

CMB

- Cosmic microwave background contains snapshot of universe at about 400,000 years.
- Useful because physics is linear, so we can link observations and theory very easily (unlike nonlinear physics in galaxies etc.)
- Key information contained in power spectrum.

CMB PS

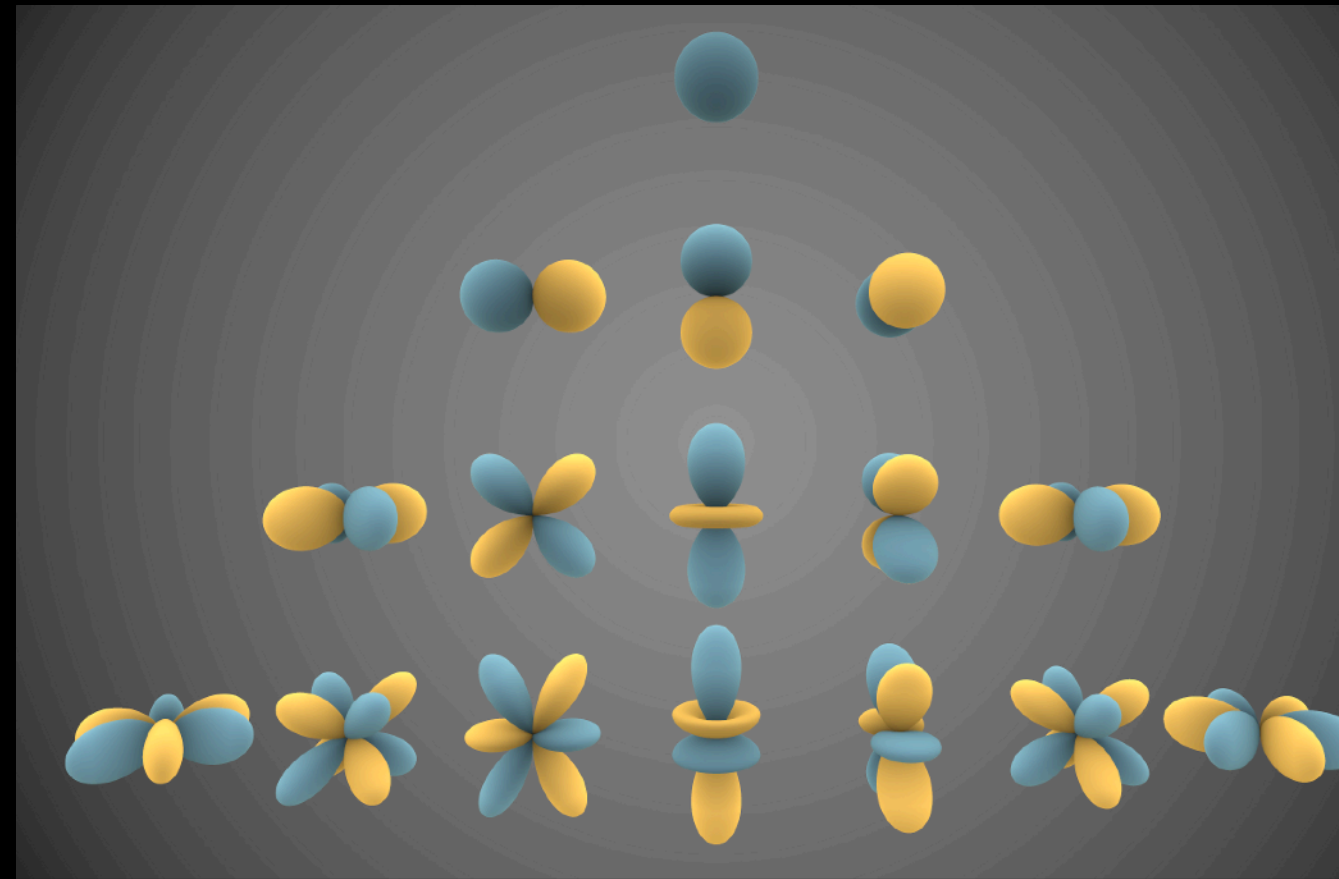
- We've met power spectra so far in Fourier transforms.
- Different on a spherical sky. Normal modes become *spherical harmonics*.
- $T(\theta, \phi) = \sum_l \sum_m Y_{lm}(\theta, \phi) a_{lm}$. a_{lm} are coefficients (like $F(k)$) and Y_{lm} are orthogonal basis functions, analogous to sine waves.
- $Y_{lm}(\theta, \phi) \propto P_l^m(\cos(\theta)) \exp(im\phi)$ where P_l^m are associated Legendre polynomials. l, m are integers, $|m| \leq l$
- Usual normalization is such that $\iint Y_{lm} Y_{l'm'} d^2\Omega = \delta(l-l') \delta(m-m')$

