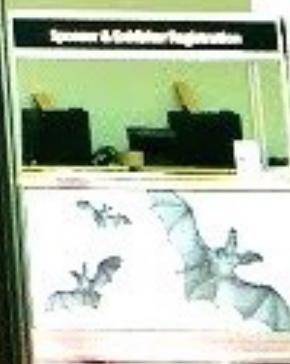


# Computational Physics

## Fourier Transforms

“Always code  
as if the guy  
who ends up  
maintaining  
your code will  
be a violent  
psychopath  
who knows  
where you live”

-Martin Golding



# Fourier Transforms

- Functions can be represented in many different ways
- We normally use “real” space -  $f(x)$
- Generally, arbitrarily many transforms exist to represent functions in different spaces -  $F(y) = Af(x)$  for some matrix  $A$  and some new variable  $y$ .  
Iff  $A$  is invertible,  $f(x) = A^{-1}F(y)$
- One important basis nature has picked out is complex exponentials/sines and cosines. Fundamental across physics, particularly quantum mechanics.

# Fundamental Definition

- $F(k) = \int f(x) * \exp(-2\pi i k x) dx$  (where  $k = l/\omega$ )
- Integral gets rid of  $x$ , replaces with  $k$ . New function has amplitude and phase as a function of  $k$ .
- Quantum mechanics - de Broglie says  $p = \hbar k$ . So, Fourier transform position to get momentum.
- Fourier transform electric field  $E(t)$  to get frequency spectrum.
- Fourier transform to get fast correlations, convolutions, many other things.

# DFT (Discrete FT)

- Computers don't do continuous. Not enough RAM for starters...
- Function exists over finite range in  $x$  at finite number of points.
- If input function has  $n$  points, output can only have  $n$  k's.
- Gives rise to discrete Fourier Transform (DFT)
- $F(k) = \sum f(x) \exp(-2\pi i k x / N)$  for  $N$  points and  $0 \leq k < N$
- What would DFT of  $f(0)=1$ , otherwise  $f(x)=0$  look like?
- What would DFT of  $f(x)=1$  look like?
- DFTs have subtle behaviours not seen in continuous, infinite FTs.

# Inverse

- One way to think about DFT is as a matrix multiply.
- $F(k) = Af$ ,  $A_{mn} = \exp(-2\pi i mn/N)$
- But look:  $A_{mn} = A_{nm}$ , so matrix is symmetric.
- Also, columns are orthogonal under conjugation:  
 $\sum \exp(-2\pi i k x) \exp(2\pi i k' x) = \sum \exp(2\pi i (k' - k)x)$ . N if  $k' = k$ , otherwise 0.
- So,  $A^{-1} = 1/N * \text{conj}(A)$ . IFT =  $1/N \sum F(k) \exp(2\pi i k x)$ .
- Get back to where we started by just doing another DFT with a sign flip, then divide by # of data points.
- Alternative: divide by  $\sqrt{N}$  in both DFT and IFT, (not standard computationally)

# Numpy Complex

```
import numpy
def exp_prod(m,n,N):
    #define imaginary unity
    J=numpy.complex(0,1)
    #now rest of code is just like for real numbers
    x=numpy.arange(0.0,N)*2*J*numpy.pi/N
    return numpy.sum(numpy.exp(-1*x*m)*numpy.exp(x*n))
if __name__=="__main__":
    print exp_prod(0,0,8)
    print exp_prod(2,4,8)
    print exp_prod(3,3,8)
    print exp_prod(0,7,8)
```

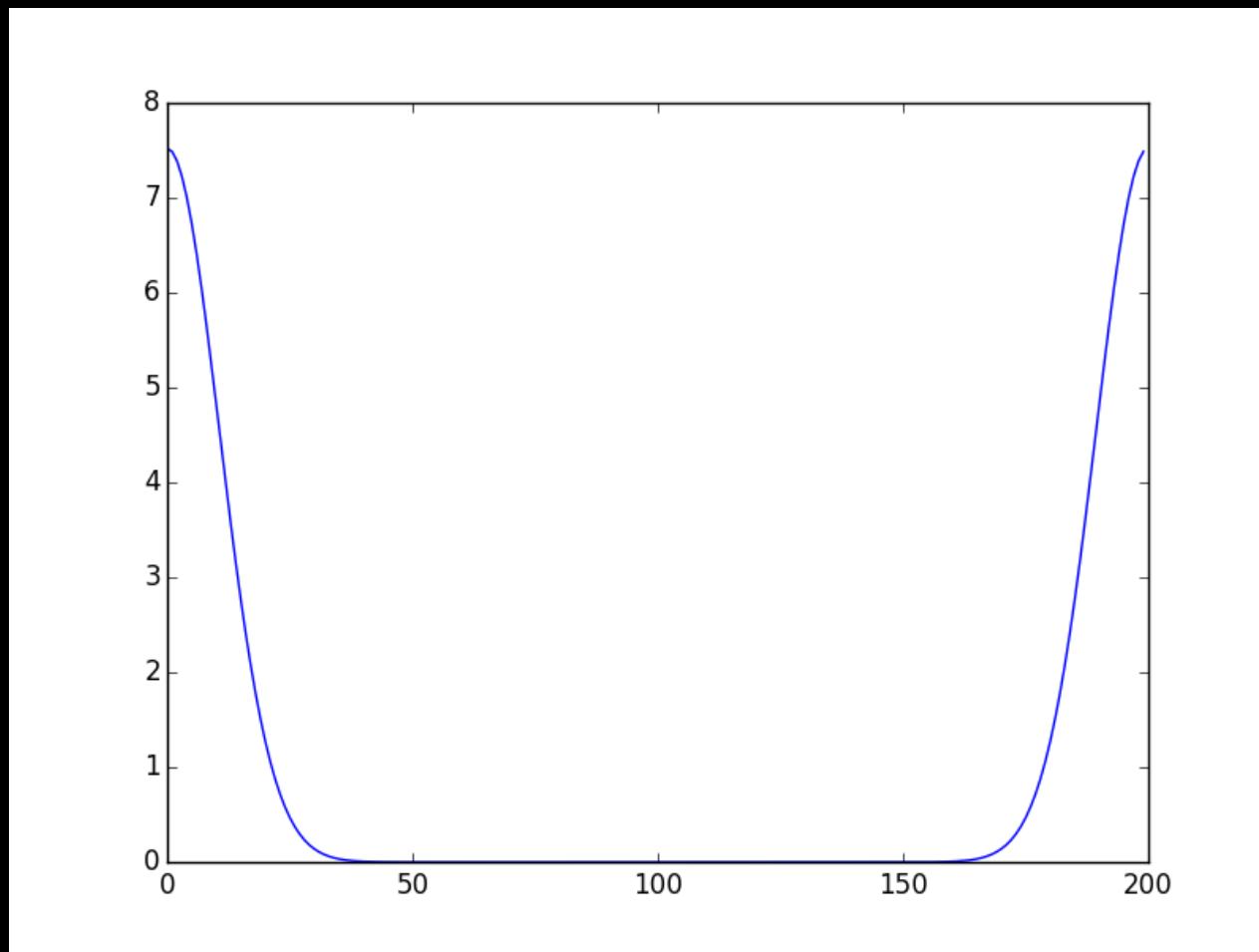
```
Jonathans-MacBook-Pro:lecture4 sievers$ python dft_columns.py
(8+0j)
(-4.28626379702e-16+4.4408920985e-16j)
(8+0j)
(3.44169137634e-15-1.11022302463e-15j)
Jonathans-MacBook-Pro:lecture4 sievers$
```

- Let's check orthogonality, need complex #'s.
- `numpy.complex(re,im)` will make a complex #
- `numpy` functions usually defined for complex #'s.

# DFTs with Numpy

- Numpy has many Fourier Transform operations
- (for reasons to be seen) they are called *Fast Fourier Transforms* - FFT is one way of implementing DFTs.
- FFT's live in a submodule of numpy called FFT
- `xft=numpy.fft.fft(x)` takes DFT
- `x=numpy.fft.ifft(x)` takes inverse DFT
- Numpy normalizes such that `f==fft(ifft(f))==ifft(fft(f))`

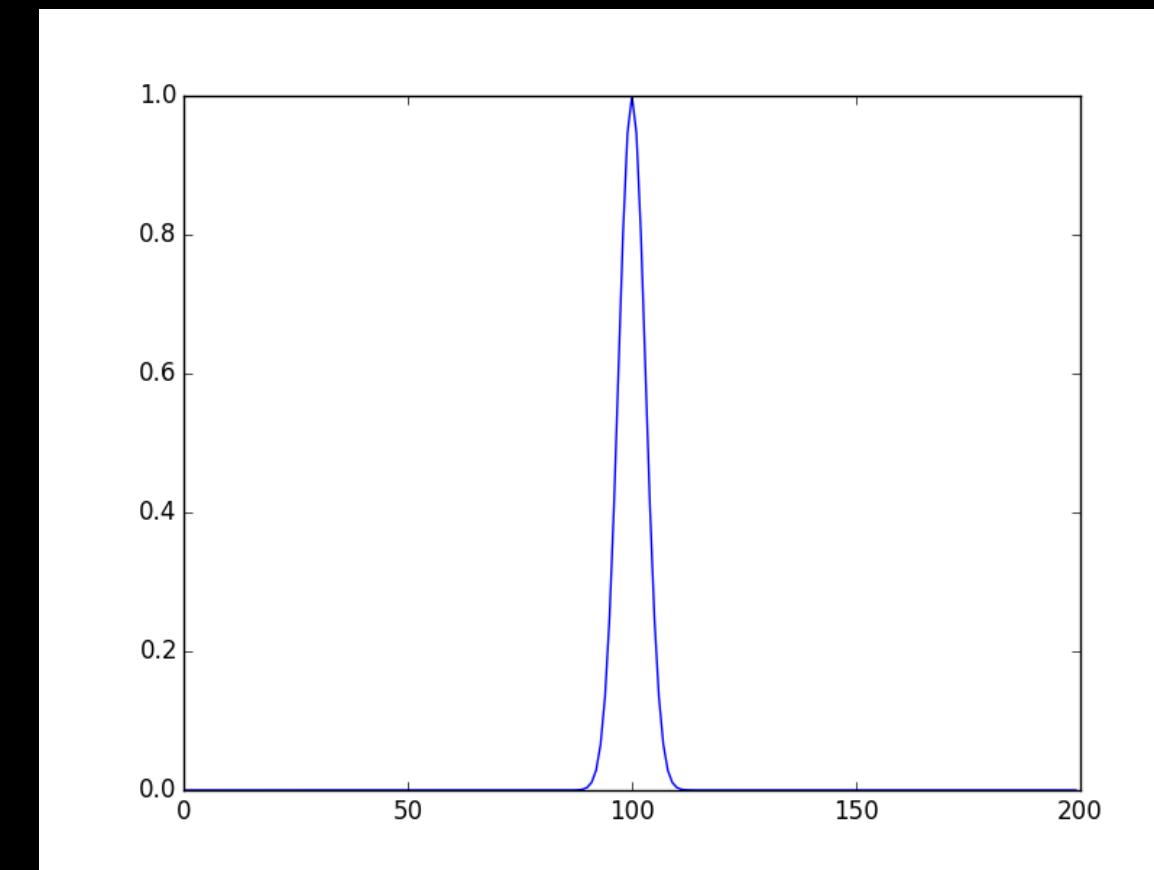
# DFT in Action



- Right: input Gaussian
- Top: DFT of the Gaussian

```
import numpy
from matplotlib import pyplot as plt

x=numpy.arange(-10,10,0.1)
y=numpy.exp(-0.5*x**2/(0.3**2))
yft=numpy.fft.fft(y)
plt.plot(numpy.abs(yft))
plt.savefig('gauss_dft')
plt.show()
```



