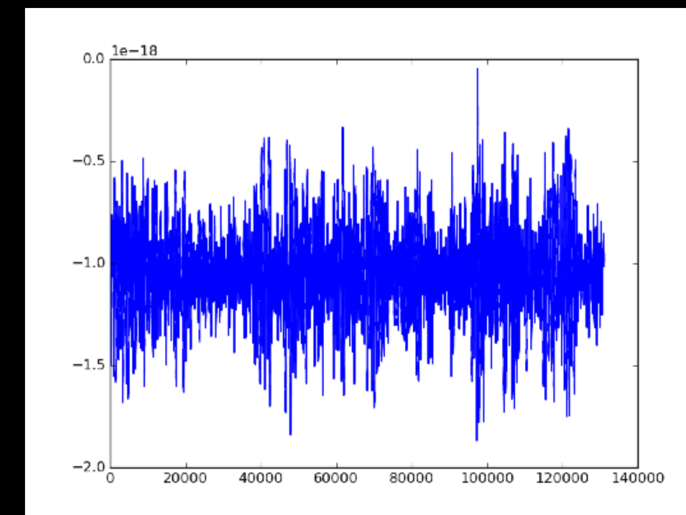
Lecture 5

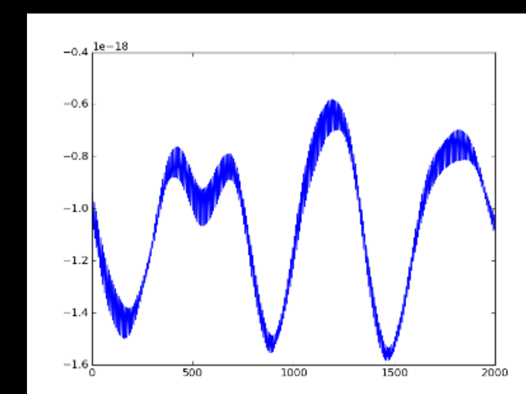
Correlated noise ctd., stationary noise, non-linear least-squares

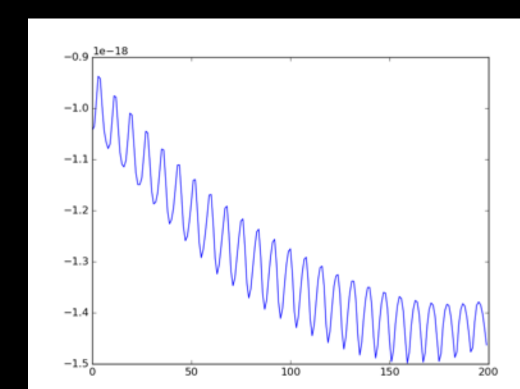
Correlated Noise

- So far, we have assumed that the noise is independent between data sets.

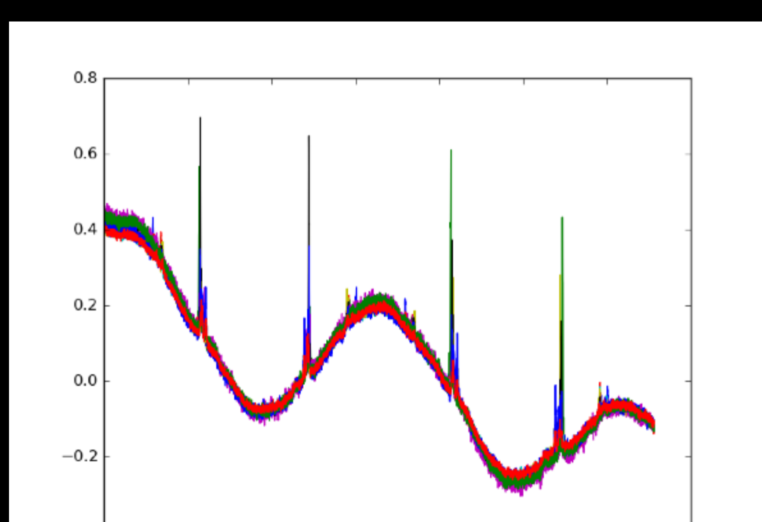
- Life is sometimes that kind, but very often not. We need tools to deal with this.