Associated Legendre “Polynomials”

- May not surprise you to hear that P_l^m can be found with recurrence relations.
- $(l-m+1)P_{l+1}^m(x) = (2l+1)xP_l^m(x) - (l+m)P_{l-1}^m(x)$, with $|m| \leq l$, $|x| \leq 1$
- Can get started with $P_{l+1}^{l+1} = -(2l+1)(1-x^2)^{1/2}P_l^l(x)$, $P_0^0(x) = 1$
- $P_1^1 = -(2 \cdot 0 + 1)(1-x^2)^{1/2}P_0^0 = -(1-x^2)^{1/2}$.
- If m even, these are polynomials (but not if m odd)
- $P_l^0 = l^{\text{th}}$ Legendre polynomial

Spherical Harmonics

- With ability to calculate P_l^m , we could calculate first few spherical harmonics.
- What is angular dependence of $m=0$ modes? Can a_{l0} be complex?
- Y_{l-m} is rotation of Y_l^m . Like FT, for real sky, $Y_{l-m}=Y_l^m^*$. So, usually only specify non-negative a_{lm} .
- Healpix is standard astro package to work with spherical harmonics
- `healpy.map2alm` converts map of sky to a_{lm} .
- `healpy.alm2map` converts a_{lm} s to map



(from Wikipedia)

