PHYS 357 Midterm. You have 3 hours from when you start, due by 11:59 PM on Wednesday October 23

1. For matrices A and B, would you expect that $\exp(\mathrm{A})\exp(\mathrm{B}) = \exp(\mathrm{A}+\mathrm{B})$? Please answer for the following three cases.

   A: If A and B are general matrices.

   If $\exp(A)\exp(B) = \exp(A+B)$ then it also equals $\exp(B)\exp(A)$, but that won't in general be true unless A and B commute, since the first expression has terms like $A^j B^k$, while the last has terms like $B^j A^k$.

   B: If A and B are Hermitian.

   Still not true, being Hermitian doesn't help with commuting.

   C: if A and B commute.

   Now they are the same, because $A^j B^k = B^k A^J$. Equivalently, we have $V\exp(\Lambda_A)V^{-1}V\exp(\Lambda_B)V^{-1} = V\exp(\Lambda_A+\Lambda_B)V^{-1} = \exp(A+B)$.

   For each case, please justify your answer mathematically.

2. Some operator A has an eigenstate $|\Psi\rangle$ with eigenvalue λ, so $\mathrm{A}|\Psi\rangle = \lambda|\Psi\rangle$. Let's assume there is another operator B that anti-commutes with A: $\{\mathrm{A},\mathrm{B}\} \equiv \mathrm{AB}+\mathrm{BA} = 0$. Show that if $\mathrm{B}|\Psi\rangle$ is non-zero, it is also an eigenstate of A and and determine its eigenvalue.

   Take AB+BA and apply to $|\Psi\rangle$, which gives $AB|\Psi\rangle + BA|\Psi\rangle = 0$, $AB|\Psi\rangle + B\lambda|\Psi\rangle = 0$, pull λ in front of B and we have

   $$A(B|\Psi\rangle) = -\lambda B|\Psi\rangle$$

so $B|\Psi\rangle$ is an eigenstate of A, with eigenvalue $-\lambda$. The existence of an operator that anti-commutes guarantees every state has a companion state with negative eigenvalue.

3. Your professor promised to show you at some point in time why a rotation matrix has the form $\exp(-iJ\theta/\hbar)$ and is not allowed to have an overall phase. That time has now come. We will do this for a spin-1/2 particle, where we saw the rotation matrix about an arbitrary axis must have the form

$$R_n(\theta) = \exp(i\delta)\begin{bmatrix} 1 & 0 \\ 0 & \exp(i\theta) \end{bmatrix} \tag{1}$$

as long as we are working in the basis defined by that axis. *i.e.* if we are rotating about the $n-$axis, then the two states in R_n are $|+n\rangle$ and $|-n\rangle$.

A: Show that δ must be proportional to θ (reminder - a rotation by $n\theta$ is the same as n rotations by θ about the same axis).

Same argument we saw in class. If $R(m\theta) = R(\theta)^m$ and R is unitary, then looking at eigenvalues we have $\exp(i\delta(m\theta)) = \exp(im\delta(\theta))$, which means δ is a linear function of θ. Any constant term must be zero, because $R(\theta)^m$ picks up m copies of the constant, while $R(m\theta)$ only picks up one. They can't be equal unless they're both zero.

Note - one can also Taylor expand $\delta(\theta)$, plug in, and see that the linear term is allowed but the others are not.

B: Show that the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in the $|+z\rangle, |-z\rangle$-basis swaps $+z$ and $-z$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}$$

Since this swapped entries, then the state $(|+z\rangle, |-z\rangle)$ goes to $(|-z\rangle, |+z\rangle)$ - they are indeed swapped.

C: The key step in getting the correct overall phase is noting that a rotation by θ about the $|+z\rangle$-axis is the same as a rotation by $-\theta$ about the $-z$-axis$^?$. Show that fact, along with the results from parts a) and b), requires $\delta = -\theta/2$.

We have

$$\exp(i\delta)\begin{bmatrix}1 & 0\\ 0 & \exp(i\theta)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\exp(-i\delta)\begin{bmatrix}1 & 0\\ 0 & \exp(-i\theta)\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \exp(-i\delta)\begin{bmatrix}\exp(-i\theta) & 0\\ 0 & 1\end{bmatrix}$$

For those to be the same, the elements must agree. Upper left term requires $i\delta = -i(\delta+\theta)$, $2\delta = -\theta$, or $\delta = -\theta/2$. Bottom left requires $i(\delta+\theta) = -i\delta$, $2\delta = -\theta$, and once again we have $\delta = -\theta/2$. We can have an internally consistent rotation only if we set the overall phase this way, which is why the observational data agree with this phase.