Right:LIGO data, with varying levels of zoom.
Left: detector timestreams from Mustang 2 camera @GBT



Fortunately…

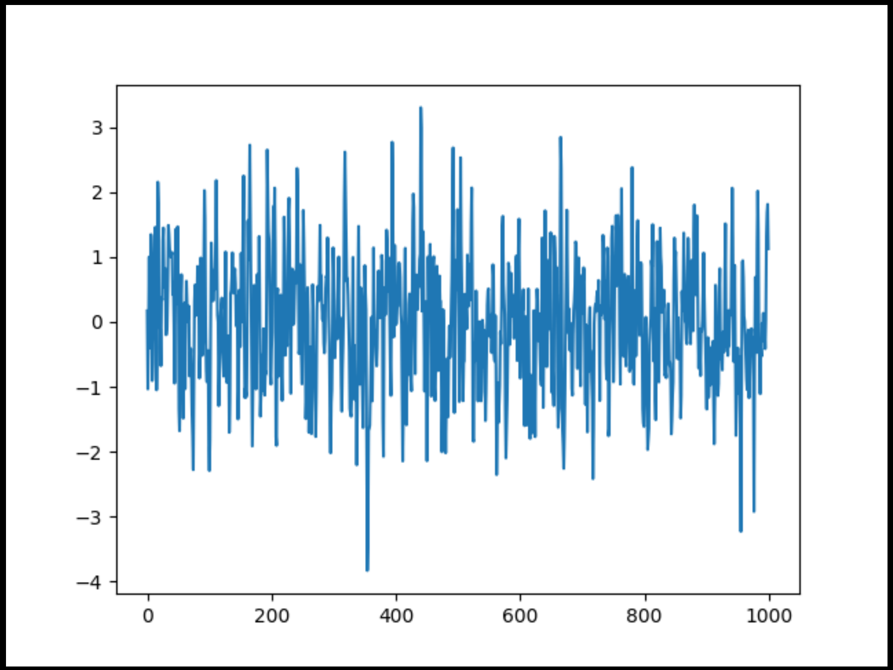
- Linear algebra expressions for χ^2 already can handle this.
- Let V be an orthogonal matrix, so $VV^T=V^TV=I$, and $d-Am=r$ (for residual)
- $\chi 2=r^TN^{-1}r=r^TV^TVN^{-1}V^TVr$. Let $r\to Vr$, $N\to VNV^T$, and $\chi 2$ expression is unchanged in new, rotated space.
- Furthermore, (fairly) easy to show that $\langle N_{ij}\rangle=\langle r_ir_j\rangle$.
- So, we can work in this new, rotated space without ever referring to original coordinates. Just need to calculate noise covariances N_{ij}.