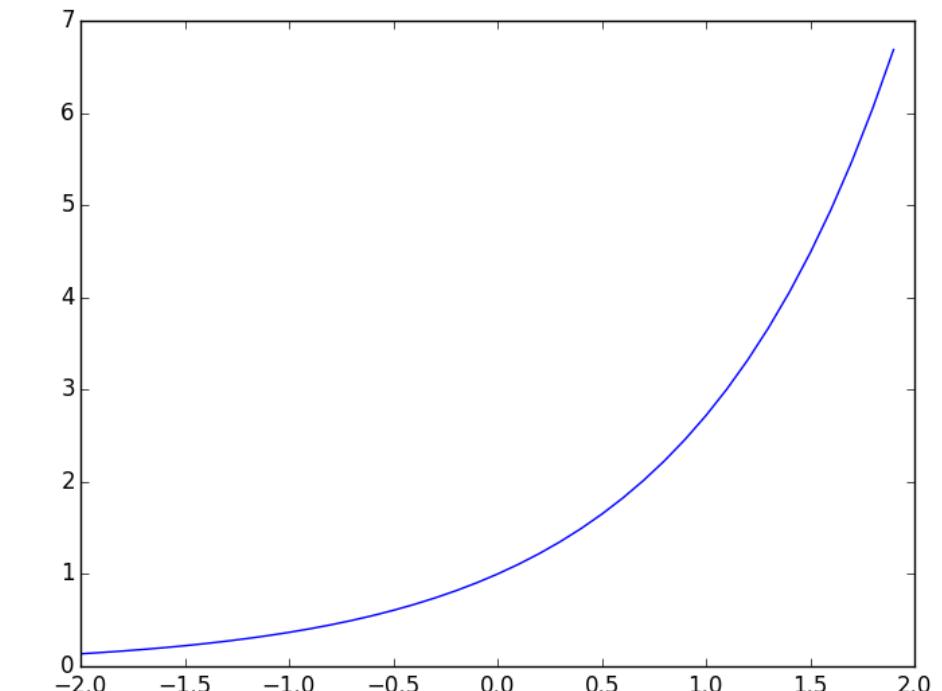
# Periodicity

- $f(x) = \sum F(k) \exp(2\pi i k x / N)$
- What is  $f(x+N)$ ?  $\sum F(k) \exp(2\pi i k (x+N) / N)$
- $= \sum F(k) \exp(2\pi i k) \exp(2\pi i k x / N).$
- $\exp(2\pi i k) = 1$  for integer  $k$ , so  $f(x+N) = f(x)$
- DFT's are periodic - they just repeat themselves ad infinitum.

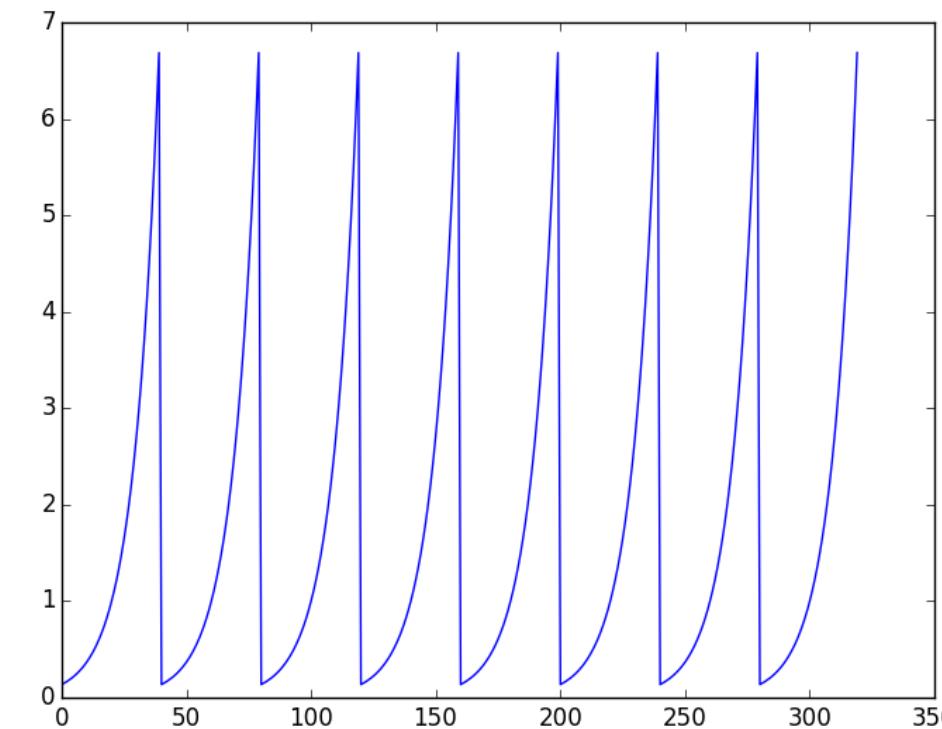
# Periodicity

```
import numpy
from matplotlib import pyplot as plt

x=numpy.arange(-2,2,0.1)
y=numpy.exp(x)
plt.plot(x,y)
plt.savefig('fft_exp')
plt.show()
yy=numpy.concatenate((y,y))
yy=numpy.concatenate((yy,yy))
yy=numpy.concatenate((yy,yy))
plt.plot(yy)
plt.savefig('fft_exp_repeating')
plt.show()
```



- You may think you're taking top transform. You're not - you're taking the bottom one.
- In particular, jumps from right edge to left will strongly affect DFT



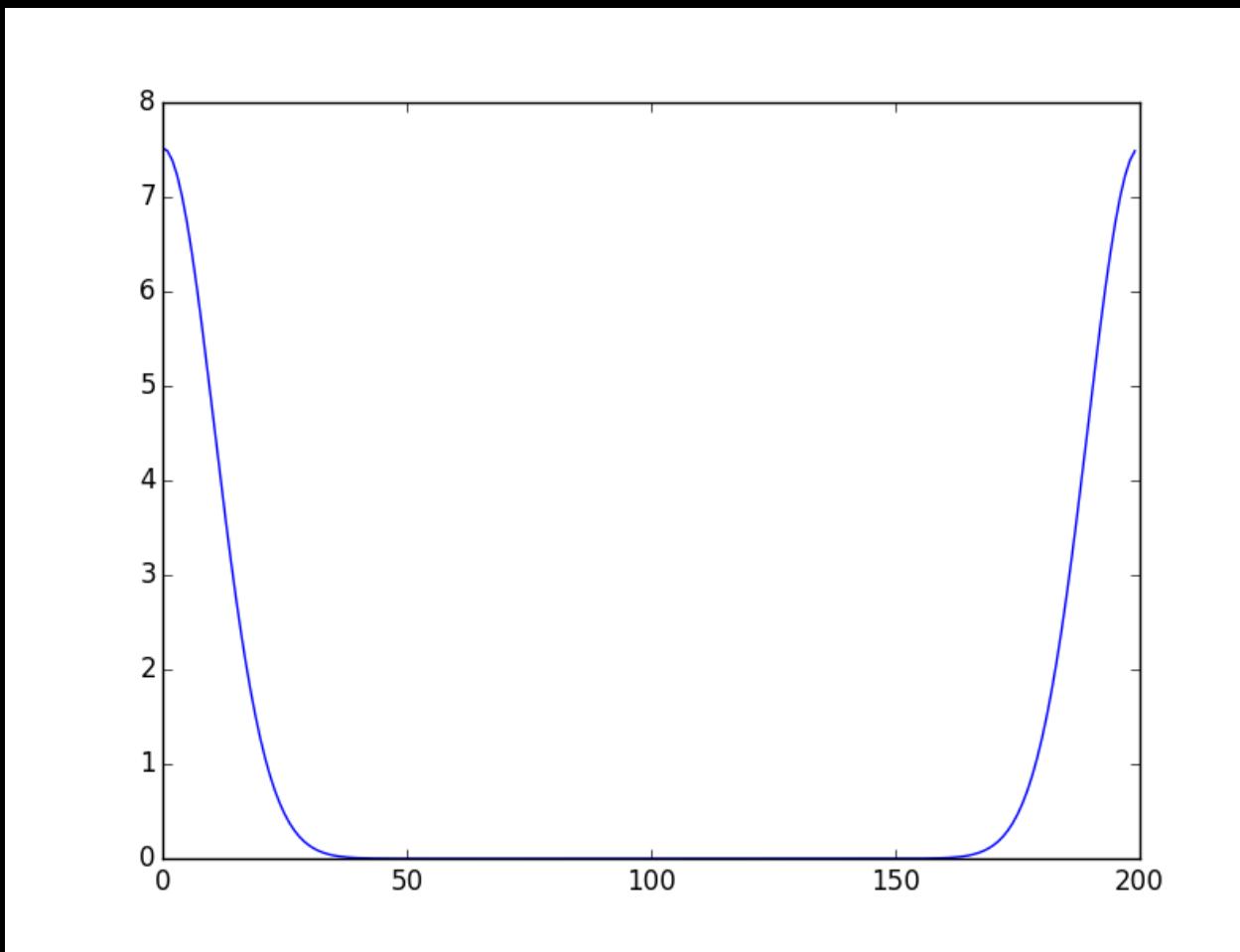
# Aliasing

- $f(x) = \sum F(k) \exp(2\pi i kx/N)$
- What if I had higher frequency,  $k > N$ ? let  $k^* = k - N$  (i.e.  $k^*$  low freq.)
- $f(x) = \sum F(k) \exp(2\pi i (k^* + N)x/N) = \sum F(k) \exp(2\pi i x) \exp(2\pi i k^* x/N)$
- for  $x$  integer, middle term goes away:  $\sum F(k^* + N) \exp(2\pi i k^* x/N)$
- High frequencies behave exactly like low frequencies - power has been *aliased* into main frequencies of DFT.
- Always keep this in mind! Make sure samples are fine enough to prevent aliasing.