4. Consider a two-state system with operator A that has eigenvalues λ_1 and λ_2. Consider a state (c_1, c_2) with amplitudes c_1 and c_2 to be in the eigenstates with eigenvalues λ_1 and λ_2. **A:** What is the expectation value of the operator A? Please express in terms of $p_1 = c_1^* c_1$, $p_2 = c_2^* c_2$,λ_1 and λ_2?

Pretty obviously, the answer is $p_1\lambda_1 + p_2\lambda_2$.

B: What is the uncertainty in a measurement of A, again in terms of the same variables.

$$< A^2 >= p_1\lambda_1^2 + p_2\lambda_2^2$$

$$< A^2 > - < A >^2 = p_1\lambda_1^2 + p_2\lambda_2^2 - (p_1\lambda_1 + p_2\lambda_2)^2$$

Multiplying out and grouping terms by λ's, we see:

$$\lambda_1^2(p_1 - p_1^2) + \lambda_2^2(p_2 - p_2^2) - 2\lambda_1\lambda_2 p_1 p_2$$

We know that $p_1 + p_2 = 1$ so $p_1 - p_1^2 = p_1(1 - p_1) = p_1 p_2$. Take advantage of that to get

$$\lambda_1^2 p_1 p_2 + \lambda_2^2 p_1 p_2 - 2\lambda_1 \lambda_2 p_1 p_2 = p_1 p_2 (\lambda_1 - \lambda_2)^2$$

C: Show the three conditions under which the uncertainty is zero, and show that is otherwise greater than zero. Hint: you might want to convince yourself that $p_1 - p_1^2 = p_2 - p_2^2$.

Once we've written the uncertainty in that compact form, for it to be zero requires either $p_1 = 0$, $p_2 = 0$, or $\lambda_1 - \lambda_2 = 0$, or $\lambda_1 = \lambda_2$. If either $p_1 = 0$ or $p_2 = 0$ then we're in a pure state of A. If $\lambda_1 = \lambda_2$ then every state is a pure state of A. In all cases, if we have a pure state, the uncertainty is zero. For the general case where we have a mixed state, $p_1 > 0$ and $p_2 > 0$, and $(\lambda_1 - \lambda_2)^2 > 0$, so the product of all three must be larger than zero.

5. **A:** Working in the $x-$basis, sketch out where the non-zero elements of the J_x raising and lowering operators are for some modest value of j (you may assume that, as usual, the m's are in decreasing order). I don't need to see the actual numbers, just where the non-zero entries are. If you feel the need to be concrete, you may write out the $j = 2$ case.

Since we move state m to $m + 1$, the entries immediately above the diagonal are the only non-zero entries for the raising operator. This only depends on writing the raising operator for the basis in which we're working. The raising operator for J_x in the $x-$basis looks just like the raising operator for J_z in the z-basis. Since the lowering is the adjoint of the raising, then the non-zero lowering entries are immediately below the diagonal.

B: Continuing to work in the $x-$basis, show where the non-zero entries of J_y and J_z are.

Since we get J_y and J_z from adding/subtracting the raising/lowering operators, the entries immediately above/below the diagonals are non-zero, the rest (including the diagonals) are zero.

C: Given the form of the operators from part B, explain why $< J_y >=< J_z >= 0$ for *any* pure state of J_x.

We know that

$$\langle j, m|_x \, A \, |j, m'\rangle_x$$

picks out the m, m' entry of A if A is expressed in the $x-$basis. For a pure state, we only have one non-zero m, and $m' = m$. In otherwords, a pure state of J_x picks out a diagonal entry of A. Since J_y, J_z have zeros on the diagonal, a pure state of J_x always gives zero expectation for J_y and J_z.

Bonus: Again, looking at the form of the angular momentum operators, what is the minimal condition on a general wave function expressed in the $x-$basis to have a non-zero mean value of J_y or J_z?

Since the only non-zero entries are just off the diagonal, that means that a state must have non-zero amplitudes for $|j, m\rangle$ and $|j, m+1\rangle$ for at least one value of m.