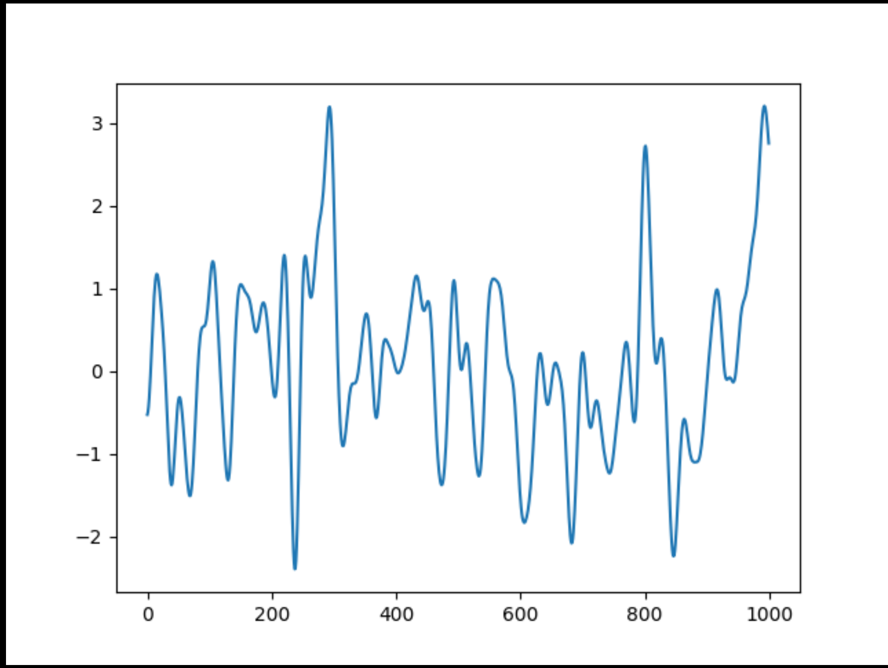
Generating Correlated Noise

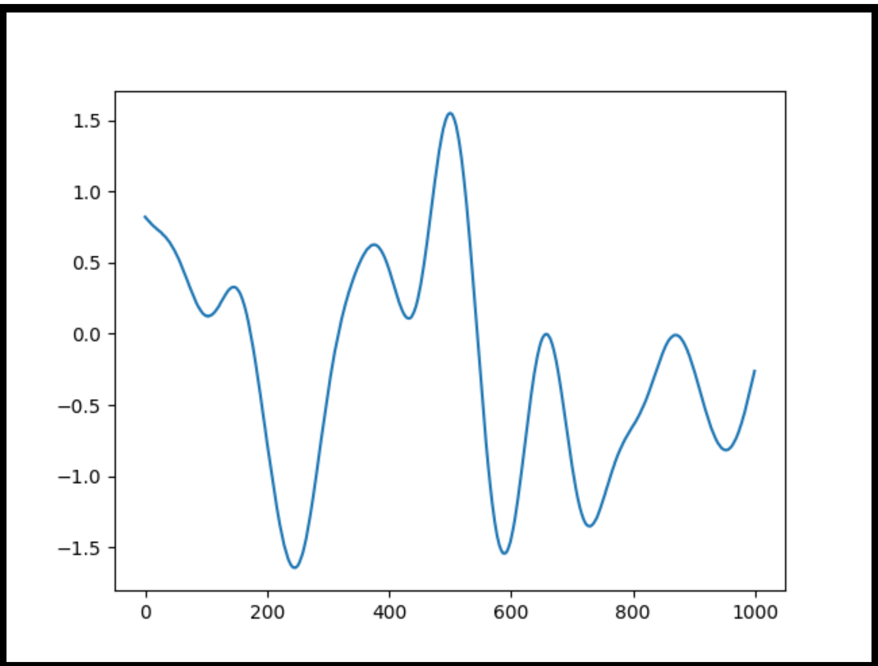
- Say we have a noise matrix N and want to create realizations from it. How do we do this?
- Same trick in reverse. If $d_{new}=Vd_{old}$, then I can generate d_{old} and rotate to get d_{new}.
- We can pick any matrix that diagonalizes N, since we know how to generate uncorrelated data.
- A particularly useful one is Cholesky (LU equivalent for positive-definite): $N=LL^T$. We can generate simulated data just by taking L*g*, where *g* is a vector of zero-mean, unit-variance Gaussian random deviates.
- Often a good idea to generate many simulations, average their outer products, and check empirical matrix agrees with expected.

Gaussian Correlated Noise

- Look at gauss_corrnoise.py
- $<f(x)f(x+dx)>=\exp(-x^2/2\sigma^2)$
- top: σ=1, middle σ=10, bottom σ=50