# Negative Frequencies

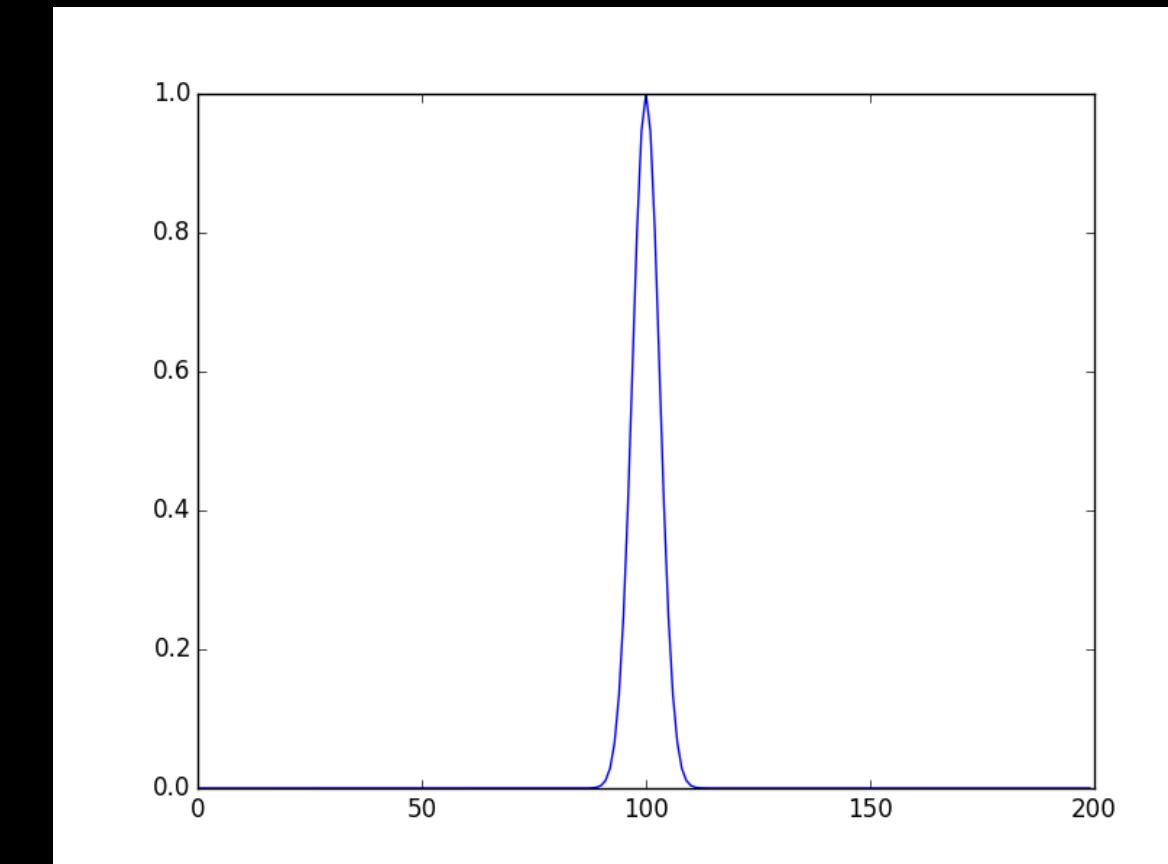
- All frequencies that are  $N$  apart behave identically
- DFT has frequencies up to  $(N-1)$ .
- Frequency  $(N-1)$  equivalent to frequency  $(-1)$ . You will do better to think of DFT as giving frequencies  $(-N/2, N/2)$  than frequencies  $(0, N-1)$
- *Sampling theorem:* if function is band-limited - highest frequency is  $v$  - then I get full information if I sample twice per frequency,  $dt=1/(2v)$ . Factor of 2 comes from aliasing.

# DFT in Action, Redux



```
import numpy
from matplotlib import pyplot as plt

x=numpy.arange(-10,10,0.1)
y=numpy.exp(-0.5*x**2/(0.3**2))
yft=numpy.fft.fft(y)
plt.plot(numpy.abs(yft))
plt.savefig('gauss_dft')
plt.show()
```



- FFT makes more sense now - negative frequencies have been aliased to high frequency.

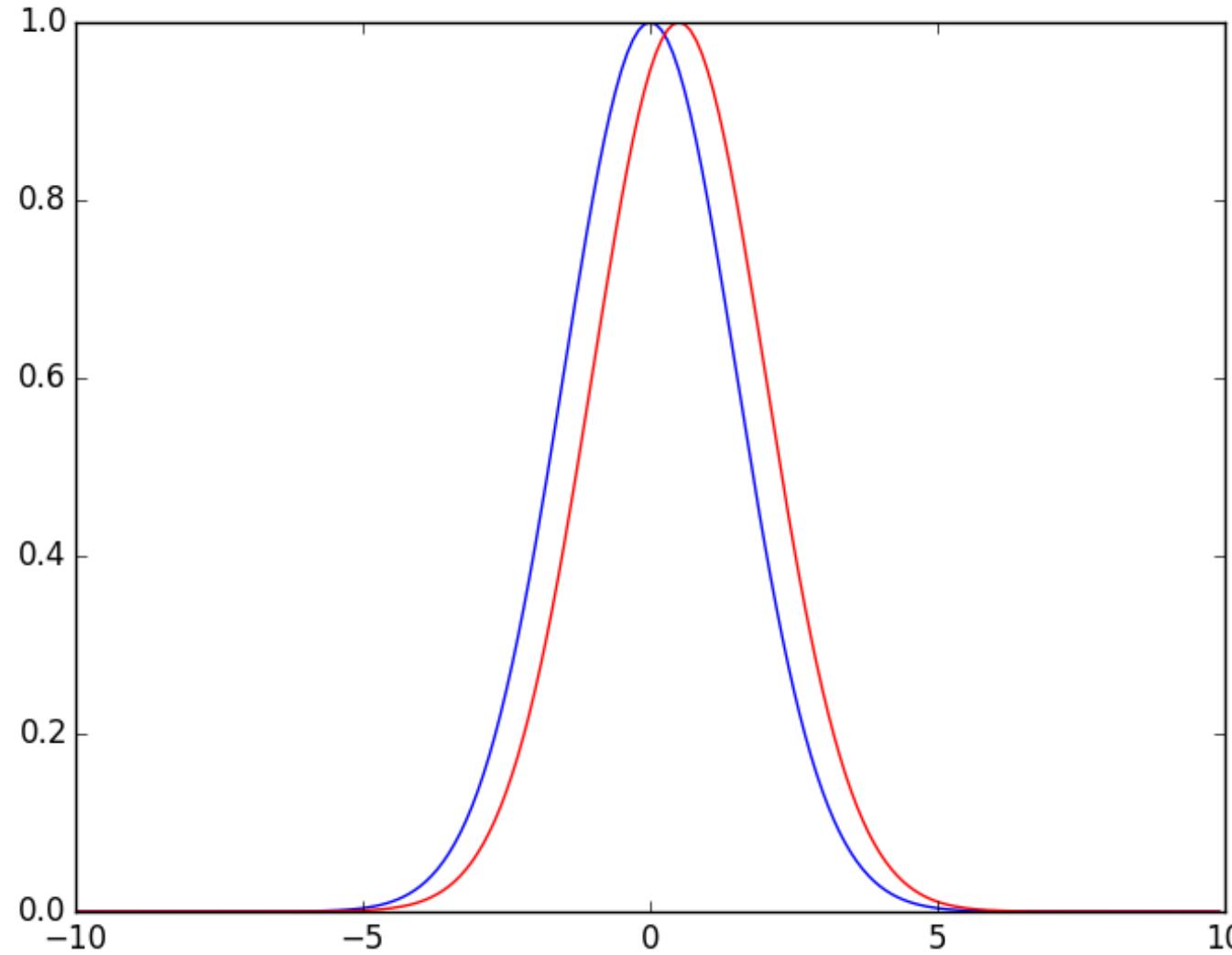
# Flipping

- What is DFT of  $f(-x)$ ?
- $\sum f(-x) \exp(-2\pi i k x / N)$ ,  $x^* = -x$ ,  $\sum f(x^*) \exp(-2\pi i k (-x^*) / N)$
- $DFT(f(-x)) = \sum f(x) \exp(2\pi i k x / N) = \text{conj}(F(k))$

# Shifting

- What is  $\text{FFT}(x+dx)$ ?  $\sum f(x+dx) \exp(-2\pi i k x / N)$ .
- $x^* = x + dx$ :  $F(k) = \sum f(x^*) \exp(-2\pi i k (x^* - dx) / N)$
- $F(k) = \exp(2\pi i k dx / N) \sum f(x^*) \exp(-2\pi i k x^* / N)$
- So, just apply a phase gradient to the DFT to shift in  $x$

# Shifting Example



```
import numpy
from matplotlib import pyplot as plt

x=numpy.arange(-10,10,0.1)
y=numpy.exp(-0.5*x**2/(1.5**2))
N=x.size
kvec=numpy.arange(N)
yft=numpy.fft.fft(y)
J=numpy.complex(0,1)
dx=5.0;
yft_new=yft*numpy.exp(-2*numpy.pi*J*kvec*dx/N)
y_new=numpy.real(numpy.fft.ifft(yft_new))
plt.plot(x,y)
plt.plot(x,y_new,'r')
plt.savefig('shifted_gaussian')
plt.show()
```