Healpy to Plot Modes

```
import numpy as np
from matplotlib import pyplot as plt
import healpy
#let's make a plot of individual Ylm's
l=5
m=2

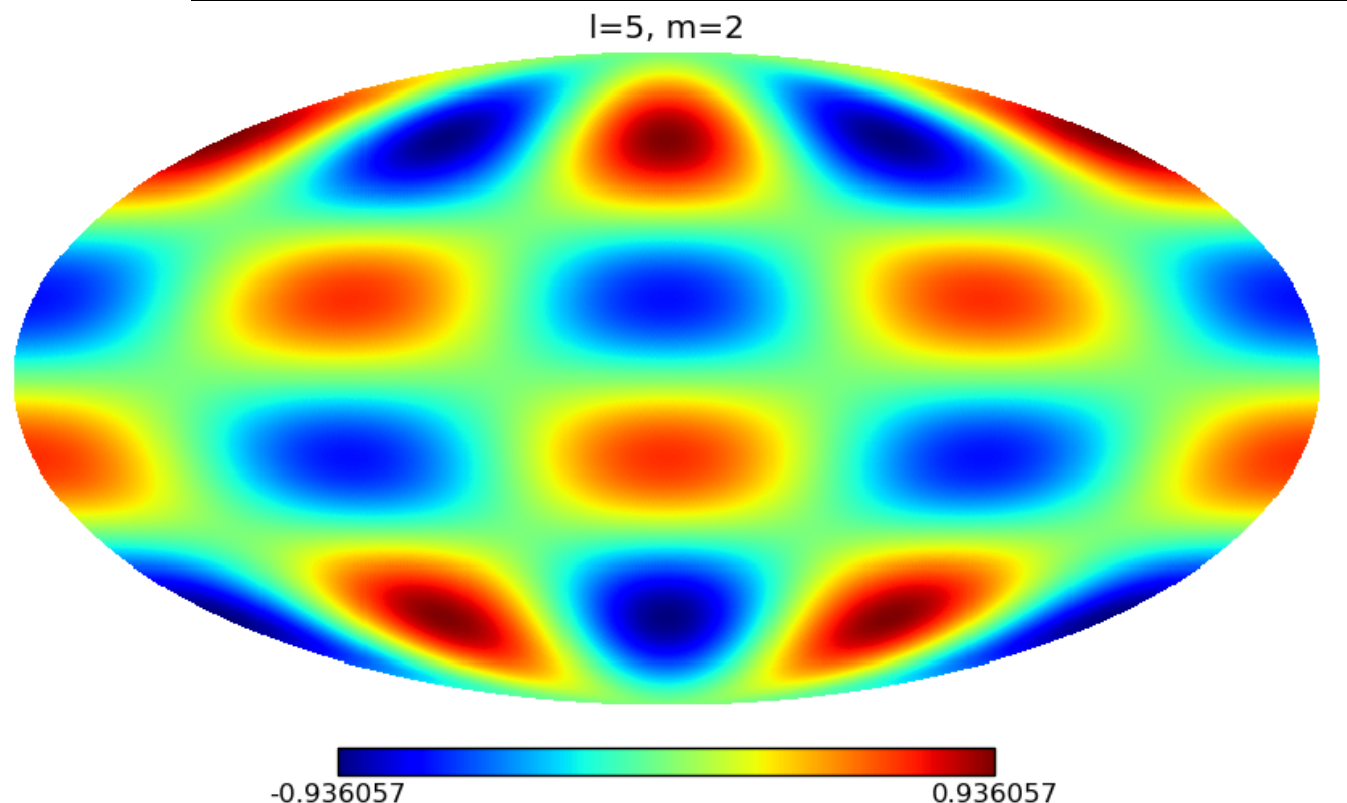
nlm=(l+1)*(l+2)/2 #where does this come from?

#healpix indexing is such that all the m=0 modes are
#first, m=1 modes are next, etc. use this bit to brute-force
#find where in healpix ordering our requested mode will
icur=0
for mm in range(m):
    icur=icur+(l+1-mm)
icur=icur+(l-m)

alm=np.zeros(nlm,dtype='complex')
alm[icur]=1.0
nside=256
map=healpy.alm2map(alm,nside)
plt.ion()
healpy.mollview(map)
```

Warning!

$\text{alm2map}(\text{map2alm}(x)) \neq x$
Unlike Fourier transforms



CMB PS Ctd

- CMB should be Gaussian, with all information contained in power spectrum.
- $\langle a_{lm}^2 \rangle = C_l$, so variance independent of m .
- How would you measure C_l given a noiseless map of the full sky?
- If C_l were constant up to some l_{\max} , what would standard deviation of the sky be?