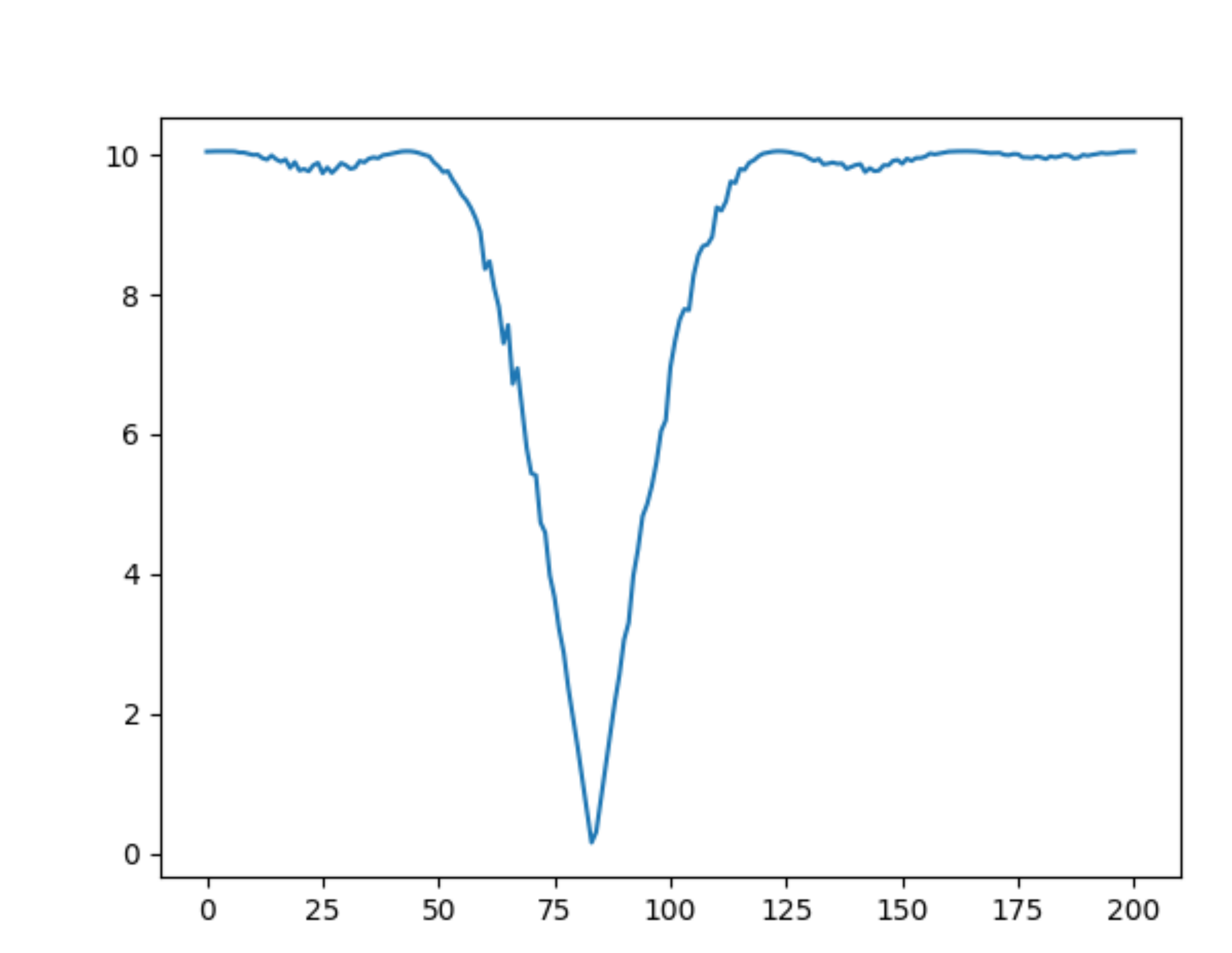
60 Hz Example

- In class we saw 60 Hz noise with a pure cos term. Can we do generic 60 Hz noise $\cos(120\pi t+\phi)$?
- It might look nonlinear, but we can turn into $\cos(\phi)\cos(120\pi t)+\sin(\phi)\sin(120\pi t)$, which is linear.
- How about errors? Let's say we fit sine waves, how should errors behave as function of frequency?

```
Ninv=np.linalg.inv(N)
nuvec=np.linspace(35,95,201)
myerrs=0.0*nuvec #this will be the correct error bar
myerrs2=0*myerrs #this will be the error bar ignoring our correlated noise
Ninv2=np.linalg.inv(np.diag(np.diag(N)))
for i in range(len(nuvec)):
    nu=nuvec[i]
    phi=2*np.pi*np.random.rand(1)
    myvec=np.cos(2*np.pi*nu*t+phi)
    lhs=myvec.T@Ninv@myvec
    myerrs[i]=np.sqrt(1/lhs)
    lhs2=myvec.T@Ninv2@myvec
    myerrs2[i]=np.sqrt(1/lhs2)

plt.clf()
plt.plot(myerrs2/myerrs) #plot the ratio of the bad to correct errors
plt.show()
plt.savefig('error_ratio.png')
```

Right: ratio of error bars assuming noise is white vs. accounting for 60 Hz. You really want to know if you have correlated noise!



Stationary Noise

- A common case for correlated noise is noise depends only on separation of points, $N_{ij}=f(|i-j|)$
- This is called *stationary* noise, since on average, noise properties don't change with time.
- We can invoke the power of Fourier transforms to enormously simplify stationary noise.
- Note - computer takes the discrete Fourier transform (DFT): $F(k)=\sum f(x)\exp(-2\pi ikx/N)$. I suggest you commit this to memory.
- IDFT: $f(x)=1/N \sum F(k)\exp(2\pi ikx/N)$
- Note: x,k integers, go from 0 to N-1.
- Also note: $\sum\exp(-2\pi ikx/N)=0$ for integer k,x, unless k=0, in which case it's N. This is the discrete equivalent of a Dirac-δ