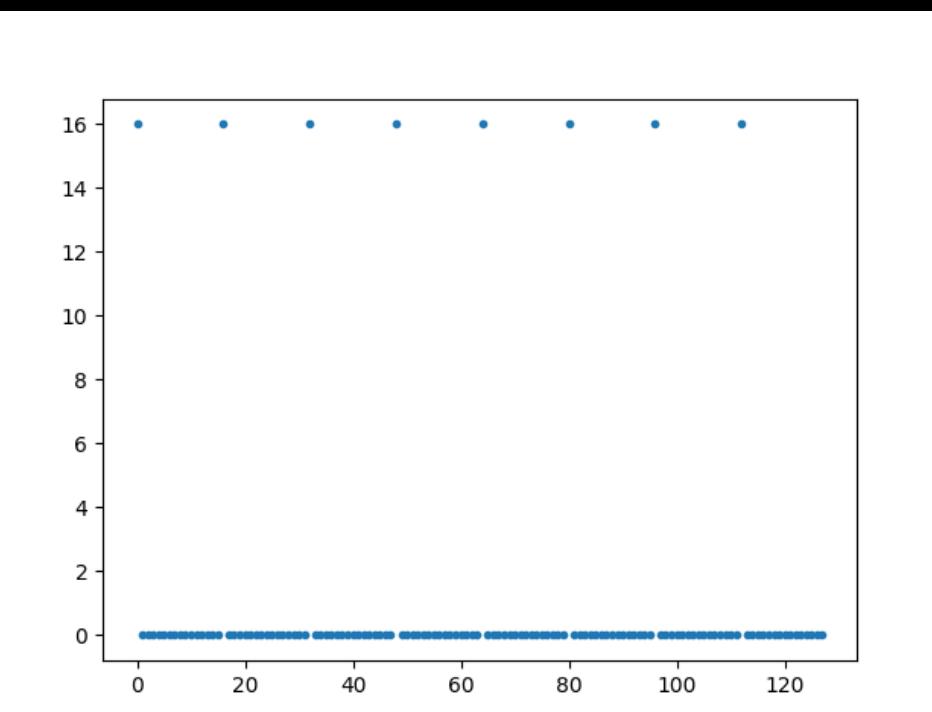
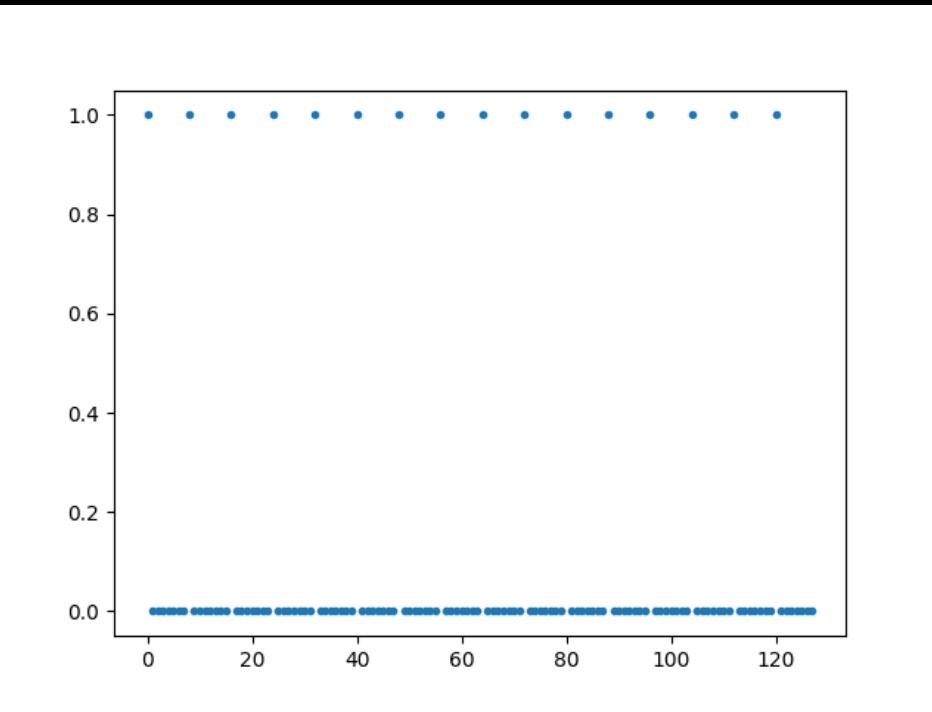
# Real Data Symmetry

- If I know  $F(k)$ , what is  $F(N-k)$  if  $f(x)$  is real?
- $F(N-k)=F(-k)$  (from alias theorem)
- $F(-k)=\sum f(x) \exp(-2\pi i (-k)x/N)$ . let  $x^*=-x$
- $F(-k)=\sum f(-x^*) \exp(2\pi i kx^*/N) = \text{conj}(F(k))$  by flipping
- So, if  $f(x)$  is real,  $F(k)=\text{conj}(F(N-k))$
- If  $N$  even,  $F(N/2)=\text{conj}(F(N/2))$ , so  $F(N/2)$  must be real.

```
>>> x=numpy.random.randn(8)
>>> xft=numpy.fft.fft(x)
>>> for xx in xft:
...     print xx
...
(-4.53568815727+0j)
(-0.174046761579+2.08827239558j)
(2.15348308858+2.32162497273j)
(-0.423040513854-3.72126858798j)
(2.75685372591+0j)
(-0.423040513853+3.72126858798j)
(2.15348308858-2.32162497273j)
(-0.174046761579-2.08827239558j)
>>>
```

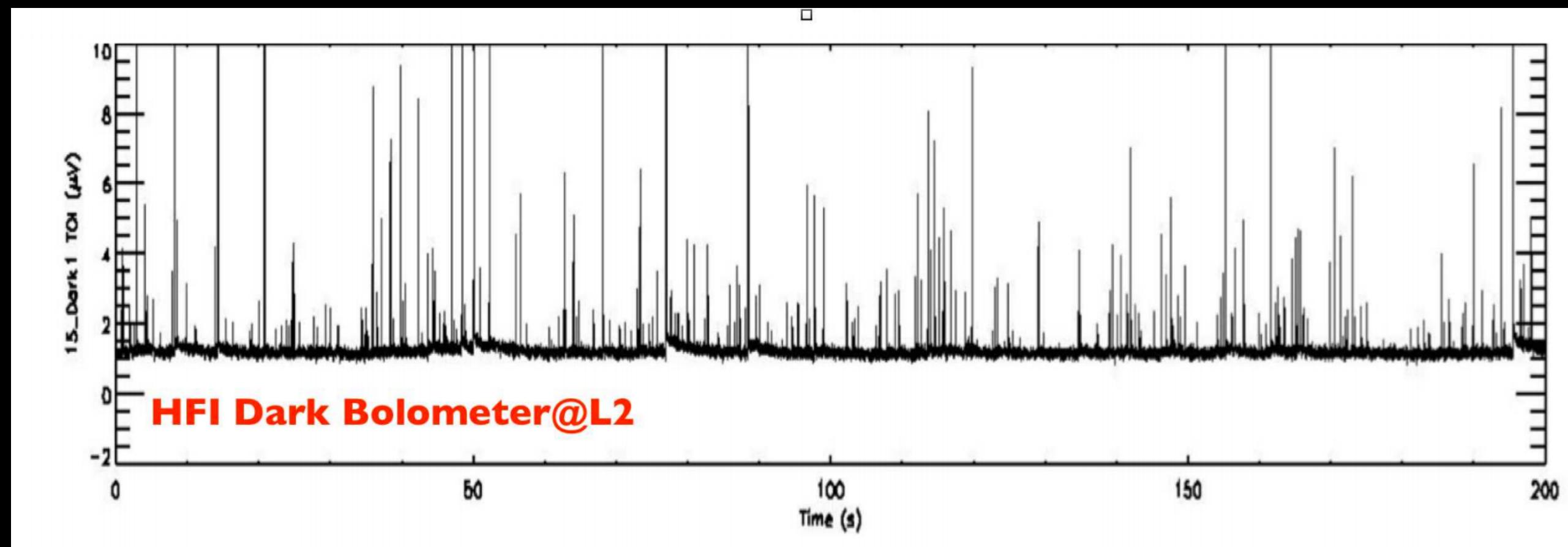
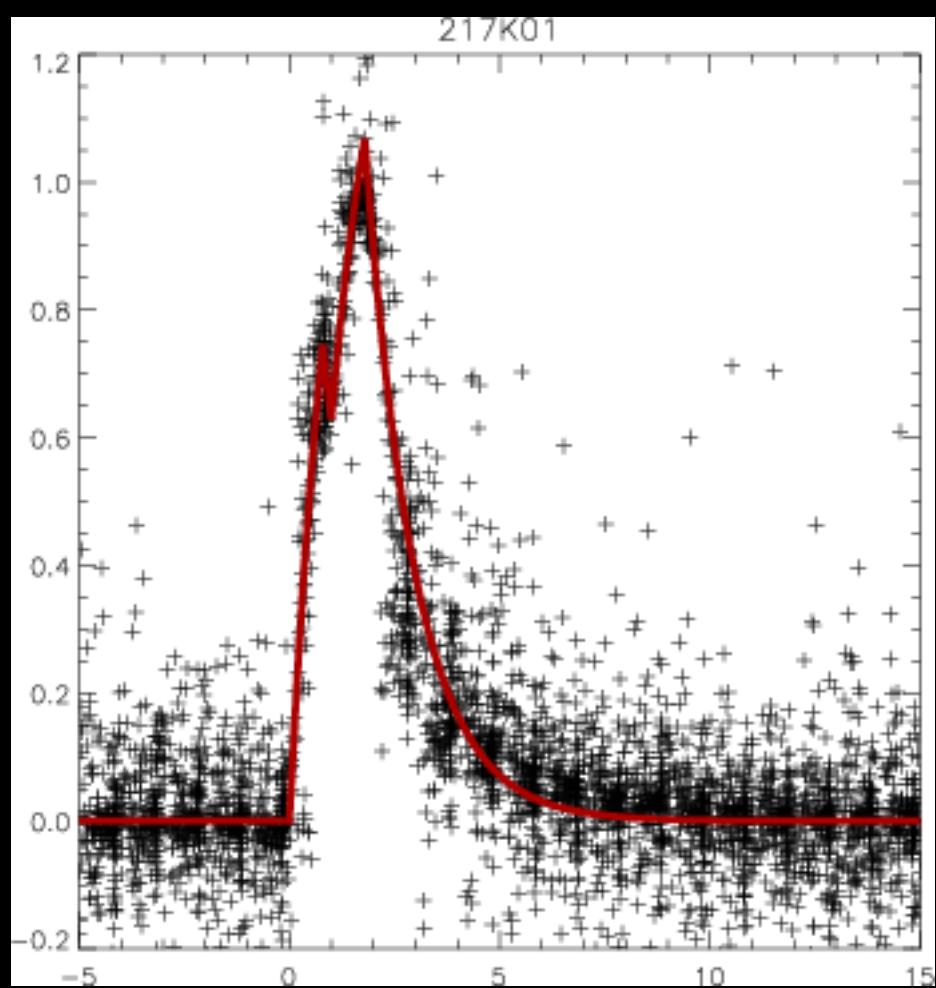
# Picket Fence

- Let's say  $f=0$ , except every  $a$ 'th element is 1. Let  $x=ax'$ , where  $x'$  goes from 0 to  $N/a$ .
- $F(k)=\sum \exp(-2\pi ikax'/N)$ , sum goes from 0 to  $N/a$ .
- Let  $M=N/a$ , gives:  $F(k)=\sum \exp(-2\pi ikx'/M)$ , which is just a delta function
- However,  $k$  goes from 0 to  $N$ , so we get  $N/M=a$  copies of the delta function.
- DFT of spikes every  $a$  samples gives  $a$  evenly-spaced delta functions.