CMB PS Ctd

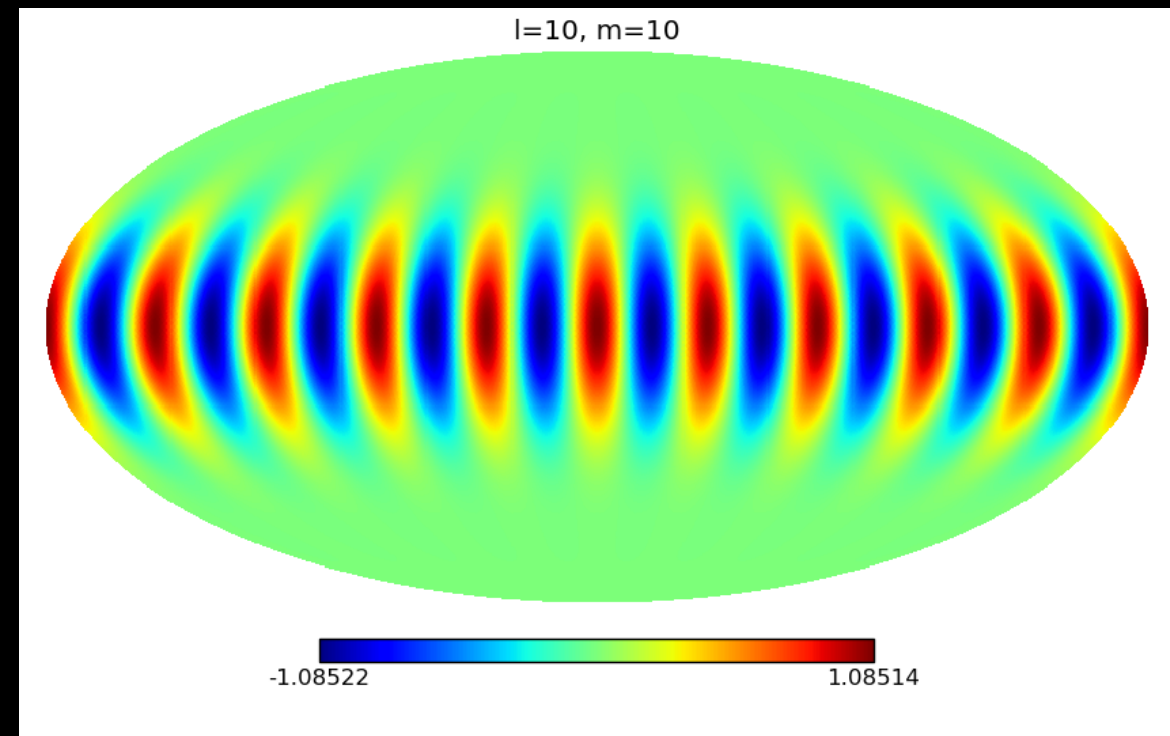
- CMB should be Gaussian, with all information contained in power spectrum.
- $\langle a_{lm}^2 \rangle = C_l$, so variance independent of m .
- How would you measure C_l given a noiseless map of the full sky?
 - Take the spherical harmonic transform, average a_{lm}^2 over m .
- If C_l were constant up to some l_{\max} , what would standard deviation of the sky be?
 - at each l , have $2l+1$ modes. $\sum (2l+1) = l(l+1) + l = l(l+2)$
 - but, spread over sky. Approximately $l^2/2\pi$

CMB PS again

- So, CMB PS usually reported not as C_l , but as $l(l+1)C_l/2\pi$, often confusingly called \mathcal{C}_l , but sometimes D_l or B_l .
- But, $\sqrt{l(l+1)C_l/2\pi}$ tells you about what fluctuations look like for broad band in power spectrum.
- Many experiments don't see full sky, but instead observe small patches.
- On small patches, flat sky approximation becomes good, $F(k)$ proportional to C_l , with k scaled to match l for patch size.