Stationary Noise 2

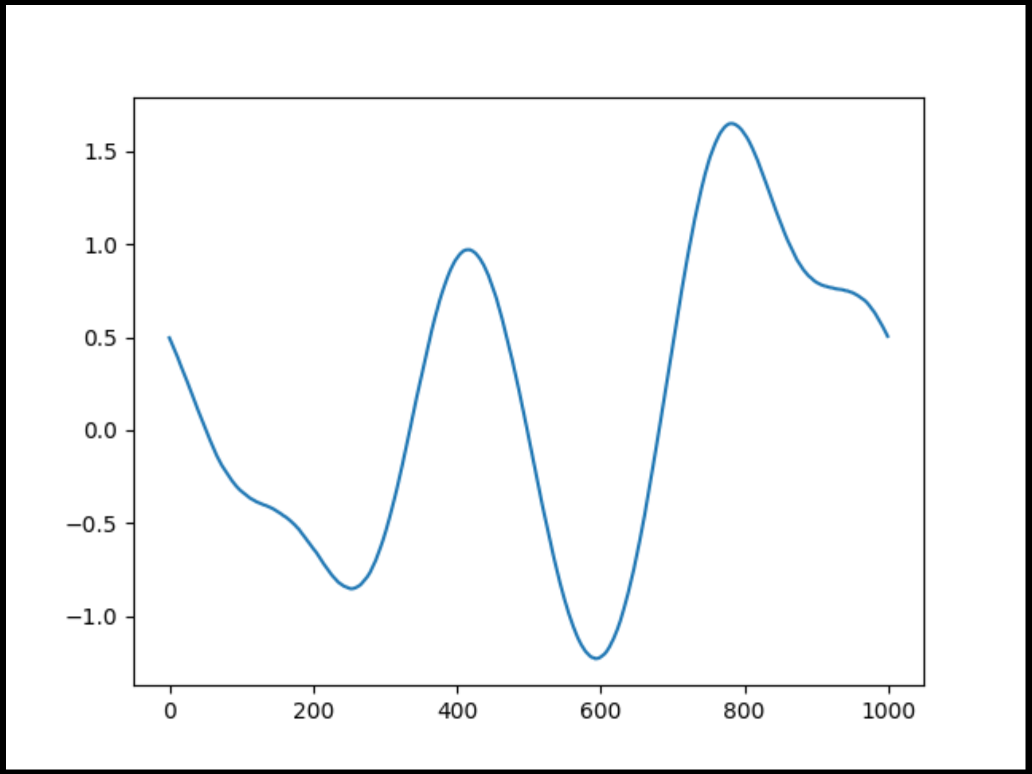
- Say we have noise where $\langle f(x)f(x+dx)\rangle = g(dx)$ (not of x)
- Fourier space: $\langle F(k)F^*(k')\rangle = \langle \sum f(x)\exp(-2\pi ikx/N)\sum f(x')\exp(2\pi ik'x'/N)$
- Can swap x' for x+dx, since sum is over all points, can sum over dx:
- $\langle F(k)F^*(k')\rangle = \langle \sum f(x)\exp(-2\pi ikx/N)\sum f(x+dx)\exp(2\pi ik'(x+dx)/N)\rangle$
- Reorder sum over x then dx: $\langle \sum \exp(2\pi ik'dx/N)\sum f(x)f(x+dx)\exp(2\pi ix(k'-k)/N\rangle$
- Now $\langle f(x)f(x+dx)\rangle = g(dx)$ (by assumption), can come out. $\sum \exp(2\pi ik'dx/N)g(dx)\sum \exp(2\pi ix(k'-k)/N)$
- Interior goes to N for k'=k, left with $N\sum g(dx)\exp(2\pi ikdx/N)\delta_{kk'}$, so Fourier transform of noise is diagonal.
- Further, variance of F(k) given by Fourier transform of correlation function g(dx).

Correlated Noise via DFT

```
import numpy as np
from matplotlib import pyplot as plt

n=1000
x=np.fft.fftfreq(n)*n
sig=100
mycorr=np.exp(-0.5*x**2/sig**2)
myps=np.fft.rfft(mycorr)

dat=np.random.randn(n)
datft=np.fft.rfft(dat)
dat_corr=np.fft.irfft(datft*np.sqrt(myps))
```



- Code is much shorter!
- And much (much) faster!