Top:  $f(x)$   
Bottom:  $F(k)$

# Cosmic Rays from Planck Satellite



# Convolution Theorem

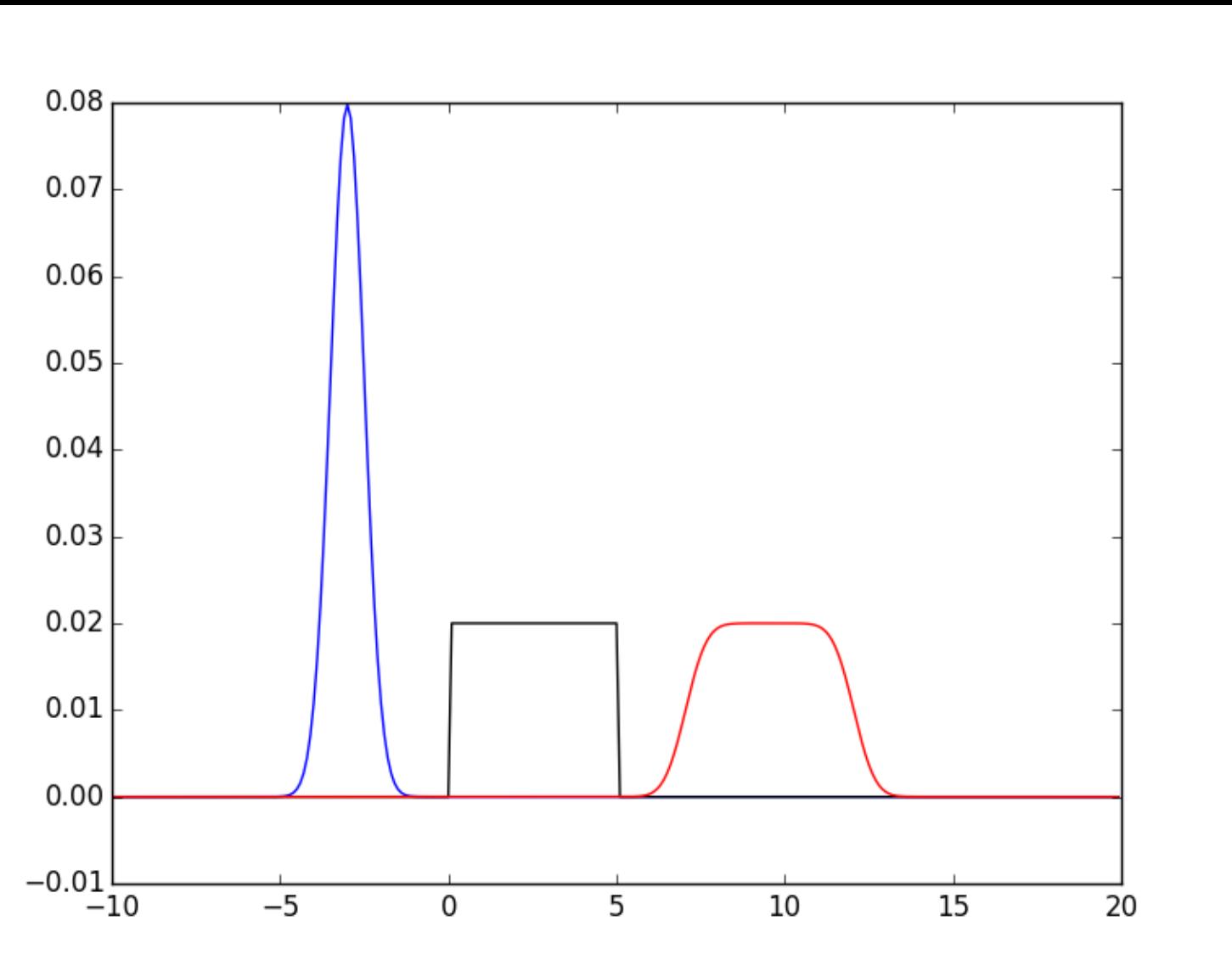
- Convolution defined to be  $\text{conv}(y) = f \otimes g = \int f(x)g(y-x)dx$
- $\sum_x \sum F(k) \exp(2\pi i k x) \sum \text{conj}(G(k')) \exp(2\pi i k' x) \exp(-2\pi i k' y/N)$
- Reorder sum:  $\sum \sum F(k) \text{conj}(G(k')) \exp(-2\pi i k' y/N) \sum_x \exp(2\pi i (k+k')x)$
- equals zero unless  $k'=-k$ . Cancels one sum, conjugates G
- $f \otimes g = \sum F(k) G(k) \exp(2\pi i ky/N) = \text{ift}(\text{dft}(f)^* \text{dft}(g))$
- So, to convolve two functions, multiply their DFTs and take the IFT

# Convolution Example

```
from numpy import arange,exp,real
from numpy.fft import fft,ifft
from matplotlib import pyplot as plt
def conv(f,g):
    ft1=fft(f)
    ft2=fft(g)
    return real(ifft(ft1*ft2))

x=arange(-10,20,0.1)
f=exp(-0.5*(x+3)**2/0.5**2)
g=0*x;g[ (x>0)&(x<5) ]=1
g=g/g.sum()
f=f/f.sum()
h=conv(f,g)

plt.plot(x,f,'b')
plt.plot(x,g,'k')
plt.plot(x,h,'r')
plt.savefig('convolved')
plt.show()
```



# Fast Fourier Transform

- How many operations does a DFT take?
- Have an  $N$  by  $N$  matrix operating on a vector of length  $N$  - clearly  $N^2$  operations, right?
- Nope! Otherwise we'd never use them. What's actually going on?
- Note  $DFT = \sum f(x) \exp(-2\pi i kx/N) = \sum f_{\text{even}}(x) \exp(-2\pi i k(2x)/N)$   
 $+ \sum f_{\text{odd}}(x) \exp(-2\pi i k(2x+1)/N)$
- $= F_{\text{even}} + \exp(-2\pi i k/N) F_{\text{odd}}$ . Let  $W_k = \exp(-2\pi i k/N)$
- if  $k > N/2$ , then  $k^* = k - N/2$  and  $DFT = F_{\text{even}} + \exp(-2\pi i k^*/N + i\pi) F_{\text{odd}} = F_{\text{even}} - W_k F_{\text{odd}}$ .

# FFT cont'd

- So  $F(k) = F_{\text{even}}(k) + W_k F_{\text{odd}}(k)$  ( $k < N/2$ ) or  $F_{\text{even}}(k) - W_k F_{\text{odd}}(k)$  ( $k \geq N/2$ )
- So, can get *all* the frequencies if I have 2 half-length FFTs.
- Well, just do the same thing again. FFT of a single element is itself.
- This algorithm works for arrays whose length is a power of 2
- Popularized by Cooley/Tukey in early computer days. Later found to go back to Gauss in 1805. Changes computational work from  $n^2$  to  $n \log n$ .

# Sample FFT

- Routine uses *recursion* - function calls itself. Recursion can be very powerful, but also easy to goof.
- `numpy.concatenate` will combine arrays - note that they have to be passed in as a tuple, hence the extra set of parenthesis
- Modern FFT routines deal with arbitrary length arrays. Fastest Fourier Transform in the West (FFTW) standard packaged - usually used by numpy.

```
from numpy import concatenate,exp,pi,arange,complex
def myfft(vec):
    n=vec.size
    #FFT of length 1 is itself, so quit
    if n==1:
        return vec
    #pull out even and odd parts of the data
    myeven=vec[0::2]
    myodd=vec[1::2]

    nn=n/2;
    j=complex(0,1)
    #get the phase factors
    twid=exp(-2*pi*j*arange(0,nn)/n)

    #get the dfts of the even and odd parts
    eft=myfft(myeven)
    oft=myfft(myodd)

    #Now that we have the partial dfts, combine them with
    #the phase factors to get the full DFT
    myans=concatenate((eft+twid*oft,eft-twid*oft))
    return myans
```

```
>>> import myft
>>> x=numpy.random.randn(32)
>>> xft1=numpy.fft.fft(x)
>>> xft2=myft.myfft(x)
>>> print numpy.sum(numpy.abs(xft1-xft2))
2.33937690259e-13
>>>
```

# In Practice

- Should you write your own FFT code? (no)
- Should you understand what is going on under the hood? (yes)

```
import numpy
import time

n=2**16
x=numpy.random.randn(n)
t1=time.time();y=numpy.fft.fft(x);t2=time.time();t_ref=t2-t1
x=numpy.random.randn(n+1) #this is a prime
t1=time.time();y=numpy.fft.fft(x);t2=time.time();t_plus1=t2-t1
x=numpy.random.randn(n+2) #this is has largest factor 331
t1=time.time();y=numpy.fft.fft(x);t2=time.time();t_plus2=t2-t1
x=numpy.random.randn(n+14) #this is has largest factor 23
t1=time.time();y=numpy.fft.fft(x);t2=time.time();t_plus14=t2-t1
print 'Reference time was ',t_ref
print 'Extending by one increased time by a factor of ',t_plus1/t_ref
print 'Extending by two increased time by a factor of ',t_plus2/t_ref
print 'Extending by 14 increased time by a factor of ',t_plus14/t_ref
```