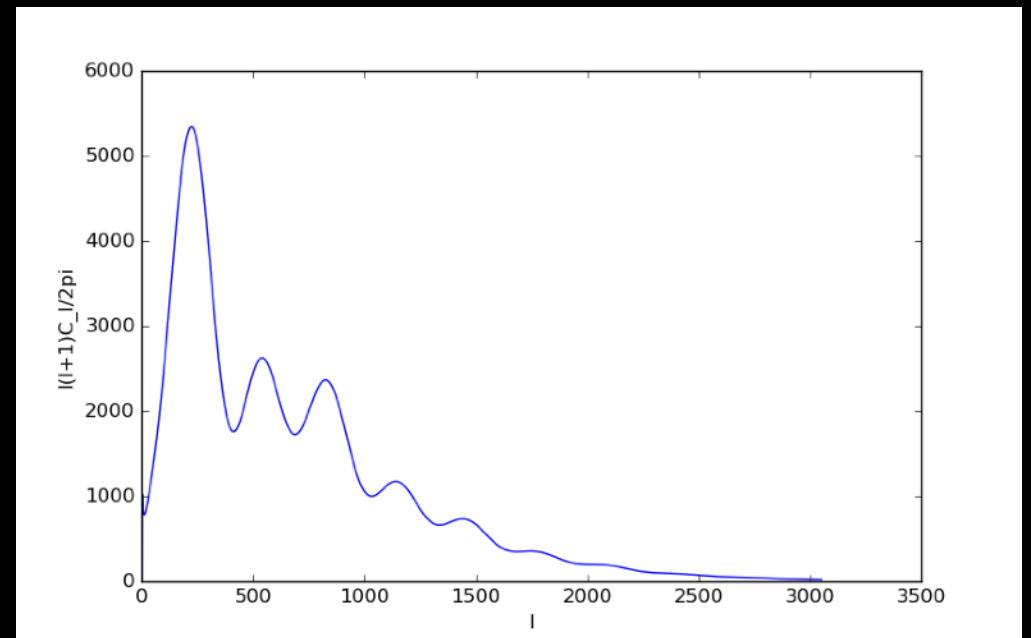
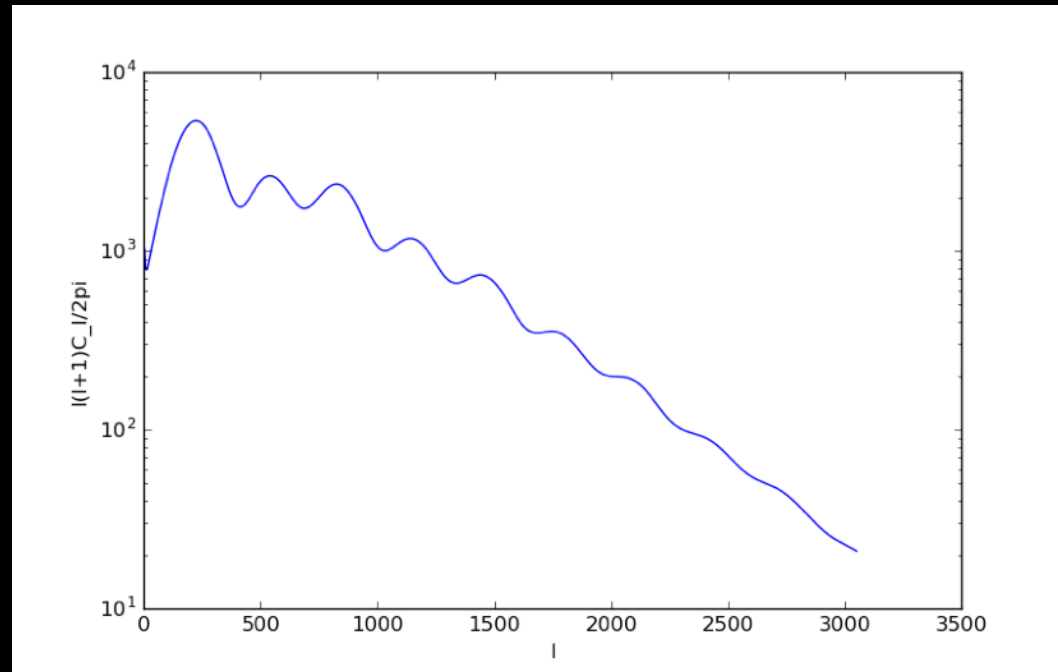
Scaling of k with l

- Y_l^m has l periods across equator.
- Box that is x radians across will have one period if $l=2\pi/x$
- But, $k=1$ will have one period as well.
So, $l \sim 2\pi k/x$
- By counting modes, can get scaling of power spectrum amplitude as well.
- What happened to our resolution in l by going to a small patch of sky?



CMB Predicted PS

- Standard code to model CMB power spectra is CAMB (available from Antony Lewis's github: <https://github.com/cmbant>)
- Top: log scale. Bottom: linear scale.



Sensitivity Needed

- Power spectrum max at $l \sim 200$, where $l(l+1)C_l/2\pi = 5300 \mu K^2$.
- What are typical fluctuation amplitudes?
- How many independent patches would I need to measure?
- In 1 second, Planck has a sensitivity of $\sim 10 \mu K$. How long would I need to measure full sky to SNR ~ 1 ?