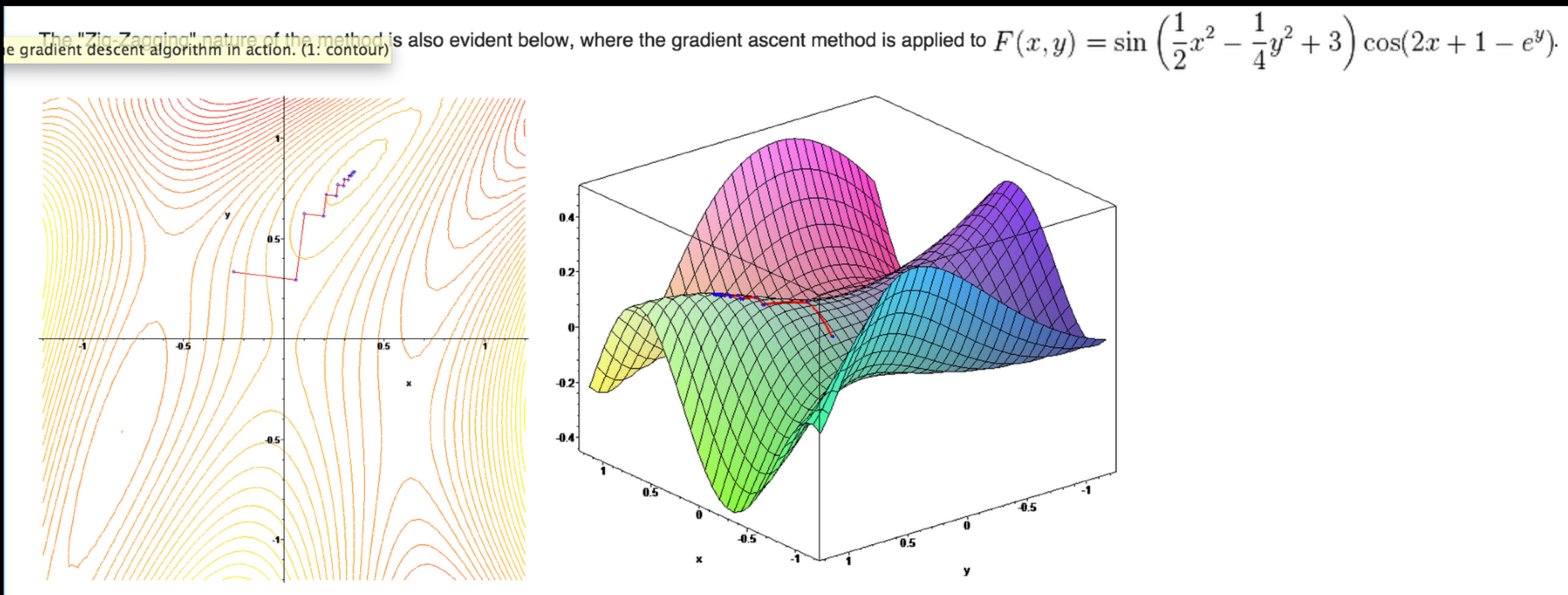
Correlated Noise Summary

- In real life, we frequently have noise that is correlated between data points.
- Our entire least-squares framework carries over, as long as $N_{ij}=\langle n_i n_j \rangle$
- We can generate correlated noise using e.g. Cholesky (or eigenvalues, but Cholesky faster)
- If the noise is stationary, the noise is diagonal in Fourier space.
- Can generate correlated stationary noise realization very fast with Fourier transforms, since $\langle F(k)^2 \rangle = \mathrm{DFT}(g(dx))$ where $g(dx)=\langle f(x)f(x+dx) \rangle$

Nonlinear Fitting

- Sometimes data depend non-linearly on model parameters
- Examples are Gaussian and Lorentzian ($a/(b+(x-c)^2)$)
- Often significantly more complicated - cannot reason about global behaviour from local properties. May be multiple local minima
- Many methods reduce to how to efficiently find the "nearest" minimum.
- One possibility - find steepest downhill direction, move to the bottom, repeat until we're happy. Called "steepest descent."
- How might this end badly?

Steepest Descent



From wikipedia.   Zigagging is inefficient.