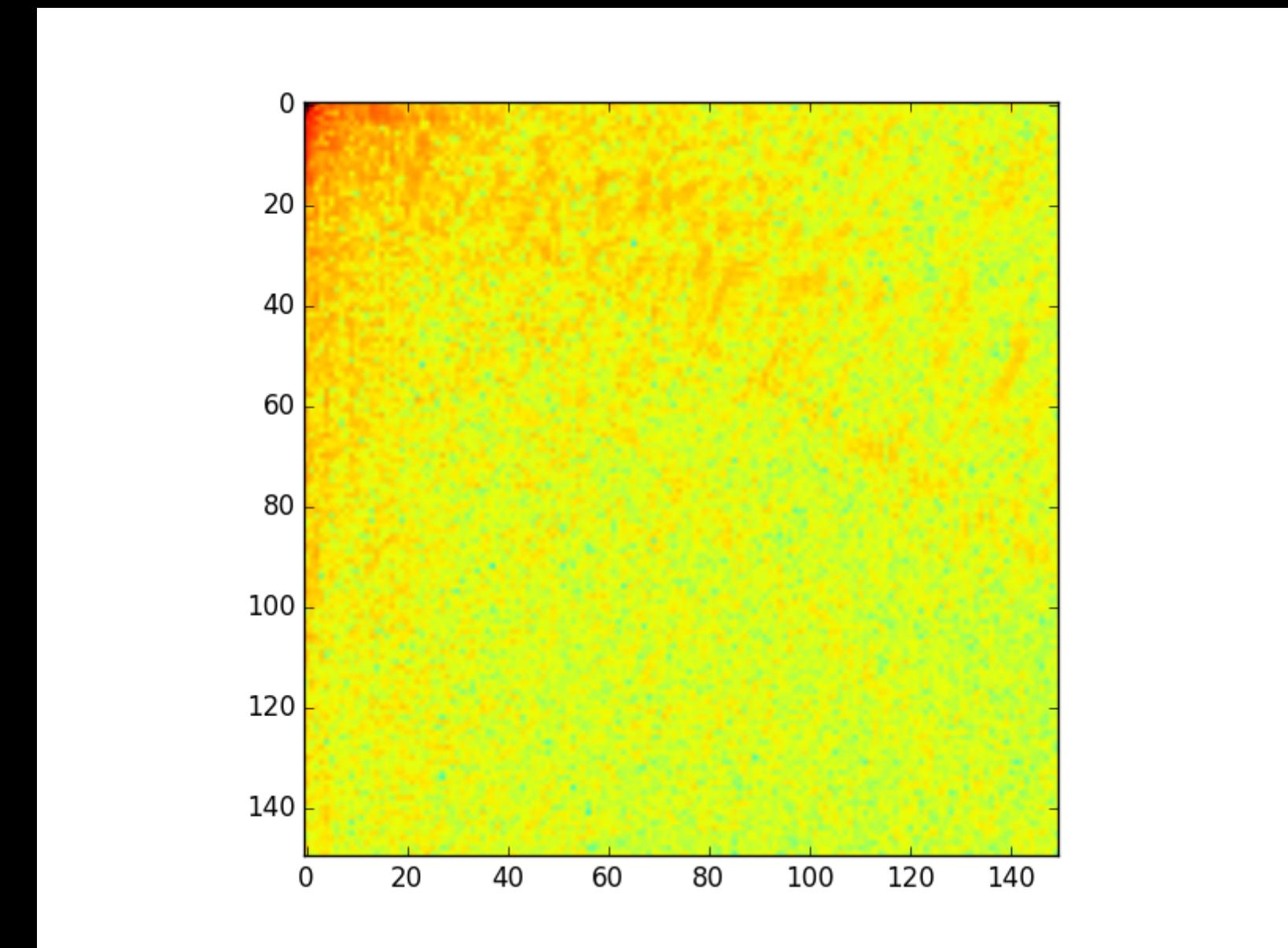
```
[>>> execfile('time_ffts.py')
Reference time was  0.00335788726807
Extending by one increased time by a factor of  2594.13178074
Extending by two increased time by a factor of  5.54963078671
Extending by 14 increased time by a factor of  1.8124112468
```

# 2D Fourier Transforms

- Fourier transform defined in 2 dimensions:
- $F(k,l) = \iint f(x,y) \exp(-ikx) \exp(-ily) dx dy$
- 2D FT's extremely common in image processing.
- JPEGs in fact are based on picking out modes from image FT's.



numpy.fft.fft2

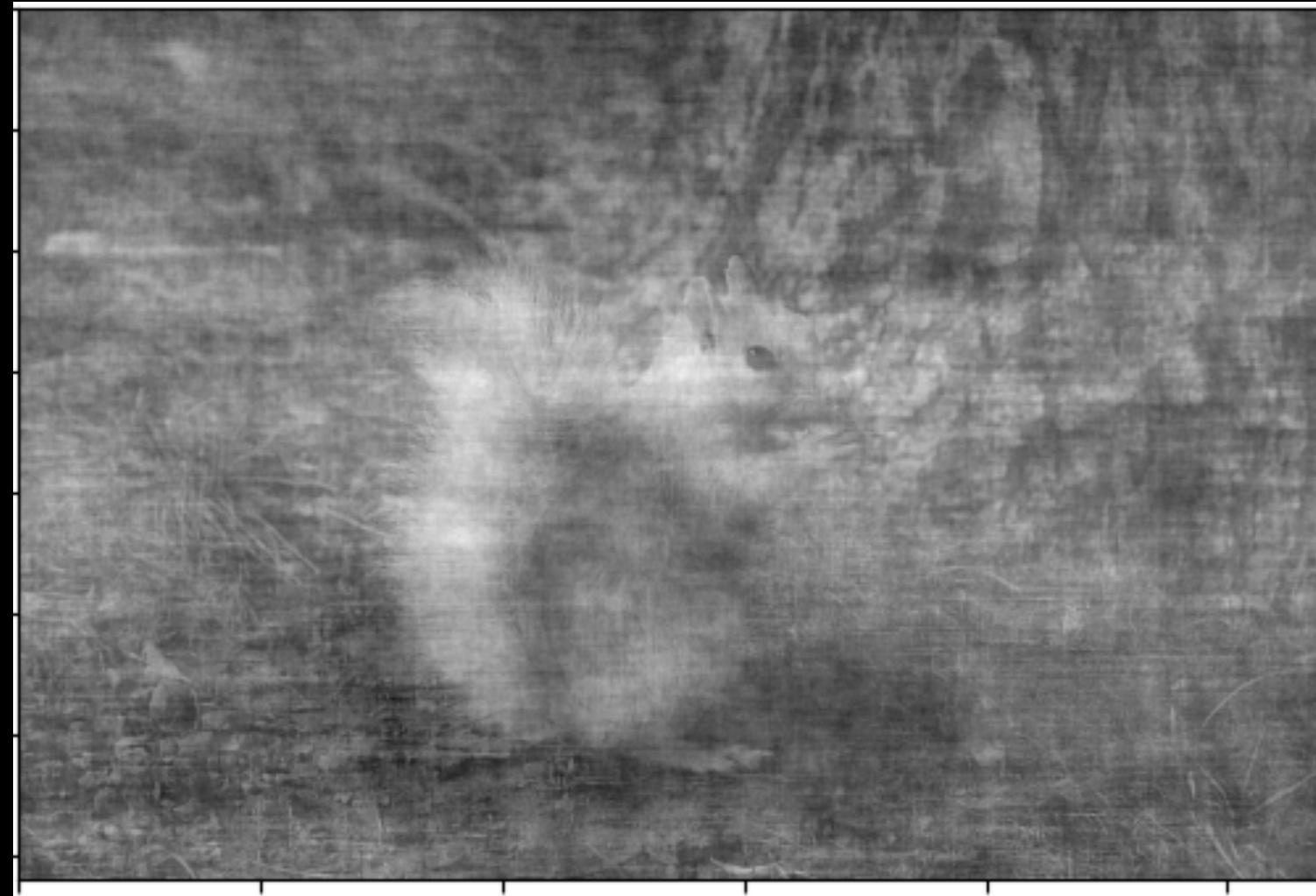


# Phase vs. Amplitude

- What contains information about an image? The phases or amplitudes?
- We'll take the phases from one image, and amplitudes from another. What do you think we'll see?



# Phase vs. Amplitude



Arts amps, Cloudberry phases



Cloudberry amps, Arts phases

# Smoothing Images

- Out-of-focus images are convolutions.
- Can defocus an image by convolving with a blurry kernel
- Let's fuzz out map by a Gaussian.

```

def get_fft_vec(n):
    vec=numpy.arange(n)
    vec[vec>n/2]=vec[vec>n/2]-n
    return vec
def smooth_map(map,npix,smooth=True):
    nx=map.shape[0]
    ny=map.shape[1]
    xind=get_fft_vec(nx)
    yind=get_fft_vec(ny)

    #make 2 1-d gaussians of the correct lengths
    sig=npix/numpy.sqrt(8*numpy.log(2))
    xvec=numpy.exp(-0.5*xind**2/sig**2)
    xvec=xvec/xvec.sum()
    yvec=numpy.exp(-0.5*yind**2/sig**2)
    yvec=yvec/yvec.sum()

    #make the 1-d gaussians into 2-d maps using numpy.repeat
    xmat=numpy.repeat([xvec],ny,axis=0).transpose()
    ymat=numpy.repeat([yvec],nx,axis=0)

    #if we didn't mess up, the kernel FT should be strictly real
    kernel=numpy.real(numpy.fft.fft2(xmat*ymat))

    #get the map Fourier transform
    mapft=numpy.fft.fft2(map)
    #multiply/divide by the kernel FT depending on what we're after
    if smooth:
        mapft=mapft*kernel
    else:
        mapft=mapft/kernel
    #now get back to the convolved map with the inverse FFT
    map_smooth=numpy.fft.ifft2(mapft)

    #since numpy gets imaginary parts from roundoff, return the real part
    return numpy.real(map_smooth)

```

smooth\_map.py



```
import numpy
from matplotlib import pyplot as plt
import smooth_map

meerkat=plt.imread('meerkat_small.jpg')

smoothed_map=numpy.zeros(meerkat.shape)
smoothed_map=numpy.zeros(meerkat.shape)
npix_smooth=3.5
npix_restore=4
for i in range(3):
    tmp=numpy.squeeze(meerkat[:, :, i])
    tmp_smooth=smooth_map.smooth_map(tmp,npix_smooth)
    smoothed_map[:, :, i]=tmp_smooth
    tmp2=smooth_map.smooth_map(tmp_smooth,npix_restore,False)
    unsmoothed_map[:, :, i]=tmp2
```

# Deconvolution

- Well, if I smear out by multiplying FT's, I can unsmear by dividing, right?
- If yes, worth billions and billions of your favo(u)rite currency. Save those fuzzy pictures...
- Maybe. Let's try smoothing image by 3.5 pixels, then unsmoothing by 4.
- What happened?