Sensitivity Needed

- Power spectrum max at $l \sim 200$, where $l(l+1)C_l/2\pi = 5300 \mu K^2$.
- What are typical fluctuation amplitudes?
 - $\sqrt{5300} \sim 70 \mu K$
- How many independent patches would I need to measure?
 - probably want to Nyquist sample, so $200^2 \times 4 = 1.6e5$
- In 1 second, Planck has a sensitivity of $\sim 10 \mu K$. How long would I need to measure full sky to SNR ~ 1 ?
 - 1 Patch takes $(10/70)^2 \sim 0.02s$. Full sky takes $1.6e5$ times longer, or about an hour.

Sensitivity Needed @ $l=2500$

- Power spectrum at $l \sim 2500$, where $l(l+1)C_l/2\pi = 70 \mu K^2$.
- What are typical fluctuation amplitudes?
 - $\sqrt{70} \sim 8.5 \mu K$
- How many independent patches would I need to measure?
 - probably want to Nyquist sample, so $2500^2 \times 4 = 2.5e7$
- In 1 second, Planck has a sensitivity of $\sim 10 \mu K$. How long would I need to measure full sky to SNR ~ 1 ?
 - 1 Patch takes $(10/8.5)^2 \sim 1.4$. Full sky takes $2.5e7$ times longer, or about $3.5e7$ seconds, or just over a year. High resolution much harder!

Detector Sensitivity

- If I have a detector that averages n photons per second, and observes for t seconds, what is the fractional uncertainty on n ?
- I get a total of nt photons, which are (probably) Poisson distributed. Variance is nt , so $\sigma = \sqrt{nt}$.
- Fractional error is then $\delta n/n = 1/\sqrt{nt}$.
- If I want more accuracy, I can either observe longer, or increase photon rate. Larger telescopes are good!

How low can we go

- Is there a limit to how high I can push photon rate n ?
- At some point, photons start to overlap. What n would I need for optical wavelengths?
- If wavelength ~ 500 nm, need $C/\lambda \sim 6e14$ photons/s.
- Could pack in $(1\text{m}/500\text{ nm})^2$ detectors/ $\text{m}^2 = 4e12$. So, would need $4e12 * 6e14 = 2.4e27$ photons/ m^2/s .
- Energy/photon = $h\nu$, or $4e-19$ joules. Total power $4e-19 * 2.4e27 = 1\text{GW}/\text{m}^2$. Camera will melt first.

What Happens When Saturate?

- In limit of many overlapping photons, best I can do is measure electric field continuously.
- How often do I get a new electric field measurement?
- If I have signal up to some ν_{\max} , correlation length goes like $1/\nu_{\max}$. We call this the *bandwidth* B .
- # of independent samples is Bt , so fractional error is just $1/\sqrt{n_{\text{samples}}}$.
- Usually refer to temperature (instead of count rate), which gives $\delta T/T = 1/\sqrt{(Bt)}$ (radiometer equation)