Worked Example

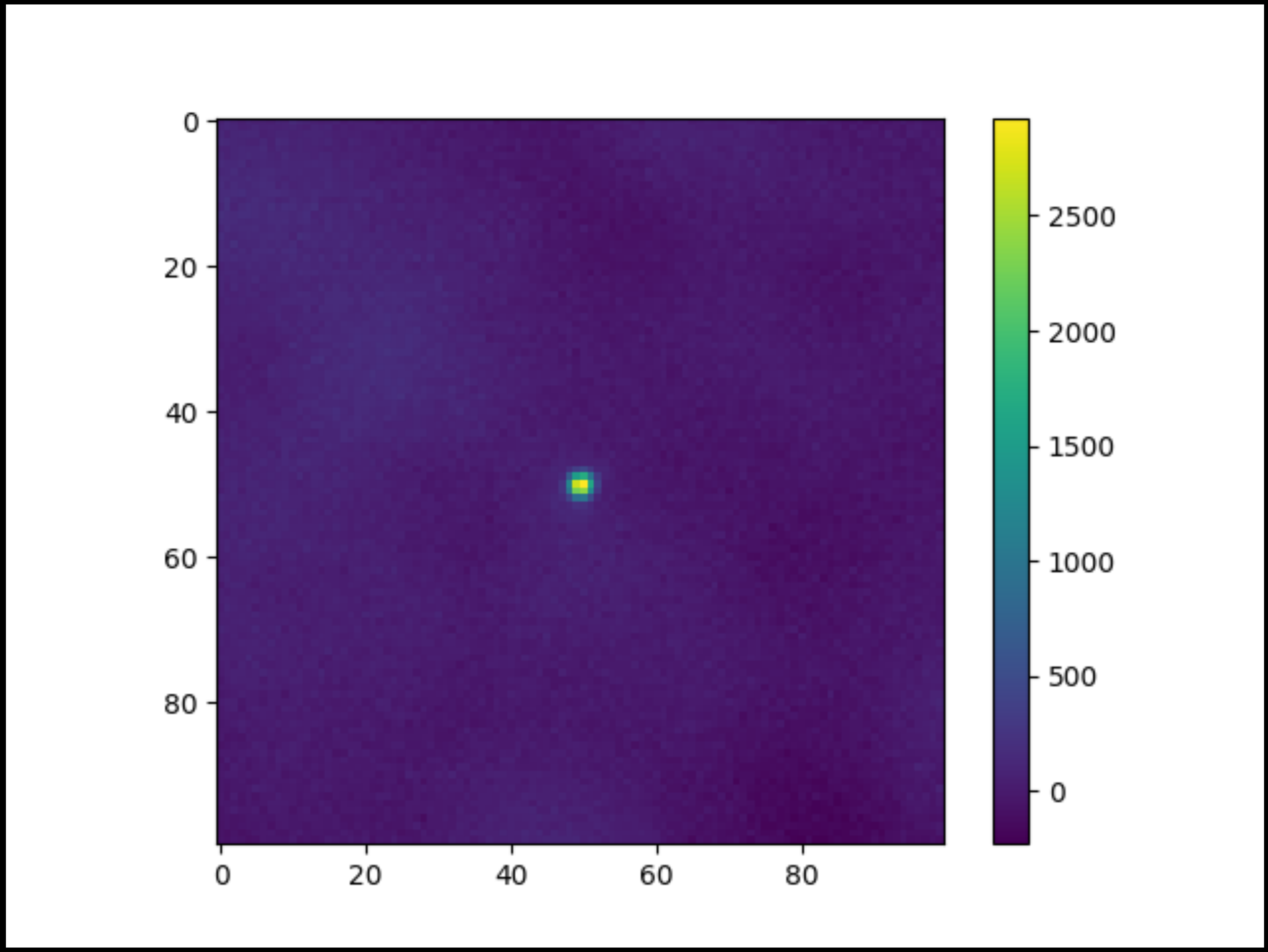
- What is the best-fit mean and error for a set of uncorrelated gaussian variables with same mean but individual errors?
- $A=$? Show that $A^TN^{-1}A=\sum(\sigma_i^{-2})$, $A^TN^{-1}d=\sum d_i/\sigma_i^2$.
- Define weights $w_i=\sigma_i^{-2}$. Then $m=\sum w_i d_i/\sum w_i$. Variance of our estimator is $1/\sum w_i$.

Worked Example 2

- Let's assume that N is constant and diagonal, and we have a single paramers.
- Show LHS = $\sum(m_i^2/\sigma^2)$
- RHS=$\sum(d_i m_i/\sigma^2)$
- Best-fit amplitude is RHS/LHS = $\sum(d_i m_i)/\sum m_i^2$
- Error=1/sqrt(RHS) = σ/sqrt($\sum m_i^2$). If there are ~n model points with value ~1, this turns into σ/sqrt(n), as roughly expected.

Example - Source in ACT Data

- Let's fit the amplitude of a source in ACT data.
- Look at find_act_flux.py.
- Let's try to guess a Gaussian, fit amplitude to it.
- Map is saved as FITS. Ability to read/write/manipulate FITS images extremely useful in astronomy!



Better: Newton's Method

- linear: $\langle d \rangle = Am$. Nonlinear: $\langle d \rangle = A(m)$ $\chi^2 = (d - A(m))^T N^{-1} (d - A(m))$
- If we're "close" to minimum, can linearize. $A(m) = A(m_0) + \partial A/\partial m * \delta m$
- Now have $\chi^2 = (d - A(m_0) - \partial A/\partial m\ \delta m)^T N^{-1} (d - A(m_0) - \partial A/\partial m\ \delta m)$
- What is the gradient?