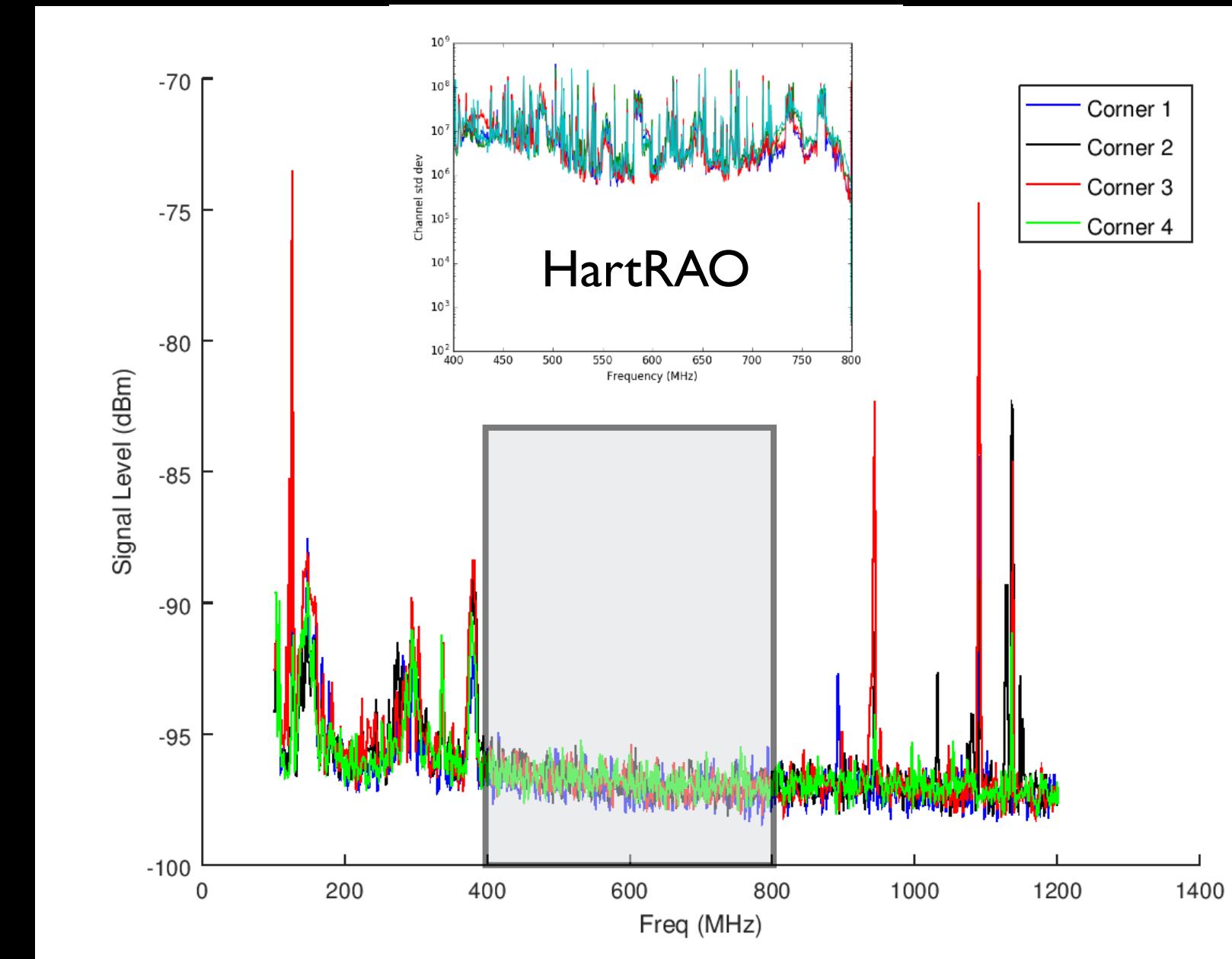
# Deconvolution



- smoothing lowers high-frequency signal
- unsmoothing *must* raise it back up.
- if there's any noise, it gets amplified by unsmoothing.
- If I smooth, then write to jpg, I round to nearest integer. Equivalent to adding noise.
- So, think those fuzzy license plates in google maps are safe?

# E field to Frequency

- Power as a function of frequency is just FT (squared) of E field
- True in ideal limit, but what happens in real life?
- We measure E field every so often, for some length of time
- Incoming field doesn't care about when we measured it. Our answer hopefully doesn't either...
- How to do this right hugely important question for any RF work (cell phones, TV, DSP, radio astronomy...)



# Windowing

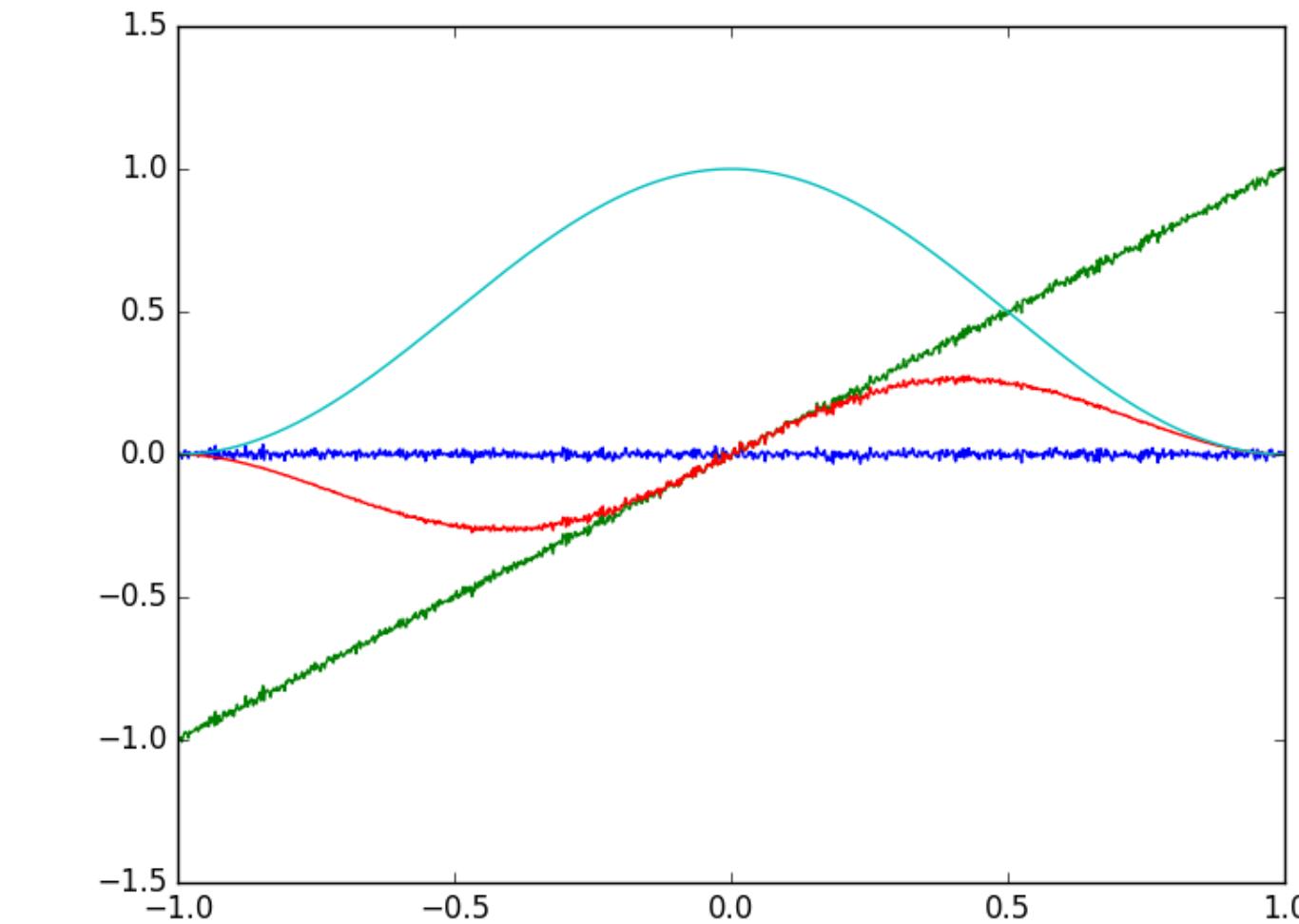
- Jumps around edges cause high frequency power in FFTs. This is a bad thing.
- Standard solution: multiply by a window that goes to zero (or some very small value) at edges.
- There are many possible windows, depending on what you want to do: Hamming, Hanning, cos... 28 listed on wikipedia page.
- If I multiply by window in real space, what have I done in Fourier space?

# Window in Real Space

Use cos window that goes to zero on edges w/derivative zero.

If I take a piece of noisy data from a smooth long-term signal, smooth part may look like similar to a line.

What does FT look like?



