-