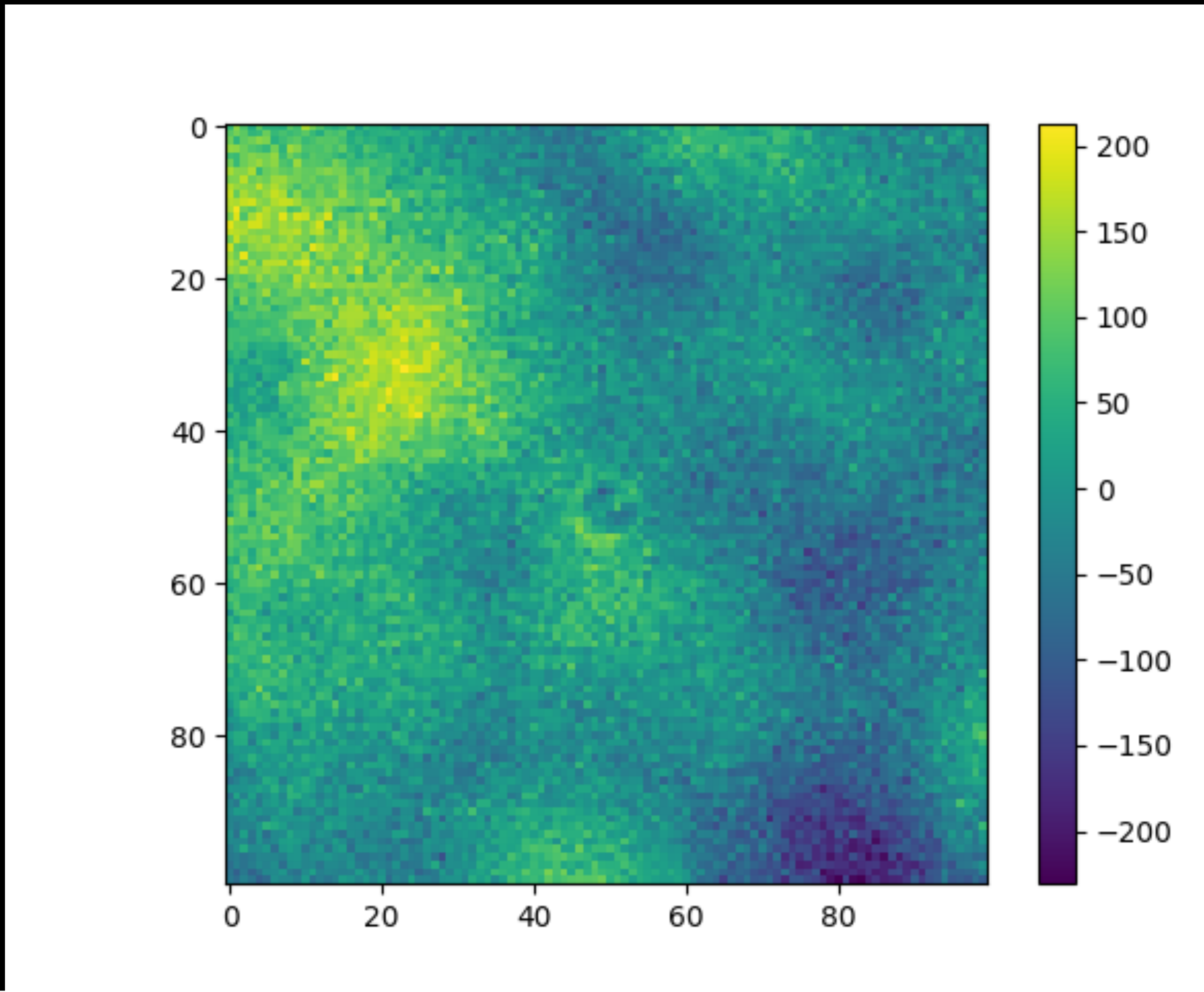
Newton's Method ctd

- Gradient trickier - $\partial A/\partial m$ depends in general on m, so there's a second derivative
- Two terms: $\nabla\chi^2=(-\partial A/\partial m)^T N^{-1}(d-A(m_0)-\partial A/\partial m\ \delta m)-(\partial^2 A/\partial m_i \partial m_j\ \delta m)^T N^{-1}(d-A(m_0)-\partial A/\partial m\ \delta m)$
- If we are near solution $d \approx A(m_0)$ and δm is small, so first term has one small quantity, second has two. Second term in general will be smaller, so usual thing is to drop it.
- Call $\partial A/\partial m$ A_m. Call $d-A(m_0)$ r. Then $\nabla\chi^2 \approx -A_m^T N^{-1}(r-A_m\delta m)$
- We know how to solve this! $A_m^T N^{-1} A_m\ \delta m = A_m^T N^{-1} r$

How to Implement

- Start with a guess for the parameters: m_0.
- Calculate model $A(m_0)$ and local gradient A_m. Gradient can be done analytically, but also often numerically.
- Solve linear system $A_m^T N^{-1} A_m \, \delta m = A_m^T N^{-1} r$
- Set $m_0 \rightarrow m_0 + \delta m$.
- Repeat until δm is "small". For χ^2, change should be << 1 (why?).

ACT Map Example



- Look at fit_act_flux_newton.py
- This implements numerical derivatives w/ Newton's method to fit a Gaussian (including sigma, dx, dy) to the ACT data.
- How should we estimate the noise, and hence the parameter uncertainties?
- Think about how we would do this accounting for the correlated noise?