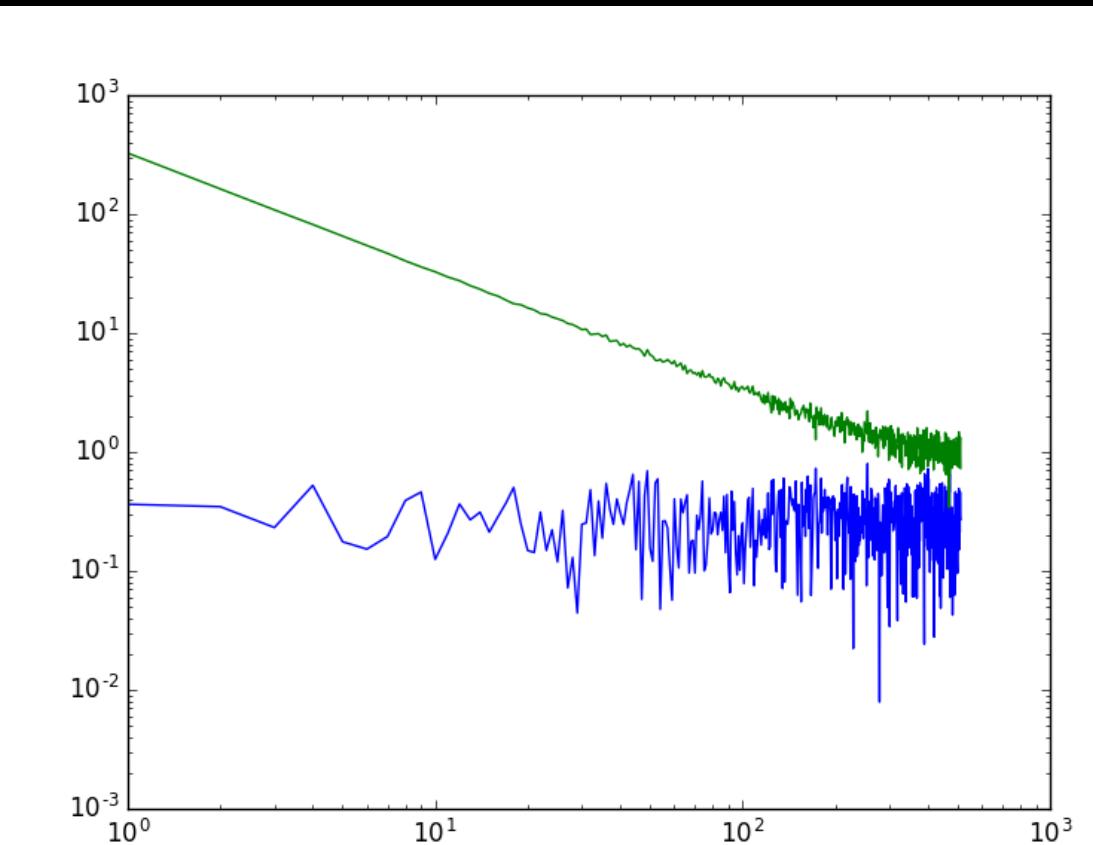
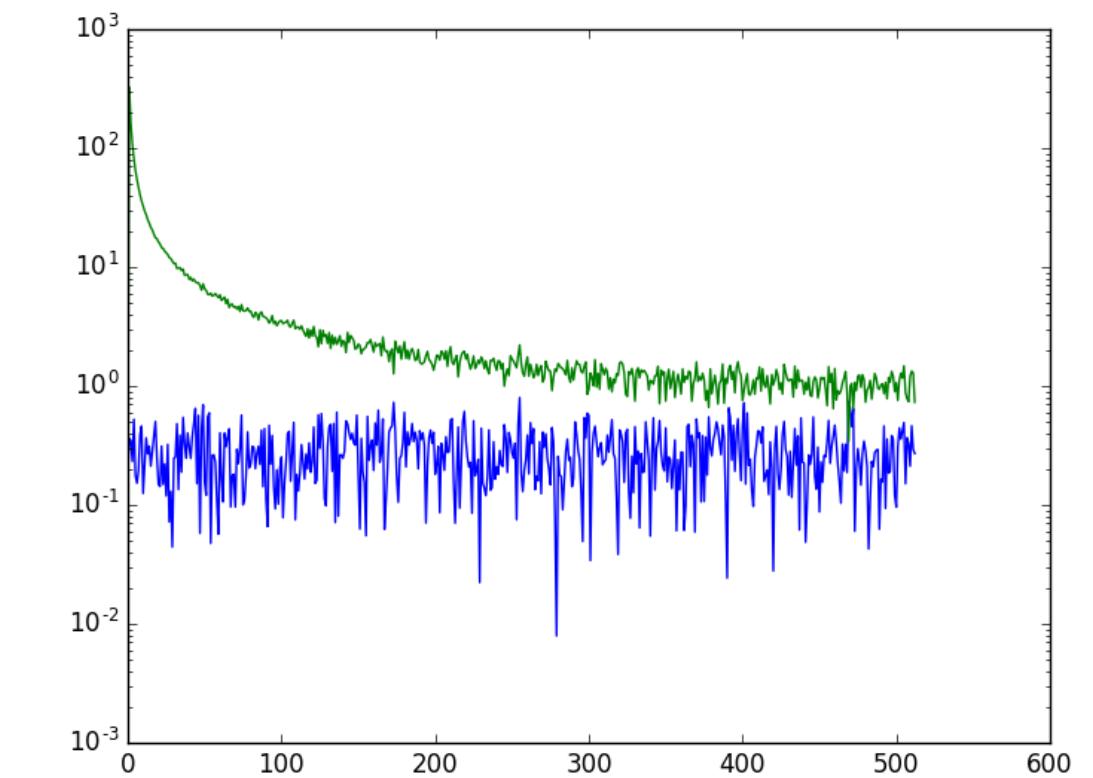
```
import numpy
from matplotlib import pyplot as plt
plt.ion();

x=numpy.arange(1024);
x=x-1.0*x.mean();x=x/x[-1]
y1=0.01*numpy.random.randn(x.size)
y2=y1+x
window=0.5*(1+numpy.cos(x*numpy.pi))
y3=y2*window
plt.clf();plt.plot(x,y1);plt.plot(x,y2);plt.plot(x,y3)
plt.plot(x,window);plt.savefig('raw_data.png')

y1ft=numpy.fft.rfft(y1)
y2ft=numpy.fft.rfft(y2)
```

# Effects of Adding Slope

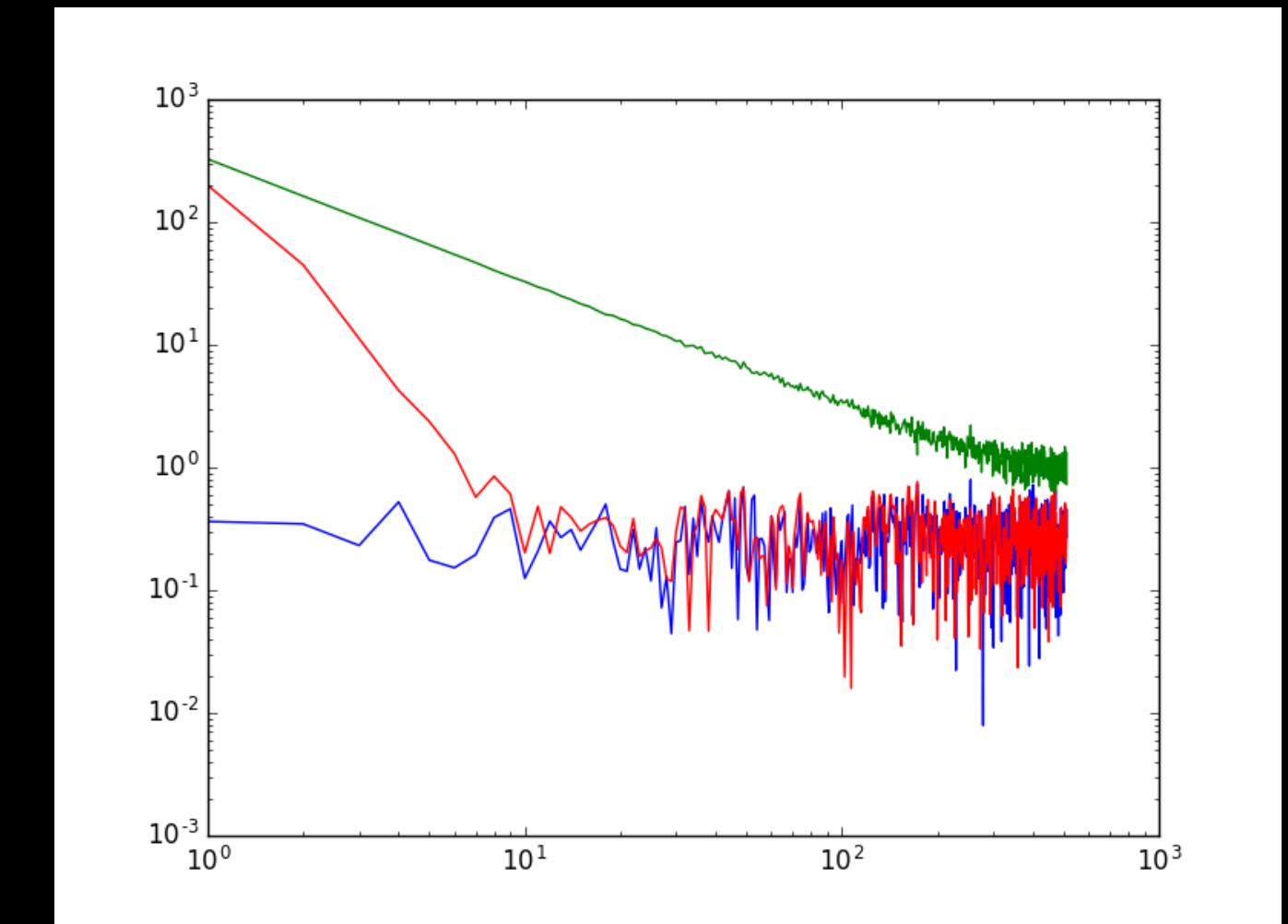
- Even though we know long-term signal is smooth, by taking piece we raise noise level in FT. This is a bad thing.
- Why does the FT look like a line in log-log space?



# Adding Window

- Multiplying the data with a slope by the window makes the high-frequency power drop back down. This is usually considered a good thing.
- Low frequency power is still large - that's real, we do have a slope in our data.
- What am I doing with that normfac thing?

Parseval's theorem: FT is a unitary rotation, so length before/after must be the same.  
Windowing removes power, so scale back up by average amount of windowing loss.



```
window=0.5*(1+numpy.cos(x*numpy.pi))
y3=y2*window
#why am I doing this normfac thing?
normfac=numpy.sqrt(numpy.mean(window**2))
y3ft=numpy.fft.rfft(y3)
plt.plot(numpy.abs(y3ft/normfac));
plt.savefig('window_log.png')
```

# More FT asides

- What is the Fourier transform of a slope?
- What is the Fourier transform of a triangle ( $f(x) = 1 - \text{abs}(x)$  for  $-1 < x < 1$ )?
- What is the (expected) Fourier transform of random noise?
- We will be looking at plots that show the amplitude of the Fourier transforms against wavelength. The variance of the FT is called the *power spectrum*, and is fundamental in many areas of electronics, physics, astronomy...

# Stationary Noise

- Say we have noise where  $\langle f(x)f(x+dx) \rangle = g(dx)$  (not of  $x$ )
- This is called stationary - noise at some point depends on noise we got at nearby points, but not on absolute time of when we measured
- Fourier space:  $\langle F(k)F^*(k') \rangle = \langle \sum f(x)\exp(-2\pi i k x/N) \sum f(x')\exp(2\pi i k' x'/N) \rangle$
- Can swap  $x'$  for  $x+dx$ , since sum is over all points, can sum over  $dx$ :
- $\langle F(k)F^*(k') \rangle = \langle \sum f(x)\exp(-2\pi i k x/N) \sum f(x+dx)\exp(2\pi i k'(x+dx)/N) \rangle$
- Reorder sum over  $x$  then  $dx$ :  $\langle \sum \exp(2\pi i k' dx/N) \sum f(x)f(x+dx)\exp(2\pi i x(k'-k)/N) \rangle$
- Now  $\langle f(x)f(x+dx) \rangle = g(dx)$  (by assumption), can come out.  $\sum \exp(2\pi i k' dx/N) g(dx) \sum \exp(2\pi i x(k'-k)/N)$
- Interior goes to  $N$  for  $k'=k$ , left with  $N \sum g(dx) \exp(2\pi i k dx/N) \delta_{kk'}$ , so Fourier transform of noise is diagonal.
- Further, variance of  $F(k)$  given by Fourier transform of correlation function  $g(dx)$ .