Shot noise vs. continuous

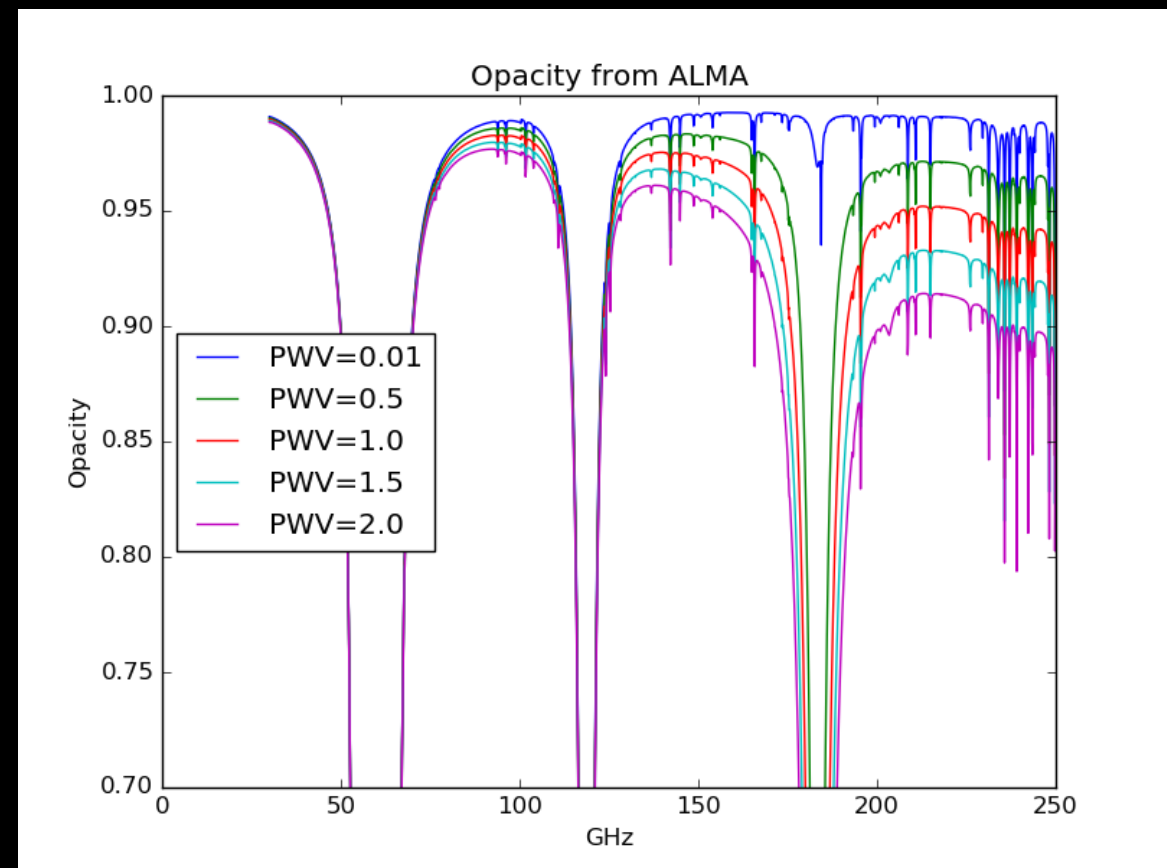
- As I crank up intensity, fractional sensitivity increases until I saturate at continuous limit.
- Shot noise absolute error: $\delta n = n / \sqrt{nt} \sim n^{1/2}$. If I want to measure a star (fixed δn) and add noise, my error on star goes like $\sqrt{\text{noise}}$
- Continuous: $\delta T = T / \sqrt{(Bt)} \sim T$. If I increase noise power, my error scales linearly (not as $\sqrt{\text{noise}}$).

Comparison to Black Body

- Where will we transition?
- If staring at black-body radiation, $B_\nu = 2h\nu^3/c^2(\exp(x)-1)$, where $x=h\nu/kT$.
- Photon occupation number is $1/(\exp(x)-1)$, so far to left of BB peak, we will be in continuous, and far to right shot noise.
- For CMB, $x=1$ at $\nu=50$ GHz. Radio always in continuous limit.

Ground-based CMB

- At typical CMB frequencies, O_2 and H_2O block some wavelengths.
- Oxygen lines can't do much about. Water can be avoided -> go high, and dry. South Pole, Chilean Atacama best places so far. (Tibet, Greenland...)
- Plot at right shows opacity from ALMA site as function of precipitable water vapor. What frequencies would you use?



Ground-based Sensitivity

- Let's pick 90 GHz window. Have 30 GHz window.
- For PWV=1mm (decent day in Chile), opacity is 0.025.
- Call temperature 270K - emission equivalent to $0.025 \times 270 = 7\text{K}$ + 3K CMB $\sim 10\text{K}$ noise signal.
- For 1sec, $\delta T = 10\text{K} / \sqrt{30 \times 10^9} = 6 \times 10^{-5} = 60 \mu\text{K}$.
- However... $x=1.6$, pushes noise to $\sim 80 \mu\text{K}$.
- Other sources of noise contribute, plus CMB not in full continuous limit. Typical ground-based limit more like 300-500 μK .